Physically Plausible Wrench Decomposition for Multieffector Object Manipulation

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Abstract—When manipulating an object with multiple effectors such as in multidigit grasping or multiagent collaboration, forces and torques (i.e., wrench) applied to the object at different contact points generally do not fully contribute to the resultant object wrench, but partly compensate each other. The current literature, however, lacks a physically plausible decomposition of the applied wrench into its manipulation and internal components. We formulate the wrench decomposition as a convex optimization problem, minimizing the Euclidean norms of manipulation forces and torques. Physical plausibility in the optimization solution is ensured by constraining the internal and manipulation wrench by the applied wrench. We analyze specific cases of three-fingered grasping and 2-D beam manipulation, and show the applicability of our method to general object manipulation with multiple effectors. The wrench decomposition method is then extended to quantification of measures that are important in evaluating physical human–human and human–robot interaction tasks. We validate our approach via comparison to the state of the art in simulation and via application to a human–human object transport study.

Index Terms—Cooperative manipulators, force decomposition, grasping, haptics and haptic interfaces, internal force, physical human–robot interaction.

I. INTRODUCTION

EITHER for moving an object or stabilizing it against external force such as gravity, supporting the object from several contact points is often an effective solution in object manipulation. When multiple effectors share the load of a rigid object, a certain object state needs to be attained not only by the force and torque (i.e., wrench) that specify desired manipulation, but also by the wrench compensating those from the other effectors. Decomposition of an applied wrench into manipulation wrench, which potentially causes motion, and internal wrench, which is compensated, is of interest in the present paper.

When multiple robotic effectors jointly control an object through rigid grasps, internal wrench is often undesired as it produces stress inside the object [3], [4]. However, a certain level of internal force may be desirable, for example when sufficient friction has to be generated to securely grasp an object on a slippery surface [5], [6]. Furthermore, internal wrench can serve as a source for haptic information exchange among decentralized systems such as in physical human–human interaction (pHHI) and human–robot interaction (pHRI) in which control disagreement [7]–[9] and action intention [10] need to be understood through the wrench perceived at the interaction. Thus, accurate wrench decomposition is imperative to analyses of multieffector object manipulation.

In the robotics case, the common approach is to use a pseudoinverse of the grasp matrix to compute the manipulation wrench the effectors need to apply to achieve a desired object state [3], [5], [11]. The grasp matrix relates applied wrench to the resultant wrench acting at the center of mass (CoM) of the object [12]. Internal forces, which lie in the null-space of the grasp matrix and consequently do not influence the object acceleration [13], are added to the manipulation forces according to a task requirement [14], [15]. Kumar and Waldron interpret the difference of forces projected onto the connection lines of the interaction points as internal force. They show that this internal force is zero if the Moore–Penrose pseudoinverse is used to compute applied forces for three fingered grasping [16]. Further extensions of the pseudoinverse wrench decomposition have been successfully used for wrench synthesis, e.g., the virtual linkage model [17] for humanoid robots in complex multicontact situations [18].

However, such pseudoinverse solutions do not differentiate applied wrench in terms of how it leads to motion or object stress. Yoshikawa and Nagai [19] were among the first to recognize that the internal force based on the pseudoinverse solutions does not show how tight an object is grasped. They instead used heuristics for a physically more plausible definition of internal forces in a precision grip, such that forces can only push but not pull.

1Note that in mechanics, wrench that exists inside an object and resists external wrench is termed internal wrench, e.g., [1], [2]. Here, we follow the common terminology of the manipulation community and use internal wrench to refer to the compensated external wrench component.
Groten et al. [20] build upon [19] and present force decomposition for the analysis of pHHI and pHRI tasks, though their application is limited to two effectors and one-dimensional (1-D) cases [8].

The lack of a generally applicable wrench decomposition method has led to task specific definitions, with a focus on obtaining, e.g., disagreement measures tailored to the task of interest rather than physically plausible results. In [21], the 1-D force decomposition solution of [20] was extended to the plane to evaluate a shared control strategy of a mobility assistance robot. Different force decompositions that allow to analyze human five fingered grasping were proposed in [22] and [23]. An alternative, but also task-specific approach without physical plausibility considerations, was recently presented in [24], where minimum-jerk trajectories were used as a human motion model to decompose applied forces during a simple dyadic object transport task.

An important step toward physically plausible wrench decomposition was recently taken by Schmidts et al. in [25], by introducing force decomposition constraints motivated by mechanical work. The wrench decomposition solution for two effectors proposed in [26] satisfies the proposed constraints of [25]. Erhart and Hirche recently suggested a different decomposition approach for cooperative object manipulation that also includes the application of torque in [27] and is based on kinematic constraint violation of desired accelerations as presented in [28]. One of the main findings of their works is the existence of infinite different pseudoinverses of the grasp matrix that specify desired load shares of the effectors, although their computation of internal wrench does not necessarily comply with the constraints of [25].

In order to overcome the case specificities and lack of physical plausibility in existing approaches, this study contributes the following.

1) An extension of the force constraints proposed by [25] to the application of torque.
2) A reformulation of the optimization proposed by [25] based on physical plausibility considerations yielding a convex optimization problem.
3) Derivation of analytic solutions for special cases.
4) Wrench measures for analysis of pHRI and pHHI tasks.

The result is a physically plausible wrench decomposition into manipulation and internal components for rigid object manipulation. Our wrench decomposition method extracts internal wrench, for the first time, in a form generalizable to realistic settings such as when quantifying haptic communications in pHHI tasks in Section IV and apply them to simulation examples in Section V and a pHHI experiment in Section VI. In Section VII, we discuss limitations and possible extensions of our work. Section VIII concludes the paper.

II. PROBLEM FORMULATION

In this paper, we address the problem of decomposing the wrench applied by n effectors to a rigid object into its motion and internal stress-inducing components in a physically plausible manner.

A. Background

We consider a rigid object as depicted in Fig. 1 with its object-fixed coordinate system \{o\} at the CoM. All vectors throughout this paper are given in this coordinate system, unless stated otherwise. Force \( f_1 \in \mathbb{R}^3 \) and torque \( t_i \in \mathbb{R}^3 \) at the \( i \)th effector position at \( r_i \in \mathbb{R}^3 \) are combined to the wrench vector \( h_i = [f_i^\top \ t_i^\top]^\top \). The grasp matrix \( G \in \mathbb{R}^{6 \times 6n} \) [12] relates the applied wrench \( h = [h_1^\top \cdots h_n^\top]^\top \in \mathbb{R}^{6n} \) to the resultant object wrench \( h_o = [f_o^\top \ t_o^\top]^\top \in \mathbb{R}^6 \) such that

\[
h_o = Gh
\]

with

\[
G = \begin{bmatrix} I_{3 \times 3} & 0_{3 \times 3} & \cdots & I_{3 \times 3} & 0_{3 \times 3} \\
S(r_1) & I_{3 \times 3} & \cdots & S(r_n) & I_{3 \times 3}
\end{bmatrix}
\]

where \( I_{3 \times 3}, 0_{3 \times 3} \in \mathbb{R}^{3 \times 3} \) are identity and zero matrices, and \( S(\cdot) \in \mathbb{R}^{3 \times 3} \) is the skew-symmetric matrix carrying out the cross-product operation: \( S(a) b = a \times b \in \mathbb{R}^3 \). In the following, we refer to the torque induced by the applied force \( f_i \) as

\[
t_{f,i} = S(r_i) f_i
\]

and to the resultant torque induced by each effector as

\[
t_{o,i} = t_{f,i} + t_i.
\]

B. SoA in Wrench Decomposition

Wrench decomposition refers to splitting the applied wrench \( h \) into manipulation wrench \( h_M = [h_{M,1}^\top \cdots h_{M,n}^\top]^\top \in \mathbb{R}^{6n} \) and internal wrench \( h_I = [h_{I,1}^\top \cdots h_{I,n}^\top]^\top \in \mathbb{R}^{6n} \)

\[
h = h_M + h_I.
\]

The internal wrench lies in the null-space of the grasp matrix, and consequently it does not produce any resultant wrench.
Fig. 2. 1-D examples to illustrate the problem of wrench decomposition based on pseudoinverses: (1) Effector 1 (left) takes over the complete load \( f_1 = f_{o1} \), (2) effector 1 and 2 equally share the load, (3) effector 2 applies an opposing force that is compensated. (a) Applied forces \( f_i \) and resultant object force \( f_{o1} \), (b) physically plausible wrench decomposition with manipulation forces \( f_{M,i} \) and internal forces \( f_{i1} \), (c) wrench decomposition based on pseudoinverses with fixed load share yields manipulation forces \( f_{G+M,1} = f_{G+M,2} = 0.5 f_{o1} \) and internal forces \( f_{G+i,1} = f_i - f_{G+M,i} \) with \( i = 1,2 \).

\[
0_{6\times 1} = G h_1. \quad \text{The manipulation wrench } h_M \text{ is responsible for the resultant object wrench } h_o.
\]

\[
h_o = G h_M = G h.
\]

The SoA in wrench decomposition is to use a pseudoinverse of the grasp matrix \( G^+ \) to compute the manipulation wrench, which yields the decomposition

\[
h_{G+,M} = G^+ G h \quad \text{and} \quad h_{G+,i} = (I_{6n\times 6n} - G^+ G) h. \tag{7}
\]

The Moore–Penrose pseudoinverse \( G^+ = G^\dagger \) yields the minimum norm solution for the manipulation wrench \( h_M \), as used in [14] and [17]. The Moore–Penrose pseudoinverse was contrasted with a different “nonsqueezing” pseudoinverse \( G^+ = G^\Delta \) by Walker et al. in [3], which computes manipulation wrenches that yield equal effector contributions to the resultant wrench \( h_o \).

Alternative approaches have been proposed to endow internal forces \( f_i \) with a physical meaning. The virtual linkage model by Williams and Khatib proposes to interpret internal forces as the forces that lock virtual prismatic actuators that connect the effectors [17]. Their extension to internal torques that lock virtual spherical joints is a simplification and, as stated in their work, does not lead to a physically plausible decomposition. In [31], on the other hand, internal forces are characterized as the forces that act inside a determinate truss that connects the effectors.

C. Force Decomposition in 1-D for \( n = 2 \)

As stated in [19], [25]–[27], the use of pseudoinverse methods as described above does not allow for a physically plausible wrench decomposition. We illustrate the issues by 1-D examples. Consider the beam in Fig. 2(1a) to which \( f_1 = 2 \) N is applied at the left-hand side but not at the right-hand side \( f_2 = 0 \). The resultant force that accelerates the object is \( f_o = 2 \) N. No force is compensated and \( f_1 \) fully contributes to the object acceleration. We thus conclude \( f_{M,1} = 2 \) N and \( f_{M,2} = f_{i1} = f_{i2} = 0 \) [see Fig. 2(1b)]. The solution for the manipulation force in (7), however, equally distributes the resultant wrench \( h_o = G h \) across the effectors through multiplication with the pseudoinverse \( G^+ \). For our simple example, (7) yields the same manipulation forces for the Moore–Penrose and the “nonsqueezing” pseudoinverse where \( f_{G+M,1} = f_{G+M,2} = 0.5 f_{o1} = 1 \) N. The difference to the actually applied wrench \( h \) is interpreted as the internal force where \( f_{G+i,1} = - f_{G+1.2} = 1 \) N [see Fig. 2(1c)]. Thus, the decomposition is physically implausible; although no force is applied at \( r_2 \), this decomposition method claims that a force of \( f_{G+i,2} = -1 \) N at \( r_2 \) is compensated. Fig. 2(2) and (3) show two additional examples of applied forces that lead to the same resultant force \( f_o = 2 \) N. From the examples in Fig. 2, we observe the following.

1) The pseudoinverse solutions decompose applied forces based on the assumption of fixed equal load shares and thus yield internal force \( f_{i1} \neq 0 \) whenever \( f_i \neq 0 \).

2) A physically plausible force decomposition should only yield nonzero internal force, when forces are applied into opposing directions, e.g., \( f_2 = - f_{i1} = -1 \) N [see Fig. 2(3)]. Different load shares [see Fig. 2(1) and (2)] that do not lead to force compensation should yield zero internal forces \( f_{i1} = 0 \).

Based on the above observations, we propose analogously to [20] to compute internal forces in 1-D for effectors \( i = 1, 2 \) by

\[
f_{i1} = \frac{1}{2} \text{sgn}(f_i)(|f_1| + |f_2| - |f_1 + f_2|). \tag{8}
\]

Note that, for wrench synthesis, the Moore–Penrose pseudoinverse \( G^\dagger \) yields desired wrenches \( h^d = G^\dagger h^d \) for given desired resultant wrenches \( h^d_i \), which result in zero internal wrenches \( h_1 = 0 \). The main drawback of \( G^\dagger \) is the fixed load shares among effectors, which do not allow for a physically plausible analysis of measured wrench \( h \). As shown in [27] for a simple example, the “nonsqueezing” pseudoinverse \( G^\Delta \) can yield desired wrenches \( h^d \) that are not free of internal wrenches \( h_1 \neq 0 \). Erhart and Hirche derived a parametrized pseudoinverse that represents infinite different load shares, which will yield zero internal wrench [27]. Based on the Gauss’ principle, they computed applied effector wrenches given desired effector accelerations and object and effector kinematics and dynamics. Motivated by the reasoning that internal wrench occurs whenever desired effector accelerations violate kinematic constraints, they proposed to compute internal wrench similarly to the effector wrenches in [27], but by exclusively considering the effector constraints [4]. However, the internal wrench computation in [4] yields results that differ from our proposed physically plausible wrench decomposition.\(^6\)

\(^6\)Consider the example displayed in Fig. 2(1a), wherein desired effector accelerations \( x_{1d}^2 = 1 \) m/s\(^2\) and \( x_{2d}^2 = 5 \) m/s\(^2\), effector masses \( m_{11} = m_{21} = 1 \) kg and object mass \( m_{o1} = 1 \) kg result in applied forces \( f_1 = 2 \) N and \( f_2 = 0 \). However, internal wrench computed according to [4] yields \( f_{i1} = -1 \) N. Thus, the internal force \( f_{i1} \) exceeds the applied force \( f_2 \). See [4], [27] for details.
D. Problem Statement for Physically Plausible Wrench Decomposition

Internal wrench is defined to lie in the null space of the grasp matrix. Thus, the virtual work by the internal wrench $h_I$ needs to be zero for any virtual displacement of the object [13] or of the effectors that satisfy the kinematic constraints [27]. We agree with above definitions but add further restrictions for physical plausibility through the following definition of internal wrench $h_I$.

Definition 1: A physically plausible internal wrench $h_I$ lies in the null space of the grasp matrix $G_{h_I}$ and the components $h_{I,i}$ of the effectors $i = 1, \ldots, n$ obey the constraints

$$\left\| f_{I,i} \right\| \leq -f_i^T a, \quad \left\| f_{I,i} \right\| \leq -f_i^{\top} a$$

(9)

$$\left\| t_{f1,i} \right\| \leq t_{f1,i}^{\top} t_{f1,i}$$

(10)

$$\left\| t_{I,i} \right\| \leq t_{I,i}^{\top} t_{I,i}$$

(11)

where $\| \cdot \|$ denotes the Euclidean norm.

As $f_{I,i}$ and its corresponding $f_{M,i}$ enclosing an angle $\geq 90^\circ$. Variation of the direction of $a$ changes the direction of possible compensation, as illustrated in Fig. 3(b).

All directions of $a$ have in common that for maximum compensation, i.e., maximum Euclidean internal force norm $\| f_{I,i} \|$, the internal force $f_{I,i}$, and its corresponding manipulation force $f_{M,i}$, enclose a $90^\circ$ angle. Consequently, all physically plausible force decompositions $f_I$ are bounded by the dashed circle inscribed in Fig. 3(c). In 3-D, the circular constraint extends to a sphere. As compensation can only occur in the opposite direction to $a$, we can replace $a$ with the negative normalized internal force $a = -f_{I,i} \parallel f_{I,i} \parallel^{-1}$ in (12) and obtain the constraint (9). The force inequality in (9) was first introduced by [25]. In Appendix A, we show that although the proposed circular constraint is required for a physically plausible wrench decomposition, it does not obey work constraints as stated in [25].

Fig. 3(d) and (e) shows 2-D examples for constraint (10) with respect to force induced torque. Force $f_I$ (left) results in a torque $t_{f1,i}$ at the CoM around the negative $z$-axis (right), which again does not fully contribute to the resultant object acceleration, but is fully (d) or partly (e) compensated by an opposing torque. The opposing torque is illustrated by an ideal torsional spring with axis $a$ such that the torsional spring can only generate opposing torque around its axis $a$. The Euclidean norm of the internal torque $t_{f1,i}$ is upper bounded by the projection of the applied force induced torque $t_{f1,i}$ onto $a$ in negative direction

$$\left\| t_{f1,i} \right\| \leq -t_{f1,i}^{\top} a.$$  

(13)

For the 2-D cases in Fig. 3(d) and (e), this results in an additional constraint: the band constrains the internal force $f_{I,i}$ such that it cannot induce a higher torque around the negative $z$-axis than the applied force $f_I$ can induce. In 3-D, the constraint forms a cylinder spanned by the vector $r_I$ and the applied force $f_I$ in force space. In torque space, the constraint for force induced torque is a circle in 2-D and a sphere in 3-D. As torque compensation can only occur around the opposite direction of $a$, we replace $a$ with the normalized internal torque $a = -t_{f1,i} \parallel t_{f1,i} \parallel^{-1}$ in (13), and obtain constraint (10). Analogously, constraint (11) for internal torque $t_{I,i}$ can be derived, which forms a circle in 2-D and a sphere in 3-D.

Complementary to Definition 1, we can also define physically plausible manipulation wrench.

Definition 2: A physically plausible manipulation wrench $h_M$ achieves the object wrench $h_o = G h_M$ and the components $h_{M,i}$ of the effectors $i = 1, \ldots, n$ obey the constraints

$$\left\| f_{M,i} \right\| \leq f_{I,i}^{\top} f_{M,i}$$

(14)

$$\left\| t_{fM,i} \right\| \leq t_{f1,i}^{\top} t_{fM,i}$$

(15)

$$\left\| t_{M,i} \right\| \leq t_{I,i}^{\top} t_{M,i}$$

(16)

where $\| \cdot \|$ denotes the Euclidean norm.

Proposition 1: The constraints (9)–(11) are equivalent to constraints (14)–(16).

Proof: See Appendix B.

Fig. 4 illustrates the implications of the manipulation-based physical plausibility definition in 2-D for applied force $f_i$.

Within the null space of the grasp matrix $G$, Definition 1 and equivalently Definition 2, further restrict the internal wrench solutions to obey $3n$ constraints for physical plausibility. Still, infinite wrench decomposition solutions exist. As we are interested in decomposing applied wrench into manipulation wrench $h_M$, which is necessary to produce the resultant object wrench $h_o$, and the part of the applied wrench, which was compensated $h_I$, we formulate our problem as follows.

Problem 1: Decompose a given applied wrench $h$ into manipulation wrench $h_M$ and internal wrench $h_I$ for a given grasp matrix $G$ with $h = h_o + h_I$, such that the manipulation wrenches $h_{M,i}$ applied by effectors $i = 1, \ldots, n$ represent a set of forces and torques of minimum Euclidean norm required to achieve a resultant object wrench $h_o = G h_M$, and such that the internal wrench $h_I$ and the manipulation wrench $h_M$ are physically plausible according to Definition 1 and Definition 2, respectively.

III. WRENCH DECOMPOSITION AS AN OPTIMIZATION PROBLEM

We propose that the solution to Problem 1 can be formulated as a convex scalarized multiobjective optimization that mini-
Fig. 3. Illustration of physically plausible internal force in 2-D: (a), (b) Linear springs of axis $a$ partly compensate applied force $f_i$ in two different directions. (c) Variation of compensation axis $a$ yields to the circular constraint for physically plausible internal force $f_I$. Two different example decompositions where the torque induced by $f_i$ is (d) completely compensated ($|t_{f,i,z}| = |t_{f_I,i,z}|$) and (e) partly compensated, but to the same extent ($|t_{f,i,z}| > |t_{f_I,i,z}|$). The restriction that internal torque cannot exceed the torque induced by $f_i$ yields a band parallel to $r_i$ as additional constraint in 2-D force space, which is equivalent to a circular constraint in 2-D torque space. 1-D torque arrows along $z$ are shown side by side for better visibility.

Fig. 4. Illustration of a physically plausible manipulation force in 2-D: Examples for physically implausible (a), (c) and plausible (b), (d) force decompositions. (a) Manipulation force $f_{M,i}$ violates the circular force constraint, i.e., the linear acceleration produced by $f_{M,i}$ is not attainable by the applied force $f_i$; the Euclidean norm of the manipulation force $f_{M,i}$ exceeds the projection of the applied force $f_i$ onto the manipulation force $f_{M,i}$.

(b) Manipulation force $f_{M,i}$ violates the band shaped force induced torque constraint, i.e., the rotational acceleration of the object $\{o\}$ produced by the manipulation force $f_{M,i}$ (force induced torque $t_{f,M,i}$) is not attainable by the applied force $f_i$ (force induced torque $t_{f,i}$): $|t_{f,M,i,z}| > |t_{f,i,z}|$.

By minimizing a manipulation wrench $h_M$ dependent cost function $J$ for a given applied wrench $h$:

\[
\text{minimize} \quad J = \sum_{i=1}^{n} (1-w) \| f_{M,i} \|^2 + sw \| t_{f,M,i} \|^2 + w \| t_M \|^2
\]

subject to

\[
Gh_M = Gh, \tag{18}
\]

\[
f_{M,i}^T f_{M,i} \leq f_i^T f_{M,i}, \tag{19}
\]

\[
t_{f,M,i}^T t_{f,M,i} \leq t_{f,i}^T t_{f,M,i}, \tag{20}
\]

\[
t_{M,i}^T t_{M,i} \leq t_i^T t_{M,i}, \tag{21}
\]

\[
i = 1, \ldots, n
\]

where $s = \{0, 1\}$ includes or excludes the manipulation torques induced through forces $t_{f,M,i}$ (3) in the cost $J$. The scalarized multiobjective cost function $J$ yields the Pareto-optimal points associated with a weighting $w \in [0, 1]$ between the objectives of Euclidean norm minimization of manipulation forces and torques [32]. As forces and torques are of different units, a plausible weighting $w$ must be selected. The choice of including ($s = 1$) or excluding ($s = 0$) the force induced torque $t_{f,M,i}$ in the cost function relates to this issue. We discuss the effects of weighting $w$ and selection parameter $s$ in the following sections in greater detail.

The inequality constraints (19)--(21) ensure a physically plausible decomposition as stated formally in the following theorem.

**Theorem 1:** A physically plausible wrench decomposition according to Definition 1 must obey the inequality constraints (19)--(21).

**Proof:** See proof of Proposition 1 in Appendix B with intermediate result (37). ■

The computation of a physically plausible force decomposition has been written as an optimization problem in [25], but as a nonconvex maximization of internal force $J = f_{I}^T f_{I}$ with $f_{I} = [f_{I,1}^T \ldots f_{I,n}^T]^T$ subject to the inequality constraint (19). Based on Definition 1, we complete the force constraints
by also considering force induced torque through inequality constraint (20). Inequality constraint (21) further extends the constraints to the application of torques. In summary, a total of $3n$ inequality constraints must be met for a physically plausible wrench decomposition according to Problem 1.

For some special cases, maximization of internal wrench $J = h_i^T h_i$ as proposed in [25] and minimization of manipulation wrench according to (17), both subject to constraints (18)–(21), yield the same solution. However, as we show by our examples in the following sections, maximization of $J = h_i^T h_i$ does not generally comply with Problem 1.

The complexity of the convex optimization problem defined in (17)–(21) rises with the number of effectors $n$. However, analytical solutions can be found for some special cases as presented in the following.

### A. Special Case: A Point Mass

**Proposition 2:** The optimization problem (17)–(21) has the following analytical solutions for a point mass

$$f_{M,i} = \theta_{f,i} \max \left( \frac{f_i^T f_o}{\| f_o \|^2}, 0 \right) f_o, \quad t_{M,i} = \theta_{t,i} \max \left( \frac{t_i^T t_o}{\| t_o \|^2}, 0 \right) t_o$$

with $\theta_{f,i} \in [0, 1]$ and $\theta_{t,i} \in [0, 1]$ such that $f_o = \sum f_{M,i}$ and $t_o = \sum t_{M,i}$, independent of $s, w$ in (17).

**Proof:** See Appendix C.

Fig. 5(a) and (b) illustrate the point mass solution for forces $f_i$ applied by three effectors $i = 1, 2, 3$. The same holds for torques. The weighting factor $\theta_{f,i}$ in (22) determines the extent to which projected forces $f_{M,i}$ point into the same direction as the resultant force $f_o$ belong to manipulation force. Infinite solutions for $\theta_{f,i}$ can lead to the same cost $J$, e.g., the resultant force $f_o = [4 \ 2 \ 0]^T$ can be formed through manipulation forces $f_{M,1} = [2 \ 1 \ 0]$ and $f_{M,2} = [2 \ 1 \ 0]$ or through $f_{M,1} = [3 \ 1.5 \ 0]$ and $f_{M,2} = [1 \ 0.5 \ 0]$ [displayed in Fig. 5(b)]. A parsimonious selection for $\theta_{f,i} = \theta(x = f_{f,i})$ from an analysis point of view is

$$\theta(x) = 1 - \frac{A_x - B_x}{A_x + B_x}, \quad A_x = \sum_{i=1}^n \| x_i \|, \quad B_x = \| \sum_{i=1}^n x_i \|$$

which yields $\| f_{M,i} \| \propto \| f_{f,i} \|$ for same direction of $f_{M,i}$ and $f_o$. Note that $\theta(x = f_{f,i})$ is equal for all effectors.

Fig. 5(c) displays the solution for a maximization of internal force $J = f_{f,i}^T f_i$ as proposed in [25] also subject to (18)–(21). The cost function $J = f_{f,i}^T f_i$ leads to solutions on the circular force components that are not parallel to the resultant force $f_o$, and consequently to manipulation forces of greater Euclidean norm than necessary. (d) Pseudoinverse-based decomposition with $G^+ = G^T G$ for point masses result in equal manipulation forces for all effectors that violate the force constraints. Only (a) and (b) represents a physically plausible wrench decomposition according to Problem 1.

### B. Special Case: Three-Fingered Grasping

Fig. 6 displays an example presented in [19] for a three-fingered grasp. Frictional point contact was assumed, such that each finger only applies force, but no torque. Fig. 6(a) shows that the force decomposition based on the heuristics given in [19] violates the force constraints for force induced torque (20). Fig. 6(b) and (c) shows optimization solutions according to (17)–(21). While for $s = 0$, the cost does not include torque and consequently the solution is independent of weighting $w$, for $s = 1$ weighting has an effect. Based on the results of Fig. 6(b) and (c), we recommend to set $s = 0$. Intuitively, it makes more sense to minimize the Euclidean norms of force that needs to be applied than accepting forces of higher Euclidean norms as long as these forces have a minimum effect on torque production.
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\[ \alpha = 0 \]

\[ n \]

\[ M = 0 \]

\[ \alpha = 0 \]

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\( \theta_{t,i} = \theta(x = t_{o.t.i}) \) in (23), where \( t_{o.t.i} \) is the projection of \( t_{o,i} \) in (4) onto the resultant torque \( t_o \). Note that \( \sum_{i=1}^{n} \alpha_{f,i} = \sum_{i=1}^{n} \alpha_{o,i} = 1 \).

Above load shares were introduced in [26] for \( n = 2 \). The force load share \( \alpha_{f,i} \), related to the weighting introduced in [33] for precise object positioning and to the assistance level in shared control for pHRI [34]. For the 1-D case and two effectors, Groten et al. [20] computed the force load share as \( \alpha_{f,i} = \frac{t_{i.o}}{t_o} \). For the general 3-D case, we cannot use the manipulation force \( f_{M,i} \) and torque \( t_{o,M,i} \) at the CoM to compute load share, but we need to relate applied forces to the CoM as in (25). This is due to the fact that manipulation wrench \( h_M \) still contains parts that can cancel on force or torque level (see for example Figs. 6 and 7).

**B. Energy Share**

In addition to above load share, the energy transfer among the effectors and the object can be of interest (see, e.g., [35] for a 1-D analysis). For a lossless system, the change in object energy is equal to the sum of the agents’ energy flows \( E_o = \sum_{i=1}^{n} E_i \). Effector \( i \) can cause a change in translational and rotational energy \( E_i = E_{\text{lin},i} + E_{\text{rot},i} = f_{i}^T \dot{p}_o + t_{i.o}^T \omega_o \). The energy flow transferred between the effectors, without influencing the object energy \( E_o \), can be calculated similarly to internal forces in the 1-D case (8) \( \dot{E}_i = \frac{1}{2} (\sum_{i=1}^{n} |E_i| - |\sum_{i=1}^{n} E_i|) \). Similar to the load share, we define the parameter energy share of effector \( i \) for the complete energy flow

\[
\beta_i = \theta_{E,i} \max \left( \frac{E_i}{E_o}, 0 \right)
\]

and for rotational and translational energy flows

\[
\beta_{\text{lin},i} = \theta_{E_{\text{lin},i}} \max \left( \frac{E_{\text{lin},i}}{E_{\text{lin},o}}, 0 \right), \quad \beta_{\text{rot},i} = \theta_{E_{\text{rot},i}} \max \left( \frac{E_{\text{rot},i}}{E_{\text{rot},o}}, 0 \right)
\]

with \( \theta_{E_{\text{lin/rot}},i} = \theta(x = E_{\text{lin/rot}}) \) in (23).

**C. Disagreement**

Internal wrench can indicate disagreement [7], [8], [36] and allow to communicate intention through the haptic channel [10]. However, previous works were limited to 1-D cases. In order to compare internal wrench within a trial or among different trials, the sum of Euclidean norms of internal force and torque can serve as a measure of disagreement in translational and rotational directions

\[
F_1 = \frac{1}{2} \sum_{i=1}^{n} \| f_{i.t.} \|, \quad T_1 = \frac{1}{2} \sum_{i=1}^{n} \| t_{o.t.} \|.
\]

As a combined measure for translation and rotation, we propose the measure relative cost \( \gamma \)

\[
\gamma = 1 - \frac{J(h_M)}{J(h)}.
\]

The cost function (17) is evaluated twice, once at its minimum \( J(h_M) \) and another at its maximum \( J(h) \). The relative cost returns values \( \gamma \in [0, 1] \), where \( \gamma = 1 \) signifies maximum disagreement, i.e., \( h_o = 0 \) and \( h = h_1 \), and \( \gamma = 0 \) signifies no disagreement in the sense that the complete applied wrench was needed to produce the resultant wrench \( h_o \), i.e., \( h = h_M \). The need for an interpretable measure \( \gamma \) strengthens our recommendation not to choose extreme values for \( w \) but rather \( w = 0.5 \) and \( s = 0 \).

**V. ANALYSIS OF SIMULATED MANIPULATION TASKS**

In real pHHI and pHRI tasks, the internal state of human agents (i.e., the control disagreement) cannot be precisely and systematically controlled, and the lack of ground truth impedes an interpretation of the results. Thus, we first use simulations to evaluate the proposed method, and assess the quality of the wrench decomposition solutions, before we apply them to a real pHHI task in Section VI. Based on the relevant use cases discussed in the introduction, we chose two different simulation scenarios: shared control of a mobility assistance robot [21] and an object transport task [24], [26]. For multidigit grasping examples, see [25]. The MATLAB/Simulink implementation of both simulations and their analyses can be found in the multimedia attachment. In the following, we use agents to refer to effectors to highlight their autonomy in contrast to centralized controllers for multieffect grasping.

We compare the proposed wrench decomposition to the following SoA approaches.

1) PM: Point mass approximation [26].
2) \( G^1 \): Moore–Penrose pseudoinverse, e.g., [5].
3) \( G^* \): “Nonsqueezing” pseudoinverse [3].
4) VL: Virtual linkage model [17].

Based on the applied wrench \( h(t) \) in simulation, we first computed the internal wrench \( h_1(t) \) based on the proposed and above SoA wrench decompositions. For the particular simulation scenarios, the proposed wrench decomposition was independent of optimization parameters \( s \in [0, 1] \) and \( w \in [0, 1] \). From \( h_1(t) \), the proposed measures for disagreement \( F_1(t), T_1 \) and \( \gamma \) in (28) and (29) were obtained. We furthermore computed the load shares \( \alpha_{f,1} \) and \( \alpha_{o,1} \) in (25) and the energy shares \( \beta_{\text{lin},1}, \beta_{\text{rot},1}, \) and \( \beta_i \) in (26) for agent 1. All computations were solely based on the observed \( h(t) \), i.e., we assumed not to have any knowledge on a desired trajectory, controllers or load sharing strategies.

**A. Shared Control of A Mobility Assistance Robot**

Let us consider a walker that can actively support an elderly human during walking. Inspired by [21], we examined two scenarios (see Fig. 9):

1) The walker (agent 2) generated torque to support the human (agent 1) during turning;
2) The walker generated opposing forces to avoid an obstacle.

\( ^4 \)Although the cost function differs and the force induced torque constraint is missing, we expect qualitatively similar results as in [25] for our proposed wrench decomposition.

\( ^5 \)We set the load and energy shares to NaN where otherwise meaningless, e.g., \( \alpha_{f,1} = \text{NaN} \) when \( f_o \approx 0 \).
1) Computation of Applied Wrench $h$: The agents determined the necessary object wrench $h_o$ to track the desired trajectories through a combination of equal inverse dynamics and impedance controllers. We computed the wrench to be applied at the human interaction point based on the reduced Moore–Penrose pseudoinverse $\tilde{h}_{G[1]} = C_{1}^\dagger h_o$, but then assigned all pure torque to the walker: $t_2 = t_{G[1]}, t_1 = 0$. For obstacle avoidance, the walker applied an additional force $f_{\text{obs},2x} = -\left(\frac{C_{\text{obs}}}{C_{\text{obs}} + C_{\text{max}}} - \frac{1}{C_{\text{max}}}\right)\dot{p}_0$ when approaching obstacles ($C_{\text{obs}} < 0$). Obstacle avoidance was active when the distance to the obstacle $C_{\text{obs}}$ (inflated by $0.5l$ of the walker length) was smaller than $C_{\text{max}} = 2\, m$.

2) Results Collaborative Turning: For the collaborative turning task, the agents agreed on the same trajectory $p$ [see Fig. 10(a)], while the human (agent 1) applied the necessary forward force $f_{1x}$ [see Fig. 10(b)] and the walker (agent 2) the torque $t_{1z}$ [see Fig. 10(c)]. Fig. 10(d) and (e) shows that only the point mass approximation and our proposed optimization yield the correct result of zero disagreement: $F_{1} = T_{1} = 0$. The pseudo-inverse based methods assume fixed equal load shares on force and torque level. In this case, however, agent 1 took over the complete load share on force level ($\alpha_{f,1} = \beta_{lin,1} = 1$) and agent 2 on torque level ($\alpha_{t,1} = \beta_{rot,1} = 0$ in Fig. 10(f)).

3) Results Obstacle Avoidance: During the obstacle avoidance scenario, the human (agent 1) intended to move from $p_{0z} = 0$ to $p_{obs} = 3\, m$ along the trajectory $p_{0,1z}(t)$ displayed in Fig. 11(a). The active obstacle avoidance through counter-acting forces $f_{2x}$ stops the walker in front of the obstacle: $p_{oz}(t) < p_{\text{obs}}(t)$. Fig. 11(c) and (d) shows the disagreement measures $F_{1}$ and $\gamma$. As for the turning scenario, the point mass approximation and our proposed optimization yield the same $F_{1}$. Note that the point mass approximation yields valid solutions for this setup, because the interaction point of the walker coincides with the CoM. The other decomposition methods inflate disagreement $F_{1}$ due to their underlying assumptions. The peak in disagreement $F_{1}$ and $\gamma$ and the switch from $\alpha_{f,1} = \beta_{lin,1} = 1$ to $0$ [see Fig. 11(e)] at $t = 2.1\, s$ occur when the applied forces of the agents reach equal values: for $t < 2.1\, s$, agent 1 dominates accelerating the walker, for $t > 2.1\, s$ agent 2 dominates decelerating the walker.

B. Collaborative Object Transport

In simulation, two agents transported a beam from a start to a goal configuration in 2-D as displayed in Figs. 12 and 13(a). Thus, a phase of pure rotation was followed by a phase of combined rotation and translation, and a phase of pure translation. We furthermore varied how the agents share the load and to which extent forces or torques were applied to induce the required object torque for rotation. Throughout the simulation, the agents agreed on the same trajectory and used the same controller parameters. Thus, we expect the analysis to reveal zero disagreement $F_{1} = T_{1} = 0$.

1) Computation of Applied Wrench $h$: The agents determined the necessary object wrench $h_o$ to track the desired trajectory through a combination of equal inverse dynamics and
impedance controllers. The applied wrench was computed from the necessary object wrench $h_o$ based on the parametrized pseudoinverse of [27] for two agents

$$h_{G+M} = G^+_M h_o = \begin{bmatrix} m_1^*(m_o^*)^{-1} I_{3\times 3} & m_1^*(J_o^*)^{-1}S(r_1)^\top \\ 0_{3\times 3} & J_1^*(J_o^*)^{-1} S(r_1)^\top \\ m_2^*(m_o^*)^{-1} I_{3\times 3} & m_2^*(J_o^*)^{-1}S(r_2)^\top \\ 0_{3\times 3} & J_2^*(J_o^*)^{-1} \end{bmatrix} h_o$$

(30)

with virtual masses $m_i^*$ and moment of inertias $J_i^*$ with $i = 1, 2$ as parameters, which have to obey

$$m_0^* = \sum_{i=1}^{n=2} m_i^*,$$

(31)

$$J_0^* = \sum_{i=1}^{n=2} J_i^* + \sum_{i=1}^{n=2} S(r_i) m_i^* S(r_i)^\top,$$

(32)

$$\sum_{i=1}^{n=2} r_i m_i^* = 0_{3\times 1}.$$

(33)

From the last equality (33) follows $m_1^* = m_2^*$ for a symmetric beam as in Figs. 2 and 7. We further set $J_1^* = J_2^* = I_{3\times 3}$ kg m^2 and vary $m_1^*$ between 1 and 4 kg as displayed in Fig. 13(b). Variation of the virtual masses $m_i^*$ regulates to which extent torque $t_o,i$ is induced by $t_i$ or $f_i$. For $m_1^* = 1$ kg, the parametrized pseudoinverse $G_M^+$ is equal to the Moore–Penrose pseudoinverse $G^+$, which yields the minimum norm solution for $h$. For increasing $m_1^*$, the required torque $t_o,i$ is induced to a higher extent through applied force $f_i$ than applied torque $t_i$.

Due to the restriction on $m_1^* = m_2^*$, the parametrized pseudoinverse $G_M^+$ cannot be used to design a desired load share but it yields balanced load sharing among the agents. As presented

Fig. 11. Analysis of the simulated assisted obstacle avoidance task: (a) Actual $p_{oi}$ and planned $p_{oi}^{\text{pl}}$ trajectory and inflated obstacle border $p_{oi,\text{infl}}$, (b) applied forces $f_{ix}$ by agents $i = 1, 2$ ($f_{iy} = t_{iz} = 0$), (c) disagreement on force level based on SoA wrench decompositions PM, $G^+$, $G^+_M$, and VL, and our proposed optimization (Opt), (d) disagreement $\gamma$, (e) load share $\alpha_{i,1}$ and energy share $\beta_1$. High forces required for deceleration in front of the obstacle are

Fig. 12. Beam motion for the simulated 2-D transport task: Beam of length $L = 2$ m, mass $m_o = 1$ kg, and moment of inertia $J_o = \frac{1}{12} m_o L_o^2 = 1$ kg m^2, subject to viscous friction on translation $f_d = -d_{\omega} \omega_o z$ with $d_{\omega} = 1$ N s/m and rotation $t_d = -d_{\omega} \omega_o$ with $d_{\omega} = 1$ N m s. Phase of pure rotation (black), followed by phase of combined rotation and translation (blue), followed by phase of pure translation (red). Agent positions at their initial and final positions in gray.

Fig. 13. Analysis of the simulated 2-D beam transport task: (a) Trajectory with $p_{oi,0} = 0$, (b) parameter $m_i^*$ of pseudoinverse [27] and 1-D load share $\alpha_{i,1}$ [8], (c) applied forces $f_{ix}, f_{iy}$ and (d) torques $t_{iz}$ in the plane by agents $i = 1, 2$, (e) disagreement on force and (f) on torque level based on SoA wrench decompositions PM, $G^+$, $G^+_M$, and VL, and our proposed 2-D beam wrench decomposition (B), (g) load shares $\alpha_{i,1}$ and $\alpha_{1,1}$ and energy share $\beta_1$. Only the proposed 2-D beam wrench decomposition (B) consistently yields the correct result $F_i = T_1 = 0$. 

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in [8], we varied the desired load share $\alpha_{d,i}^f$ along the redundant $x$-direction of the beam [see Fig. 13(b)]. This was done by further modifying the $x$-values $f_{G+M,x}$ of the computed wrench $h_{G+M}$ from (30) in the null space of the grasp matrix $\text{Ker}(G) = [1 \ 0_{1 \times 5} - 1 \ 0_{1 \times 5}]$ according to

$$h = h_{G+M} + (-f_{G+M,x} + 2\alpha_{d,i}^f f_{G+M,x}) \text{Ker}(G).$$

(34)

Thus, for $\alpha_{d,i}^f = 0.5$, we kept $h = h_{G+M}$ and consequently $f_{1x} = f_{2x}$. In contrast, e.g., for $\alpha_{d,i}^f = 1$, agent 1 would take over the complete load in $x$-direction.

2) Results: In simulation, the two agents applied the wrench $h(t)$ displayed in Fig. 13(c) and (d) to track the desired trajectory and achieve the desired load share displayed in Fig. 13(a) and (b), respectively. Fig. 13(e) and (f) show the results for the disagreement $F_1$ and $T_1$ in (28). Our proposed wrench decomposition yields the correct result of zero disagreement between the agents.6 The point mass approximation proposed in [26] neglects that forces also induce torque for the computation of $f_{M,i}$. As a consequence, opposing forces that were applied to induce torque are interpreted as internal force, which results in $F_1 \neq 0$ during rotation. Wrench decomposition according to the Moore–Penrose pseudoinverse $G^+$ only results in zero disagreement when the agents use $G^+$ to compute $h_1$ and $h_2$. This is the case for $m^*_i = 1$ kg during rotation and $\alpha_{d,i}^f = 0.5$ during translation. Similar to the Moore–Penrose pseudoinverse-based wrench decomposition, the nonsqueezing pseudoinverse-based wrench decomposition of [3] only yields zero internal force and torque, when $h = G^+_2 h_2$ holds. For the simulation under consideration, this was only the case during the last second, i.e., pure translation and equal load sharing $\alpha_{d,i}^f = 0.5$. Wrench decomposition according to the virtual linkage model of [17] assumes that rotation around the $z$-axis should be caused by forces instead of torques and interprets any applied torque along $z$ as internal torque. Furthermore, according to the virtual linkage model, internal force only occurs along the $x$-direction of the beam. Thus, $F_1 = 0$ during pure rotation. However, the virtual linkage model essentially computes the axial force in the center of the beam and assigns its absolute value to $F_1$, which results in $F_1 \neq 0$ for load distributions $\alpha_{d,i}^f \neq 0$.

Fig. 13(g) shows the load and energy shares of (25) and (26) for agent 1. The load share $\alpha_{d,i}^f$ distributes the demanded object force along the redundant $x$-direction and is therefore restricted to 1-D. Consequently, $\alpha_{f,1} = \alpha_{d,i}^f$ only during pure translation. The energy share $\beta_1$ combines the force and torque load shares in one measure.

VI. ANALYSIS OF A HUMAN–HUMAN OBJECT TRANSPORT TASK

In this section, we contrast the internal wrench estimated by different decomposition methods during a real pHHI task in order to illustrate how key behavioral measures for pHHI and PHRI are sensitive to a decomposition method. The results will demonstrate that our proposed method is more resilient to the inflation of the disagreement index than the others as postulated in the simulation work. Furthermore, we calculate the load share index to characterize the underlying coordination dynamics of the working pair. The coordination dynamics of the working pair was partially controlled by means of a task instruction to the participants. Causes for a nonzero disagreement measure during pHHI range from walking motion of the participants, over decision making, and to differing intended trajectories.

A. Methods

In this study, 12 pairs of two male participants carried a steel beam (mass $m = 7.7$ kg) from a start to a final platform located between obstacles (see Fig. 14). The study was designed to examine how humans haptically reach to consensus about how to reach the target configuration. Thus, the participants were prohibited from making conversations or intentional communication using their body such as hand gesture.

The experiment was a within-subject design with three levels. The independent variable was the guiding method. In the one-guide condition, one of the two partners was assigned the leader role and was always given an instruction about how the beam had to be oriented on the final platform. In the two-guide condition, both participants were told about the orientation of the beam at the final platform. In the free-guide condition, no instructions were given.

The side at which the participants stood on the platform was counterbalanced and quasi-randomly assigned in each trial. The experimental conditions were block-randomized and the participants performed 10 trials per condition, which resulted in a total of 30 trials per pair. We recorded the applied wrench $h$ using two JR3 force/torque sensors (JR3, Inc., Woodland, CA, USA) mounted between the agents’ handles and the beam. An Oqus motion capture system (Qualisys, Göteborg, Sweden) recorded position and orientation of the beam.

B. Results

1) Disagreement: The beam was kept horizontal during the transportation task. This allowed an application of the efficient 2-D beam wrench decomposition implementation introduced in Section III-C, which yielded results close to the optimization-based solution with $s = 0$ and $w = 0.5$. We observed an inflation of disagreement/compensation on force and torque level for the SoA decomposition approaches, which is in line with our simulation results and the observations for multidigit

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6The decompositions proposed in [20] and [27] yield zero disagreement as well, but are restricted to 1-D or require knowledge of desired velocities with the associated problems outlined in the problem formulation, respectively.
A. Uniqueness of the Wrench Decomposition Solution

Wrench decomposition aims at splitting applied wrench into force and torque load share, for which we present a comparison via a repeated-measures ANOVA in the following. The first factor was the method of decomposition and the second factor was the guiding instruction. The analysis showed the main effect of decomposition/compensation methods (PM, G, G, and VL) inflated the disagreement measure. Differing guide to disagreement relations (T1(Free-Guide) > T1(Two-Guide) > T1(One-Guide) for PM and G, T1(One-Guide) > T1(Two-Guide) > T1(Free-Guide) for G and T1(One-Guide) > T1(Free-Guide) > T1(Two-Guide) for VL and the proposed decomposition) confirms the need for a physically plausible wrench decomposition for interpretable results. The error bar indicates one standard error.

2) Load Share: Joint density estimation of force and torque load share revealed the coordination strategies of the interacting partners as their share indexes sum to 1, e.g., \( \alpha_f_1 + \alpha_f_2 = 1 \). In this way, we calculated the portion of which agent 1 (the leading partner during One-Guide) is classified into one of the quadrants.

In order to evaluate how the coordination patterns of the load share index was affected by our experimental manipulation, we ran one-way repeated-measures ANOVA on percentage of time for which the joint density of the force and torque load share fell in the dominant quadrant [see Fig. 16(c)]. The analysis indicates that there is a main effect of guide on the coordination, \( F(2, 22) = 9.219, p < .005 \). The analysis suggests that the participants formed a dominant-passive coordination strategy more often for one-guide (31.39% ± 9.99) than two-guide (25.20% ± 9.49) or free-guide (26.45% ± 8.23).

VII. DISCUSSION OF LIMITATIONS

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VII. DISCUSSION OF LIMITATIONS

A. Uniqueness of the Wrench Decomposition Solution

Wrench decomposition aims at splitting applied wrench into force and torque load share, for which we present a comparison via a repeated-measures ANOVA in the following. The first factor was the method of decomposition and the second factor was the guiding instruction. The analysis showed the main effect of decomposition/compensation methods (PM, G, G, and VL) inflated the disagreement measure. Differing guide to disagreement relations (T1(Free-Guide) > T1(Two-Guide) > T1(One-Guide) for PM and G, T1(One-Guide) > T1(Two-Guide) > T1(Free-Guide) for G and T1(One-Guide) > T1(Free-Guide) > T1(Two-Guide) for VL and the proposed decomposition) confirms the need for a physically plausible wrench decomposition for interpretable results. The error bar indicates one standard error.
ms, and can thus be directly used for realtime haptic interaction control. In contrast, computation of $B_{0.5}$ for the beam transport experiment in Section VI required an average time of 0.4 s using CVX with MATLAB R2015a and solver SeDuMi v1.34 [39] on a desktop pc. For the general 3-D case, the computational cost increased as follows with the number of effectors: \( t(n = 3) = 0.8 \) s, \( t(n = 4) = 1.1 \) s, \( t(n = 10) = 2.1 \) s. Note that CVX is a modeling framework that allows for convenient solving of convex optimization problems written in natural MATLAB syntax, taking over the effort, among others, of transformation into solvable form and the choice of an appropriate solver. Significant speed-up can be achieved by using more efficient commercial solvers [40], [41] and by splitting the solver up into an initialization routine that is performed once and a real-time routine that efficiently solves instances of the same problem [42]. Also, for many interaction scenarios, wrench decomposition can be approximated by analytic solutions. Here, we projected the human–human transport task in Section VI into the 2-D plane and applied the analytic pTiTC solution. Also, for the mobility assistance scenario in Section V-A, the point mass approximation as an analytic solution was found.

### C. Wrench Analysis and Wrench Synthesis

In this work, we focussed on deriving a physically plausible wrench decomposition for the analysis of general manipulation tasks. The proposed wrench decomposition can now be readily applied to pHRI tasks, e.g., to compare different wrench synthesis approaches to control the agents’ applied wrench in simulation [8], [21], [27], we refrained from analyzing a real-world pHRI task: the added complexity of an uncontrollable human agent and the need for wrench synthesis would impair our goal of fully understanding and evaluating the capabilities of our physically plausible wrench decomposition. Instead, we examined the proposed wrench decomposition on three levels.

1) “Snap shots”: They visually illustrate the method, covering the range of simple 1-D to the general 3-D cases.
2) Simulations: They allow for controlled disagreement and thus interpretable results.
3) The pHRI study: It exemplifies the application of the derived measures to real world interaction tasks.

For wrench synthesis, common pseudoinverse approaches can be straightforwardly applied, if equal load share and a fixed force induced torque to applied torque relation are acceptable. The parametrized pseudoinverse proposed in [27] only partly alleviates above restrictions, i.e., for the beam transport task only the induced torque to applied torque relation was adjustable, while the load share between the agents remained fixed. The null space approach of [8] allows to choose a desired load share along a redundant direction. Nonetheless, their approach is currently limited to 1-D, ignoring rotation, with the result of not directly relating to our proposed general load share measures. The derivation of a general wrench synthesis method that achieves a desired load share or controls internal wrench for haptic communication is an interesting and challenging topic that we would like to explore in our future work. Such wrench synthesis applied to robot control will allow more accurate tuning of the robot to the user behavior and intention in pHRI.

### VIII. Conclusion

The proposed wrench decomposition allows for the first time to separate applied wrench into internal and manipulation wrench for general rigid objects manipulated by multiple effectors, while ensuring physically plausible results. We define manipulation wrench as the wrench with minimum Euclidean norm to produce the resultant object wrench. Physical plausibility is achieved by constraining the internal and manipulation wrenches by the applied wrench. The proposed optimization is convex and has an intuitive analytic solution for a point mass. The solution for a 2-D beam requires optimization only for one special case, which can be approximated through an analytic solution. The efficient 2-D beam implementation can potentially be used for real-time control and analysis for various 2-agent object manipulation tasks. Applications in example measures such as load and energy share are defined based on the analytic point mass solution. The extent to which the applied wrench is not used for manipulation, but, e.g., for communication or to express disagreement, can be characterized by the wrench decomposition-based relative cost and Euclidean internal force and torque norms. Simulated mobility assistance and object transport scenarios showed that our method was able to correctly evaluate the control disagreement based on the measured wrench unlike other existing methods. Finally, we illustrated the potential of the derived wrench measures to study aspects as decision making, dominance, and specialization during haptic interaction via an exemplary application to a human–human object transport experiment. How to extend the presented wrench decomposition to wrench synthesis that realizes desired load and energy shares or internal wrench for communication remains an open question, which we are interested in examining in future work.

### APPENDIX

#### A Work versus Force Constraints

Schmidts et al. derived the force constraint (9) based on the requirement that a manipulation force \( f_{M,i} \) cannot do more mechanical work than the projection of the corresponding applied force \( f_i \) onto the manipulation force (see [25, Lemma 1]). In the following, we show that for work computations, the applied force \( f_i \) instead of its projection onto \( f_{M,i} \) needs to be considered. Work constraints that ensure that a manipulation force cannot do more work than its corresponding applied force can be formulated as

\[
0 \leq f_{M,i}^\top nds \leq f_i^\top nds, \tag{35}
\]

\[
0 \leq f_{fM,i}^\top qd\phi \leq f_i^\top qd\phi \tag{36}
\]
for an infinitesimal translational displacement $dn = nds \in \mathbb{R}^3$ with $\|n\| = 1$ and an infinitesimal rotational displacement $dq = g d\phi \in SE(3)$ with $\|q\| = 1$. However, above work constraints are not equivalent to the circular force constraint (9), as illustrated for an example decomposition in Fig. 17. In order to ensure that the work of $f_{\text{M},i}$ is bounded by the work of $f$, the current direction of translational velocity $n = \hat{\rho}_i/\|\hat{\rho}_i\|$ and rotational velocity $\omega = \omega_o/\|\omega_o\|$ of $\{o\}$ have to be taken into account. In this work, we refrain from requiring a manipulation wrench to obey work constraints (35) and (36). The resultant object wrench $h_o$ could also needed to withstand an external force such as gravity, which might come along with zero velocity. Our aim is to use wrench decomposition to analyze the extent to which the wrench applied at the individual effectors $h_i$ effects the resultant object wrench $h_o$, and how much of it is compensated, independent of the current object velocity. An important result of above considerations is that $h_i$ and not $h_{\text{M}}$, needs to be used to compute energy measures as illustrated in the case of energy share in Section IV.

### B Proof of Proposition 1

Proof: Multiplication of the inequalities (9)–(11) with the respective Euclidean norms $\|x_i\|$ with $x_i = \{f_{ti}, t_{fi}, t_i\}$ on both sides and insertion of $\|x_i\|^2 = x_i^T x_i$ and $x_1 = x - x_M$ of (5) yields

$$x_M^T x_M \leq \sqrt{\lambda} x_M$$

(37)

with pairs $(x, x_M) = \{(f_{\text{M},i}, t_{fi,\text{M}}, t_i, t_{M,i})\}$. Insertion of $x_M^T x_M = \|x_M\|^2$ and rearrangements yield the constraints (14)–(16).

### C Proof of Proposition 2

Proof: For a point mass $(t_{fi,\text{M}} = 0_{3 \times 1})$, the optimization problem (17)–(21) can be solved separately for forces and torques, with analogous results. The Lagrangian for the minimization of manipulation force is

$$L = \sum_{i=1}^n \|f_{\text{M},i}\| + \lambda^T (f_o - \sum_{i=1}^n f_{\text{M},i})$$

$$+ \sum_{i=1}^n \mu_i (f_{\text{M},i} f_{\text{M},i} - f_i^T f_{\text{M},i})$$

(38)

with three Lagrange multipliers concatenated in $\lambda \in \mathbb{R}^3$ and $n$ Kuhn–Tucker multipliers $\mu = [\mu_1, \ldots, \mu_n] \in \mathbb{R}^n$. For $\mu_i = 0$

$$\nabla_{f_{\text{M},i}} L = \frac{f_{\text{M},i}}{\|f_{\text{M},i}\|} - \lambda.$$  

(39)

From (39) and because $f_o = \sum_{i=1}^n f_{\text{M},i}$ we see that every nonzero manipulation force has to point into the same direction as the resultant force $f_o$

$$\frac{f_{\text{M},i}}{\|f_{\text{M},i}\|} = \frac{f_o}{\|f_o\|}.$$  

(40)

A unique solution exists for the special case $f_{\text{M},i} f_{\text{M},i} = f_i^T f_{\text{M},i}$ for all $i = 1, \ldots, n$. In this case, the manipulation forces $f_{\text{M},i}$ are equal to the projections of the applied forces $f_i$ onto the resultant force $f_o$: $f_{\text{M},i} = (f_o^T f_o)^{-1} f_o^T f_{\text{M},i}$.

Note that this solution only exists if all $f_i$ projections onto $f_o$ point along $f_o$. From (40), it follows that $f_{\text{M},i} = 0_{3 \times 1}$ if $\text{sgn}(f_o^T f_o) < 0$. This is equivalent to force compensation along $f_o$, with the consequence that a unique solution might not exist for $n > 2$. The family of solutions with equal minimum cost $J$ can be described via (22). The solutions (22) are the global minimum due to the convexity of the optimization problem.

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**References**


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