Electrical Three Phase Circuit Analysis Using Quaternions

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Abstract—Nowadays, reactive power and power factor in unbalanced circuits and/or with the presence of harmonics are still being investigated by several researchers. In this context, quaternions, whose use has recently been intensified in the electrical engineering field, is presented as an alternative tool to analyze electrical quantities in single and three-phase circuits. However, studies solving three-phase circuits by quaternions are not yet in literature, nor a complete electrical description with this tool. This article, therefore, presents single and balanced three-phase quantities expressed as quaternions. Results of an analysis of a series RLC single and balanced three-phase circuits are presented. It is important to notice that this tool can also be employed to represent three-phase power under unbalanced situations and with the presence of harmonics.

Index Terms—Instantaneous Active and Reactive Power, Quaternion Electrical Quantities, Three-Phase Circuits.

I. INTRODUCTION

The study of electrical quantities such as voltage, current, impedance and power has classically begun with time domain analysis, in which solving circuits involves differential equations [1]. Therefore, in order to simplify the analysis, phasors were introduced as an electrical circuit analysis tool. However, power factor and reactive power representations for three-phase unbalanced systems and/or with the presence of harmonics are not yet well established, although there is a rising need for these definitions in order to develop better power conditioning [2], [3], [4], [5], [6].

On the other hand, there is a mathematical tool named quaternions, whose use has recently been intensified in the electrical engineering field. Its use in this context was first made by James Clerk Maxwell in his treatise on electricity and magnetism [7]. However, after vectorial calculus advent by J. Willard Gibbs [8], Maxwell’s theory was rewritten in the usual vector form [9]. The use of quaternion in this field was abandoned, but nowadays it has been rescued by several power systems researchers [6], [10], [11], [12], [13].

By means of quaternions it is possible to completely define three-phase reactive power under unbalanced conditions and/or in presence of harmonics. Hence an efficient active power filter can be designed, as presented in [6]. In [11], a quaternion frequency estimator algorithm is presented and the results are then compared to those obtained by traditional algorithms based on the Clarke orthogonal decomposition. In [6] and [11], the unified framework provided by quaternion structure shows itself appropriate for three-phase unbalanced circuits under sinusoidal and non-sinusoidal conditions, since both zero and negative sequence are taken into account.

It is also possible to identify articles that analyze single-phase voltage, current, impedance and power expressed as quaternions [10], [12]. For three-phase circuits, a few articles [6], [11] use quaternions to represent voltages, currents and power, however quantities are not completely described, nor do the articles present the relationships between electrical quaternion definitions and their properties with an analogy to the traditional mathematical tool: phasors.

In order to fill this gap, this article presents definitions and an analysis of quaternions single and three-phase quantities. For this purpose, results of both series RLC, single and balanced three-phase, circuits are presented. Using this tool, steady state representation (the only one phasors can represent) is preserved as well as the instantaneous, that carries out unbalance and/or harmonics information [6].

It is noteworthy that this tool may also be employed in the representation and circuit solving of both balanced and unbalanced three-phase power, as well as in the presence of harmonics, but for the sake of simplicity and aiming to fill literature gap, this article will only solve balanced circuits.

Since there is a lack of systematic studies of electrical quaternion definitions and properties, this study aims at contributing to the development of methods that will make use of quaternions to solve problems related to power quality.

Initially, quaternions and their properties and characteristics are presented. Next, electrical single and three-phase quantities such as voltage, current, impedance and power are defined as quaternions. For three-phase circuits a direct analogy of the use of phasors in a single-phase circuit is obtained. Concepts are developed with a single and three-phase balanced series RLC circuits. Finally, conclusions drawn from this study are presented.

II. QUATERNION DEFINITION AND PROPERTIES

Quaternions are a set of 4 real numbers belonging to a hyper complex space, denoted by ℍ, and can be represented as

\[ \mathbf{A} = w + x\mathbf{q}_1 + y\mathbf{q}_2 + z\mathbf{q}_3, \]

where \( w, x, y \) and \( z \) are real numbers and \( \mathbf{q}_1, \mathbf{q}_2 \) and \( \mathbf{q}_3 \) the orthonormal basis of this space.
It is useful to break the quaternion into two distinct elements, its real part (\(A\), denoted by tilde) and its vectorial part (\(\tilde{A}\)), i.e.,

\[
\tilde{A} = w
\]

\[
\hat{A} = xq_1 + yq_2 + zq_3.
\]  

(2) (3)

Despite different concepts and operations, quaternions are frequently representing vectors. In this article, usual vector notation (with an arrow above the element) will represent the operation of extracting the vectorial part out of a quaternion.

A. Product

The main equation for the product, as defined by Hamilton, is

\[
q_1^2 = q_2^2 = q_3^2 = q_1q_2q_3 = -1.
\]  

(4)

Therefore, product between two arbitrary quaternions (\(A\) and \(B\)) is

\[
AB = (\tilde{A}\hat{B} - \tilde{B}\hat{A}) + \hat{A}\hat{B} + \hat{B}\hat{A} + \tilde{A}\times\tilde{B},
\]  

(5)

where \(\tilde{A}\cdot\tilde{B}\) represents the inner product and \(\tilde{A}\times\tilde{B}\) the outer product. It is noteworthy that quaternion algebra is non-commutative, since

\[
\tilde{A}\times\tilde{B} = -\tilde{B}\times\tilde{A}.
\]  

(6)

Notice that if \(\tilde{A} = \tilde{B} = 0\), the product is reduced to

\[
AB = -\tilde{A}\cdot\tilde{B} + \tilde{A}\times\tilde{B}
\]  

(7)

that is the difference between their outer and inner product.

Table 1 summarizes these product rules.

B. Conjugate

Quaternion conjugate is defined as

\[
A^* = \tilde{A} - \tilde{\hat{A}},
\]  

(8)

and an important property follows

\[
(AB)^* = B^*A^*.
\]  

(9)

C. Norm

The norm is equivalently to euclidean vector norm, and is expressed as

\[
|A| = \sqrt{w^2 + x^2 + y^2 + z^2}
\]  

(10)

and as a consequence

\[
|A|^2 = AA^* = A^*A.
\]  

(11)

D. Inverse

The inverse of a quaternion is intuitively defined as

\[
A^{-1}A = AA^{-1} = 1.
\]  

(12)

Left multiplying (12) second equality by \(A^*\) and applying (11), the inverse can be calculated as

\[
A^{-1} = \frac{A^*}{|A|^2}.
\]  

(13)

E. Rotations, Derivatives and Integrals

Hamilton’s objective when he proposed quaternions was to represent rotation of vectors in tridimensional space in a similar way complex numbers represent in a plane. One of the main goals was to find a way to rotate those vectors in space. With four elements written in the form of quaternions, rotation of \(V\) around direction \(n\) (unit norm) by an angle \(\theta_r\) is

\[
V_r = RVR^*
\]  

(14)

where \(V_r\) is the rotated result and

\[
R = \cos \left(\frac{\theta_r}{2}\right) + n\sin \left(\frac{\theta_r}{2}\right)
\]  

(15)

is the rotation quaternion, which also has unit norm.

Quaternion derivatives and integrals in relation to a scalar, e.g. time, are defined as follows

\[
\frac{dA(t)}{dt} = \frac{dw(t)}{dt} + \frac{dx(t)}{dt}q_1 + \frac{dy(t)}{dt}q_2 + \frac{dz(t)}{dt}q_3
\]  

(16)

\[
\int A(t)dt = \int w(t)dt + \left(\int x(t)dt\right)q_1 + \left(\int y(t)dt\right)q_2 + \left(\int z(t)dt\right)q_3
\]  

(17)

and rules, such as the chain rule, applies normally.

If \(A\) describes a rotation movement, it can be expressed as in (14), and its derivative can be rewritten as

\[
\frac{dA(t)}{dt} = n\frac{d\theta_r(t)}{dt}A
\]  

(18)

and analogously the integral is

\[
\int A(t)dt = n\left(\frac{d\theta_r(t)}{dt}\right)^{-1}A.
\]  

(19)

F. Polar Representation

Similar to phasors, quaternions can have a polar representation. Taking the rotation quaternion from (15), it can be rewritten as

\[
R = e^{n\theta_r/2}.
\]  

(20)

For a detailed explanation on quaternions, reader is referred to [14].
III. SINGLE-PHASE ELECTRICAL QUATERNIONS

In this section, single-phase circuit quantities will be modeled in the form of quaternions. Results for a series RLC circuit are presented and discussed.

According to [12], generic voltage and current RMS signals can be represented in a plane, such as \( q_1 q_2 \), as follows

\[
V = V\cos(\theta_v)q_1 + V\sin(\theta_v)q_2, \tag{21}
\]
\[
I = I\cos(\theta_i)q_1 + I\sin(\theta_i)q_2. \tag{22}
\]

Therefore, single-phase power quaternion is defined as

\[
S = VI^* \tag{23}
\]

and after some algebraic work,

\[
S = P + Q q_3 = VI\cos(\theta_v - \theta_i) + VI\sin(\theta_v - \theta_i)q_3 \tag{24}
\]

where active power is clear to be the scalar part and reactive power to be the vectorial part (which actually is a single complex number, similarly to phasors).

In sinusoidal operation, current and charge are orthogonal signals, thus it is intuitive to define each of them in orthogonal axis. Moreover, charge and current derivative are parallel. So it follows [10]

\[
I(t) = i(t)q_2 \tag{25}
\]
\[
\int i(t)dt = \int i(t)dt q_1 \tag{26}
\]
\[
\frac{di(t)}{dt} = \frac{di(t)}{dt} q_1. \tag{27}
\]

A. Single-Phase Series RLC Circuit

Consider now a series RLC circuit. Applying Kirchhoff’s voltage law with quaternions and considering signals in the time domain, voltage is

\[
V(t) = RI(t) + \frac{1}{C} \int I(t)dt + L \frac{di(t)}{dt}. \tag{28}
\]

Applying (25), (26) and (27),

\[
V(t) = Ri(t)q_2 + \left( \frac{1}{C} \int i(t)dt + L \frac{di(t)}{dt} \right) q_1. \tag{29}
\]

Since the circuit is under sinusoidal operation, taking the RMS value for voltage and current results in

\[
V = \left( Rq_2 + \left( \frac{1}{wC} - wL \right) q_1 \right) I, \tag{30}
\]

where \( V \) is the RMS voltage quaternion and \( I \) is the RMS current (which is a scalar).

Based on (30), [12] defines the impedance quaternion \( Z_B \) as voltage RMS quaternion divided by current RMS (scalar value), i.e.

\[
Z_B = Rq_2 + \left( \frac{1}{wC} - wL \right) q_1. \tag{31}
\]

Quaternion impedance is in the same plane as voltage and current. Also both resistive and reactive components are multiplied by complex entities. With this definition, power quaternion can be expressed as follows,

\[
S = (Z_B I) I^*. \tag{32}
\]

It is notable that current RMS appears multiplying current quaternion. This last expression and (31) are very different from those obtained with phasors.

It would be more intuitive to define impedance as the relation between the same type elements, i.e., voltage RMS quaternion and current RMS quaternion. Therefore, this article proposes

\[
Z = VI^{-1} \tag{33}
\]

and for the RLC circuit in analysis,

\[
Z = R + \left( wL - \frac{1}{wC} \right) q_3, \tag{34}
\]

where real part represents resistive elements in the circuit and vectorial part, in analogy to phasors imaginary part, represents energy storage elements.

Power quaternion for this circuit will be

\[
S = VI^* = ZI^*, \tag{35}
\]
\[
S = Z|I|^2 \tag{36}
\]
\[
S = RI^2 + \left( wL - \frac{1}{wC} \right) I^2 q_3. \tag{37}
\]

Notice how (37) is similar to (24), i.e. its real part represents active power and its vectorial part represents reactive power. Additionally, with the new impedance definition proposed, power can be calculated as impedance times current norm squared as in (36) in a similar way to phasors.

IV. THREE-PHASE ELECTRICAL QUATERNIONS

Three-phase voltage quaternion, as in [6] and [11], is usually denoted by

\[
V(t) = v_a(t)q_1 + v_b(t)q_2 + v_c(t)q_3, \tag{38}
\]

where

\[
v_a(t) = V_a\cos(\omega t) \tag{39}
\]
\[
v_b(t) = V_b\cos(\omega t - \frac{2\pi}{3} + \phi_b) \tag{40}
\]
\[
v_c(t) = V_c\cos(\omega t + \frac{2\pi}{3} + \phi_c) \tag{41}
\]

represents instantaneous voltages of each phase with reference to a neutral point, and \( \omega \) is the electrical frequency in rad/s. If voltages are balanced, \( \phi_b = \phi_c = 0 \) and \( V_a = V_b = V_c \).

Analogously current is denoted by

\[
I(t) = i_a(t)q_1 + i_b(t)q_2 + i_c(t)q_3, \tag{42}
\]

where \( i_a(t) \), \( i_b(t) \) and \( i_c(t) \) refer to line currents flowing through phases a, b and c, respectively. Notice that these quaternions are time varying and can be directly computed from current and voltage samples.

An alternate way to describe the voltage and/or current not yet presented in literature is by standard rotations formulae...
([11] described voltage as rotations, however not in the standard quaternion rotation formula). A 3D plot of the locus described by the voltage quaternion in a period is shown in Fig. (1). It can be seen, that voltage is contained in a 2D plane (as it were expected since it is constituted with a 2D function family - sin and cossine). Furthermore, it describes a circle (in balanced situations). So this article proposes rewriting the voltage as

$$ V(t) = R(t)V(t_0)R^{-1}(t), \quad (43) $$

where

$$ R(t) = \cos \left( \frac{wt}{2} \right) + n \sin \left( \frac{wt}{2} \right), \quad (44) $$

n is the plane direction and $t_0$ is an arbitrary instant to initialize the rotation movement.

Notice that, because n is always orthogonal to V(t), the rotation equation can be rewritten as

$$ V(t) = V(t_0)e^{nwt}. \quad (45) $$

Under unbalanced conditions, the voltage quaternion describes an ellipse, which can be described by the combination of a clockwise and a counterclockwise rotation elements [11]. So voltage can be expressed as

$$ V(t) = V^+(t_0)e^{nwt} + V^-(t_0)e^{-nwt}. \quad (46) $$

Quaternion n can be determined in several ways, one of them is simply the cross product between voltage in consecutive moments, i.e. the vectorial part of their quaternion product, and then normalizing it, i.e.

$$ n = \frac{V(t)V(t + \Delta t)}{|V(t)||V(t + \Delta t)|}. \quad (47) $$

In Fig. 2 it is clear that the direction of n is affected by voltage unbalance. For the current locus, however, the direction is fixed for Y or Δ loads, since $i_a + i_b + i_c = 0$.

For balanced voltages,

$$ n = \frac{1}{\sqrt{3}} (q_1 + q_2 + q_3). \quad (48) $$

In consequence, the derivative of voltage quaternion is

$$ V'(t) = n\omega V(t), \quad (49) $$

and for current likewise. Similar to phasors in single-phase, (49) transform derivatives into products. Therefore, using this property, this article proposes representing energy storage elements as an impedance quaternion as it will be shown in the next subsection.

Power quaternion defined accordingly to [6] is

$$ S(t) = V(t)I(t) = -p_{abc}(t) + Q(t), \quad (50) $$

where $p_{abc}$ is the instantaneous three-phase active power and instantaneous reactive power is

$$ Q = \vec{S} = q_aq_1 + q_bq_2 + q_cq_3, \quad (51) $$

$$ q_a = v_bi_c - v_ci_b, \quad (52) $$

$$ q_b = v_ci_a - v_ai_c, \quad (53) $$

$$ q_c = v_ai_b - v_bi_a, \quad (54) $$

and all the terms are time varying (the term “t” is omitted to simplify notation). Quaternion Q is equivalent to instantaneous reactive power representation via vectors, as described in [15]. It is noteworthy that, in linear and balanced three-phase circuits, both instantaneous active and reactive power are constant. Hence, power variations (for a constant load) brings out unbalance and harmonics information [6].

In contrast to [6], this article proposes

$$ S = VI^* = p_{abc} - Q, \quad (55) $$

the reason for this will become clear in the next subsection. Researches involving quaternions in the electrical context can easily be applied with this new power definition (since there is only a sign change).

For unbalanced situations, active and reactive power have an oscillatory behavior due to voltage and/or load unbalance. Additionally, the mean of the norm of these components is equal to those represented by phasors. So it is clear that power represented via phasors is contained in the quaternion representation.
With these definitions, current can be easily decomposed into components parallel \(I_p\) and orthogonal \(I_q\) to voltage, as in [6]. As a result, power can be written as
\[
S = V (I_p^* + I_q^*) .
\] (56)

Since, voltage and current quaternions real parts are zero, \(V^* = -V\) and \(I^* = -I\) and the active power is clear to be
\[
p_{abc} = VI_p^* ,
\] (57)
and the reactive power \(Q_n\) as proposed in this article is
\[
Q_n = -Q = VI_q^* .
\] (58)

So parallel and orthogonal components are
\[
I_p = -V^{-1}p_{abc}\text{ and } I_q = V^{-1}Q ,
\] (59)
which are equal to those obtained by [6], therefore this new power definition is compatible with [6] control theory.

Power, for balanced situations, is constant and can be written as a function of voltage and current norms and angle \(\phi_{vi}\) between them.
\[
S = |S| = |V||I|\cos(\phi_{vi}) + n\sin(\phi_{vi})
\] (60)
So power norm is
\[
|S| = |V||I|
\] (61)
where for balanced situations
\[
|V| = \sqrt{3}V, |I| = \sqrt{3}I .
\] (62)
Notice how (61) is very similar to single-phase power expression using phasors, however it does represent three-phase apparent power, i.e.
\[
|S| = 3V^2 I ,
\] (63)
where \(V\) and \(I\) are voltage RMS with respect to a neutral point and current RMS flowing through each line, respectively. Active and reactive norms are
\[
p_{abc} = |S| = |V||I|\cos(\phi_{vi}) ,
\] (64)
\[
|Q| = |S| = |V||I|\sin(\phi_{vi}) .
\] (65)
This makes clear reactive power quaternion norm equals traditional definition of three-phase reactive power, and similarly the scalar part of \(S\) is clear to be identical to active power as expected for balanced cases.

A. Three-phase series RLC balanced circuit

In this section a three-phase balanced Y load will be studied. This approach, however, can be applied to \(\Delta\) loads and balanced Y loads with ground connection. Considering the circuit presented in Fig. 3 (with a balanced voltage and load), the three-phase voltage over the capacitor will be balanced and senoidal thus it can be written as in (43), likewise the indutor current. Applying Kirchhoff’s voltage law,
\[
V = RI + LI' + \frac{1}{C} \int I dt ,
\] (66)
\[
V' = RI' + LI'' + \frac{1}{C} I ,
\] (67)
and applying (49),
\[
V = (R + n\omega L - \frac{n}{wC}) I .
\] (68)

Just as in single-phase circuits, the three-phase quaternion impedance is defined as the element relating voltage and current, more specifically, voltage right divided by current.
\[
Z = VI^{-1}
\] (69)
It is important to notice, that despite voltage and current being time variant in (69), the impedance is constant for balanced conditions and depends only on load values. All impedance association rules applies, regarding the fact that product is not commutative. For the series RLC circuit, impedance is
\[
Z = R + n\left(wL - \frac{1}{wC}\right) .
\] (70)

Therefore, power is expressed by
\[
S = Z|I|^2 .
\] (71)

Intuitively, power quaternion can be computed as impedance quaternion times current quaternion norm squared, similarly to single-phase with phasors. It is important to notice that because of the power quaternion definition proposed in this article, it has the same direction as impedance, equivalently to phasors. Moreover, it can be written as a function of voltage and impedance, by applying (55), (69), (9) and (11).
\[
S = |V|^2 (Z^{-1})^* .
\] (72)

Equation (72) is equivalent to single-phase power expression using phasors, i.e. voltage norm squared over impedance conjugate. Calculating the impedance inverse and applying conjugating operation,
\[
S = |V|^2 \frac{Z}{|Z|^2} .
\] (73)

With (73), it is clear the power direction is the same as the impedance. The power norm is equal to
\[
|S| = \frac{|V|^2}{|Z|} .
\] (74)
\[
|S| = |Z||I|^2 .
\] (75)
These relations would not be true if power definition of [6] was adopted.

With the new power definition, if circuit is inductive, reactive power will be in the same direction as inductor impedance, being clear the reactive consumption. Moreover, both power scalar sign and power vectorial direction indicates active and reactive power consumption, respectively. Additionally, quaternions provide an explanation for the direction of $Q_n$ that, to the best of our knowledge, does not exist for the vectorial representation presented in [15].

As shown, quaternions expressions for three-phase balanced circuits are very similar to those obtained when using phasors for singles phase circuits. Table II summarizes these relationships. It is possible to verify the similarities between expression and also that steady state information is preserved on both representations. Moreover, quaternions represent instantaneous information.

V. CONCLUSION

This article presented definitions and an analysis of quaternions single and three-phase quantities. Results of both series RLC single and balanced three-phase circuits were presented.

Section III showed how quaternions can be used to calculate current and active and reactive power for a single-phase circuit. A series RLC circuit was used as an example and results were compared to the expected when using the traditional tool, i.e., phasors. All quantities defined with phasors were defined analogously with quaternions, therefore it is expected that quaternions could actually perform similarly in real problems. However, it is more computationally expensive and does not bring any additional benefit for single-phase circuit analysis.

Section IV defined voltage, current and instantaneous power in the form of quaternions. It is notable that active and reactive power at an arbitrary time instant can be easily calculated from voltage and current samples at that same time. Computing reactive power with phasors requires an estimation of the phasors, which is not trivial. Moreover, since instantaneous power carries information about unbalance and harmonics, this representation can be applied to power quality studies.

Voltage and current rotation equation as suggested in this article permitted transforming derivatives into products, thus allowing a RLC circuit analysis and three-phase impedance definition.

<table>
<thead>
<tr>
<th>TABLE II</th>
<th>QUATERNION THREE-PHASE QUANTITIES COMPARED TO PHASOR SINGLE-PHASE QUANTITIES.</th>
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<tr>
<td>Voltage derivative</td>
<td>$V' = n_0 V$</td>
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<tr>
<td>Impedance</td>
<td>$Z = R + jX$</td>
</tr>
<tr>
<td>Power</td>
<td></td>
</tr>
<tr>
<td>$S = V' Z$</td>
<td>$\bar{S} = \gamma Z$</td>
</tr>
<tr>
<td>$S =</td>
<td>V'</td>
</tr>
</tbody>
</table>

With quaternions it is possible to define three-phase electrical quantities analogously to single-phase’s with phasors. Quantities, however, are represented in time domain (except for impedance), thus it is possible to define and compute instantaneous active and reactive power. It is noteworthy that, in balanced situations, phasors values, e.g. active and reactive power, can be easily obtained from quaternions values, as shown. Quaternions also provides a framework to deal with three-phases simultaneously, as in contrast to the individual phase approach. Therefore, it is expected that quaternions may be employed in substitution to phasors, since it represents three-phase quantities in a more complete way.

Finally, since unbalance and harmonics information is carried in instantaneous power, quaternions representation is adequate for quality power problems.

REFERENCES