Certain Bounds Related to Multi-Parameterized $k$-Fractional Integral Inequalities and Their Applications

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\section*{ABSTRACT} A $k$-fractional integral identity with multiple parameters is investigated. Based on this identity, some estimation-type results related to $k$-fractional integral inequalities for the first-order differentiable functions are obtained. These results are then applied to the estimation of cumulative distribution function and some other special means.

\section*{INDEX TERMS} Hadamard’s inequality, generalized $(\alpha, m)$-preinvex functions, $k$-fractional integrals.

\section*{I. INTRODUCTION} In 2013, Sarikaya et al. established the following Hadamard’s inequality utilizing Riemann–Liouville fractional integrals.

\textbf{Theorem 1} [24]: Let $g : [e_1, e_2] \to \mathbb{R}$ be a positive function with $0 \leq e_1 < e_2$ and $g \in L^1([e_1, e_2])$. If $g$ is convex on $[e_1, e_2]$, then the following inequalities for fractional integrals hold:

\begin{equation}
J_{e_1}^\mu g(x) = \frac{1}{\Gamma(\mu)} \int_{e_1}^{x} (x - \lambda)^{\mu-1} g(\lambda) d\lambda, \quad e_1 < x
\end{equation}

and

\begin{equation}
J_{e_2}^\mu g(x) = \frac{1}{\Gamma(\mu)} \int_{x}^{e_2} (\lambda - x)^{\mu-1} g(\lambda) d\lambda, \quad x < e_2.
\end{equation}

Here, $(\mu)$ is the gamma function, i.e., $\Gamma(\mu) = \int_{0}^{\infty} e^{-\lambda} \lambda^{\mu-1} d\lambda$. It is to be noted that $J_{e_1}^0 g(x) = J_{e_1}^0 g(x) = g(x)$.

Due to the extensive applications of Riemann–Liouville fractional integrals, there are many studies involving this integral operator, for example, see [4], [7], [11]–[13], [25], [27], [29] and the references therein. In the case of $\mu = 1$, the fractional integral inequality (1) reduces to the classical Hadamard’s inequality. For recent studies on Hadamard’s inequality, see, for instance, [14], [15], [18], [19], [21].

In 2012, Mubeen and Habibullah presented the following $k$-fractional integrals.

\textbf{Definition 2} [22]: Let $g \in L^1([e_1, e_2])$, the $k$-fractional integrals $kJ_{e_1}^\mu g(x)$ and $kJ_{e_2}^\mu g(x)$ of order $\mu > 0$ are defined by

\begin{equation}
kJ_{e_1}^\mu g(x) = \frac{1}{k \Gamma_k(\mu)} \int_{e_1}^{x} (x - \lambda)^{\mu-1} g(\lambda) d\lambda
\end{equation}

and

\begin{equation}
kJ_{e_2}^\mu g(x) = \frac{1}{k \Gamma_k(\mu)} \int_{x}^{e_2} (\lambda - x)^{\mu-1} g(\lambda) d\lambda
\end{equation}

respectively, where $0 \leq e_1 < x < e_2$, $k > 0$ and $\Gamma_k(\mu)$ is the $k$-gamma function defined by $\Gamma_k(\mu) = \int_{0}^{\infty} e^{-\lambda} \lambda^{\mu-1} e^{-\frac{\lambda}{k}} d\lambda, \quad \mu > 0$, along with the properties $\Gamma_k(\mu + k) = k \Gamma_k(\mu)$ and $\Gamma_k(1) = 1$.

Some recent results related to the $k$-fractional integral operators can also be found in [8], [9], [16], [26], [28], [30]. In 2016, Farid et al. presented the following $k$-fractional integral inequality.

\textbf{Theorem 3} [10]: Let $g : [e_1, e_2] \to \mathbb{R}$ be a positive mapping with $0 \leq e_1 < e_2$ and $g \in L^1([e_1, e_2])$. If $g$ is convex on $[e_1, e_2]$, then the following $k$-fractional integral inequality
with \( \mu > 0 \) and \( k > 0 \) holds:
\[
g\left(\frac{e_1 + e_2}{2}\right) \leq \frac{2^{\mu - 1} \Gamma_\mu(\mu + k)}{(e_2 - e_1)^{\mu}} \times \left[k \int_0^r \left(\frac{e_2}{e_1 + e_2}\right)^\mu g(e_2) + k \int_0^{e_1/2} g(e_1)\right] \leq g(e_1) + g(e_2)
\]

(2)

Here, our main goal is to obtain new estimation-type results associated with \( k \)-fractional integral operators. To this end, we consider the following three cases: (i) the considered mapping is generalized \((\alpha, m)\)-preinvex; (ii) the derivative of the considered mapping is bounded; (iii) the derivative of the considered mapping satisfies the Lipschitz condition.

Let us end this section by recalling some special functions and definitions as follows.

(1) The beta function:
\[
\beta(u, v) = \frac{\Gamma(u) \Gamma(v)}{\Gamma(u + v)} = \int_0^1 \lambda^{u-1} (1 - \lambda)^{v-1} d\lambda, \quad u, v > 0.
\]

(2) The hypergeometric function:
\[
\sum_{k=1}^{\infty}\frac{a^k}{k!} = \sum_{k=1}^{\infty}\frac{a^k}{k!} = \sum_{k=1}^{\infty}\frac{a^k}{k!} = \sum_{k=1}^{\infty}\frac{a^k}{k!}
\]

where \( r_2 > r_1 > 0 \) and \( |z| < 1 \).

Definition 4 [11]: A set \( K \subseteq \mathbb{R}^n \) is named invex set with respect to the mapping \( \eta: K \times K \to \mathbb{R}^n \), if \( u + t \eta(v, u) \in K \) for any \( u, v \in K \) and \( t \in [0, 1] \). The invex set \( K \) is also referred to as an \( \eta \)-connected set.

Definition 5 [5]: A set \( K \subseteq \mathbb{R}^n \) is named \( m \)-invex with respect to the mapping \( \eta: K \times K \times [0, 1] \to \mathbb{R}^n \) for certain fixed \( m \in (0, 1), r_1, r_2 \in K \) with \( r_1 < r_2 \). Assume that \( g: K \to \mathbb{R} \) is a differentiable mapping satisfying that \( g^{(r)} \) is integrable on the \( \eta \)-invex set of \( \eta(m) = \eta_x + \eta_y \), where \( m = \eta_y \) and \( \eta_x \in K \). Before stating the results, we set the following notation:

\[
\mathcal{H}_{\eta}(u, \eta, \lambda, n, x) := \frac{n + 1}{2\eta(r_2, r_1, m)} \left( (1 - \lambda)^{\eta_x(x, r_1, m)} \times \left( g(mr_1) + g(mr_1 + \eta(x, r_1, m)) \right)
\]

\[
+ \eta_x^{(r_2, r_1, m)} \left( g(mx + \eta(r_2, x, m)) + g(mx) \right) \right)
\]

\[
+ \lambda \left[ \eta_x^{(r_2, x, m)} g \left( mx + \frac{n}{n+1} \eta(x, r_1, m) \right) \right]
\]

\[
+ \eta_x^{(r_2, x, m)} \left( g \left( mx + \frac{n}{n+1} \eta(r_2, x, m) \right) \right)
\]

\[
- \left( n + 1 \right)^{\eta_x(x, r_1, m)} \times \left[ k \int_{(mr_1)}^{\eta_x(x, r_1, m)} g \left( mr_1 + \frac{n}{n+1} \eta(x, r_1, m) \right) \right]
\]

\[
+ k \int_{(mx)}^{\eta_x(x, r_1, m)} g \left( mx + \frac{n}{n+1} \eta(r_2, x, m) \right)
\]

\[
- k \int_{(mx)}^{\eta(x, r_1, m)} g \left( mx + \frac{n}{n+1} \eta(r_2, x, m) \right)
\]

\[
\]
In particular, if \( \eta(e_1, e_2, m) = e_1 - me_2 \) with \( m = 1 \) for \( e_1, e_2 \in [r_1, r_2] \), Eq. (3) reduces to

\[
\mathcal{H}(\mu, k; \lambda, n, x) := \frac{n + 1}{2} \left\{ (1 - \lambda) \left[ \frac{(x - r_1)\mu}{r_2 - r_1} g(x) + (x - r_1)\frac{\eta}{r_2 - r_1} g(r_1) + (r_2 - x)\frac{\eta}{r_2 - r_1} g(r_2) \right] + \lambda \left[ \frac{(x - r_1)\mu}{r_2 - r_1} g(x) + g \left( \frac{n + 1}{n + 1} x + \frac{1}{n + 1} x \right) \right] + \left( \frac{r_2 - x}{r_2 - r_1} g \left( \frac{n + 1}{n + 1} x + \frac{1}{n + 1} r_2 \right) \right) \right\} - (n + 1)\frac{\eta}{2(r_2 - r_1)} \\
\times k\mathcal{J}_{\mu,r_2}^\mu g \left( \frac{n + 1}{n + 1} x + \frac{1}{n + 1} r_2 \right) + k\mathcal{J}_{\mu,x}^\mu g \left( \frac{n + 1}{n + 1} x + \frac{1}{n + 1} r_1 \right) + k\mathcal{J}_{\mu,r_2}^\mu g \left( \frac{n + 1}{n + 1} x + \frac{1}{n + 1} r_2 \right) \right\}
\]

We need the succeeding lemma.

**Lemma 12**: The following \( k \)-fractional integral identity together with \( x \in (r_1, r_2) \), \( n \in \mathbb{N}^+ \), \( \lambda \in [0, 1] \), \( \mu > 0 \) and \( k > 0 \) holds:

\[
\mathcal{H}_{\eta, \mu}(\mu, k; \lambda, n, x) = \frac{\eta^\mu}{2\eta(r_2 - r_1)} \int_0^1 \left( r^\mu - \lambda \right) g \left( mr_1 + \frac{n + t}{n + 1} \eta(x, r_1, m) \right) dt - g \left( mr_1 + \frac{n + t}{n + 1} \eta(x, r_1, m) \right) \int_0^1 \left( r^\mu - \lambda \right) g \left( mx + \frac{n + t}{n + 1} \eta(x, r_1, m) \right) dt.
\]

**Proof**: Integrating by parts and changing the variable, we have that

\[
\int_0^1 \left( r^\mu - \lambda \right) g \left( mr_1 + \frac{n + t}{n + 1} \eta(x, r_1, m) \right) dt = \frac{(n + 1)(1 - \lambda)g \left( mr_1 + \frac{n + t}{n + 1} \eta(x, r_1, m) \right) \left|_0^1 \right.}{\eta(x, r_1, m)} - \int_0^1 \frac{\mu(n + 1)\eta^\mu}{k\eta(x, r_1, m)} g \left( mr_1 + \frac{n + t}{n + 1} \eta(x, r_1, m) \right) dt.
\]

Similarly, we get that

\[
\int_0^1 \left( r^\mu - \lambda \right) g \left( mx + \frac{n + t}{n + 1} \eta(x, r_1, m) \right) dt = \frac{(n + 1)(1 - \lambda)g \left( mx + \frac{n + t}{n + 1} \eta(x, r_1, m) \right) \left|_0^1 \right.}{\eta(x, r_1, m)} - \int_0^1 \frac{\mu(n + 1)\eta^\mu}{k\eta(x, r_1, m)} g \left( mx + \frac{n + t}{n + 1} \eta(x, r_1, m) \right) dt.
\]

and

\[
\int_0^1 \left( r^\mu - \lambda \right) g \left( mx + \frac{n + t}{n + 1} \eta(x, r_1, m) \right) dt = \frac{(n + 1)(1 - \lambda)g \left( mx + \frac{n + t}{n + 1} \eta(x, r_1, m) \right) \left|_0^1 \right.}{\eta(x, r_1, m)} - \int_0^1 \frac{\mu(n + 1)\eta^\mu}{k\eta(x, r_1, m)} g \left( mx + \frac{n + t}{n + 1} \eta(x, r_1, m) \right) dt.
\]
After suitable rearrangements, the desired result in (4) is obtained. This ends the proof.

**Corollary 13:** In Lemma 12, if \( \eta(e_1, e_2, m) = e_1 - me_2 \) with \( m = 1 \) for \( e_1, e_2 \in [r_1, r_2] \), then we obtain the following identity:

\[ H(\mu, k; \lambda, n, x) = \frac{(x - r_1)^{\frac{k}{m} + 1}}{2(r_2 - r_1)} \int_0^1 \left( r^\mu - \lambda \right) \left[ g'(\frac{n + t}{n + 1}x + 1 - t) \right] dt \]

where

\[ e = \left( \frac{k}{m} + 1 \right)(n + 1)^\alpha \]

\[ \frac{2n^\alpha \lambda}{(m + 1)(n + 1)^\alpha} \]

\[ + \frac{\lambda(2n + \frac{1}{n})\alpha + 1 - (n + 1)^\alpha + 1 - \mu^\alpha + 1}{(m + 1)(n + 1)^\alpha} \]

and

\[ \Psi_{\mu, k}(\lambda, n, \alpha) = \int_0^1 \left| t^\mu - \lambda \right|^\alpha \left| f(t) \right|^\frac{1}{2} \left| g(t) \right|^\frac{1}{2} dt \]

where \( f \) and \( g \) are both integrable mappings on \([a, b]\) with \( q \geq 1 \).

Using this inequality, Lemma 12, and the generalized \((\alpha, m)\)-preinvexity of \( |g'|^q \), we get that

\[ |H_{\eta_m}(\mu, k; \lambda, n, x)| \]

\[ \leq \frac{\mu}{2} \left( \frac{n^\alpha \lambda}{(m + 1)(n + 1)^\alpha} \right) \left( \int_0^1 \left| t^\mu - \lambda \right| dt \right)^{\frac{1}{2}} \left( \int_0^1 \left| f(t) \right|^\frac{1}{2} \left| g(t) \right|^\frac{1}{2} dt \right)^{\frac{1}{2}} \]

\[ + \frac{\mu}{2} \left( \frac{n^\alpha \lambda}{(m + 1)(n + 1)^\alpha} \right) \left( \int_0^1 \left| t^\mu - \lambda \right| dt \right)^{\frac{1}{2}} \left( \int_0^1 \left| f(t) \right|^\frac{1}{2} \left| g(t) \right|^\frac{1}{2} dt \right)^{\frac{1}{2}} \]

where

\[ \Gamma_{\mu, k}(\lambda) = \int_0^1 \left| t^\mu - \lambda \right| dt \]

\[ = \left( \frac{k}{\mu + k} - \lambda \right) \frac{2m^\alpha}{\mu + k} \]

\[ \Phi_{\mu, k}(\lambda, n, \alpha) = \int_0^1 \left| t^\mu - \lambda \right| \left( \frac{n + t}{n + 1} \right)^\alpha dt \]

\[ \leq \frac{\mu}{2} \left( \frac{n^\alpha \lambda}{(m + 1)(n + 1)^\alpha} \right) \left( \int_0^1 \left| t^\mu - \lambda \right| dt \right)^{\frac{1}{2}} \left( \int_0^1 \left| f(t) \right|^\frac{1}{2} \left| g(t) \right|^\frac{1}{2} dt \right)^{\frac{1}{2}} \]
and
\[ I_k = \int_0^1 \left| t^\frac{\mu}{n} - \lambda \right| ^q g' \left( m x + \frac{n + t}{n + 1} (r_2, x, m) \right) \left| \left( \frac{n + t}{n + 1} \right) ^a g(x) \right| ^q dt \]
\[ \leq \int_0^1 \left| t^\frac{\mu}{n} - \lambda \right| \left( \frac{m (1 - \left( \frac{n + t}{n + 1} \right) ^a)}{\left( n + 1 \right) ^a} \right) g(x) \, dt \]
\[ + \left( \frac{n + t}{n + 1} \right) ^a g(x) \, dt. \]

Hence the proof is completed. \( \square \)

**Remark 16:** In Theorem 15, taking \( \eta(e_1, e_2, m) = e_1 - me_2 \) with \( m = 1 \) for \( e_1, e_2 \in [r_1, r_2] \), and choosing \( \lambda = 0, k = 1 = n \) together with \( \alpha = 1 \), one gets Theorem 3 established by Mihai and Mitroi in [20]. Furthermore, taking \( \mu = 1 \), one has Theorem 3 presented by Latif in [17].

**Corollary 17:** In Theorem 15, choosing \( \eta(e_1, e_2, m) = e_1 - me_2 \) with \( m = 1 \) for \( e_1, e_2 \in [r_1, r_2] \), and taking \( x = \frac{r_1 + r_2}{2}, n = 1 \), one has the following \( k \)-fractional integral inequality for \( \alpha \)-convex mappings:
\[ \left| \frac{2^\frac{\mu}{n} - 1}{\left( r_2 - r_1 \right) ^{\frac{\mu}{n} - 1}} \mathcal{H} \left( \mu, k; 1, 1, \frac{r_1 + r_2}{2} \right) \right| \left| g' \left( r_1 \right) \right| ^\eta \]
\[ \leq \left| \left( 1 - \lambda \right) \left[ g \left( \frac{r_1 + r_2}{2} \right) + \frac{g(r_1) + g(r_2)}{2} \right] \right| ^\frac{\mu}{n} \]
\[ + \lambda \left[ g \left( \frac{3r_1 + r_2}{4} \right) + \frac{g(r_1) + 3g(r_2)}{4} \right] \]
\[ - \frac{2^\frac{\mu}{n} - 1}{\left( r_2 - r_1 \right) ^\frac{\mu}{n} - 1} \left[ k \mathcal{J} _{r_1} ^\mu g \left( \frac{3r_1 + r_2}{4} \right) \right] \]
\[ + k \mathcal{J} _{r_1} ^\mu g \left( \frac{3r_1 + r_2}{4} \right) \]
\[ + k \mathcal{J} _{r_1} ^\nu g \left( \frac{r_1 + r_2}{2} \right) \]
\[ + k \mathcal{J} _{r_1} ^\mu g \left( \frac{r_1 + r_2}{2} \right) \]
\[ \leq \frac{r_2 - r_1}{8} \chi ^{\frac{n - 1}{n}} \left[ \left( \chi - \Phi_\mu,k(1, \alpha) \right) \right] ^\eta \]
\[ + \Phi_\mu,k(1, \alpha) \left| g' \left( \frac{r_1 + r_2}{2} \right) \right| ^\eta \]
\[ + \left| \left( \chi - \Psi_\mu,k(1, \alpha) \right) g' \left( r_1 \right) \right| \]
\[ + \left| \left( \chi - \Psi_\mu,k(1, \alpha) \right) g' \left( r_2 \right) \right| \]
\[ + \left( \chi - \Phi_\mu,k(1, \alpha) \right) g' \left( \frac{r_1 + r_2}{2} \right) \]
\[ + \Phi_\mu,k(1, \alpha) g' \left( \frac{r_1 + r_2}{2} \right). \]

**Remark 18:** Consider Corollary 17.

(i) Taking \( \lambda = 0 \) and \( \alpha = 1 \) yields that
\[ \left| \frac{2^\frac{\mu}{n} - 1}{\left( r_2 - r_1 \right) ^{\frac{\mu}{n} - 1}} \mathcal{H} \left( \mu, k; 0, 1, \frac{r_1 + r_2}{2} \right) \right| \]
\[ \leq \frac{r_2 - r_1}{8} \left( \frac{k}{\mu + k} \right) \left\{ \left[ \frac{k}{2\mu + 4k} \left| g'(r_1) \right| ^q \right] ^\frac{1}{q} \right\} \]
\[ + \frac{2\mu + 3k}{2\mu + 4k} \left| g' \left( \frac{r_1 + r_2}{2} \right) \right| ^\frac{1}{q} \]
\[ + \left[ \frac{2\mu + 3k}{2\mu + 4k} g'(r_1) \right] ^q + \frac{3\mu + 3k}{4\mu + 8k} \left| g' \left( \frac{r_1 + r_2}{2} \right) \right| ^\frac{1}{q} \]
\[ + \left[ \frac{3\mu + 5k}{4\mu + 8k} \right] \left| g'(r_1) \right| ^q + \frac{3\mu + 5k}{4\mu + 8k} \left| g'(r_1) \right| ^q \]
\[ + \left[ \frac{3\mu + 5k}{4\mu + 8k} \right] \left| g'(r_1) \right| ^q + \frac{3\mu + 5k}{4\mu + 8k} \left| g'(r_1) \right| ^q \]
\[ + \frac{3\mu + 5k}{4\mu + 8k} \left| g'(r_1) \right| ^q + \frac{3\mu + 5k}{4\mu + 8k} \left| g'(r_1) \right| ^q \].

(ii) Taking \( \lambda = 1 \) and \( \alpha = 1 \) claims that
\[ \left| \frac{2^\frac{\mu}{n} - 1}{\left( r_2 - r_1 \right) ^{\frac{\mu}{n} - 1}} \mathcal{H} \left( \mu, k; 1, 1, \frac{r_1 + r_2}{2} \right) \right| \]
\[ \leq \frac{r_2 - r_1}{8} \left( \frac{\mu}{\mu + k} \right) \left\{ \left[ \frac{k + 3k}{4\mu + 8k} \left| g'(r_1) \right| ^q \right] ^\frac{1}{q} \right\} \]
\[ + \frac{3\mu + 5k}{4\mu + 8k} \left| g' \left( \frac{r_1 + r_2}{2} \right) \right| ^\frac{1}{q} \]
\[ + \left[ \frac{3\mu + 5k}{4\mu + 8k} \right] \left| g'(r_1) \right| ^q + \frac{3\mu + 5k}{4\mu + 8k} \left| g'(r_1) \right| ^q \]
\[ + \frac{3\mu + 5k}{4\mu + 8k} \left| g'(r_1) \right| ^q + \frac{3\mu + 5k}{4\mu + 8k} \left| g'(r_1) \right| ^q \]
\[ + \frac{3\mu + 5k}{4\mu + 8k} \left| g'(r_1) \right| ^q + \frac{3\mu + 5k}{4\mu + 8k} \left| g'(r_1) \right| ^q \].

(iii) Taking \( \lambda = \frac{1}{3} \) or \( \frac{1}{2} \), we have similar results mentioned above.

The following result holds for generalized \((\alpha, m)\)-preinvexity.

**Theorem 19:** Assume that \( |g'| \) for \( q > 1 \) is generalized \((\alpha, m)\)-preinvex with \( p + q^{-1} = 1 \). The following \( k \)-fractional integral inequality together with \( \mu > 0, k > 0, \lambda \in [0, 1], n \in \mathbb{N}^* \) and \( x \in (r_1, r_2) \) holds:
\[ \mathcal{H} _{n,m} \left( \mu, k; \lambda, n, x \right) \]
\[ \leq \frac{1}{\tau_1 ^\frac{\mu}{n} } \left\{ \left| \eta ^{\frac{\mu}{n} + 1} (x, r_1, m) \right| \right\} \]
\[ \times \left[ \left( m(1 - \tau_2 (n, \alpha)) |g'(r_1)| ^q + \tau_2 (n, \alpha) |g'(x)| ^q \right) ^\frac{1}{q} \right] \]
\[ + \left( m(1 - \tau_3 (n, \alpha)) |g'(r_1)| ^q + \tau_3 (n, \alpha) |g'(x)| ^q \right) ^\frac{1}{q} \]
\[ + \left| \frac{\eta ^{\frac{\mu}{n} + 1} (r_2, x, m) }{2^\eta (r_2, r_1, m) } \right| \]
\[ \left( m(1 - T_3(n, \alpha)) \right| g'(x) \right|^q + T_3(n, \alpha) \right| g'(r_2) \right|^q \right) ^{\frac{1}{q}} \\
+ \left( m(1 - T_2(n, \alpha)) \right| g'(x) \right|^q + T_2(n, \alpha) \right| g'(r_2) \right|^q \right) ^{\frac{1}{q}} \right) \\
\right\}, \quad (7) \]

where

\[ T_1(\mu, k, \lambda, p) = \begin{cases} \frac{k}{\mu p + k}, & \lambda = 0, \\
\frac{2 \mu p \frac{\mu p + k}{\mu p + k} - \lambda}{\mu p + k}, & 0 < \lambda < 1, \\
\frac{\mu p + k}{\mu} \left( k \mu, p + 1 \right), & \lambda = 1, \\
n_2(n, \alpha) = \int_0^1 \left( n + \frac{t}{n + 1} \right) ^{\alpha} dt = \frac{1}{(\alpha + 1)(n + 1)^{\alpha}}. \\
\end{cases} \]

\[ T_3(n, \alpha) = \int_0^1 \left( \frac{1 - t}{n + 1} \right) ^{\alpha} dt = \frac{1}{(\alpha + 1)(n + 1)^{\alpha}}. \]

\[ \text{Proof:} \text{ From Lemma 12, utilizing the Hölder inequality and the generalized} \ (\alpha, m) \text{-preinvexity of} \ |g'|^q, \text{we get that} \]

\[ \left| \mathcal{H}_{\eta_{\mu}}(\mu, k; \lambda, n, x) \right| \leq \frac{1}{2 \left[ \eta(r_2, r_1, m) \right]} \left( \int_0^1 \left| t - \lambda \right|^p dt \right) ^{\frac{1}{p}} \left[ (J_1) ^{\frac{1}{q}} + (J_2) ^{\frac{1}{q}} \right] \\
+ \frac{\eta^{\frac{1}{n}+1}(r_2, x, m)}{2 \left[ \eta(r_2, r_1, m) \right]} \left( \int_0^1 \left| t - \lambda \right|^p dt \right) ^{\frac{1}{p}} \left[ (J_3) ^{\frac{1}{q}} + (J_4) ^{\frac{1}{q}} \right]. \]

where

\[ J_1 = \int_0^1 \left| g'(m x + n + \frac{t}{n + 1} \eta(x, r_1, m)) \right|^q dt \]
\[ J_2 = \int_0^1 \left| g'(m x + n + \frac{t}{n + 1} \eta(r_1, x, m)) \right|^q dt \]
\[ J_3 = \int_0^1 \left| g'(m x + n + \frac{t}{n + 1} \eta(r_2, x, m)) \right|^q dt \]
\[ J_4 = \int_0^1 \left| g'(m x + n + \frac{t}{n + 1} \eta(r_2, x, m)) \right|^q dt \]

and

\[ J_1 = \int_0^1 \left| g'(m x + n + \frac{t}{n + 1} \eta(x, r_1, m)) \right|^q dt \]
\[ J_2 = \int_0^1 \left| g'(m x + n + \frac{t}{n + 1} \eta(r_1, x, m)) \right|^q dt \]
\[ J_3 = \int_0^1 \left| g'(m x + n + \frac{t}{n + 1} \eta(r_2, x, m)) \right|^q dt \]
\[ J_4 = \int_0^1 \left| g'(m x + n + \frac{t}{n + 1} \eta(r_2, x, m)) \right|^q dt \]

When \( \lambda = 0 \), we have that

\[ \int_0^1 \left| t - \lambda \right|^p dt = \frac{k}{\mu p + k} \]
In particular, taking $\alpha = 1$, one gets that
\[
\left| \frac{2^{\mu - 1}}{(r_2 - r_1)^{\beta - 1}} K_1 \left( \mu, k; \lambda, \frac{r_1 + r_2}{2} \right) \right| 
\leq \frac{r_2 - r_1}{8} T_1^k (\mu, k, \lambda, p) 
\times \left\{ \begin{array}{l}
\left[ \frac{1}{4} |r'(r_1)|^q + \frac{3}{4} g\left( \frac{r_1 + r_2}{2} \right)|^q \right]^{\frac{1}{q}} \\
+ \left[ \frac{3}{4} |g'\left( r_1 \right)|^q + \frac{1}{4} |g\left( \frac{r_1 + r_2}{2} \right)|^q \right]^{\frac{1}{q}} \\
+ \left[ \frac{3}{4} g\left( \frac{r_1 + r_2}{2} \right)|^q + \frac{1}{4} |g'(r_2)|^q \right]^{\frac{1}{q}} \\
+ \left[ \frac{1}{4} |g\left( \frac{r_1 + r_2}{2} \right)|^q + \frac{3}{4} |g'(r_2)|^q \right]^{\frac{1}{q}} \end{array} \right\}.
\]

**Remark 22:** In Corollary 21, taking $\lambda = 0$, $\frac{1}{2}$ and 1 respectively, one obtains the similar results.

Our next result is about an estimation of the upper bound of $k$-fractional integral inequality through products of two generalized ($\alpha, m$)-preinvex mappings.

**Theorem 23:** Let $K \subseteq \mathbb{R}$ be an open $m$-invex subset with respect to $\eta : K \times \mathbb{R} \times (0, 1) \to \mathbb{R}$ for certain fixed $m \in (0, 1), r_1, r_2 \in K$ with $0 \leq r_1 < r_2$. If $g : K \to (0, +\infty)$ and $h : K \to (0, +\infty)$ are both generalized ($\alpha, m$)-preinvex, then the following inequality holds:
\[
\frac{2^{\mu - 1} \Gamma_k(\mu + k)}{\eta^\mu (r_2, r_1, m)} \left[ k \mathcal{J}_1^{\mu} (\eta(r_2, r_1, m)) \right] - \left( gh \right)(mr_1) 
\leq \left[ \begin{array}{l}
1 - 2L_1 + L_2 - 2L_3 + 2L_4 m^2 g(r_1) h(r_1) \\
+ [L_1 - L_2 + L_3 - L_4] [mg(r_1) h(r_2) + mg(r_2) h(r_1)] \\
+ [L_2 + L_4] g(r_2) h(r_2), \end{array} \right]
\]
where
\[
L_1 = \frac{\mathcal{F}_1[-\alpha, \frac{\mu}{k}; \frac{\mu}{k}; + 1; \frac{1}{2}]}{2},
\]
\[
L_2 = \frac{\mathcal{F}_1[-2\alpha, \frac{\mu}{k}; \frac{\mu}{k}; + 1; \frac{1}{2}]}{2},
\]
\[
L_3 = \frac{\mu}{2^{\alpha + 1}(\mu + ka)},
\]
\[
L_4 = \frac{\mu}{2^{\alpha + 1}(\mu + 2ka)}.
\]

**Proof:** Since $g$ and $h$ are generalized ($\alpha, m$)-preinvex, one has that
\[
\frac{2^{\mu - 1} \Gamma_k(\mu + k)}{\eta^\mu (r_2, r_1, m)} 
\times k \mathcal{J}_1^{\mu} (\eta(r_2, r_1, m)) (gh)(mr_1 + \eta(r_2, r_1, m)) 
= \frac{\mu 2^{\mu - 1}}{k \eta^\mu (r_2, r_1, m)} 
\times \mathcal{J}_1^{\mu} (\eta(r_2, r_1, m)) (gh)(mr_1 + \eta(r_2, r_1, m)).
\]

Similarly, we have that
\[
\frac{2^{\mu - 1} \Gamma_k(\mu + k)}{\eta^\mu (r_2, r_1, m)} \left[ k \mathcal{J}_1^{\mu} (\eta(r_2, r_1, m)) \right] - \left( gh \right)(mr_1) 
\leq \left[ \begin{array}{l}
1 - 2L_1 + L_2 - 2L_3 + 2L_4 m^2 g(r_1) h(r_1) \\
+ [L_1 - L_2 + L_3 - L_4] [mg(r_1) h(r_2) + mg(r_2) h(r_1)] \\
+ [L_2 + L_4] g(r_2) h(r_2), \end{array} \right]
\]
Adding both sides of these two inequalities correspondingly, we obtain the desired result in (8). This ends the proof.  

**Remark 24:** In Theorem 23, taking $\eta(r_2, r_1, m) = r_2 - mr_1$ with $m = 1$ and choosing $\alpha = 1$ and $h(x) = 1$, we obtain the right part of the inequality (2.1) in Theorem 15 presented by Farid et al. in [10].

Another $k$-fractional integral inequality involving products of two generalized ($\alpha, m$)-preinvex functions is obtained as follows.

**Theorem 25:** With the same assumptions in Theorem 23, one has that
\[
\frac{\Gamma_k(\mu + k)}{2^{\mu - 1} \eta^\mu (r_2, r_1, m)} \left[ k \mathcal{J}_1^{\mu} (\eta(r_2, r_1, m)) \right] - \left( gh \right)(mr_1) 
\leq \left[ \begin{array}{l}
1 - 2\Theta_1 + \Theta_2 - 2\Theta_3 + 2\Theta_4 m^2 g(r_1) h(r_1) \\
+ [\Theta_1 - \Theta_2 + \Theta_3 - \Theta_4] [mg(r_1) h(r_2) + mg(r_2) h(r_1)] \\
+ [\Theta_2 + \Theta_4] g(r_2) h(r_2), \end{array} \right]
\]
where
\[
\Theta_1 = \frac{\mu \beta \mu}{2k}, \\
\Theta_2 = \frac{\mu}{2k}, \\
\Theta_3 = \frac{\mu}{2k}, \\
\Theta_4 = \frac{\mu}{2k}.
\]
Proof: The proof of Theorem 25 is analogous to that of in Theorem 23 and is omitted. □

Remark 26: In Theorem 25, taking \( \eta(r_2, r_1, m) = r_2 - mr_1 \) with \( m = 1 \), and choosing \( k = 1 = \alpha \), we have Theorem 15 presented by Chen in [3].

III. FURTHER ESTIMATION RESULTS

If the considered mapping \( g' \) is bounded, then we have the following result.

Theorem 27: If there exist constants \( r < R \) such that \( -\infty < r \leq g'(z) \leq R < \infty \) for all \( z \in \mathcal{K} \), then the following inequality together with \( \mu > 0, k > 0, \lambda \in [0, 1], n \in \mathbb{N}^* \) and \( x \in (r_1, r_2) \) holds:

\[
\left| \mathcal{H}_{\eta_0}(\mu, k; \lambda, n, x) \right| \leq \frac{(R - r)(|\eta^{\mu+1}(x, r_1, m)| + |\eta^{\mu+1}(r_2, x, m)|)}{2|\eta(r_2, r_1, m)|} \\
\times \left[ \frac{k}{\mu + k} - \lambda + 2\lambda \frac{\mu}{\mu + k} \right].
\]

(10)

Proof: From Lemma 12, one has that

\[
\mathcal{H}_{\eta_0}(\mu, k; \lambda, n, x) = \frac{\eta^{\mu+1}(x, r_1, m)}{2|\eta(r_2, r_1, m)|} \left\{ \int_0^1 (t^\mu - \lambda) \right. \\
\times \left[ g'(mr_1 + \frac{n + t}{n + 1} \eta(x, r_1, m)) - \frac{r + R}{2} \right] \, dt \right. \\
\left. - \int_0^1 \left[ g'(mx + \frac{n + t}{n + 1} \eta(x, x, r_1, m)) - \frac{r + R}{2} \right] \, dt \right\} \\
- \frac{\eta^{\mu+1}(r_2, x, m)}{2|\eta(r_2, r_1, m)|} \left\{ \int_0^1 (t^\mu - \lambda) \\
\times \left[ g'(mr_1 + \frac{n + t}{n + 1} \eta(x, r_2, m)) - \frac{r + R}{2} \right] \, dt \right. \\
\left. - \int_0^1 \left[ g'(mx + \frac{n + t}{n + 1} \eta(x, r_2, x, m)) - \frac{r + R}{2} \right] \, dt \right\}. 
\]

Using the inequality \( r \leq g'(mr_1 + \frac{n + t}{n + 1} \eta(x, r_1, m)) \leq R \), we have that

\[
r - \frac{r + R}{2} \leq g'(mr_1 + \frac{n + t}{n + 1} \eta(x, r_1, m)) - \frac{r + R}{2} \leq R - \frac{r}{2},
\]

which implies that

\[
\left| g'(mr_1 + \frac{n + t}{n + 1} \eta(x, r_1, m)) - \frac{r + R}{2} \right| \leq \frac{R - r}{2}.
\]

Similarly, we get that

\[
\left| g'(mr_1 + \frac{n + t}{n + 1} \eta(x, r_1, m)) - \frac{r + R}{2} \right| \leq \frac{R - r}{2},
\]

and

\[
\left| g'(mx + \frac{n + t}{n + 1} \eta(x, r_2, m)) - \frac{r + R}{2} \right| \leq \frac{R - r}{2}.
\]

Therefore

\[
|\mathcal{H}_{\eta_0}(\mu, k; \lambda, n, x)| \leq \frac{1}{2|\eta(r_2, r_1, m)|} \left\{ \int_0^1 (t^\mu - \lambda) \right. \\
\times \left[ g'(mr_1 + \frac{n + t}{n + 1} \eta(x, r_1, m)) - \frac{r + R}{2} \right] \, dt \right. \\
\left. + \int_0^1 \left[ g'(mx + \frac{n + t}{n + 1} \eta(x, r_1, m)) - \frac{r + R}{2} \right] \, dt \right\} \\
+ \frac{|\eta^{\mu+1}(r_2, x, m)|}{2|\eta(r_2, r_1, m)|} \left\{ \int_0^1 (t^\mu - \lambda) \\
\times \left[ g'(mr_1 + \frac{n + t}{n + 1} \eta(x, r_2, m)) - \frac{r + R}{2} \right] \, dt \right. \\
\left. + \int_0^1 \left[ g'(mx + \frac{n + t}{n + 1} \eta(x, r_2, m)) - \frac{r + R}{2} \right] \, dt \right\} \\
\leq (R - r)(|\eta^{\mu+1}(x, r_1, m)| + |\eta^{\mu+1}(r_2, x, m)|) \\
\times \left[ \frac{k}{\mu + k} - \lambda + 2\lambda \frac{\mu}{\mu + k} \right].
\]

This ends the proof. □

Corollary 28: In Theorem 27, choosing \( \eta(e_1, e_2, m) = e_1 - me_2 \) with \( m = 1 \) for \( e_1, e_2 \in [r_1, r_2] \) and taking \( \lambda = 0, n = 1 \) and \( x = \frac{r_1 + r_2}{2} \), we have that

\[
\left| \frac{2^{\mu-1}}{(r_2 - r_1)^{\mu-1}} \mathcal{H}(\mu, k; 0, 1, \frac{r_1 + r_2}{2}) \right| \\
= \left| g\left(\frac{r_1 + r_2}{2}\right) + g(r_1) + g(r_2) - \frac{2^{\mu-1}}{(r_2 - r_1)^{\mu-1}} \right. \\
\times \left[ k\mathcal{J}_r^\mu g\left(\frac{3r_1 + r_2}{4}\right) + k\mathcal{J}_r^\mu g\left(\frac{3r_1 + r_2}{4}\right) \right] \\
\left. + k\mathcal{J}_r^\mu g\left(\frac{3r_1 + r_2}{4}\right) + k\mathcal{J}_r^\mu g\left(\frac{3r_1 + r_2}{4}\right) \right| \\
\leq k(R - r)(r_2 - r_1) \frac{4}{4(\mu + k)}.
\]

(11)
In particular, taking \( \mu = 1 = k \), we obtain that
\[
\left| g\left( \frac{r_1 + r_2}{2} \right) + g(r_1) + g(r_2) - \frac{2}{r_2 - r_1} \int_{r_1}^{r_2} g(x) dx \right| \\
\leq \frac{(R - r)(r_2 - r_1)}{8}.
\] (12)

Our next goal is another estimation-type result when the considered mapping \( g' \) satisfies Lipschitz condition.

**Theorem 29:** If \( g' \) satisfies Lipschitz condition on \( K \) for some \( L > 0 \), then the following inequality together with \( \mu > 0, k > 0, \lambda \in [0, 1], n \in \mathbb{N}^+ \) and \( x \in (r_1, r_2) \) holds:
\[
\left| \mathcal{H}_{\eta_{\mu}}(\mu, k; \lambda, n, x) \right| \\
\leq \frac{L(|\eta_\mu^{k+2}(x, r_1, m)| + |\eta_\mu^{k+2}(r_2, x, m)|)}{2(n+1)|\eta(r_2, r_1, m)|} \\
\times \left[ \frac{2\mu\lambda^{k+1} + 2k}{\mu + 2k} + \frac{(n-1)(2\mu\lambda^{k+1} + k)}{\mu + k} - n\lambda \right].
\] (13)

**Proof:** From Lemma 12, we get that
\[
\mathcal{H}_{\eta_{\mu}}(\mu, k; \lambda, n, x) \\
= \frac{\eta_\mu^{k+1}(x, r_1, m)\int_0^1 (t_\mu^\lambda - \lambda) \left\{ g' \left( mr_1 + \frac{n + t}{n + 1} \eta(x, r_1, m) \right) \\
- g \left( mr_1 + \frac{1}{2} \eta(x, r_1, m) \right) \right\} dt \\
- \frac{\eta_\mu^{k+2}(r_2, x, m)\int_0^1 (t_\mu^\lambda - \lambda) \left\{ g' \left( mx + \frac{n + t}{n + 1} \eta(r_2, x, m) \right) \\
- g \left( mx + \frac{1}{2} \eta(r_2, x, m) \right) \right\} dt \right\} \\
\times \frac{\eta_\mu^{k+1}(r_2, x, m)}{2n\eta(r_2, r_1, m)} \\
\times \left\{ \int_0^1 |t_\mu^\lambda - \lambda| \left| g' \left( mr_1 + \frac{n + t}{n + 1} \eta(x, r_1, m) \right) \\
- g \left( mr_1 + \frac{1}{2} \eta(x, r_1, m) \right) \right| dt \\
+ \int_0^1 |t_\mu^\lambda - \lambda| \left| g' \left( mx + \frac{n + t}{n + 1} \eta(r_2, x, m) \right) \\
- g \left( mx + \frac{1}{2} \eta(r_2, x, m) \right) \right| dt \right\}
\]
Since \( g' \) satisfies Lipschitz condition on \( K \) for some \( L > 0 \), we have that
\[
\left| g' \left( mr_1 + \frac{n + t}{n + 1} \eta(x, r_1, m) \right) - g' \left( mr_1 + \frac{1}{2} \eta(x, r_1, m) \right) \right| \\
\leq L \left| mr_1 + \frac{n + t}{n + 1} \eta(x, r_1, m) - \left( mr_1 + \frac{1}{2} \eta(x, r_1, m) \right) \right| \\
= L |\eta(x, r_1, m)| \left( \frac{2t + n - 1}{n + 2} \right).
\]

Similarly, we obtain that
\[
\left| g' \left( mr_1 + \frac{1 - t}{n + 1} \eta(x, r_1, m) \right) - g' \left( mr_1 + \frac{1}{2} \eta(x, r_1, m) \right) \right| \\
\leq L |\eta(x, r_1, m)| \left( \frac{2t + n - 1}{n + 2} \right),
\]
\[
\left| g' \left( mx + \frac{n + t}{n + 1} \eta(r_2, x, m) \right) - g' \left( mx + \frac{1}{2} \eta(r_2, x, m) \right) \right| \\
\leq L |\eta(r_2, x, m)| \left( \frac{2t + n - 1}{n + 2} \right)
\]
and
\[
\left| g' \left( mx + \frac{1 - t}{n + 1} \eta(r_2, x, m) \right) - g' \left( mx + \frac{1}{2} \eta(r_2, x, m) \right) \right| \\
\leq L |\eta(r_2, x, m)| \left( \frac{2t + n - 1}{n + 2} \right).
\]

Therefore,
\[
\mathcal{H}_{\eta_{\mu}}(\mu, k; \lambda, n, x) \\
\leq \frac{|\eta_\mu^{k+1}(x, r_1, m)|}{2|\eta(r_2, r_1, m)|} \\
\times \left\{ \int_0^1 |t_\mu^\lambda - \lambda| \left| g' \left( mr_1 + \frac{n + t}{n + 1} \eta(x, r_1, m) \right) \\
- g \left( mr_1 + \frac{1}{2} \eta(x, r_1, m) \right) \right| dt \\
+ \int_0^1 |t_\mu^\lambda - \lambda| \left| g' \left( mx + \frac{n + t}{n + 1} \eta(r_2, x, m) \right) \\
- g \left( mx + \frac{1}{2} \eta(r_2, x, m) \right) \right| dt \right\}
\]
\[
\times \frac{|\eta_\mu^{k+2}(r_2, x, m)|}{2n|\eta(r_2, r_1, m)|} \\
\times \left\{ \int_0^1 |t_\mu^\lambda - \lambda| \left| g' \left( mr_1 + \frac{n + t}{n + 1} \eta(x, r_1, m) \right) \\
- g \left( mr_1 + \frac{1}{2} \eta(x, r_1, m) \right) \right| dt \\
+ \int_0^1 |t_\mu^\lambda - \lambda| \left| g' \left( mx + \frac{n + t}{n + 1} \eta(r_2, x, m) \right) \\
- g \left( mx + \frac{1}{2} \eta(r_2, x, m) \right) \right| dt \right\}
\]
\[
\leq \frac{L(|\eta_\mu^{k+2}(x, r_1, m)| + |\eta_\mu^{k+2}(r_2, x, m)|)}{2(n+1)|\eta(r_2, r_1, m)|} \\
\times \left[ \frac{2\mu\lambda^{k+1} + 2k}{\mu + 2k} + \frac{(n-1)(2\mu\lambda^{k+1} + k)}{\mu + k} - n\lambda \right].
\]
The proof is completed. \( \square \)
Corollary 30: In Theorem 29, choosing \( \eta(e_1, e_2, m) = e_1 - me_2 \) with \( m = 1 \) for \( e_1, e_2 \in [r_1, r_2] \) and taking \( \lambda = 0, n = 1 \) and \( x = \frac{r_1 + r_2}{2} \), we have that
\[
\left| \frac{2^{\frac{2}{\mu} - 1}}{(r_2 - r_1)^{\frac{1}{\mu}} - 1} H_k \left( \mu, k; 0, 1, \frac{r_1 + r_2}{2} \right) \right|
\[
= \left| g \left( \frac{r_1 + r_2}{2} \right) + \frac{g(r_1) + g(r_2)}{2} - \frac{2^{\frac{2}{\mu} - 1}}{(r_2 - r_1)^{\frac{1}{\mu}}} \right|
\times \left[ k \mathcal{J}_{\alpha} \left( \frac{3r_1 + r_2}{4} \right) + k \mathcal{J}_{\alpha} \left( \frac{3r_1 + r_2}{4} \right) \right]
\leq \frac{k L (r_2 - r_1)^2}{8(\mu + 2k)}.
\]
In particular, taking \( \mu = 1 = k \) yields that
\[
\left| g \left( \frac{r_1 + r_2}{2} \right) + \frac{g(r_1) + g(r_2)}{2} - \frac{2}{r_2 - r_1} \int_{r_1}^{r_2} g(u) du \right|
\leq \frac{L (r_2 - r_1)^2}{24}.
\]

IV. APPLICATIONS
A. PROBABILITY DENSITY FUNCTIONS
Let \( \tau : [r_1, r_2] \rightarrow [0, 1] \) be the probability density function of a continuous random variable \( X \) with the cumulative distribution function
\[
F(x) = \Pr(X \leq x) = \int_{r_1}^{x} \tau(t) dt.
\]
Using the fact that \( E(X) = \int_{r_1}^{r_2} t dF(t) = r_2 - \int_{r_1}^{r_2} F(t) dt \), we get the following results.

Proposition 31: In Theorem 15, taking \( \eta(e_1, e_2, m) = e_1 - me_2 \) with \( m = 1 \) for \( e_1, e_2 \in [r_1, r_2] \), and choosing \( \lambda = 0, \mu = 1 = k \) together with \( \alpha = 1 = n \), one gets the following inequality
\[
\Pr(X \leq x) + \frac{r_2 - x}{r_2 - r_1} \left( \frac{2}{r_2 - r_1} (r_2 - E(X)) \right)^{\frac{1}{3}}
\leq \left( \frac{x - r_1}{4(r_2 - r_1)} \right) \left[ \frac{1}{6} |\tau(r_1)|^q + \frac{5}{6} |\tau(x)|^q \right]^{\frac{1}{3}}
+ \left( \frac{2}{3} |\tau(r_1)|^q + \frac{1}{3} |\tau(x)|^q \right)^{\frac{1}{3}}
+ \left( \frac{1}{3} |\tau(x)|^q + \frac{2}{3} |\tau(r_2)|^q \right)^{\frac{1}{3}}.
\]
In particular, taking \( q = 1 \) yields that
\[
\Pr(X \leq x) + \frac{r_2 - x}{r_2 - r_1} \left( \frac{2}{r_2 - r_1} (r_2 - E(X)) \right)^{\frac{1}{3}}
\leq \left( \frac{x - r_1}{4(r_2 - r_1)} \right) \left[ |\tau(r_1)| + |\tau(x)| \right]
+ \left( \frac{2}{3} |\tau(r_1)| + \frac{1}{3} |\tau(x)| \right)^{\frac{1}{3}}
+ \left( \frac{1}{3} |\tau(x)| + \frac{2}{3} |\tau(r_2)| \right)^{\frac{1}{3}}.
\]

Proposition 32: In Theorem 15, taking \( \eta(e_1, e_2, m) = e_1 - me_2 \) with \( m = 1 \) for \( e_1, e_2 \in [r_1, r_2] \), and choosing \( \lambda = 1, \mu = 1 = k \) together with \( \alpha = 1 = n \), we have the following inequality
\[
\Pr(X \leq \frac{r_1 + x}{2}) + \frac{r_2 - x}{r_2 - r_1} \Pr(X \leq \frac{x + r_2}{2})
- \frac{r_2 - x}{r_2 - r_1} (r_2 - E(X)) \leq \frac{(x - r_1)^2}{8} \left[ \left( \frac{1}{3} |\tau(r_1)|^q + \frac{2}{3} |\tau(x)|^q \right)^{\frac{1}{3}}
+ \left( \frac{2}{3} |\tau(r_1)| + \frac{1}{3} |\tau(x)| \right)^{\frac{1}{3}}
+ \left( \frac{1}{3} |\tau(x)| + \frac{2}{3} |\tau(r_2)| \right)^{\frac{1}{3}}. \right.
\]
In particular, taking \( q = 1 \), we have that
\[
\Pr(X \leq \frac{r_1 + x}{2}) + \frac{r_2 - x}{r_2 - r_1} \Pr(X \leq \frac{x + r_2}{2})
- \frac{r_2 - x}{r_2 - r_1} (r_2 - E(X)) \leq \frac{(x - r_1)^2}{8} \left[ |\tau(r_1)| + |\tau(x)| \right]
+ \left( \frac{1}{3} |\tau(x)| + \frac{2}{3} |\tau(r_2)| \right)^{\frac{1}{3}}.
\]

Proposition 33: In Theorem 19, taking \( \eta(e_1, e_2, m) = e_1 - me_2 \) with \( m = 1 \) for \( e_1, e_2 \in [r_1, r_2] \), and choosing \( \lambda = 0, \mu = 1 = k \) together with \( \alpha = 1 = n \), we have the following inequality
\[
\Pr(X \leq x) + \frac{r_2 - x}{r_2 - r_1} \left( \frac{2}{r_2 - r_1} (r_2 - E(X)) \right)^{\frac{1}{3}}
\leq \left( \frac{x - r_1}{r_2 - r_1} \right) \left[ \frac{1}{4} |\tau(r_1)|^q + \frac{3}{4} |\tau(x)|^q \right]^{\frac{1}{3}}
+ \left( \frac{2}{3} |\tau(r_1)| + \frac{1}{3} |\tau(x)| \right)^{\frac{1}{3}}
+ \left( \frac{1}{3} |\tau(x)| + \frac{2}{3} |\tau(r_2)| \right)^{\frac{1}{3}}.
\]

Proposition 34: In Theorem 19, taking \( \eta(e_1, e_2, m) = e_1 - me_2 \) with \( m = 1 \) for \( e_1, e_2 \in [r_1, r_2] \), and choosing \( \lambda = 1, \mu = 1 = k \) together with \( \alpha = 1 = n \), we obtain the following inequality
\[
\Pr(X \leq \frac{r_1 + x}{2}) + \frac{r_2 - x}{r_2 - r_1} \Pr(X \leq \frac{x + r_2}{2})
- \frac{r_2 - x}{r_2 - r_1} (r_2 - E(X)) \leq \frac{(x - r_1)^2}{8} \left[ |\tau(r_1)| + |\tau(x)| \right]
+ \left( \frac{1}{3} |\tau(x)| + \frac{2}{3} |\tau(r_2)| \right)^{\frac{1}{3}}.
\]
\[
\left( \frac{1}{p+1} \right)^{\frac{1}{2}} \left\{ \frac{1}{4} \left[ \frac{1}{4} \left( r_1 \right)^{q} + \frac{3}{4} \left( r_2 \right)^{q} \right] \right. \\
\left. + \frac{3}{4} \left[ \frac{1}{4} \left( r_1 \right)^{q} + \frac{1}{4} \left( r_2 \right)^{q} \right] \right\} + \frac{(r_2 - r_1)^2}{4} \left\{ \frac{3}{4} \left| \left( r_1 \right)^{q} + \frac{1}{4} \left| \left( r_2 \right)^{q} \right| \right\} \right\}.
\]

### B. SPECIAL MEANS

Let us recall certain means as follows.

1. The arithmetic mean:
   \[ A(e_1, e_2) = \frac{e_1 + e_2}{2}, \quad e_1, e_2 \in \mathbb{R}. \]

2. The harmonic mean:
   \[ H(e_1, e_2) = \frac{2}{\frac{1}{e_1} + \frac{1}{e_2}}, \quad e_1, e_2 \in \mathbb{R} \setminus \{0\}. \]

3. The logarithmic mean:
   \[ L(e_1, e_2) = \frac{e_2 - e_1}{\ln|e_2| - \ln|e_1|}, \quad \text{where} \quad e_1, e_2 \in \mathbb{R}, \quad |e_1| \neq |e_2|, \quad e_1, e_2 \neq 0. \]

4. The generalized logarithmic mean:
   \[ L_s(e_1, e_2) = \left[ \frac{e_2^{s+1} - e_1^{s+1}}{(s+1)(e_2 - e_1)} \right]^{\frac{1}{s}}, \quad \text{where} \quad e_1, e_2 \in \mathbb{R}, \quad s \in \mathbb{Z} \setminus \{-1, 0\}, \quad e_1 \neq e_2. \]

**Remark 37:** Certain applications based on the obtained results to trapezoidal formulae can also be provided, and we omit the details.

### V. CONCLUSION

Based on a new identity with multiple parameters, we have obtained certain estimation-type results pertaining the \( k \)-fractional integral inequality for the first-order differentiable mappings. More results can be deduced by choosing different mappings \( \eta \) and the special parameter values such as \( \mu, k, n \) and \( \lambda \). It is an interesting topic to apply these estimations to random variables and to special means.

### REFERENCES

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