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Function Weighted Quasi-Metric Spaces and Fixed Point Results

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ABSTRACT Hereafter, the concept of a function weighted quasi-metric space is introduced. A necessary topology on this new structure is considered. A condition that ensures the uniqueness of the limit of a sequence in such a space is provided. A relation between the bi-convergent sequences and the bi-Cauchy ones are proved. Certain classes of operators with respect to their fixed point properties are investigated, having in view this framework. Examples that support our results are also included.

INDEX TERMS Contractions, fixed point, function weighted quasi-metric space.

I. INTRODUCTION

Fixed point theory can be formulated as the search for a solution to the equation $Fx = x$, F being a self-mapping on a non-empty set X. It is not a stretch to say that this theory was axiomatically initiated by Banach [7], who gave an affirmative answer by equipping X with a norm $\| \cdot \|$ and restricting F as a contraction (formally, there exists $\kappa \in [0, 1)$ so that $\|Fq-Fp\| \leq \kappa \|q-p\|$). Indeed, the Banach's solution is unique for the mentioned equation. The researchers have proposed either different structures or distinct criteria on mappings or both to improve and extend this research field. Having in view this aspect, characterizations of this classic principle have been proved in many abstract spaces. All these abstract structures have attracted the attention of researchers, who made studies of them from various points of view. In [2] the quasi-metric spaces are introduced, while in [3] a study on the completeness of such spaces is done. In [4] Suzuki mappings are considered in the setting of modular vector spaces. Reference [5] refers to modular metric spaces. The same structure is used in [6] with respect to some Meir-Keeler type contractions. Reference [7] and later [8] have in view the notion of b-metric spaces, while in [9] some Volterra integral inclusions are considered in this setting. A more general space, the quasi-b-metric like space has been used in [10] to develop Ulam stability results; also in [11] b-rectangular metric spaces are a fruitful framework for Banach type theorems. In [12] a generalized metric space was introduced, the so-called G-metric space, which proved to be a suitable setting to study Φ-mappings [14], or $(\psi - \phi)$-weakly contractive mappings [13]. Partial metric spaces were introduced in [15], while in [16] fixed point results in the sense of Berinde are developed. The notion was generalized even further in [17] to quasi-partial metric spaces. Ordered metric spaces are used in [18] to obtain results on cyclic nonlinear contractions. [19]–[21], or [22] refer to properties of fuzzy metric spaces and probabilistic metric spaces.

In this paper, we restrain ourselves only to $\mathcal{F}$-metric spaces, or function weighted metric spaces that were defined very recently [23]. Our main goal is to refine this notion by removing the symmetry condition, and to investigate the existence of a fixed point in this new structure, which we call quasi-$\mathcal{F}$-metric spaces.

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II. PRELIMINARIES

For the sake of the completeness and self-contained text, we recall the definition of Jleli and Samet [23]. In this regard, we need too fundamental classes of functions that we shall consider in the paper.

A function $f : (0, +\infty) \to \mathbb{R}$ is logarithmic-like if each sequence $\{\tau_n\} \subset (0, +\infty)$ satisfies

$$\lim_{n\to+\infty} \tau_n = 0 \text{ if and only if } \lim_{n\to+\infty} f(\tau_n) = -\infty. \qquad (LF)$$

$f : (0, +\infty) \to \mathbb{R}$ is a non-decreasing function if

$$0<\sigma < \tau \text{ implies } f(\sigma) \leq f(\tau). \qquad (ND)$$

The symbol $\mathfrak{F}$ represents the set of all functions that are non-decreasing (in symbol, (ND)) and logarithmic-like (in symbol, (LF)).

Example 2.1: *The following functions belong to* $\mathfrak{F}$

(i) $f\colon (0,\infty)$, $f(t) = \ln t$;

(ii) $f\colon (0,\infty)$, $f(t) = -e^{\frac{1}{t^p}}$, *where p is an odd positive integer;*

(iii) $f\colon (0,\infty)$, $f(t) = -\frac{1}{t^p}$, *where p is an odd positive integer.*

Henceforward, we presume that X is a nonempty set.

By the help of auxiliary functions of $\mathfrak{F}$, Jleli and Samet [23] defined a new metric, more precisely a function weighted metric space. Indeed, in this new metric definition, Jleli-Samet [23] replaced the standard triangle inequality by an inequality obtained by the use of a function from the set $\mathfrak{F}$ as in the next lines.

Definition 2.1: *Consider* $\delta\colon X\times X \to [0,+\infty)$ *a given mapping, for which there exist* $f \in \mathfrak{F}$ *and a constant* $\mathcal{C} \in [0,+\infty)$ *such that*

(Δ_1) $\delta(s,t) = 0$ *if and only if* $s = t$, *for* $s,t \in X$ *(self-distance axiom);*

(Δ_2) *for all* $s,t \in X$, *we have* $\delta(s,t) = \delta(t,s)$ *(symmetry axiom);*

(Δ_3) *for any pair* $s,t \in X$ *and for any* $\kappa \in \mathbb{N}$, $\kappa \geq 2$, *we have*

$$\delta(s,t) > 0 \implies f(\delta(s,t)) \leq f\left(\sum_{i=1}^{\kappa-1} \delta(u_i,u_{i+1})\right) + \mathcal{C},$$

for every $(u_i)_{i=1}^{N} \subset X$ *with* $(u_1,u_\kappa) = (s,t)$ *(generalized function* $\mathcal{F}$*-weighted triangle inequality axiom).*

Then, δ *is designated as "a function weighted metric" or "*$\mathcal{F}$*-metric", and the couple* (X,δ) *is named as "a function weighted metric space" or "*$\mathcal{F}$*-metric space".*

Herein after, instead of "a $\mathcal{F}$-metric space", we choose to use the words "a function weighted metric space".

Note that the only difference between a "standard metric space" and "a function weighted metric space" is the last axiom. More precisely, in "a function weighted metric space" the standard axiom "the triangle inequality" has been replaced by a new axiom, namely "the generalized $\mathcal{F}$-weighted triangle inequality". Based on this remark, we deduce that any metric on X is "a function weighted metric space" on X by letting $f(x) = \ln x$ for the axiom (Δ_3). Indeed, based on the triangle inequality, for all distinct $s,t \in X$ and each $\kappa \in \mathbb{N}$, $\kappa \geq 2$, and any $(\upsilon_i)_{i=1}^{\kappa} \subset X$ with $(\upsilon_1,\upsilon_\kappa) = (s,t)$, we get

$$d(s,t) > 0 \implies \ln(d(s,t)) \leq \ln\left(\sum_{i=1}^{\kappa-1} d(\upsilon_i,\upsilon_{i+1})\right),$$

since $d(s,t) \leq \sum_{i=1}^{\kappa-1} d(\upsilon_i,\upsilon_{i+1})$, and $f(x) = \ln x$ is non-decreasing. Here, we take $\mathcal{C} = 0$.

III. FUNCTION WEIGHTED QUASI-METRIC SPACES

We begin by introducing a new notion, namely "a function weighted quasi-metric" as follows.

Definition 3.1: *Consider* $\delta_q\colon X\times X \to [0,+\infty)$ *a given mapping for which there exist* $f \in \mathfrak{F}$ *and a constant* $\mathcal{C} \in [0,+\infty)$ *so that the conditions* (Δ_1) *and* (Δ_3) *from the definition of a function weighted metric are fulfilled. Then,* δ_q *is designated as "a function weighted quasi-metric" on* X. *Moreover, the couple* (X,δ_q) *is called a function weighted quasi-metric space.*

It can be observed that a function weighted quasi-metric space (X,δ_q) naturally induces another function weighted quasi-metric space, $\delta_q^{-1}\colon X\times X \to [0,\infty)$, $\delta_q^{-1}(s,t) = \delta_q(t,s)$. Moreover, there is a function weighted quasi-metric space which can be associated to these function weighted quasi-metric spaces, namely

$$\delta_q^*\colon X\times X \to [0,\infty), \quad \delta_q^*(s,t) = \max\{\delta_q(s,t), \delta_q^{-1}(s,t)\}.$$

Regarding the discussion above, we conclude that any quasi-metric is a function weighted quasi-metric by choosing $f(t) = \ln t$ for the axiom (Δ_3), with $\mathcal{C} = 0$.

Next, we give some examples of function weighted quasi-metric spaces which do not form function weighted metric spaces.

Example 3.1: *Let* X *denote the set of natural numbers. Consider* $\delta_q\colon X\times X \to [0,+\infty)$ *be the mapping*

$$\delta_q(s,t) = \begin{cases} 0, & \text{if } s = t; \\ (s-t)^2 + s, & \text{if } s,t \in \{0,1,2\},\ s \neq t; \\ |s-t| + s, & \text{otherwise.} \end{cases} \tag{1}$$

Obviously, δ_q *satisfies axiom* (Δ_1). *However,* δ_q *is not symmetric, since*

$$\delta_q(1,2) = 2 \neq 3 = \delta_q(2,1).$$

Let us prove that δ_q *fulfills the generalized triangle inequality. Consider* $s,t \in X$, $s \neq t$, *and* $(\upsilon_i)_{i=1}^{\kappa} \subset X$, *where* $\kappa \in \mathbb{N}$, $\kappa \geq 2$, *and* $(\upsilon_1,\upsilon_\kappa) = (s,t)$. *Without loss of generality we may assume that* $\upsilon_i \neq \upsilon_{i+1}$, $i = \overline{1,\kappa-1}$. *Let*

$$I = \{i = 1,2,\ldots,\kappa-1 : (\upsilon_i,\upsilon_{i+1}) \in \{0,1,2\}\times\{0,1,2\}$$

and

$$J = \{1,2,\ldots,\kappa-1\} \setminus I.$$

There are two possible cases, as in the following.

Case I: $(s,t) \notin \{0,1,2\}\times\{0,1,2\}$. *The following relations are checked*

$$\begin{aligned} &\delta_q(s,t) \\ &= |s-t| + s \\ &\leq \sum\nolimits_{i=1}^{\kappa-1}(|\upsilon_{i+1}-\upsilon_i| + \upsilon_i) \\ &= \sum\nolimits_{i\in I}(|\upsilon_{i+1}-\upsilon_i| + \upsilon_i) + \sum\nolimits_{j\in J}(|\upsilon_{j+1}-\upsilon_j| + u_j) \\ &\leq \sum\nolimits_{i\in I}((\upsilon_{i+1}-\upsilon_i)^2 + \upsilon_i) + \sum\nolimits_{j\in J}(|\upsilon_{j+1}-\upsilon_j| + \upsilon_j) \\ &= \sum\nolimits_{i=1}^{N-1} \delta_q(\upsilon_i,\upsilon_{i+1}). \end{aligned}$$

Case II: $(s, t) \in \{0, 1, 2\} \times \{0, 1, 2\}$. *Here, we find*

$$\begin{aligned}
&\delta_q(s, t)\\
&= |s - t|^2 + s\\
&\le 2(|s - t| + s)\\
&\le 2\left(\sum_{i\in I}(|\upsilon_{i+1} - \upsilon_i| + \upsilon_i) + \sum_{j\in J}(|\upsilon_{j+1} - \upsilon_j| + \upsilon_j)\right)\\
&\le 2\left(\sum_{i\in I}(|\upsilon_{i+1} - \upsilon_i|^2 + \upsilon_i) + \sum_{i\in J}(|\upsilon_{j+1} - \upsilon_j| + \upsilon_j)\right)\\
&= 2\sum_{i=1}^{N-1} \delta_q(\upsilon_i, \upsilon_{i+1}).
\end{aligned}$$

Taking advantage of the previous analysis, we obtain that for each $s, t \in X$, *each* $\kappa \in \mathbb{N}$, $\kappa \ge 2$, *and each* $(\upsilon_i)_{i=1}^{\kappa} \subset X$ *with* $(\upsilon_1, \upsilon_\kappa) = (s, t)$,

$$\delta_q(s, t) \le 2\sum_{i=1}^{\kappa-1} \delta_q(\upsilon_i, \upsilon_{i+1}), \tag{2}$$

as long as $s \ne t$. *This implies, for* $s \ne t$, *that*

$$\ln(\delta_q(s, t)) \le \ln\left(\sum_{i=1}^{\kappa-1} \delta_q(\upsilon_i, \upsilon_{i+1})\right) + \ln 2.$$

By taking $f(t) = \ln t$, *for* $t > 0$, *and* $\mathcal{C} = \ln 2$, *it follows that axiom* (Δ_3) *is fulfilled. Therefore,* δ_q *is a quasi* $\mathcal{F}$*-metric on* X.

Thus the example above indicates that "a function weighted quasi-metric" is a genuine notion.

Definition 3.2: $\delta_q{}^*: X \times X \to [0, +\infty)$ *is a function weighted relaxed-quasi-metric if it satisfies* (Δ_1), *and additionally,*

$(\Delta_3{}^*)$ *There is* $\lambda \ge 1$ *so that for all* $s, t \in X$, *and each* $\kappa \in \mathbb{N}$, $\kappa \ge 2$, *and each* $(\upsilon_i)_{i=1}^{\kappa} \subset X$ *with* $(\upsilon_1, \upsilon_\kappa) = (s, t)$, *the inequality below is satisfied*

$$\delta_q{}^*(s, t) \le \lambda\sum_{i=1}^{\kappa-1} \delta_q{}^*(\upsilon_i, \upsilon_{i+1}).$$

Note that $\delta_q{}^*$ satisfies (Δ_3) with $f(x) = \ln x$, $x > 0$, and $\mathcal{C} = \ln\lambda$. Hence, the notion of "a function weighted quasi-metric" on X is more general than that of "a function weighted relaxed-quasi-metric" on X.

Remark 3.1: Here, on account of (2), *the mapping* δ_q *defined by* (1) *is a function weighted relaxed-quasi-metric on* X *with* $\lambda = 2$.

The following example proves that the class of "function weighted quasi-metrics" is wider than that of "function weighted relaxed-quasi-metrics".

Example 3.2: Consider X *as the set of natural numbers, and the distance* $\delta_q: X \times X \to [0, +\infty)$,

$$\delta_q(s, t) = \begin{cases} 0, & \text{if } s = t,\\ \exp|s - t| + s, & \text{if } s \ne t,\end{cases} \tag{3}$$

for $s, t \in X$. *Obviously, axiom* (Δ_1) *is checked.*

In the following, we prove that δ_q *is not a function weighted relaxed-quasi- metric. Presume that* δ_q *fulfills condition* (Δ_3^*) *of Definition 3.2, with* $\lambda \ge 1$. *By utilizing this condition, we get*

$$\delta_q(2n, 0) \le \lambda\left(\delta_q(2n, n) + \delta_q(n, 0)\right), \quad n \in \mathbb{N}^*.$$

This leads us to

$$\exp(2n) + 2n \le \lambda(2\exp(n) + 3n), \quad n \in \mathbb{N}^*,$$

or

$$\exp(2n) - 2\lambda\exp(n) \le 3\lambda n - 2n, \quad n \in \mathbb{N}^*.$$

By dividing by $\exp(2n) - \lambda\exp n$, *and by taking* $n \to +\infty$ *in the inequality obtained, it follows a contradiction. Therefore,* δ_q *is not a function weighted relaxed-quasi- metric.*

Next, we check that δ_q *is a function weighted quasi-metric. Take*

$$f(t) = -\frac{1}{t}, \quad t > 0,$$

which belongs to $\mathfrak{F}$.

In order to verify axiom (Δ_3), *take two distinct arbitrary points* $s, t \in X$. *For each* $\kappa \in \mathbb{N}$, $\kappa \ge 2$, *and each* $(\upsilon_i)_{i=1}^{\kappa} \subset X$ *with* $(\upsilon_1, \upsilon_\kappa) = (s, t)$, *the following relations hold true*

$$\begin{aligned}
&1 + f\left(\sum_{i=1}^{\kappa-1} \delta_q(\upsilon_i, \upsilon_{i+1})\right) - f(\delta_q(s, t))\\
&= 1 - \frac{1}{\displaystyle\sum_{\substack{i=1\\ \upsilon_{i+1}\ne\upsilon_i}}^{\kappa-1} (\exp|\upsilon_{i+1} - \upsilon_i| + \upsilon_i)} + \frac{1}{\exp|s - t| + s}\\
&\ge 1 - 1 + \frac{1}{\exp|s - t| + s}\\
&\ge 0.
\end{aligned}$$

Hence, we have proved that

$$f(\delta_q(s, t)) \le f\left(\sum_{i=1}^{\kappa-1} \delta_q(\upsilon_i, \upsilon_{i+1})\right) + 1,$$

and, consequently, δ_q *satisfies axiom* (Δ_3) *with* $f(t) = -\frac{1}{t}$, $t > 0$, *and* $\mathcal{C} = 1$. *Then* δ_q *is a function weighted quasi-metric.*

Recall that a distance function $\delta_q: X \times X \to [0, \infty)$ is called quasi-b-metric (see e.g. [10], [11]) if

(Δ_1^{**}) $\delta_q(s, t) = 0 \Leftrightarrow s = t$;

(Δ_3^{**}) $\delta_q(s, t) \le \lambda[\delta_q(s, v) + \delta_q(v, t)]$,

for all $s, t, v \in X$, where $\lambda \ge 1$.

We claim that there exist function weighted quasi-metric spaces which do not fulfill all the axioms of quasi b-metric spaces. It is straightforward that hypothesis (Δ_3^{**}) is less general than hypothesis (Δ_3^*). Looking at the other part, any function weighted relaxed-quasi-metric is a quasi b-metric. The following indicates that the reverse does not hold.

Example 3.3: Take $X = [0, 1]$, *with the distance*

$$\delta_q: X \times X \to [0, +\infty), \delta_q(s, t) = \begin{cases} (s - t)^2 + s^2, & s \ne t;\\ 0, & s = t.\end{cases}$$

It is clear that δ_q *is a quasi b-metric on* X, *where* $\lambda = 2$. *Suppose that there exists* $\mathcal{C} \in [0, +\infty)$ *and* $f \in \mathfrak{F}$ *in a way* *that δ_q satisfies (Δ_3). Set $n \in \mathbb{N}^*$, and define*

$$\upsilon_i = \frac{i}{n}, \quad i = 1, 2, \ldots, n-1.$$

On the account of (Δ_3), we derive that $f(\delta_q(0,1)) \leq f\left(\delta_q(0,\upsilon_1) + \delta_q(\upsilon_1,\upsilon_2) + \ldots + \delta_q(\upsilon_{n-1},1)\right) + \mathcal{C}$, $n \in \mathbb{N}^*$, i.e.,

$$f(1) \leq f\left(\frac{1}{n} + \frac{(n-1)(2n-1)}{6n(n-1)^2}\right) + \mathcal{C}, \quad n \in \mathbb{N}^*.$$

Keeping in mind the properties of the mapping f, we have

$$f(1) \leq \lim_{n\to+\infty} f\left(\frac{1}{n} + \frac{(n-1)(2n-1)}{6n(n-1)^2}\right) + \mathcal{C} = -\infty,$$

a contradiction.

Remark 3.2: Example 3.2 indicates that δ_q defined in (3) is a function weighted quasi-metric. On the other hand, a function weighted metric does not form a function weighted relaxed-quasi-metric. Indeed, it is easily observed that δ_q is not a quasi-b-metric.

IV. SOME PROPERTIES OF THE TOPOLOGY OF FUNCTION WEIGHTED QUASI-METRIC SPACES

This section aims to define a topology by means of function weighted quasi-metric spaces, and study its characteristics.

Let (X, δ_q) be a function weighted quasi-metric space. For $x \in X$, the right centered ball at x, and of radius $\varrho > 0$ is the set

$$B_r(s, \varrho) = \{t \in X : \delta_q(s,t) < \varrho\};$$

respectively, the left centered ball at x, and of radius $\varrho > 0$ is

$$B_l(s, \varrho) = \{t \in X : \delta_q(t,s) < \varrho\}.$$

Our next purpose is to define the convergence in the setting offered by function weighted quasi-metric spaces.

Definition 4.1: Let $\{s_n\}$ be a sequence in a function weighted quasi-metric space (X, δ_q). The sequence $\{s_n\}$ is right-convergent (respectively, left-convergent) to $s \in X$ if $\lim_{n\to+\infty} \delta_q(s,s_n) = 0$ (respectively, $\lim_{n\to+\infty} \delta_q(s_n,s) = 0$). A sequence $\{s_n\}$ is bi-convergent (or, simply, convergent) to $s \in X$ if

$$\lim_{n\to+\infty} \delta_q(s,s_n) = 0 = \lim_{n\to+\infty} \delta_q(s_n,s).$$

With regard to the limit of such a sequence in a function weighted quasi-metric space, the uniqueness property is satisfied, as follows from the next proposition.

Proposition 4.1: Let (X, δ_q) be a function weighted quasi-metric space, and $\{s_n\} \subset X$. If $s, t \in X$ such that

$$\lim_{n\to+\infty} \delta_q(s,s_n) = \lim_{n\to+\infty} \delta_q(s_n,t) = 0,$$

then $s = t$.

Proof: Let $s \neq t$ be points from X, such that

$$\lim_{n\to+\infty} \delta_q(s,t_n) = \lim_{n\to+\infty} \delta_q(s_n,t) = 0.$$

Using the generalized triangle property, we get that there exists a pair $(f, \mathcal{C}) \in \mathfrak{F} \times [0,+\infty)$ such that

$$f(\delta_q(s,t)) \leq f(\delta_q(s,t_n) + \delta_q(s_n,t)) + \mathcal{C}, \quad \text{for all } n.$$

By means of axiom (Δ_3) and property (LF), it follows

$$\lim_{n\to+\infty} f(\delta_q(s,s_n) + \delta_q(s_n,t)) + \mathcal{C} = -\infty,$$

a contradiction. Thus, we have $s = t$. □

The next stage is to define the notion of a Cauchy sequence in such generalized metric spaces.

Definition 4.2: Consider that (X, δ_q) is a function weighted quasi-metric space, and $\{s_n\}$ a sequence in X. $\{s_n\}$ is a right-Cauchy sequence (respectively, a left-Cauchy sequence) if $\lim_{\substack{n,m\to+\infty\\ m\geq n}} \delta_q(s_n,s_m) = 0$ (respectively, $\lim_{\substack{n,m\to+\infty\\ m\geq n}} \delta_q(s_m,s_n) = 0$). The sequence $\{s_n\}$ is bi-Cauchy (or, simply, Cauchy) if it is both left and right-Cauchy.

A function weighted quasi-metric space (X, δ_q) is called right-complete if every right-Cauchy sequence in X is right-convergent to $x \in X$. Analogously, we define left-completeness. (X, δ_q) is bi-complete (or, in short, complete) if it is both left and right-complete.

Example 4.1: On $X = \mathbb{N}$ consider the function weighted quasi-metric $\delta_q : X \times X \to [0,+\infty)$ defined in (3). It has been proved (see Example 3.2) that (X, δ_q) is a function weighted quasi-metric space with respect to $f(t) = -\frac{1}{t}$, $t > 0$, and $\mathcal{C} = 1$. We focus now on the completeness of this space. Consider $\{s_n\} \subset X$ a Cauchy sequence, that is

$$\lim_{\substack{n,m\to+\infty\\ m\geq n}} \delta_q(s_n,s_m) = \lim_{\substack{n,m\to+\infty\\ m\geq n}} \delta_q(s_m,s_n) = 0.$$

Hence, there exists $\kappa \in \mathbb{N}$ for which

$$\delta_q(s_n,s_m) < \frac{1}{2}, \quad n, m \geq \kappa, m \geq n.$$

Presume that there are $n, m \geq N$, $m \geq n$, so that $s_n \neq s_m$. It follows

$$1 \leq \exp(|s_n - s_m|) + s_n^2 = \delta_q(s_n,s_m) < \frac{1}{2},$$

a contradiction. Hence $s_n = s_\kappa$, for all $n \geq \kappa$, which compels

$$\lim_{n\to+\infty} \delta_q(s_\kappa,s_n) = 0,$$

so $\{s_n\}$ converges to s_κ. The proof has been completed.

Proposition 4.2: Let (X, δ_q) be a function weighted quasi-metric space. If $\{s_n\} \subset X$ is bi-convergent, then it is bi-Cauchy.

Proof: Consider that δ_q is a function weighted quasi-metric with regard to $(f, \mathcal{C}) \in \mathfrak{F} \times [0,+\infty)$, and $x \in X$, for which

$$\lim_{n\to+\infty} \delta_q(s,s_n) = \lim_{n\to+\infty} \delta_q(s,s_n) = 0.$$

Let $\varepsilon > 0$. Property (Δ_3) compels there exists $\delta > 0$ with the property

$$0 < t < \delta \text{ implies } f(t) < f(\varepsilon) - \mathcal{C}.$$

Moreover, there exists $\kappa \in \mathbb{N}$ such that

$$\delta_q(s_n, s) + \delta_q(s, s_m) < \delta, \quad n, m \geq \kappa. \tag{4}$$

Let $m \geq n \geq \kappa$. If $s_m = s_n$, obviously $\delta_q(x_n, x_m) = 0 < \varepsilon$.

Suppose now that $s_m \neq s_n$. By the use of (4), it follows that

$$0 < \delta_q(s_n, s) + \delta_q(s, s_m) < \delta,$$

hence

$$f(\delta_q(s_n, s) + \delta_q(s_m, s)) < f(\varepsilon) - \mathcal{C}.$$

The second property of δ_q leads to

$$f(\delta_q(s_n, s_m)) \leq f(\delta_q(s_n, s) + \delta_q(s, s_m)) + \mathcal{C} < f(\varepsilon),$$

and we obtain $\delta_q(s_n, s_m) < \varepsilon$. It follows that

$$\delta_q(s_n, s_m) < \varepsilon, \quad n, m \geq \kappa,$$

that is

$$\lim_{\substack{n,m\to+\infty \\ m\geq n}} \delta_q(s_n, s_m) = 0.$$

Similarly, it can be proved that $\lim_{\substack{n,m\to+\infty \\ n\geq m}} \delta_q(s_n, s_m) = 0$. Hence, $\{s_n\}$ is bi-Cauchy. □

V. AN ANALOG OF BANACH'S FIXED POINT THEOREM ON FUNCTION WEIGHTED QUASI-METRIC SPACES

In the following, we use the framework provided by function weighted quasi-metric spaces to state and prove a Banach contraction type result.

Theorem 5.1: Let T be a self-mapping on a function weighted quasi-metric space (X, δ_q). If (X, δ_q) is bi-complete, and there exists $k \in (0, 1)$ such that

$$\delta_q(Ts, Tt) \leq k\delta_q(s, t), \quad s, t \in X, \tag{5}$$

then T possess a unique fixed point $s^* \in X$.

Proof: Suppose that δ_q is a function weighted quasi-metric with respect to the pair $(f, \mathcal{C}) \in \mathfrak{F}\times[0, +\infty)$. Let $\varepsilon > 0$. Then there exists $\delta > 0$ for which the next implication holds true

$$0 < t < \delta \text{ implies } f(t) < f(\varepsilon) - \mathcal{C}. \tag{6}$$

Let $s_0 \in X$, and define $s_{n+1} = Ts_n, n \in \mathbb{N}$. If there is an index n^* such that $\delta_q(s_n^*, s_{n^*+1}) = 0$, then s_{n^*} is a fixed point of T. Without loss of generality, in the sequel we may assume that $\delta_q(s_n, s_{n+1}) > 0$. From inequality (5),

$$\delta_q(s_n, s_{n+1}) \leq k^n\delta_q(s_0, s_1), \quad n \in \mathbb{N},$$

we get

$$\sum_{i=n}^{m-1} \delta_q(s_i, s_{i+1}) \leq \frac{k^n}{1-k}\delta_q(s_0, s_1), \quad m > n.$$

But

$$\lim_{n\to+\infty} \frac{k^n}{1-k}\delta_q(s_0, s_1) = 0,$$

so there exists $K \in \mathbb{N}$ for which

$$0 < \frac{k^n}{1-k}\delta_q(s_0, s_1) < \delta, \quad n \geq K.$$

By the use of both (6) and $(\mathcal{F}_1)$, we obtain, for $m > n \geq K$,

$$f\left(\sum_{i=n}^{m-1} \delta_q(s_i, s_{i+1})\right) \leq f\left(\frac{k^n}{1-k}\delta_q(s_0, s_1)\right) < f(\varepsilon) - \mathcal{C}.$$

Using (Δ_3), the inequality

$$\delta_q(s_n, s_m) > 0, \quad m > n \geq K$$

implies that

$$f(\delta_q(s_n, s_m)) \leq f\left(\sum_{i=n}^{m-1} \delta_q(s_i, s_{i+1})\right) + \mathcal{C} < f(\varepsilon)$$

hold true, so $\delta_q(s_n, s_m) < \varepsilon, \quad m > n \geq \kappa$. We have proved that $\{s_n\}$ is a right-Cauchy sequence. Analogously, by revising the order of the pairs (s_{i+1}, s_i) in the related expression above, we conclude also that $\{s_n\}$ is a left-Cauchy sequence and hence, it is a Cauchy sequence. Recall that (X, δ_q) is bi-complete, therefore there exists $s^* \in X$ such that $\{s_n\}$ is convergent to s^*, i.e.

$$\lim_{n\to+\infty} \delta_q(s^*, s_n) = 0. \tag{7}$$

Similarly, it can be proved that there exists t^* such that $\lim_{n\to\infty} \delta_q(s_n, t^*) = 0$. By Proposition 4.1, it follows $s^* = t^*$.

Let us prove now that x^* is a fixed point of T. Presume $\delta_q(Ts^*, s^*) > 0$. By (Δ_3), we have

$$f(\delta_q(Ts^*, s^*)) \leq f(\delta_q(Ts^*, s_{n+1}) + \delta_q(s_{n+1}, s^*)) + \mathcal{C}, n \in \mathbb{N}.$$

Furthermore,

$$f(\delta_q(Ts^*, s^*)) \leq f(k\delta_q(s^*, s_n) + \delta_q(s_{n+1}, s^*)) + \mathcal{C}, \quad n \in \mathbb{N}.$$

From (Δ_3), we obtain

$$\lim_{n\to+\infty} f(k\delta_q(s^*, s_n) + \delta_q(s_{n+1}, s^*)) + \mathcal{C} = -\infty,$$

a contradiction. As a conclusion, we have $\delta_q(Ts^*, s^*) = 0$, so $s^* = Ts^*$.

As a last stage, we indicate the uniqueness. Let $s, t \in X$ two distinct fixed points of T. The contraction condition implies

$$\delta_q(s^*, t^*) = \delta_q(Ts^*, Tt^*) \leq k\delta_q(s^*, t^*) < \delta_q(s^*, t^*),$$

a contradiction. Accordingly, we deduce that $s^* \in X$ is the unique fixed point of T. □

Theorem 5.2: Let T be a self-mapping on a function weighted quasi-metric space (X, δ_q). Assume (X, δ_q) is bi-complete, and there can be found $\mu \in (0, \frac{1}{2})$ such that

$$\delta_q(Ts, Tt) \leq \mu[\delta_q(s, Ts) + \delta_q(t, Tt)], \quad s, t \in X. \tag{8}$$

Then T possess a unique fixed point $s^* \in X$.

Proof: By analogy to the proof of Theorem 5.1, we derive an iterative sequence $\{s_n\}$ whose successive terms are distinct, that is, $\delta_q(s_n, s_{n+1}) > 0$. From inequality (8),

$$\delta_q(s_n, s_{n+1}) \leq \mu[\delta_q(s_{n-1}, s_n) + \delta_q(s_n, s_{n+1})], \quad n \in \mathbb{N}.$$

After an adequate evaluation, we find

$$\delta_q(s_n, s_{n+1}) \leq k\delta_q(s_{n-1}, s_n), \quad n \in \mathbb{N},$$

where $k = \frac{\mu}{1-\mu} < 1$. Thus, we have

$$\delta_q(s_n, s_{n+1}) \leq k^n\delta_q(s_0, s_1), \quad n \in \mathbb{N}.$$

Furthermore, by using the corresponding arguments as in the proof of Theorem 5.1, we are lead to the conclusion that $\{s_n\}$ is a Cauchy sequence. Recall that (X, δ_q) is bi-complete, therefore there exists $s^* \in X$ such that $\{s_n\}$ converges to s^*, i.e.

$$\lim_{n\to+\infty} \delta_q(s^*, s_n) = 0 = \lim_{n\to+\infty} \delta_q(s_n, s^*) = 0.$$

Let us prove now that s^* is a fixed point of T. Presume $\delta_q(s^*, Ts^*) > 0$. By (8), we have

$$\delta_q(Ts_n, Ts^*) \leq \mu\delta_q(s_n, Ts_n) + \mu\delta_q(s^*, Ts^*),$$

which, by having in view also that $\delta_q(s^*, Ts^*) \leq \delta_q(s^*, Ts_n) + \delta^*(Ts_n, Ts^*)$, and taking $n \to +\infty$, leads to

$$\delta_q(s^*, Ts^*) \leq \mu\delta_q(s^*, Ts^*),$$

so $Ts^* = s^*$.

As a last step, we indicate the uniqueness. Let s^*, $t^* \in X$ two distinct fixed points of T. The contraction condition implies

$$\delta_q(s^*, t^*) = \delta_q(Ts^*, Tt^*) \leq \mu[\delta_q(s^*, Ts^*) + \delta_q(t^*, Tt^*)] = 0,$$

a contraction. Accordingly, we deduce that uniqueness of the fixed point of T. □

VI. CONCLUSION

The concept of a function weighted quasi-metric space has been obtained by dropping the symmetry condition from $\mathcal{F}$-metric spaces. Topological aspects and fixed point properties are studied in this setting. As further developments, we intend to have in view modifying the first condition from the definition of a function weighted quasi-metric space and do a research on how this change reflects on the topological structure, and also obtain some fixed point results in this framework.

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REFERENCES

[1] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations inégrales," *Fund. Math.*, vol. 3, no. 1, pp. 133–181, 1922.
[2] W. A. Wilson, "On quasi-metric spaces," *Amer. J. Math.*, vol. 53, no. 1, pp. 675–684, 1931.
[3] D. Doitchinov, "On completeness in quasi-metric spaces," *Topology Appl.*, vol. 30, no. 2, pp. 127–148, 1988.
[4] A. Bejenaru and M. Postolache, "On Suzuki mappings in modular spaces," *Symmetry-Basel*, vol. 11, no. 3, 2019, Art. No. 319.
[5] V. V. Chistyakov, "Modular metric spaces, I: Basic concepts," *Nonlinear Anal., Theory, Methods Appl.*, vol. 72, pp. 1–14, Jan. 2010.
[6] U. Aksoya, E. Karapınar, I. Erhan, and V. Rakočević, "Meir-keeler type contractions modular metric spaces," *Filomat*, vol. 32, no. 10, pp. 3697–3707, 2018.
[7] I. A. Bakhtin, "The contraction mapping principle in almost metric spaces," *Fac. Math. Inform., Plovdiv Univ.*, vol. 30, pp. 26–37, 1986.
[8] S. Czerwik, "Contraction mappings in *b*-metric spaces," *Acta Math. Inform. Universitatis Ostraviensis*, vol. 1, no. 1, pp. 5–11, 1993.
[9] M. U. Ali, T. Kamran, and M. Postolache, "Solution of Volterra integral inclusion in b-metric spaces via new fixed point theorem," *Nonlinear Anal. Model. Control*, vol. 22, no. 1, pp. 17–30, 2017.
[10] H. Alsulami, S. Gulyaz, E. Karapınar, and I. Erhan, "An Ulam stability result on quasi-*b*-metric-like spaces," *Open Math.*, vol. 14, no. 1, pp. 1087–1103, 2016.
[11] Z. D. Mitrović, "A note on a Banach's fixed point theorem in b-rectangular metric space and b-metric space," *Math. Slovaca*, vol. 68, no. 5, pp. 1113–1116, 2018.
[12] Z. Mustafa and B. Sims, "A new approach to generalized metric spaces," *J. Nonlinear Convex Anal.*, vol. 7, no. 2, pp. 289–297, 2006.
[13] H. Aydi, M. Postolache, and W. Shatanawi, "Coupled fixed point results for (ψ, ϕ)-weakly contractive mappings in ordered *G*-metric spaces," *Comput. Math. Appl.*, vol. 63, no. 1, pp. 298–309, 2012.
[14] W. Shatanawi, "Fixed point theory for contractive mappings satisfying open image in new window ψ-maps in open image in new window G-metric spaces," *Fixed Point Theory Appl.*, vol. 2010, Dec. 2010, Art. no. 181650.
[15] S. G. Matthews, "Partial metric topology," *Ann. New York Acad. Sci.*, vol. 728, no, 1, pp. 183–197, 1994.
[16] W. Shatanawi and M. Postolache, "Coincidence and fixed point results for generalized weak contractions in the sense of Berinde on partial metric spaces," *Fixed Point Theory Appl.*, vol. 2013, no. 1, 2013, Art. no. 54.
[17] E. Karapınar, I. Erhan, and A. Öztürk, "Fixed point theorems on quasi-partial metric spaces," *Math. Comput. Model.*, vol. 57, pp. 2442–2448, May 2013.
[18] W. Shatanawi and M. Postolache, "Common fixed point results for mappings under nonlinear contraction of cyclic form in ordered metric spaces," *Fixed Point Theory Appl.*, vol. 2013, Mar. 2013, Art. no. 60.
[19] A. George and P. Veeramani, "On some results in fuzzy metric spaces," *Fuzzy Sets Sys.*, vol. 64, Jun. 1994, pp. 395–399.
[20] O. Kramosil and J. Michálek, "Fuzzy metrics and statistical metric spaces," *Kybernetika*, vol. 11, no. 5, pp. 336–344, 1975.
[21] K. Menger, B. Schweizer, and A. Sklar, "On probabilistic metrics and numerical metrics with probability," *Czechoslovak Math. J.*, vol. 9, no. 3, pp. 459–466, 1959.
[22] Z. Shi-Sheng, "Basic theory and applications of probabilistic metric spaces (I)," *Appl. Math. Mech.*, vol. 9, no. 2, pp. 123–133, 1998.
[23] M. Jleli and B. Samet, "On a new generalization of metric spaces," *J. Fixed Point Theory Appl.*, vol. 20, p. 128, Sep. 2018.

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