Angle-Polarization-Range Estimation Using Sparse Polarization Sensitive FDA-MIMO Radar With Co-prime Frequency Offsets

JUTING WANG1, SHENGLI JIANG2
1Department of Mechanical and Electronic Engineering, Chizhou University, Chizhou 247000, China (e-mail: jtwang767@sina.com)
2School of Electronic Information and Electrical Engineering, Hefei Normal University, Hefei 230601, China (e-mail: slijiang1979@163.com)
Corresponding author: Shengli Jiang (e-mail: slijiang1979@163.com).
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ABSTRACT Polarization sensitive MIMO radar with frequency diverse array (PSFDA-MIMO) can provide additional target polarization information. It has a better target localization, tracking, classification, and recognition performance, compared with FDA-MIMO radar. In this paper, a sparse polarization sensitive FDA-MIMO radar with co-prime frequency offsets (SPS-CopFDA-MIMO radar) is proposed and a fast parameter estimation algorithm based on propagator method is developed. In receiver, the SPS-CopFDA-MIMO radar introduces full component electromagnetic vector sensor (EMVS) spaced much farther apart than a half-wavelength to achieve aperture extension, therefore provides high accuracy and unambiguous estimates of angle and polarization. In transmitter, the SPS-CopFDA-MIMO radar utilizes co-prime frequency offsets to enhance range resolution without decreasing the maximum unambiguous range. In the proposed algorithm, the angle, polarization, and range are estimated successively. First, polarization diversity is utilized to resolve the phase ambiguity induced by the sparse receiving array. Then, the polarization parameters are estimated based on these estimated angle parameters. Finally, a two-step phase unwrapping processing is proposed to get the high resolution and unambiguous range estimates. The Cramér-Rao bounds for angle, polarization and range are derived. The performance of the proposed SPS-CopFDA-MIMO radar technique is also validated by computer simulation results.

INDEX TERMS MIMO radar, frequency diverse array, sparse vector-sensor array, co-prime frequency offsets, phase unwrapping, parameter estimation

I. INTRODUCTION

In recent years, MIMO radar with frequency diverse array (FDA-MIMO) has received an increased investigation [1-6]. Xu et al. proposed an unambiguous method for joint range and angle estimation using FDA-MIMO radar [2]. In [3], an adaptive range-angle-Doppler processing approach was proposed to suppress the ambiguous ground clutter for airborne FDA-MIMO radar, utilizing additional degrees of freedoms (DOFs) in the range dimension. In [4], a sparse model was adapted to design the transmitting beamspace for FDA-MIMO radar. Xiong analyzed the theoretical performance of range–angle estimation algorithm based on the multiple signal classification (MUSIC) method for FDA-MIMO radar [5].

The FDA exploits a uniform frequency offset, and there is a tradeoff between the range resolution and maximum unambiguous range. Co-prime sampling attracts much attention in the past few years [7-15], which can offer enhanced DOFs for parameter estimation. Integrating co-prime sampling and FDA can enhance range resolution without decreasing the maximum unambiguous range. In [16], Qin proposed a FDA radar using both co-prime arrays and co-prime frequency offsets, and developed an angle-range estimation algorithm based on compressive sensing approach. In [17], a planar FDA-MIMO array with co-prime frequency offsets was proposed, and three different parameter estimation algorithms based on MUSIC method were developed.

Most of the works mentioned above utilized scalar an-
tennas, and the radar can only get the corresponding scalar measurements of the reflected electromagnetic signals. As we know, the reflected signals consist of electric and magnetic components. Polarization information combined with angle, range and frequency information can improve the performance of target localization, tracking, classification, and recognition [18-30]. In [18-25], the reception polarization diversity technique was addressed to measure both horizontal and vertical components of the targets’ echo, utilizing electromagnetic vector-sensor (EMVS) array in receiver. The full-polarization matched-illumination technique is another subject of considerable interest. It can control the transmit waveform polarization to match the polarization of targets, utilizing EMVS in both transmitter and receiver [26-30]. We will address the reception polarization diversity technique in FDA-MIMO radar in this work.

Recently, Li introduced a polarization sensitive FDA-MIMO radar (PSFDA-MIMO radar) to extract angle-range-polarization information [24], where the crossed dipoles were used in receiver and scalar antennas were deployed in transmitter. A successive algorithm was also developed to reduce the computational complexity, based on the estimating signal parameter via rotational invariance techniques (ESPRIT) [31]. However, the ESPRIT-based algorithm in [24] did not consider the cyclical ambiguity of phase difference between two adjacent transmit sensors in FDA-MIMO array, which may lead to the failure of range estimation if phase ambiguity occurs.

In order to improve the performance of PSFDA-MIMO radar, we propose a sparse polarization sensitive FDA-MIMO radar with co-prime frequency offsets (SPS-CopFDA-MIMO radar) and develop a computationally efficient angle-range-polarization estimation algorithm based on the propagator method [32, 33]. In order to enhance array resolution and parameter estimation precision, we utilize sparse full component EMVS array in receiver. The sparse EMVS array consists of three orthogonal electric dipoles and another three orthogonal magnetic loops, co-located in space with the intersensor spacing beyond a half-wavelength [19, 20]. Meanwhile, in order to improve the range resolution, we utilize coprime frequency offsets in transmitter, without decreasing the maximum unambiguous range. In the proposed algorithm, we get the unambiguous and accurate parameter estimates successively. First, we extract the unambiguous but low accurate direction cosine estimates from the full component vector sensor. Then, based on these estimates, we resolve the angle ambiguity of the sparse array to obtain the precise and unambiguous angle estimates and polarization parameters. Finally, we develop a two-step phase unwrapping processing to get the high resolution and unambiguous range estimates.

The main contributions of this work are summarized as follows:
(a) The resolution and parameter estimation precision of the PSFDA-MIMO radar is improved, by exploiting both the sparse EMVS array and co-prime frequency offsets. The Cramér-Rao bounds (CRBs) for the angle, polarization and range are also derived.
(b) A computationally efficient algorithm based on the propagator method is proposed to estimate the angle, polarization, and range successively.
(c) A novel phase disambiguation procedure for range estimation is proposed to solve the phase ambiguity problem in search-free algorithms, thus the proposed algorithm is more robust for range estimation than Li’s method.

Notation: Throughout the paper, scalars are denoted by lowercase letters. Bold lowercase letters are used for vectors and uppercase letters for matrices. $\ast$, $T$, $H$ and $\dagger$ represent complex conjugate, the transpose, conjugate transpose, and the pseudo inverse, respectively. $\text{Tr}$ denotes trace and $\text{Re}$ denotes real parts. $\otimes$ symbolizes the Kronecker product, and $\odot$ denotes the Hadamard product. $I_m$ denotes the identity matrix and $0_{m \times n}$ represents the $m \times n$ zero matrix.

II. SIGNAL MODEL OF SPS-COPFDA-MIMO RADAR

Fig. 1 depicts the configuration for the SPS-CopFDA-MIMO radar. A uniform linear scalar array with $M = M_t + N_t$ antennas is utilized in transmitter, and a uniform rectangular sparse EMVS array with $N = N_y^r \times N_z^r$ antennas is utilized in receiver. The transmitting linear array is divided into two subarrays. Specifically, the first subarray consists of $M_t$ elements and the second subarray consists of $N_t$ elements, where $M_t$ and $N_t$ are co-prime integers [7, 16]. The carrier frequency of the $m$-th transmitting element is

$$f_m = \begin{cases} f_0 + m N_t \Delta f & m = 0, \cdots, M_t - 1 \\ f_0 + (m - M_t) M_t \Delta f & m = M_t, \cdots, M - 1 \end{cases}$$

(1)

where $f_0$ and $\Delta f$ denote the reference carrier frequency and frequency increment, respectively, with $\Delta f < f_0$.

Without loss of generality, the transmitting array is aligned along the y-axis, with the inter-element spacing $d_x = \lambda/2$, where $\lambda$ denotes the wavelength corresponding to the reference carrier frequency defined in (1). The receiving sparse EMVS array is in $x-y$ plane, and the $(n_y^r, n_z^r)$-th EMVS is located at $(n_y^r d_x, n_z^r d_y)$, where the inter-sensors spacing $d_y$ of the receiving array is larger than $\lambda/2$.

The transmitting waveform of the $m$-th element is expressed as

$$s_m = \phi_m(t) e^{j2\pi f_m t} \quad 0 \leq t \leq T_0$$

(2)

where $T_0$ is the pulse duration and $\phi_m(t)$ denotes the baseband waveform. We assume the transmitting waveforms to satisfy the orthogonal condition [1,2]

$$\int_0^{T_0} \phi_m(t) \phi_n^*(t - \tau) e^{j2\pi \Delta f (m-n) t} dt = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

(3)

where $\tau$ denotes the time shift.

Let us consider $K$ static targets located in the far field. The parameter set of the $k$-th target’s elevation angle, azimuth angle, auxiliary polarization angle, polarization phase difference, and range is represented by $\Theta_k = (\theta_k, \varphi_k, \gamma_k, \eta_k, r_k)$.
The transmitting spatial steering vector $a_k^t$, the transmitting range steering vector $d_k^r$, and the receiving spatial steering vector $a_k^r$ are respectively given as

$$a_k^t = \left[ 1, e^{j\pi v_k}, \ldots, e^{j\pi(M-1)v_k} \right]^T$$

$$d_k^r = \left[ 1, e^{-j4\pi N_i\Delta f_{rk}/c}, \ldots, e^{-j4\pi(M_i-1)\Delta f_{rk}/c}, 1, e^{-j4\pi M_i\Delta f_{rk}/c}, \ldots, e^{-j4\pi(N_i-1)M_i\Delta f_{rk}/c} \right]^T$$

$$a_k^r = \left[ 1, e^{j2\pi d_{rk}/\lambda}, \ldots, e^{j2\pi(N_i-1)d_{rk}/\lambda} \right]^T \otimes \left[ 1, e^{j2\pi d_{rk}/\lambda}, \ldots, e^{j2\pi(N_i-1)d_{rk}/\lambda} \right]^T$$

Then the joint angle-range-polarization steering vector can be expressed as

$$a_k = c_k \otimes [a_k^t \otimes d_k^r] \otimes a_k^r$$

For the $l$-th pulse, after down-conversion and matched-filtering, the $6MN \times 1$ received signal vector can be represented as

$$x(l) = \sum_{k=1}^{K} \rho_k(l) a_k + n(t)$$

$$= A \rho(l) + n(t), \quad l = 1, 2, \ldots, L$$

where $A = [a_1, a_2, \ldots, a_K] \in C^{6MN \times K}$, $\rho(l) = [\rho_1(l), \rho_2(l), \ldots, \rho_K(l)]^T \in C^{K \times 1}$ with $\rho_k(l)$ denoting the complex reflection coefficient of the $k$-th target at the $l$-th pulse. All the targets are assumed to be Swerling II type, and $n(t)$ is the additive white Gaussian noise vector with zero mean. Furthermore, the noise is assumed to be statistically independent with $\rho_k(l)$. Then, the covariance matrix of the received signal vector $x(l)$ is given by

$$R = E[x(l)x^H(l)] = AR_s A^H + \sigma_n^2 I_{6MN}$$

where $\sigma_n^2$ denotes the noise variance, $\sigma_{\rho k}^2$ denotes the variance of $\rho_k(l)$, and $R_s = \text{diag}[\sigma_{\rho 1}^2, \ldots, \sigma_{\rho K}^2]$ denotes the signal covariance matrix.

The objective of this paper is to estimate the angle, polarization, and range parameters $\Theta_k = (\theta_k, \varphi_k, \gamma_k, \eta_k, \tau_k)$ based on the signal model of the proposed SPS-CopFDA-MIMO radar in (12).

### III. ALGORITHM DEVELOPMENT

#### A. ESTIMATION OF THE UNAMBIGUOUS AND LOW ACCURATE DIRECTION COSINES

In order to exploit the propagator method for parameter estimation, we first partition the matrix $A$ into $A = [A_1^T, A_2^T]^T$, where $A_1$ and $A_2$ denote the first $K$ rows and the remaining $6MN - K$ rows of $A$, respectively. From the signal model given by (12), it can be found that $A_1$ is a full rank matrix [32, 33]. Then a unique linear operator $P_A$ with dimension $K \times (6MN - K)$ is defined as

$$P_A^H A_1 = A_2$$

**FIGURE 1:** Configuration of the SPS-CopFDA-MIMO radar

where $\theta_k \in [0, \pi/2]$, $\varphi_k \in [0, 2\pi)$, $\gamma_k \in [0, \pi/2)$, $\eta_k \in (-\pi, \pi)$, $\tau_k \in (0, c/2\Delta f)$, and $c$ denotes the velocity of electromagnetic wave propagation. For the $k$-th target, the $6 \times 1$ EMVS steering vector $c_k = [c_{k,1}, c_{k,2}, \ldots, c_{k,6}]^T$ can be expressed as [18]

$$c_k = \Xi_k q_k = \begin{bmatrix} \sin \gamma_k \cos \theta_k \cos \varphi_k e^{j\eta_k} - \cos \gamma_k \sin \varphi_k \\ \sin \gamma_k \cos \theta_k \sin \varphi_k e^{j\eta_k} + \cos \gamma_k \cos \varphi_k \\ - \sin \gamma_k \sin \theta_k e^{j\eta_k} \\ - \cos \gamma_k \cos \theta_k \cos \varphi_k - \sin \gamma_k \sin \varphi_k e^{j\eta_k} \\ - \cos \gamma_k \cos \theta_k \sin \varphi_k + \sin \gamma_k \cos \varphi_k e^{j\eta_k} \\ \cos \gamma_k \sin \theta_k \end{bmatrix}$$

where $\Xi_k$ and $q_k$ denote the response of a single EMVS and the receive polarization vector, each of which can be specified as

$$\Xi_k = \begin{bmatrix} \cos \theta_k \cos \varphi_k & -\sin \varphi_k \\ \cos \theta_k \sin \varphi_k & \cos \varphi_k \\ -\sin \theta_k & 0 \\ -\sin \varphi_k & -\cos \theta_k \cos \varphi_k \\ \cos \varphi_k & -\cos \theta_k \sin \varphi_k \\ \sin \theta_k \end{bmatrix}$$

$$q_k = \begin{bmatrix} \sin \gamma_k e^{j\eta_k} \\ \cos \gamma_k e^{j\eta_k} \end{bmatrix} \triangleq \begin{bmatrix} q_{k,1} \\ q_{k,2} \end{bmatrix}$$

The Poynting vector for the $k$-th target can be expressed as

$$P_k = \begin{bmatrix} u_k \\ v_k \\ w_k \end{bmatrix} = \begin{bmatrix} \sin \theta_k \cos \varphi_k \\ \sin \theta_k \sin \varphi_k \\ \cos \theta_k \end{bmatrix}$$

where $u_k$, $v_k$, and $w_k$ denote the direction cosines along the x-axis, y-axis, and z-axis, respectively.
Let \( P = [I_K, P_A^T]^T \), we have
\[
P A_1 = A
\]
Define the following six selection matrices
\[
J_i^r = g_i \otimes I_M \otimes I_N, \quad i = 1, 2, \ldots, 6 \tag{16}
\]
where \( g_i \) is a \( 1 \times 6 \) row vector of all zeros except a 1 at the \( i \)-th position. We have
\[
P_i^r = J_i^r P, \quad i = 1, 2, \ldots, 6 \tag{17}
\]
Let \([e_k^T, h_k^T]^T = c_k / c_{k,1} \), where \( e_k \) and \( h_k \) are two \( 3 \times 1 \) vectors. Then, but low accurate estimates of the x-axis and y-axis direction cosines \( \hat{u}_k^\text{coarse} \) and \( \hat{c}_k^\text{coarse} \) can be obtained as [19] (detailed in Appendix A)
\[
\begin{bmatrix}
\hat{u}_k^\text{coarse} \\
\hat{c}_k^\text{coarse} \\
\hat{c}_k^\text{coarse}
\end{bmatrix} = \frac{\hat{e}_k}{\|\hat{e}_k\|} \times \frac{\hat{h}_k^*}{\|\hat{h}_k\|} \tag{18}
\]
where \( \hat{e}_k \) and \( \hat{h}_k \) can be obtained from the eigenvalues of the estimates of \( (P_1^r)^T P_i^r \), and \( \times \) denotes the vector cross product.

**B. ESTIMATION OF THE UNAMBIGUOUS AND HIGH ACCURATE ANGLE PARAMETERS**

Define the following four selection matrices
\[
J_1^r = I_6 \otimes I_M \otimes \left\{ I_{N^0_0-1,0(N^0_{N-1})1} \otimes I_{N^1_0} \right\} \tag{19}
\]
\[
J_2^r = I_6 \otimes I_M \otimes \left\{ 0(N^0_{N-1})1 \otimes I_{N^1_0} \right\} \tag{20}
\]
\[
J_3^r = I_6 \otimes I_M \otimes \left\{ I_{N^0_0} \otimes I_{N^0-1_0,0(N^0_{N-1})1} \right\} \tag{21}
\]
\[
J_4^r = I_6 \otimes I_M \otimes \left\{ I_{N^0_0} \otimes I_{N^0-1_0,0(N^0_{N-1})1} \right\} \tag{22}
\]

Let \( P_1^r = J_1^r P, \quad P_2^r = J_2^r P, \quad P_3^r = J_3^r P, \) and \( P_4^r = J_4^r P \). Then, we can get the high accurate but cyclically ambiguous y-axis and x-axis direction cosine estimates as (detailed in Appendix B)
\[
\hat{v}_k^\text{fine} = \frac{\arg \left[ \Phi_{\bar{y}_k}^r (k, k) \right]}{2 \pi d_r / \lambda} \tag{23}
\]
\[
\hat{u}_k^\text{fine} = \frac{\arg \left[ \Phi_{\bar{y}_k}^r (k, k) \right]}{2 \pi d_r / \lambda} \tag{24}
\]
where \( \arg [z] \) denotes the principal argument of the complex number \( z \) in the range of \([-\pi, \pi]\), \( \Phi_{\bar{y}_k}^r \) and \( \Phi_{\bar{y}_k}^r \) are the estimates of \( \Phi_{\bar{y}_k}^r \) and \( \Phi_{\bar{y}_k}^r \). By using (18), (23), and (24), the unambiguous and high accurate y-axis and x-axis direction cosine estimates can be obtained as [20]
\[
\hat{v}_k = \hat{v}_k^\text{fine} + \hat{n}_1 \frac{\lambda}{d_r} \tag{25}
\]
\[
\hat{u}_k = \hat{u}_k^\text{fine} + \hat{n}_2 \frac{\lambda}{d_r} \tag{26}
\]
where \( \hat{n}_1 = \arg \min_{n_1} \left| \hat{v}_k^\text{coarse} - \hat{v}_k^\text{fine} - n_1 \frac{\lambda}{d_r} \right| \) and \( \hat{n}_2 = \arg \min_{n_2} \left| \hat{u}_k^\text{coarse} - \hat{u}_k^\text{fine} - n_2 \frac{\lambda}{d_r} \right| \).

Using (25) and (26), we can estimate the elevation angle and azimuth angle of the \( k \)-th signal as
\[
\hat{\theta}_k = \arcsin \left( \sqrt{\hat{u}_k^2 + \hat{v}_k^2} \right) \tag{27}
\]
\[
\hat{\phi}_k = \arg \left( \hat{u}_k + j \hat{v}_k \right) \tag{28}
\]

**C. ESTIMATION OF POLARIZATION PARAMETERS**

Substituting the estimated parameters \( \hat{\theta}_k, \hat{\phi}_k, \) and \( \hat{e}_k \) into (4), we can estimate the polarization parameters as
\[
\hat{\gamma}_k = \arctan \left( \frac{\hat{q}_k,1}{\hat{q}_k,2} \right) \tag{29}
\]
\[
\hat{\eta}_k = \arg \left( \hat{q}_k,1 \right) - \arg \left( \hat{q}_k,2 \right) \tag{30}
\]
where
\[
\begin{bmatrix}
\hat{q}_k,1 \\
\hat{q}_k,2
\end{bmatrix} = \begin{bmatrix}
\hat{e}_k^\dagger \hat{e}_k
\end{bmatrix} \tag{31}
\]

**D. ESTIMATION OF RANGE WITH TWO-STEP PHASE UNWRAPPING PROCESSING**

For the first transmitting subarray with inter-element frequency increment \( N_t \Delta f \), we define the following two selection matrices
\[
J_1^r = I_6 \otimes \left[ I_{M_t-1,0(N_t-1)1} \otimes I_{N_r-1} \right] \otimes I_N \tag{32}
\]
\[
J_2^r = I_6 \otimes \left[ 0(N_t-1)1 \otimes I_{M_t-1,0(N_t-1)1} \otimes I_N \right] \otimes I_N \tag{33}
\]
Similarly, for the second transmitting subarray with inter-element frequency increment \( M_t \Delta f \), we define another two selection matrices as
\[
J_3^r = I_6 \otimes \left[ 0(N_t-1)1 \otimes I_{N_t-1,0(N_t-1)1} \otimes I_{N_r} \right] \otimes I_N \tag{34}
\]
\[
J_4^r = I_6 \otimes \left[ 0(N_t-1)1 \otimes I_{N_t-1} \otimes I_{N_r} \right] \otimes I_N \tag{35}
\]

Let \( P_1^r = J_1^r P, \quad P_2^r = J_2^r P, \quad P_3^r = J_3^r P, \) and \( P_4^r = J_4^r P \). In a way similar to that in Appendix B, we have
\[
(P_1^r)^T P_1^r = A_1 \Phi_{\bar{y}_1}^r A_1^\dagger \tag{36}
\]
\[
(P_2^r)^T P_2^r = A_1 \Phi_{\bar{y}_2}^r A_1^\dagger \tag{37}
\]
where
\[
\Phi_{\bar{y}_1}^r = \text{diag} \left\{ e^{j \pi v_1 - j 4 \pi N_t \Delta f r_1 / c}, e^{j \pi v_2 - j 4 \pi N_t \Delta f r_2 / c}, \ldots, e^{j \pi v_K - j 4 \pi N_t \Delta f r_K / c} \right\} \tag{38}
\]
\[
\Phi_{\bar{y}_2}^r = \text{diag} \left\{ e^{j \pi v_1 - j 4 \pi M_t \Delta f r_1 / c}, e^{j \pi v_2 - j 4 \pi M_t \Delta f r_2 / c}, \ldots, e^{j \pi v_K - j 4 \pi M_t \Delta f r_K / c} \right\} \tag{39}
\]

Then, \( \Phi_{\bar{y}_1}^r \) and \( \Phi_{\bar{y}_2}^r \) can be derived from the eigenvalues of \( (P_1^r)^T P_1^r \) and \( (P_3^r)^T P_3^r \), respectively. With the estimates of \( (P_1^r)^T P_1^r \) and \( (P_3^r)^T P_3^r \), we can get the \( \arg \left[ \Phi_{\bar{y}_1}^r (k, k) \right] \) and \( \arg \left[ \Phi_{\bar{y}_2}^r (k, k) \right] \), respectively.

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For the first transmitting subarray, let \( r_k^1 = r_k \) denotes the range of \( k \)-th target and \( r_k^1 = c/(2N_t \Delta f) \) denotes the maximum unambiguous range. Then, \( r_k^1 \) can be written as
\[
    r_k^1 = r_k^p + n_k r_n^p, \quad n_k = 0, 1, \ldots, N_t - 1
\]  
where \( r_k^p \in (0, c/(2N_t \Delta f)) \) denotes the principal range of the first transmitting subarray.

Similarly, for the second transmitting subarray, let \( r_k^2 = r_k \) denotes the range of \( k \)-th target and \( r_k^2 = c/(2M_t \Delta f) \) denotes the maximum unambiguous range. Then, \( r_k^2 \) can be written as
\[
    r_k^2 = r_k^p + m_k r_m^p, \quad m_k = 0, 1, \ldots, M_t - 1
\]  
where \( r_k^p \in (0, c/(2M_t \Delta f)) \) denotes the principal range of the second transmitting subarray.

Equation (38) and (40) show that \( \arg[\Phi_k^1(k,k)] \) is determined by the direction cosine \( v_k \) and the principal range \( r_k^p \), and (39) and (41) show that \( \arg[\Phi_k^2(k,k)] \) is determined by the direction cosine \( v_k \) and the principal range \( r_k^p \). This coupling relationship between the angle and range parameters will result in phase ambiguity in range estimation.

Specifically, as for \( \arg[\Phi_k^1(k,k)] = \arg(\pi v_k - 4\pi N_t \Delta f r_k^p/c), \) since \( \theta_k \in [0, \pi/2], \varphi_k \in [0, 2\pi), \) then \( \pi v_k \) belongs to \((-\pi, \pi), \) and \( 4\pi N_t \Delta f r_k^p/c \) is within the range of \((0, 2\pi).\) Thus, the phase \( \pi v_k - 4\pi N_t \Delta f r_k^p/c \) will lie in \((-3\pi, \pi).\) The functional relationship among \( \Phi_k^1(k,k), 4\pi N_t \Delta f r_k^p/c, \) and \( \pi v_k \) is shown in Fig. 2 (a), and the functional relationship among \( \arg[\Phi_k^1(k,k)], 4\pi N_t \Delta f r_k^p/c, \) and \( \pi v_k \) is shown in Fig. 2 (b). When \( \pi v_k - 4\pi N_t \Delta f r_k^p/c \) lies in the range of \((-3\pi, -\pi),\) phase ambiguity will occur.

Similarly, as for \( \arg[\Phi_k^2(k,k)] = \arg(\pi v_k - 4\pi M_t \Delta f r_k^p/c), \) when \( \pi v_k - 4\pi M_t \Delta f r_k^p/c \) lies in the range of \((-3\pi, -\pi),\) phase ambiguity will also occur.

In order to get the unambiguous range estimates, we propose a two-step phase unwrapping method. First, we use the direction cosine estimates obtained from the receiving array, to get unambiguous estimates of \( r_k^p \) and \( r_k^p.\) Then, based on the co-prime property, we estimate \( n_k \) and \( m_k \) to finally get the unique accurate range estimates. The phase unwrapping processing is given as bellow.

1) First step: estimation of \( r_k^p \) and \( r_k^p \)

For the first transmitting subarray, since \( r_k^p \in (0, c/(2N_t \Delta f)), \) we have
\[
    \pi v_k - 4\pi N_t \Delta f r_k^p/c < \pi v_k
\]  
In order to determine whether it exists phase ambiguity or not, we can compare \( \arg[\Phi_k^1(k,k)] \) with \( \pi v_k. \)

If \( \arg[\Phi_k^1(k,k)] \geq \pi v_k, \) there exists phase ambiguity. It means
\[
    \pi v_k - 4\pi N_t \Delta f r_k^p/c \in (-3\pi, \pi)
\]  
and
\[
    \arg[\Phi_k^1(k,k)] = \pi v_k - 4\pi N_t \Delta f r_k^p/c + 2\pi
\]  
Then, \( r_k^p \) can be estimated as
\[
    r_k^p = \frac{\pi v_k - \arg[\Phi_k^1(k,k)] + 2\pi}{4\pi N_t \Delta f/c}
\]  
Otherwise, there does not exist phase ambiguity, i.e.
\[
    \pi v_k - 4\pi N_t \Delta f r_k^p/c \in (-\pi, \pi)
\]  
and
\[
    \arg[\Phi_k^1(k,k)] = \pi v_k - 4\pi N_t \Delta f r_k^p/c
\]
In this unambiguous case, \( r_k^{P1} \) can be estimated as
\[
\hat{r}_k^{P1} = \frac{\pi \hat{v}_k - \arg \left[ \Phi_k^1 (k,k) \right]}{4\pi N_t \Delta f/c}
\]
(48)

So, we have
\[
\hat{r}_k^{P1} = \begin{cases} 
\frac{\pi \hat{v}_k - \arg \left[ \Phi_k^1 (k,k) \right] + 2\pi}{4\pi N_t \Delta f/c} & \text{arg} \left[ \Phi_k^1 (k,k) \right] \geq \pi \hat{v}_k \\
\frac{\pi \hat{v}_k - \arg \left[ \Phi_k^1 (k,k) \right]}{4\pi N_t \Delta f/c} & \text{arg} \left[ \Phi_k^1 (k,k) \right] < \pi \hat{v}_k 
\end{cases}
\]
(49)

Similarly, for the second transmitting subarray, we can get \( \hat{r}_k^{P2} \) as
\[
\hat{r}_k^{P2} = \begin{cases} 
\frac{\pi \hat{v}_k - \arg \left[ \Phi_k^2 (k,k) \right] + 2\pi}{4\pi M_t \Delta f/c} & \text{arg} \left[ \Phi_k^2 (k,k) \right] \geq \pi \hat{v}_k \\
\frac{\pi \hat{v}_k - \arg \left[ \Phi_k^2 (k,k) \right]}{4\pi M_t \Delta f/c} & \text{arg} \left[ \Phi_k^2 (k,k) \right] < \pi \hat{v}_k 
\end{cases}
\]
(50)

2) Second step: estimation of \( n_1 \) and \( m_1 \)

With the estimates of the principal ranges \( \hat{r}_k^{P1} \) and \( \hat{r}_k^{P2} \), two ambiguous range estimates of the \( k \)-th target from each subarray can be expressed
\[
\hat{r}_k^{(1)}(n_t) = \hat{r}_k^{P1} + \frac{m_t c}{2 \Delta f N_t}, \ n_t = 0, 1, \cdots, N_t - 1
\]
(51)
\[
\hat{r}_k^{(2)}(m_t) = \hat{r}_k^{P2} + \frac{m_t c}{2 \Delta f M_t}, \ m_t = 0, 1, \cdots, M_t - 1
\]
(52)

With the co-prime property between \( M_t \) and \( N_t \), there is only a unique pair \( \{\hat{n}_t, \hat{m}_t\} \) that satisfies the following equation
\[
\{\hat{n}_t, \hat{m}_t\} = \arg \min_{0 \leq n_t \leq N_t-1, 0 \leq m_t \leq M_t-1} |\hat{r}_k^{(1)}(n_t) - \hat{r}_k^{(2)}(m_t)|
\]
(53)

Then, the range of the \( k \)-th target can be estimated as
\[
\hat{r}_k = \frac{\hat{r}_k^{(1)}(\hat{n}_t) + \hat{r}_k^{(2)}(\hat{m}_t)}{2}
\]
(54)

E. CALCULATION OF THE PROPAGATOR AND PARAMETERS PAIRING

In this subsection, we will show the calculation of the propagator based on \( \varphi (l) \). We first calculate the estimated data correlation matrix
\[
\hat{R} = \frac{1}{L} \sum_{l=1}^{L} x(l) x^H(l)
\]
(55)

Then, partitioning \( \hat{R} \) into \( \hat{R} = [\hat{R}_1^T, \hat{R}_2^T]^T \), where \( \hat{R}_1 \) and \( \hat{R}_2 \) denote the first \( K \) rows and the remaining \( 6MN - K \) rows of \( \hat{R} \), respectively, we have
\[
P_A^H \hat{R}_1 = \hat{R}_2
\]
(56)

Based on (56), \( P_A \) can be estimated as
\[
\hat{P}_A = \left( \hat{R}_1 \hat{R}_1^H \right)^{-1} \hat{R}_1 \hat{R}_2^H
\]
(57)

From the definitions of \( P_i^c, P_i^r \), and \( P_i^t \), we get
\[
\hat{P}_i^c = J_i^c \hat{P}_i, \ i = 1, 2, \ldots, 6
\]
\[
\hat{P}_i^r = J_i^r \hat{P}_i, \ i = 1, 2, 3, 4
\]
\[
\hat{P}_i^t = J_i^t \hat{P}_i, \ i = 1, 2, 3, 4
\]
(58)

<table>
<thead>
<tr>
<th>Step</th>
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<tr>
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</tr>
<tr>
<td></td>
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<td>Step6</td>
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<td>The 1st step of phase unwrapping for range estimation: get the unambiguous principal range estimates ( \hat{r}_k^{P1} ) and ( \hat{r}_k^{P2} ) using (49) and (50)</td>
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<td>Step8</td>
<td>The 2nd step of phase unwrapping for range estimation: obtain the unambiguous range estimate ( \hat{r}_k ) using (54)</td>
</tr>
</tbody>
</table>

where
\[
\hat{P} = [I_K, \hat{P}_A]^T
\]
(59)

From (36), (37), (76), (81), and (82), it can be found that \( (P_1^c)^i, (P_2^c)^i, (P_3^c)^i, (P_c^c)^i, (P_1^r)^i, (P_2^r)^i, (P_3^r)^i \) \ (i = 2, 3, ..., 6), \( (P_1^t)^i, (P_2^t)^i, (P_3^t)^i \) have the same set of eigenvectors. However, the estimates of these matrices from finite data may have different sets of eigenvectors, and the eigenvectors associated with the same target may be in different columns. For brevity, we only illustrate how to match the range estimation of the same target. Let \( \hat{A}_{1,1} \) and \( \hat{A}_{1,3} \) denote the estimates of \( A_1 \) derived from eigen-decomposition of \( \hat{P}_A^r \) and \( \hat{P}_A^t \), respectively. For the \( k \)-th column of the matrix product \( \hat{A}_{1,1} J_{k,1,3}^{-1} \), let \( k \) denote the row indice of the matrix elements with the largest absolute values. It is inferred that the eigenvectors constituting the \( k \)-th column of \( \hat{A}_{1,1} \) and the \( k \)-th column of \( \hat{A}_{1,3} \) correspond to the same target \( [33] \).

For clarity, the processing block diagram and the flow chart of the proposed algorithm are summarized in Fig. 3 and Table 1, respectively.

IV. PERFORMANCE ANALYSIS

In this section, we first analyze the computational complexity for the proposed algorithm. Then, we derive the CRBs for the angle, polarization and range to study the performance lower bound of SPS-CopFDA-MIMO radar.

A. COMPUTATIONAL COMPLEXITY

The major computational complexities of the proposed algorithm come from the following aspects:

1) The complex multiplication operations required for estimating the correlation matrix \( R \) are \( O \left( L(6MN)^2 \right) \)

2) The complex multiplication operations required for estimating propagator \( P \) are \( O \left( (6MN)^2 K + 6MNK^2 + K^3 \right) \)

3) The complex multiplication operations required
The correlation matrix estimation using (55)

\[ \text{the propagator estimation using (59)} \]

\[ \text{Sparse receiving array processing using (23) and (24)} \]

\[ \text{Transmitting sub-array processing using (36) and (37)} \]

\[ \text{The 1st step of phase unwrapping for range estimation via (49) and (50)} \]

\[ \text{The 2nd step of phase unwrapping for range estimation via (54)} \]

\[ \hat{\psi}_d \]

\[ \hat{\psi}_n \]

FIGURE 3: Processing block diagram of the proposed algorithm.

The Fisher information matrix for the vector \( \psi \) is given by

\[ \text{FIM}_{p,q} = L \text{Tr} \left( \frac{\partial R}{\partial \psi_p} R^{-1} \frac{\partial R}{\partial \psi_q} R^{-1} \right) \]

(65)

In this paper, we are interested in the desired parameters \( \psi_d \). Similar to the derivation in [34], we can get the \( \psi_d \)-block of the CRB = \( FIM^{-1} \) expressed in (66) at the top of next page, where \( 1_{5 \times 1} = [1, 1, 1, 1, 1]^T \), and

\[ D = \left[ a'_{1,\theta}, \cdots, a'_{K,\theta}, a'_{1,\varphi_1}, \cdots, a'_{K,\varphi_1}, a'_{1,\gamma_1}, \cdots, a'_{K,\gamma_1}, a'_{1,\eta_1}, \cdots, a'_{K,\eta_1}, a'_{1,r_1}, \cdots, a'_{K,r_1} \right] \]

(67)

\[ a'_{k,\theta} = \frac{\partial a_k}{\partial \theta_k}, k = 1, 2, \cdots, K \]

(68)

\[ a'_{k,\varphi_k} = \frac{\partial a_k}{\partial \varphi_k}, k = 1, 2, \cdots, K \]

(69)

\[ a'_{k,\gamma_k} = \frac{\partial a_k}{\partial \gamma_k}, k = 1, 2, \cdots, K \]

(70)

\[ a'_{k,\eta_k} = \frac{\partial a_k}{\partial \eta_k}, k = 1, 2, \cdots, K \]

(71)

\[ a'_{k,r_k} = \frac{\partial a_k}{\partial r_k}, k = 1, 2, \cdots, K \]

(72)

\[ \prod_{A}^{+} = \sigma_n^2 I_{6MN} - A A^T \]

(73)

for performing eigen-decomposition of the matrices \( (P_1^T I_{P_1}) (P_1 I_{P_1}), (P_1^T I_{P_1}) (P_2 I_{P_2}), (P_2^T I_{P_2}) (P_1 I_{P_1}), (P_2^T I_{P_2}) (P_2 I_{P_2}) \), and \( (P_4^T I_{P_4}) (P_4 I_{P_4}) \) are \( O \left[ 69MN - 18M \left( N_p^T + N_p^T \right) - 2N \right] K^2 + 18K^3 \}

\[ (4) \]

The complex multiplication operations required for estimating the polarization parameters are \( O (68K) \)

\[ \text{Thus, the total complex multiplication operations are} \]

\[ O \left\{ L(6MN)^2 + \left[ (6MN)^2 + 68 \right] K + \left[ 75MN - 18M \left( N_p^T + N_p^T \right) - 2N \right] K^2 + 19K^3 \right\} \]

\[ (60) \]

For comparison, the major complex multiplication involved in Li’s algorithm is to estimate the correlation matrix and to perform the eigendecomposition. When adapted to the SPS-CopFDA-MIMO radar, the total complex multiplication operations of Li’s algorithm are

\[ O \left\{ L(6MN)^2 + (6MN)^3 + 68K + \left[ 69MN - 18M \left( N_p^T + N_p^T \right) - 2N \right] K^2 + 18K^3 \right\} \]

\[ (61) \]

**B. THE CRAMÉR-RAO BOUNDS**

The \( 6K \times 1 \) unknown parameter vector is

\[ \psi = \left[ \psi_d^T, \psi_n^T \right]^T \]

(62)

where \( \psi_d \) and \( \psi_n \) denote the desired and nuisance parameters with

\[ \psi_d = [\theta_1, \cdots, \theta_K, \varphi_1, \cdots, \varphi_K, \gamma_1, \cdots, \gamma_K, \eta_1, \cdots, \eta_K, r_1, \cdots, r_K]^T \]

(63)

\[ \psi_n = [\sigma_{p_1}^2, \cdots, \sigma_{p_K}^2, \sigma_{n}^2]^T \]

(64)
CRB_{\phi_d} = \frac{\sigma_d^2}{2L} \text{Re} \left[ \left( D^H \prod_{A}^T D \right) \otimes \left( 1_{5 \times 1} \otimes 1_{5 \times 1}^T \right) \otimes \left( R_A A^H R^{-1} A R_A \right) \right]^{-1} \quad (66)

V. SIMULATION RESULTS

In this section, Monte Carlo simulations are carried out to evaluate the performance of the proposed SPS-CopFDA-MIMO radar and the corresponding parameter estimation algorithm. For comparison, both the SPS-CopFDA-MIMO radar and the PSFDA-MIMO radar use a rectangular EMVS array in receiver with \( N = 4 \times 4 \), and a linear uniform array in transmitter with \( d_t = \lambda/2 \) and \( M = 9 \). The number of pulses is set to \( L = 1000 \). For the PSFDA-MIMO radar, the inter-sensors spacing in receiver is set to \( d_r = 0.5a \) and the uniform frequency increment is set to \( \Delta f = 1 \) kHz, which indicates the maximum unambiguous range is 150 km. For the SPS-CopFDA-MIMO radar, the inter-sensors spacing in receiver is set to \( d_r = 3a \) and the number of two linear transmitting subarrays is set to \( N_t = 5 \) and \( M_t = 4 \), respectively. The frequency increments of the two subarrays are set to \( N_t \Delta f = 5 \) kHz and \( M_t \Delta f = 4 \) kHz, respectively. The performance metric used is the root mean square error (RMSE). RMSE of \( \theta \) is defined as

\[
\text{RMSE}_{\theta} = \sqrt{\frac{1}{K M_o} \sum_{k=1}^{K} \sum_{m=1}^{M_o} \left( \hat{\theta}_{k,m} - \theta_k \right)^2} \quad (74)
\]

where \( M_o \) denotes the number of Monte Carlo experiments, and \( \hat{\theta}_{k,m} \) denotes the estimate of \( \theta_k \) in the \( m \)-th Monte Carlo experiment. RMSE_{\phi}, RMSE_{\theta}, RMSE_{\gamma}, and RMSE_{\eta}, are defined similarly.

A. SPS-COPFDA-MIMO RADAR VERSUS PSFDA-MIMO RADAR

In this simulation, we compare the parameter estimation performance between the SPS-CopFDA-MIMO radar and the PSFDA-MIMO radar utilizing the proposed successive algorithm based on propagator method. Three targets with the following parameters are to be estimated: \((\theta_1, \varphi_1, \gamma_1, \eta_1, r_1) = (5^\circ, 8^\circ, 18^\circ, 7^\circ, 93\text{km})\), \((\theta_2, \varphi_2, \gamma_2, \eta_2, r_2) = (31^\circ, 51^\circ, 36^\circ, 45^\circ, 131\text{km})\), and \((\theta_3, \varphi_3, \gamma_3, \eta_3, r_3) = (55^\circ, 330^\circ, 54^\circ, 22^\circ, 38\text{km})\).

Fig. 4 shows RMSEs of the parameter estimates versus SNR via 500 independent trials. For comparison, Fig. 4 also gives the CRBs for SPSCopFDA-MIMO radar and PSFDA-MIMO radar. The results show that SPS-CopFDA-MIMO radar can offer enhanced estimation accuracy without decreasing the maximum unambiguous range compared with PSFDA-MIMO radar, because of the co-prime frequency offsets and larger inter sensor spacing. Meanwhile the RMSEs of the parameter estimates for the proposed algorithm are close to the CRBs.

B. PROPOSED PARAMETER ESTIMATION ALGORITHM VERSUS LI’S ALGORITHM

In this simulation, we compare the performance of the proposed algorithm with Li’s algorithm.

Firstly, the scenario without phase ambiguity is considered. Three targets with the following parameters are to be estimated: \((\theta_1, \varphi_1, \gamma_1, r_1) = (30^\circ, 330^\circ, 18^\circ, 7^\circ, 38\text{km})\), \((\theta_2, \varphi_2, \gamma_2, r_2) = (33^\circ, 8^\circ, 36^\circ, 22^\circ, 93\text{km})\), and \((\theta_3, \varphi_3, \gamma_3, r_3) = (55^\circ, 51^\circ, 54^\circ, 45^\circ, 144\text{km})\). In this scenario, \(\pi_{v_1} - 4\pi N_t \Delta f r_1^1 / c = -0.19\pi\), \(\pi_{v_2} - 4\pi M_t \Delta f r_2^1 / c = -0.95\pi\), \(\pi_{v_3} - 4\pi N_t \Delta f r_3^1 / c = -0.33\pi\), \(\pi_{v_2} - 4\pi M_t \Delta f r_2^2 / c = -0.59\pi\), \(\pi_{v_3} - 4\pi N_t \Delta f r_3^2 / c = -0.94\pi\), and \(\pi_{v_3} - 4\pi M_t \Delta f r_3^3 / c = -0.44\pi\). From (43) and (46), we can find that there does not exist phase ambiguity.

Fig. 5 shows the RMSEs of the parameter estimates versus SNR via 500 independent trials. From the five figures, we can find that the proposed algorithm can offer estimation accuracy comparable to Li’s algorithm in the case without phase ambiguity.

Secondly, the scenario with phase ambiguity is considered. Three targets with the following parameters are to be estimated: \((\theta_1, \varphi_1, \gamma_1, r_1) = (2^\circ, 330^\circ, 18^\circ, 7^\circ, 38\text{km})\), \((\theta_2, \varphi_2, \gamma_2, r_2) = (33^\circ, 8^\circ, 36^\circ, 22^\circ, 93\text{km})\), and \((\theta_3, \varphi_3, \gamma_3, r_3) = (55^\circ, 51^\circ, 54^\circ, 45^\circ, 144\text{km})\). In this scenario, \(\pi_{v_1} - 4\pi N_t \Delta f r_1^1 / c = -0.55\pi\), \(\pi_{v_1} - 4\pi M_t \Delta f r_1^2 / c = -0.04\pi\), \(\pi_{v_2} - 4\pi N_t \Delta f r_2^1 / c = -0.12\pi\), \(\pi_{v_2} - 4\pi M_t \Delta f r_2^2 / c = -0.88\pi\), \(\pi_{v_3} - 4\pi N_t \Delta f r_3^1 / c = -0.96\pi\), and \(\pi_{v_3} - 4\pi M_t \Delta f r_3^2 / c = -1.04\pi\). From (43) and (46), we can find that there exists phase ambiguity for target 3.

Fig. 6 plots angle-polarization-range estimation results of three targets with SNR = 5dB over 50 independent trials. Fig. 7 plots the RMSEs of the parameter estimates for the proposed algorithm via 500 independent trials. The results show that the proposed algorithm can effectively estimate angle, polarization, and range parameters in phase ambiguity case. However, without phase unwrapping processing, Li’s algorithm can only estimate the angle-polarization parameters effectively, but fails to estimate the range of the target 3.

From Fig. 5, Fig. 6 and Fig. 7, we can see that the proposed algorithm can effectively estimate the parameters and the RMSEs are close to the CRBs in both of the two scenarios. We can also find that Li’s algorithm is only suitable in the scenario without phase ambiguity.

Thirdly, we compare the computational complexity of the proposed method with that of Li’s method. Fig. 8 shows the complex multiplications needed for both of two methods versus the number of pulses under the condition of \( K = 3 \), \( N = 4 \times 4 \), \( N_t = 5 \) and \( M_t = 4 \). It can be seen from the
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\[ P_i^c \Phi_i = A_i \Phi_i \]

where \( \Phi_i = \text{diag}\left[ c_{1,1}/c_{1,1}, c_{2,1}/c_{2,1}, \ldots, c_{K,1}/c_{K,1} \right] \).

Using (75), we can get

\[ (P_i^c)^\dagger P_i^c = A_i \Phi_i A_i^{-1} \]

VI. CONCLUSIONS

In this paper, we propose an SPS-CopFDA-MIMO radar, which incorporates both the receiving full component EMVS array spacing beyond a half-wavelength and transmitting co-prime frequency offsets, to enhance parameter estimation precision. A computationally efficient parameter estimation algorithm is also derived. The proposed algorithm can extract unambiguous angle, range and polarization estimates successively in a computationally simple manner based on the propagator method. Computer simulation results validate the performance of the proposed SPS-CopFDA-MIMO radar technique.

APPENDIX A PROOF OF EQUATION (18)

From the definition of \( P_i^c \) shown in (17), we can get

\[ P_i^c A_1 = P_i^c A_1 \Phi_i \]

FIGURE 4: RMSEs of parameter estimation verse SNR for SPS-CopFDA-MIMO radar and PSFDA-MIMO radar. (a) RMSE of elevation angle. (b) RMSE of azimuth angle. (c) RMSE of auxiliary polarization angle. (d) RMSE of polarization phase difference. (e) RMSE of range.

\[ \Phi_i = \text{diag}\left[ c_{1,1}/c_{1,1}, c_{2,1}/c_{2,1}, \ldots, c_{K,1}/c_{K,1} \right] \].

Using (75), we can get

\[ (P_i^c)^\dagger P_i^c = A_i \Phi_i A_i^{-1} \]
Figure 5: RMSEs of parameter estimation versus SNR for the proposed algorithm versus Li’s algorithm without phase ambiguity. (a) RMSE of elevation angle. (b) RMSE of azimuth angle. (c) RMSE of auxiliary polarization angle. (d) RMSE of polarization phase difference. (e) RMSE of range.

Equation (76) establishes the relationship between $P_c^i$ and $\Phi_c^i$. $A_1$ can be obtained from the eigenvectors of $(P_c^i)^\dagger P_c^i$, and $e_k$ and $h_k$ can be extracted from the eigenvalues of $(P_c^i)^\dagger P_c^i$.

Referring back to (4), note that the vectors $e_k$ and $h_k$ are orthogonal to each other and to the Poynting vector $p_k$ of the signal, i.e.

\[
p_k = \frac{e_k}{\|e_k\|} \times \frac{h_k^*}{\|h_k\|} \quad (77)
\]

Using the estimates of $e_k$ and $h_k$, we have

\[
\begin{pmatrix}
\hat{u}_k^\text{coarse} \\
\hat{v}_k^\text{coarse} \\
\hat{w}_k^\text{coarse}
\end{pmatrix} = \hat{p}_k = \frac{\hat{e}_k}{\|\hat{e}_k\|} \times \frac{\hat{h}_k^*}{\|\hat{h}_k\|} \quad (78)
\]

Appendix B Proof of Equations (23) and (24)

From the definition of $P_r^i$, we have

\[
P_r^2 A_1 = P_r^i A_1 \Phi_u^i \quad (79)
\]

\[
P_r^i A_1 = P_r^i A_1 \Phi_u^i \quad (80)
\]
respectively. Then, we get (23) and (24) with the estimates of
and
Using (79) and (80), we can get

Equation (81) and (82) demonstrate that
and

where
and

Using (79) and (80), we can get

(b) results of auxiliary polarization angle -polarization phase difference using proposed algorithm. (c) results of azimuth-elevation using Li’s algorithm. (d) results of auxiliary polarization angle -polarization phase difference using Li’s algorithm. (e) results of range using the proposed algorithm and Li’s algorithm.

FIGURE 6: Parameter estimation results for the proposed algorithm versus Li’s algorithm in phase ambiguity scenario (a) results of azimuth-elevation estimation using proposed algorithm. (b) results of auxiliary polarization angle -polarization phase difference using proposed algorithm. (c) results of azimuth-elevation using Li’s algorithm. (d) results of auxiliary polarization angle -polarization phase difference using Li’s algorithm. (e) results of range using the proposed algorithm and Li’s algorithm.

REFERENCES
FIGURE 7: RMSEs of parameter estimation verse SNR for the proposed algorithm with phase ambiguity. (a) RMSE of elevation angle. (b) RMSE of azimuth angle. (c) RMSE of auxiliary polarization angle. (d) RMSE of polarization phase difference. (e) RMSE of range.


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FIGURE 8: Computational complexity of the proposed algorithm and Li’s algorithm versus the number of pulses.


