Bayesian Inference for Optimal Risk Hedging Strategy using Put Options with Stock Liquidity

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ABSTRACT This paper considers the problem of hedging the risk exposure to imperfectly liquid stock by investing in put options. In an incomplete market, we firstly obtain a closed-form pricing formula of the European put option with liquidity-adjustment by measure transformation. Then, an optimal hedging strategy which minimizes the Value-at-Risk (VaR) of the hedged portfolio is deduced by determining an optimal strike price for the put option. Furthermore, we provide a new perspective to estimate parameters entering the minimal VaR, since the likelihood function is analytically intractable. A Bayesian statistical method is proposed to perform posterior inference on the minimal VaR and the optimal strike price. Empirical results show that the risk hedging strategy with liquidity-adjustment differs from the hedging strategy based on Black-Scholes model. The effect of the stock liquidity on risk hedging strategy is significant. These results can provide more decision information for institutions and investors with different risk preferences to avoid risk.

INDEX TERMS Stock liquidity, Incomplete market, Risk hedging, Option pricing, Minimizing Value-at-Risk, Bayesian statistical inference

I. INTRODUCTION

Nowadays, with the rising of volatility in financial market, financial institutions and investors pay much more attention to manage the exposure to market risk of stocks, interest rates or exchange rates. Therefore, how to use risk management tool to avoid the risk has been a main research area of financial engineering and modern finance theory.

Financial derivatives, an important risk management tool, have been widely used to hedge potential losses that may be incurred in an investment. In recent decades, options as an effective tool to avoid risk have rapid development and have been commonly used to hedge downside risk in financial market. Hedging with options can avoid the undesirable scenarios, while leaving oneself the opportunity to the positive incomes. Thus, a natural idea to hedge the risk of a long position in risky asset is to buy the put option contract on the asset. A pioneering study in this area is the work of Ahn et al. [1] who studied the problem of hedging the exposure to stock price and took Value-at-Risk (VaR) as the risk measure. Ahn et al. proposed an optimal hedging strategy that minimizes the VaR of a position in stock by investing in put options under the assumptions of Black-Scholes model [2].

Since the seminal work of Ahn et al. [1], many extensions of managing the risk using options have been made. In a stochastic interest rate model, Annaert et al. [3] extended the work of Ahn et al. to optimally manage the price risk of a bond by using a bond put option. They provided an approach to determine the optimal strike price for the bond option and calibrated model parameters by using option price data. Ramponi [4] investigated the problem of hedging the exposure to risky asset by buying put options and provided an optimal hedging strategy by minimizing the VaR of the hedged portfolio under a regime-switching jump-diffusion model which can well capture the observed features in financial markets. Recently, Zhang et al. [5] studied the problem of hedging the exchange rate risk by investing in currency option under a minimal conditional VaR framework. More related researches can refer to [6]–[9].

The aforementioned literatures study the hedging strategy...
with options under a fundamental assumption that the market is complete and the underlying asset is perfectly liquid. However, these assumptions have been questioned for reasons that the underlying asset is not perfectly liquid and the imperfect liquidity occurs when a shortage or surplus of assets exists in the market. Numerous researches provide empirical evidence that market liquidity is a significant effect factor in asset pricing and risk management [10]–[13]. Therefore, an increasing number of attention has been paid to the liquidity in financial market.

The liquidity effect on stock returns was initially studied by Amihud and Mendelson [14] who proposed the liquidity premium and found that stock returns are related to the liquidity. Since then, many scholars have studied the liquidity effect on asset pricing from theoretical and practical perspectives [15]–[17]. Among them, a popular way to capture the liquidity effect on stock prices is introducing the liquidity discount factor \( L_t \) into the demand function of stocks, where \( L_t \) satisfies

\[
\frac{dL_t}{L_t} = \left( \frac{1}{2} \xi^2 \omega_t^2 - \xi \omega_t \right) dt - \xi \omega_t dW_t^{L,P},
\]

where \( \omega_t \) is the stock liquidity level, \( \xi > 0 \) is the sensitivity of stock price to liquidity level, \( L_0 = 1, W_t^{L,P} \) is a standard Brownian motion under physical measure \( P \), Brunetti and Caldarera [18] incorporated the liquidity discount factor \( L_t \) into the demand function \( D \equiv D(S_t, L_t, I_t) \), where \( S_t \) is the imperfectly liquid stock price, and \( I_t \) is the information process followed by

\[
\frac{dI_t}{I_t} = \mu_f dt + \sigma_f dW_t^{I,P},
\]

where \( \mu_f \) is the drift and \( \sigma_f \) is the diffusion, \( W_t^{I,P} \) is a standard Brownian motion. Then, under the market clearing condition, Brunetti and Caldarera proposed a liquidity-adjusted asset pricing model

\[
\frac{dS_t}{S_t} = \left( \mu + \xi \omega_t + \frac{1}{2} \xi^2 \omega_t^2 \right) dt + \xi \omega_t dW_t^{L,P} + \lambda dW_t^{I,P},
\]

where \( \omega_t > 0 \) (\( \omega_t < 0 \)) implies that the market is in shortage (surplus), \( \omega_t = 0 \) implies a perfectly liquid market (i.e., Black-Scholes economy). \( \mu \) and \( \lambda \) is part of the expected return and volatility, respectively, and \( dW_t^{L,P} dW_t^{I,P} = 0 \).

Meanwhile, a large number of literatures investigated the effect of stock liquidity on option pricing under the assumption that the underlying asset is not perfectly liquid. By applying the hedging strategy, some researchers deduced the theoretical option pricing formula with liquidity-adjustment to demonstrate the liquidity effect on option pricing, see for example [19], [20]. From the supply and demand function perspective, some scholars studied the liquidity effect on option pricing by modeling the equilibrium equalization of the underlying asset, see for example [18], [21]. Under the liquidity-adjusted asset pricing model, Feng et al. [22], [23] investigated the liquidity effect on option prices by employing a European call option pricing formula with liquidity-adjustment. They conducted the sensitivity analysis of option prices to liquidity measures and concluded that the improvement in option pricing error is significant when taking the liquidity factor into account. Recently, Li et al. [24], [25] studied the impact of stock liquidity on the pricing of Asian options and barrier options. They derived the corresponding option pricing formulas with stock liquidity and concluded that stock liquidity is a significant effect factor in option pricing. These researches have demonstrated that introducing liquidity effect into option pricing can improve the pricing performance.

Reviewing the risk hedging strategy proposed by Ahn et al., the calculation for VaR of the hedged portfolio depends on the values of risky asset and the put option. Intuitively, the liquidity effect on the prices of risky asset and the options should affect the VaR of the hedged portfolio as well. However, to the best of our knowledge, almost all literatures on hedging VaR using options are based on the hypothesis of a perfectly liquid market. Few literatures investigate the effect of stock liquidity on the risk hedging strategy. So the aim of this paper is to fill this gap.

Motivated by these insights, this paper considers the problem of hedging the price risk of imperfectly liquid stock by investing in put options on the stock. We aim to find an optimal hedging strategy which minimizes the VaR of the hedged portfolio by determining an optimal strike price for the put option in the incomplete market.

In practice, the computational problem of the optimal strike price and minimal VaR has not been actually solved, even though the analytic expression for the optimal hedging strategy is obtained, because the unknown parameters need to be estimated. Therefore, the accuracy of parameter estimation has a significant influence on financial asset pricing and risk management. The commonly used estimation methods include Bayesian statistical method and traditional statistical method (such as maximum likelihood estimation, moment estimation, etc). However, the traditional estimation method does not work well for the liquidity-adjusted asset pricing model because the likelihood function has no analytical expression. Due to the flexibility in parameter estimation, Bayesian statistical method has been widely used to perform inference on econometric models [26]–[29]. Karolyi [26] applied Bayesian method to estimate the volatility of stock return and performed posterior inference on the option price, which provided a new perspective to evaluate European options. Under a mixed normal GARCH model, Rombouts and Stentoft [27] conducted posterior inference on model parameters and evaluated the European option by using the posterior predictive density. The empirical results showed that Bayesian method is superior to traditional method in option pricing when less data is available. Recently, Gao et al. [30] studied the pricing of European options with stock liquidity by using Bayesian statistical method. Empirical results indicated that Bayesian statistical method is superior...
to traditional method in both parameter estimation and option pricing. Additionally, Tunaru and Zheng [31] adopted Bayesian method to investigate the influence of parameter estimation on asset pricing and risk management, and they found that the influence is significant.

Although many literatures study how to optimally manage the VaR of risky asset by using options, related researches have paid little attention to the method of parameter estimations. Considering the influence of parameter estimation, we use Bayesian statistical method to perform posterior inference on model parameters. This method can fully account for prior information and parameter uncertainty. As far as we know, this kind of article is rare.

In empirical analysis, we investigate the effect of stock liquidity on hedging strategy. The paper defines the liquidity as the ability to trade quickly any amount of assets at the market price without additional cost. To examine the robustness of the empirical results, two common liquidity measures are used to serve as the liquidity proxies for stock liquidity. The first liquidity measure is the return over the dollar trading volume (hereafter RDV). The second one is defined as the change in daily close price over the dollar trading volume (hereafter PDV). More details about the liquidity measures refer to [23].

The main work of this paper is summarized as follows. We study the problem of hedging the exposure to a position in stock by buying the put option in an imperfectly liquid market. By Esscher measure transforms, we firstly obtain a closed-form pricing formula of European put option in the incomplete market. This model allows for the effect of stock liquidity on risk hedging. Then, an optimal hedging strategy which minimizes the VaR of the hedged portfolio is obtained by determining an optimal strike price for the put option. Moreover, considering the influence of parameter estimations on hedging strategy, we propose a new perspective to investigate the statistical properties of the optimal strike price and the minimal VaR. A Bayesian statistical method is used to perform inference on model parameters and the optimal hedging strategy. This method can fully account for prior information and parameter uncertainty. Finally, the random walk chain Metropolis-Hastings algorithm is implemented to draw samples from posterior kernels in the empirical experiment. We find that the risk hedging strategy with liquidity-adjustment differs from the hedging strategy based on Black-Scholes model. The empirical results indicate that the effect of stock liquidity on the hedging strategy is significant.

Our paper differs from existing researches in several aspects. Firstly, we consider the hedging problem of stocks in an imperfectly liquid market. The effect of stock liquidity on risk hedging has been considered. However, almost all researches on hedging VaR using options are based on the hypothesis of a perfectly liquid market, ignoring the effect of stock liquidity on risk hedging. Secondly, we use Bayesian statistical method to perform posterior inference on model parameters. This method can fully account for prior information and parameter uncertainty. However, literatures on risk management pay little attention to how to effectively estimate model parameters, ignoring the influence of parameter estimations on risk hedging. Furthermore, we investigate the statistical properties of the optimal strike price and the minimal VaR by conducting posterior inference based on Metropolis-Hastings sampling. Unlike existing literatures usually providing only a point estimation, we provide more information about the optimal strike price and the minimal VaR from a probabilistic perspective. These results are useful for financial institutions and investors with different risk preferences to make better decisions.

The rest of the paper is constructed as follows. Section 2 describes the model setting with stock liquidity and derives a liquidity-adjusted pricing formula for European put option. Section 3 discusses the optimal hedging strategy using options with liquidity by minimizing VaR. Section 4 presents the Bayesian inference for model parameters and the minimal VaR. Section 5 illustrates an empirical experiment. Section 6 gives a conclusion.

II. MODEL SPECIFICATIONS FOR ASSET PRICING WITH STOCK LIQUIDITY

Suppose that an institution has an exposure to an imperfectly liquid stock $S_t$. The imperfect liquidity occurs when a shortage or surplus of stocks exists in financial market. To capture the liquidity effect on risk hedging, we employ the liquidity-adjusted asset pricing model proposed by Brunetti and Caldarera [18] to describe the stock price dynamics

$$\frac{dS_t}{S_t} = (\mu + \xi \omega_t + \frac{1}{2} \xi^2 \omega_t^2)dt + \xi \omega_t dW_t^{L,P} + \lambda dW_t^{I,P},$$

where $\omega_t$ is the stock liquidity level, $\omega_t > 0$ ($\omega_t < 0$) indicates that the market is in shortage (surplus), $\omega_t = 0$ means a perfectly liquid market. $\xi > 0$ is the sensitivity of the stock price to liquidity level, and $\lambda$ is a part of the volatility.

Now we consider the hedging problem in which the institution is concerned about the downside risk of stock price $S_t$ over the next periods, and decides to hedge the risk by investing in European put option on the stock. Since the underlying asset is not perfectly liquid violating the assumption of a complete market, the Black-Scholes formula is not applicable for hedging any more.

Therefore, we should derive the pricing formula for the European put option used in the hedge in the incomplete market. According to martingale pricing principle, we need to find an equivalent martingale measure with respect to the physical measure $P$.

In the incomplete market, by Esscher measure transforms [32], we find an equivalent martingale measure $Q$ defined by

$$\frac{dQ}{dP} |_{F_t} = \exp \left[ \int_0^t -\frac{1}{2} \lambda^2 (\lambda^2 + \xi^2 \omega_u^2)du + \int_0^t h \lambda dW_u^{I,P} + \int_0^t h \xi \omega_u dW_u^{L,P} \right].$$

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where \( \mathcal{F}_t \) is the filtration generated by Brownian processes, and
\[
\frac{\partial}{\partial t} + \frac{1}{2} \sigma^2 \omega_t^2 - \frac{1}{2} \tau - r = \frac{\partial^2}{\partial \omega_t^2},
\]
and \( r \) is the constant riskless interest rate.

By Girsanov theorem [33], under the equivalent martingale measure \( Q \), we have
\[
dW_t^L = dW_t^L - \frac{\lambda}{2} dt - h \xi \omega_t dt, \quad dW_t^L = dW_t^L - h \lambda d\omega_t, \quad \text{and} \quad dW_t^L dW_t^L = 0.
\]
(3)

Conditionally on the stock return process (3), we proceed to deduce the pricing formula of the European put option whose payoff is given by
\[
\text{terminal payoff} = \max\{X - S_T, 0\},
\]
where \( X \) denotes the strike price and \( T \) is the maturity for the option contract.

**Theorem 1** Assume the underlying asset is an imperfectly liquid stock \( S_t \) defined by (2). Under the martingale measure \( Q \), the price of European put option with strike price \( X \) and maturity \( T \) at time \( t \) is
\[
P_t(S_t, X, \tau, \omega_t, \lambda, \xi) = X e^{-\tau \Phi(d_1)} - S_t \Phi(d_2),
\]
(4)

where \( \tau = T - t \), \( \omega_t \) is the stock liquidity level, \( \lambda, \xi \) are defined as previously, and
\[
\begin{align*}
d_1 &= \frac{\ln \frac{X}{S_t} - \left(r - \frac{1}{2} \lambda^2\right) \tau + \frac{1}{2} \xi^2 \int_t^T \omega_u^2 du}{\sqrt{\lambda^2 \tau + \xi^2 \int_t^T \omega_u^2 du}}, \\
d_2 &= d_1 - \sqrt{\lambda^2 \tau + \xi^2 \int_t^T \omega_u^2 du}.
\end{align*}
\]

**Proof:** By the martingale pricing principle, we have
\[
P_t(S_t, X, \tau, \omega_t, \lambda, \xi) = e^{-\tau \Phi(d_1)} \mathbb{E}[\max\{X - S_T, 0\} | \mathcal{F}_t],
\]
where \( \mathbb{E}[\cdot] \) is the expectation operator under measure \( Q \), and \( \Phi(\cdot) \) is the indicator function.

The first expectation can be rewritten as
\[
\mathbb{E}[\max\{X - S_T, 0\} | \mathcal{F}_t] = \Phi(d_1),
\]
where \( \Phi(\cdot) \) is the standard normal cumulative distribution. We denote \( d_1 = (\ln \frac{X}{S_t} - \mu_s)/\sigma_s \), where
\[
\begin{align*}
\mu_s &= \ln X - \frac{1}{2} \lambda^2 T - \frac{1}{2} \xi^2 \int_t^T \omega_u^2 du, \\
\sigma_s &= \sqrt{\lambda^2 T + \xi^2 \int_t^T \omega_u^2 du}.
\end{align*}
\]

By Girsanov theorem, we denote the Radon-Nikodym derivative by
\[
\frac{dQ}{d\mathbb{P}} = \exp \left[ -\int_t^T \frac{1}{2} \lambda^2 du + \int_t^T \lambda dW^L_u + \int_t^T \xi \omega_u dW^L_u \right],
\]
and we have
\[
dW^L_t = dW^L_t - \lambda dt, \quad dW^L_t = dW^L_t - \xi \omega_t dt.
\]

Then, the second expectation can be rewritten as
\[
\mathbb{E}[\max\{X - S_T, 0\} | \mathcal{F}_t] = \mathbb{E}[\max\{X - S_T, 0\} | \mathcal{F}_t] = e^{-\tau \Phi(d_2)},
\]
where \( d_2 = d_1 - \sigma_s \).

Therefore, the price of the European put option with liquidity-adjustment is
\[
P_t(S_t, X, \tau, \omega_t, \lambda, \xi) = X e^{-\tau \Phi(d_1)} - S_t \Phi(d_2).
\]

Thus the proof is finished.

It is worth noticing that the price of the European put option on imperfectly liquid stock is adjusted depending on the stock liquidity level \( \omega_t \) and the sensitivity \( \xi \) of stock price to liquidity level. In other words, the Black-Scholes option pricing formula should be adjusted with liquidity, when the underlying asset is not perfectly liquid. The above proof process is inspired by the method in Feng et al. [23].

**III. OPTIMAL RISK HEDGING USING PUT OPTIONS BY MINIMIZING VAR**

To begin with, we recall from Ahn et al. [1] the classical hedging problem, in which an institution has an exposure to a stock. The institution is concerned about the downside risk of the stock price over the next periods, and decides to hedge the risk by buying put options under the assumptions of Black-Scholes model. However, the underlying asset is imperfectly liquid and the imperfectly liquidity occurs when a shortage or surplus of assets exists in the market. Since the imperfect liquidity violates the assumption of a complete market, the classical hedging strategy may not be applicable any more. Therefore, we consider the problem of hedging the risk of stock price in an incomplete market and adjust the classical hedging strategy with stock liquidity.

Unlike Ahn et al., we suppose an institution has an exposure to an imperfectly liquid stock \( S_t \), whose price dynamics are governed by the liquidity-adjusted asset pricing model (2). The institution decides to hedge the price risk of the stock by buying the put option \( P_t = P_t(S_t, X, \tau, \omega_t, \lambda, \xi) \).
defined by Theorem 1. This model allows for the effect of stock liquidity on risk hedging.

Therefore, the hedged portfolio consists of the stock and the put option. Assuming that investments are made at time \( t \) and are terminated at time \( t + \tau \). Denote by \( V_{t+\tau} \) the value of the portfolio at time \( t + \tau \). Then, the future value of the hedged portfolio is given by

\[
V_{t+\tau} = S_{t+\tau} + l \cdot \max\{X - S_{t+\tau}, 0\},
\]

where \( l \) denotes the number of option contract used in the hedge. From a practical perspective, the hedging cost the institution is willing to incur is limited, because raising additional funds may be difficult and costly. Thus we assume that the exposure is not fully hedged, i.e., \( 0 < l < 1 \).

If the put option finishes in-the-money, then the value of the hedged portfolio is

\[
V_{t+\tau} = (1 - l)S_{t+\tau} + lX.
\]

If the put option finishes out-of-the-money, then the value of the hedged portfolio is

\[
V_{t+\tau} = S_{t+\tau}.
\]

Now we calculate the price risk of the hedged portfolio and take VaR as the risk measure. The VaR is defined as the dollar loss in the future value in \( \tau \) periods will not exceed \( \text{VaR}_{t+\tau} \) with \((1 - \alpha)\%\) percent confidence. When computing a loss on a given investment we will take into account the time value of money, therefore the formal definition for \( \text{VaR}_{t+\tau} \) is

\[
P(V_{t} - e^{-r\tau}V_{t+\tau} > \text{VaR}_{t+\tau}) = \alpha.
\]

That is to say the probability of the potential losses greater than \( \text{VaR}_{t+\tau} \) is \( \alpha \).

As discussed by Ahn et al., there exists no optimal hedging strategy when the put option finishes out-of-the-money. Therefore, we make an assumption that the put option finishes in-the-money. Taking into account the cost of the position in put options, the present value for the \( \text{VaR}_{t+\tau} \) of the hedged portfolio over the next \( \tau \) periods is given by

\[
\text{VaR}_{t+\tau} = S_{t} + C_{t} - e^{-r\tau}[(1 - l)S_{t}e^{\eta(a)} + lX],
\]

where \( C_{t} = l \cdot P_{t} \) is the hedging cost the institution invest at time \( t \), \( l \in (0, 1) \) is the hedge ratio, \( \eta(a) = \mu_{s} + c(a)s_{s} \), and \( c(a) \) is the \( \alpha \) quantile of a standard normal distribution. Reviewing the expressions of \( \mu_{s} \) and \( \sigma_{s} \) defined in Theorem 1, we note that the \( \text{VaR}_{t+\tau} \) is adjusted depending on the stock liquidity level \( \omega_{t} \) and the sensitivity \( \xi \) of stock price to liquidity level.

Analogously to Ahn et al., we study the optimal hedging strategy that minimizes the \( \text{VaR}_{t+\tau} \) of the hedged portfolio by determining an optimal strike price for the put option \( P_{t} \) under a limited hedging cost \( C_{t} \). More precisely, the optimal strategy can be characterized by the following optimization problem

\[
\min_{l, X} \text{VaR}_{t+\tau}(\lambda, \xi, \omega_{t}) = S_{t} + C_{t} - e^{-r\tau}[(1 - l)S_{t}e^{\eta(a)} + lX],
\]

subject to \( C_{t} = l \cdot P_{t} \).

Solving this constrained optimization problem, the optimal strike price \( X^{\ast}(\lambda, \xi, \omega_{t}) \) is followed by

\[
X^{\ast}(\lambda, \xi, \omega_{t}) = \arg\min_{X} \left\{ S_{t} + C_{t} - e^{-r\tau} \left[ (1 - \frac{C_{t}}{P_{t}})S_{t}e^{\eta(a)} + \frac{C_{t}}{P_{t}}X \right] \right\} = \arg\max_{X} \left\{ \frac{X - S_{t}e^{\eta(a)}}{P_{t}} \right\}.
\]

Taking the first derivative with respect to \( X \), then we have

\[
0 = \frac{P_{t} - (X - S_{t}e^{\eta(a)})\frac{\partial \Phi(d_{2})}{\partial X}}{P_{t}^{2}}.
\]

Therefore, the optimal strike price \( X^{\ast}(\lambda, \xi, \omega_{t}) \) is given by the implicit equation

\[
e^{\eta(a)} - e^{-r\tau} = \frac{\Phi(d_{2})}{\Phi(d_{1})} = 0.
\]

Note that the optimal strike price \( X^{\ast}(\lambda, \xi, \omega_{t}) \) is adjusted depending on the liquidity level \( \omega_{t} \), parameter \( \lambda \) and the sensitivity \( \xi \) of stock price to liquidity level. Although the expression governed by the optimal strike price is obtained, the computational problem of the optimal strike price has not been actually solved, because of the unknown parameters. Thus, parameters in (10) are required to be estimated before solving \( X^{\ast}(\lambda, \xi, \omega_{t}) \). Considering the influence of parameter estimation on risk hedging, we propose a new perspective to investigate the statistical properties of \( X^{\ast}(\lambda, \xi, \omega_{t}) \) and \( \text{VaR}^{\ast}(\lambda, \xi, \omega_{t}) \), including posterior mean, standard deviation, quantiles and shape of the distribution. A Bayesian statistical method is used to perform posterior inference on parameters and the optimal hedging strategy. The proposed method can provide more decision information for institutions and investors with different risk preferences to avoid risk. To the best of our knowledge, few existing literatures consider the effect of stock liquidity and parameter uncertainty on \( X^{\ast}(\lambda, \xi, \omega_{t}) \) and \( \text{VaR}^{\ast}(\lambda, \xi, \omega_{t}) \).

### IV. BAYESIAN POSTERIOR INFERENCES FOR MODEL PARAMETERS AND MINIMAL \( \text{VaR} \)

#### A. BAYESIAN POSTERIOR INFERENCES FOR LIQUIDITY-ADJUSTED ASSET PRICING MODEL

Denote by \( y_{t} = \ln \frac{S_{t}}{S_{t-1}} \) the log-return of stock \( S_{t} \) defined by (2) between consecutive time interval, for \( t = 1, 2, ..., T \). From (3), it is easy to know \( y_{t} \) is independently distributed normal with mean \( r - \frac{1}{2} \xi_{t}^{2} \omega_{t}^{2} \) and variance \( \xi_{t}^{2} \omega_{t-1}^{2} + \lambda^{2} \).

Given the stock returns, \( y = (y_{1}, y_{2}, ..., y_{T})' \), the likelihood function is

\[
L(y; \lambda, \xi, \omega) = \prod_{t=1}^{T} \frac{1}{\sqrt{2\pi \xi_{t}^{2} \omega_{t-1}^{2} + \lambda^{2}}} e^{-\frac{(y_{t} - r + \frac{1}{2} \xi_{t}^{2} \omega_{t}^{2})^{2}}{2(\xi_{t}^{2} \omega_{t-1}^{2} + \lambda^{2})}},
\]

where \( \omega = (\omega_{0}, \omega_{1}, ..., \omega_{T-1})' \) is a vector of the liquidity level measured by liquidity proxies RDV (PDV) defined before.
By Bayesian theorem, we now turn to the prior distributions for parameters. On the basis of the known results in existing literature, we take truncated normal distributions as the prior distributions for $\lambda$ and $\xi$, respectively. Assuming that they are independent each other, thus, the joint prior density is given by

$$p(\lambda, \xi | \mu_\lambda, \sigma_\lambda, \mu_\xi, \sigma_\xi) = f_N(\lambda | \mu_\lambda, \sigma_\lambda) I_{\{\lambda > 0\}} \times f_N(\xi | \mu_\xi, \sigma_\xi) I_{\{\xi > 0\}},$$

where $f_N(\cdot | a, b)$ represents the probability density of normal distribution with mean $a$ and variance $b$. $\mu_\lambda, \sigma_\lambda, \mu_\xi, \sigma_\xi$ are hyper parameters of the prior distributions. For reducing estimation error, based on the sample information and Jeffrey’s prior, we will firstly apply the method in Gao et al. [30] to obtain the posterior distributions for $\lambda$ and $\xi$, which are further used as prior information here for determining the values of the hyper parameters.

According to Bayesian formula, the kernel of the joint posterior density is

$$p(\lambda, \xi | y, \omega) \propto p(\lambda, \xi | \mu_\lambda, \sigma_\lambda, \mu_\xi, \sigma_\xi) L(y | \lambda, \xi, \omega) \times \prod_{t=1}^{T} \frac{1}{\sqrt{\xi^2 \omega^{-2}_{t-1} + \lambda^2}} e^{-\frac{(y_t - r + \frac{1}{2} \xi^2 \omega^{-2}_{t-1} + \frac{1}{2} \lambda^{2})^2}{2(\xi^2 \omega^{-2}_{t-1} + \lambda^2)}},$$

Then, the kernels of the full conditional posterior densities are

$$p(\lambda | \xi, y, \omega) \propto \frac{1}{\sigma_\lambda} e^{-\frac{(\lambda - \mu_\lambda)^2}{2\sigma_\lambda^2}} I_{\{\lambda > 0\}} \times \prod_{t=1}^{T} \frac{1}{\sqrt{\xi^2 \omega^{-2}_{t-1} + \lambda^2}} e^{-\frac{(y_t - r + \frac{1}{2} \xi^2 \omega^{-2}_{t-1} + \frac{1}{2} \lambda^{2})^2}{2(\xi^2 \omega^{-2}_{t-1} + \lambda^2)}},$$

and

$$p(\xi | \lambda, y, \omega) \propto \frac{1}{\sigma_\xi} e^{-\frac{(\xi - \mu_\xi)^2}{2\sigma_\xi^2}} I_{\{\xi > 0\}} \times \prod_{t=1}^{T} \frac{1}{\sqrt{\xi^2 \omega^{-2}_{t-1} + \lambda^2}} e^{-\frac{(y_t - r + \frac{1}{2} \xi^2 \omega^{-2}_{t-1} + \frac{1}{2} \lambda^{2})^2}{2(\xi^2 \omega^{-2}_{t-1} + \lambda^2)}}.$$

We next proceed to conduct inferences on $\lambda$ and $\xi$ based on the statistical properties of posterior distributions. However, the posterior densities have no analytical expressions, the Markov chain Monte Carlo (MCMC) algorithm is required to simulate the posterior samples, $\lambda^j$ and $\xi^j$, $j = 1, 2, ..., N$.

### B. BAYESIAN POSTERIOR INFERENCE FOR OPTION PRICE AND OPTIMAL HEDGING STRATEGY

Now, we proceed to perform posterior inferences on the price of the put option used in the hedge. Given the values of $S_t, X, \tau, \omega_t$, the option price $P_t(S_t, X, \tau, \omega_t, \lambda, \xi)$ defined by (4) is a function of $\lambda$ and $\xi$ from a mathematical perspective. Hence, the ergodic results conducting the MCMC sampling provide a direct mechanism to extract the posterior distribution of the option price. Conditionally on the posterior samples $\lambda^j$ and $\xi^j$, $P_t^j(S_t, X, \tau, \omega_t, \lambda^j, \xi^j)$ computed from (4) can be treated as the posterior samples of the put option price, more details refer to [31]. By using Monte Carlo method, the posterior expectation of the option price is

$$E[P_t(S_t, X, \tau, \omega_t, \lambda, \xi) | y] \approx \frac{1}{N - n} \sum_{j=n+1}^{N} P_t^j(S_t, X, \tau, \omega_t, \lambda^j, \xi^j).$$

Similarly, $X^{\ast j}(\lambda^j, \xi^j, \omega_j)$ computed from (10) yields the posterior sample of the optimal strike price. By using Monte Carlo method, we have

$$E[X^{\ast} | (\lambda, \xi, \omega_t) | y] \approx \frac{1}{N - n} \sum_{j=n+1}^{N} X^{\ast j}(\lambda^j, \xi^j, \omega_j).$$

Conditionally on $\lambda^j$, $\xi^j$ and $X^{\ast j}(\lambda^j, \xi^j, \omega_j)$, from (8), we obtain the posterior samples $\text{VaR}_{\lambda^j+\tau}^j(\lambda^j, \xi^j, \omega_j)$ for the minimal VaR. Then, we perform statistical inferences on the minimal $\text{VaR}_{\lambda^j+\tau}^j(\lambda, \xi, \omega_t)$ based on the posterior samples. By using Monte Carlo method, we have

$$E[\text{VaR}_{\lambda^j+\tau}^j(\lambda, \xi, \omega_t) | y] \approx \frac{1}{N - n} \sum_{j=n+1}^{N} \text{VaR}_{\lambda^j+\tau}^j(\lambda^j, \xi^j, \omega_j).$$

Moreover, we can further perform inference on any posterior moment we are interested in as well as the confidence interval. Unlike the existing literatures usually providing only a point estimation, the proposed method provides more information about $X^{\ast} | (\lambda, \xi, \omega_t)$ and $\text{VaR}_{\lambda^j+\tau}^j(\lambda, \xi, \omega_t)$ from a probabilistic perspective. These results are useful for financial institutions and investors with different risk preferences to make better decisions. Adopting point estimation may lead to a narrow view about the risk hedging strategy with a large amount of information being ignored.

### V. NUMERICAL EXPERIMENTS

#### A. RANDOM WALK CHAIN METROPOLIS-HASTINGS ALGORITHM

Since the posterior densities have no analytical expressions, the MCMC numerical algorithm is required to simulate the posterior samples for further statistical inferences on parameters, option prices, optimal strike price and the minimal VaR. Inspired by [34], we adopt the random walk chain Metropolis-Hastings algorithm to simulate posterior samples. Denote by $\theta = (\lambda, \xi)$ the vector of unknown parameters, and $\theta^\ast$ represents the candidate sample at iteration $j$ generated by following scheme

$$\theta^\ast = \theta^{(j-1)} + \epsilon,$$

where $\epsilon$ is independently identically distributed normal random variable with mean 0. Then, the candidate sample $\theta^\ast$ is accepted with probability $\hat{\alpha}(\theta^{(j-1)}, \theta^\ast)$ given by

$$\hat{\alpha}(\theta^{(j-1)}, \theta^\ast) = \min \left\{ \frac{p(\theta^\ast | y, \omega)}{p(\theta^{(j-1)} | y, \omega)} ; 1 \right\}.$$
B. EMPIRICAL APPLICATION TO S&P 500 INDEX

In order to illustrate the optimal hedging strategy using put option, we suppose an institution is concerned about the market risk of S&P 500 index over the next τ periods, and decides to hedge the VaR by investing in the put option. We will use the liquidity-adjusted (L-A) option pricing model (4) to hedge the exposure and determine an optimal strike price for the option contract which minimizes the VaR of the hedged portfolio.

The model parameters are required to be estimated before calculating the VaR of the hedged portfolio. For fully considering the effect of parameter uncertainty, the Bayesian statistical method is used to estimate model parameters. Then, the posterior inferences on the optimal strike price $X^*(\lambda, \xi, \omega_t)$ and minimal VaR$_{t+\tau}^*$$(\lambda, \xi, \omega_t)$ are performed. We collected the daily closing price of S&P 500 index and the dollar trading volume during 20 March 2017 to 15 March 2019. The risk free interest rate $r$ is available by the LIBOR rate. The data set is taken from Reuters Datastream.

According to the liquidity measures RDV and PDV defined before, we present the liquidity levels of S&P 500 index during the sample period in Fig 1.

Conditionally on $p(\lambda|\xi, y, w), p(\xi|y, w, \omega)$, the random walk chain Metropolis-Hastings algorithm is implemented 30000 iterations and discarding the initial 10000 iterations to eliminate the effect of initial values. The convergence has been checked using Geweke convergence diagnostic. Based on the posterior samples, we conduct statistical inference on parameters and $X^*(\lambda, \xi, \omega_t)$, VaR$_{t+\tau}^*$$(\lambda, \xi, \omega_t)$, including the posterior mean, standard deviation, confidence interval, kernel density, etc.

Table 1 presents the posterior results for parameters $\lambda$ and $\xi$ under different liquidity measures RDV and PDV, respectively, including the posterior means and standard deviations and MCMC convergence diagnostics. The column labeled ‘NSE’ contains numerical standard errors. The column labeled ‘CD’ proposed by Geweke [35] indicates that the convergence of the Markov chain has been achieved. The last column represents the 95% highest posterior density interval (HPDI) for parameters. Fig 2 shows the posterior histograms and posterior kernel densities of parameters $\lambda$ and $\xi$ under different liquidity measures.

We can see that the posterior estimations of $\lambda$ under liquidity measure RDV are similar to that under liquidity measure PDV, indicating that $\lambda$ is robust to the choice of liquidity measure. The posterior estimations of $\xi$ under liquidity measure RDV slightly differ from that under liquidity measure PDV, indicating that parameter $\xi$ may be somewhat sensitive to the choice of liquidity measure. In other words, the sensitivity parameter $\xi$ is related the level of stock liquidity.

For illustrating the optimal hedging strategy using the put option with liquidity-adjustment, we assume that the hedging cost $C = $1.9, the horizon of the hedge $\tau = 70$ days, and the level of protection desired by the institution, i.e., $1 - \alpha = 0.95$. By formula (10), we perform Bayesian inference on the optimal strike price $X^*(\lambda, \xi, \omega_t)$ on the basis of the posterior samples of parameters $\lambda$ and $\xi$.

Table 2 shows the posterior estimations of the optimal strike price $X^*(\lambda, \xi, \omega_t)$ under different liquidity measures: RDV and PDV. Fig 3 shows the posterior histogram and posterior kernel density for $X^*(\lambda, \xi, \omega_t)$. Based on the aforementioned parameter values and $S_t = $2531.9, the optimal strike price ranges from $2281.4 to $2478.2 at the 95% confidence level with a mean value of $2370.1 under liquidity measure RDV. We can get the similar results under liquidity measure PDV. The posterior results are robust to different liquidity measures. Consequently, the institution should take more information (quantiles, confidence interval, kernel density, etc.) into account for better investment when deciding on an appropriate optimal strike price for the option contract. Adopting point estimation may lead to a narrow view about the risk hedging strategy with a large amount of information being ignored.

Conditionally on the posterior samples of $\lambda, \xi$ and $X^*(\lambda, \xi, \omega_t)$, we then perform posterior inference on the minimal VaR$_{t+\tau}^*$$(\lambda, \xi, \omega_t)$ of the hedged portfolio. By using Bayesian statistical method, we can obtain the posterior mean, standard deviation, quantiles, the 95%HPDI and the posterior kernel density, which provide more information about minimal VaR$_{t+\tau}^*$$(\lambda, \xi, \omega_t)$ for financial institutions. Table 3 presents the posterior estimations of the minimal VaR$_{t+\tau}^*$$(\lambda, \xi, \omega_t)$ under different liquidity measures: RDV and PDV. Fig 4 shows the posterior histogram and posterior kernel density for VaR$_{t+\tau}^*$$(\lambda, \xi, \omega_t)$.

We notice that the minimal VaR$_{t+\tau}^*$$(\lambda, \xi, \omega_t)$ centers on $232.5945$ ($234.5875$), when the optimal strike price $X^*(\lambda, \xi, \omega_t)$ centers on $2370.1$ ($2370.3$), under liquidity measure RDV (PDV). From the 95%HPDI, we can see that the minimal VaR$_{t+\tau}^*$$(\lambda, \xi, \omega_t)$ varies in a large range relative to the changes of the optimal strike price $X^*(\lambda, \xi, \omega_t)$. In other words, the institution should not only consider the mean of VaR$_{t+\tau}^*$$(\lambda, \xi, \omega_t)$ but also account for the 95%HPDI which reflects the influence of parameter uncertainty on the hedging strategy. Unlike existing literatures usually providing only a point estimation, these results can provide more information about $X^*(\lambda, \xi, \omega_t)$ and VaR$_{t+\tau}^*$$(\lambda, \xi, \omega_t)$ from a probabilistic perspective, which are useful for financial institutions with different risk preferences to make better decisions.

Furthermore, in the framework of the L-A option pricing model, we investigate the relationship between the VaR$_{t+\tau}$ and the strike price of the option contract. Fig 5 plots the variations of the VaR$_{t+\tau}$ against strike prices under hedging cost $C = $1.9, $C = $2.9 and $C = $3.9, respectively. Meanwhile, as a comparison, Fig 5 (b) shows the relationship between the VaR$_{t+\tau}$ and the strike price in the framework of Black-Scholes (B-S) model. In the framework of the L-A model, we can find that the VaR$_{t+\tau}$ decreases as the hedging cost increases; and the VaR$_{t+\tau}$ is always minimized for options with a strike price of $2395 for different hedging cost. In other words, the optimal strike price is independent of the hedging cost. In the framework of B-S model, we
get the similar conclusion. But it is worth noting that the minimal $\text{VaR}_{t+\tau}^*(\lambda, \xi, \omega_t)$ with liquidity-adjustment differs from the minimal $\text{VaR}_{t+\tau}^*(\lambda, \xi, \omega_t)$ under the B-S model. Therefore, the market liquidity should be incorporated into risk measurement and management, when the underlying asset is not perfectly liquid. Otherwise still using the risk hedging strategy based on B-S model may lead to large losses for institutions.

Furthermore, we investigate the effect of stock liquidity level on $\text{VaR}_{t+\tau}$ of the hedged portfolio. As is shown in Fig 6, the effect of the stock liquidity level on the $\text{VaR}_{t+\tau}$ is significant. These numerical results support the idea that the classical risk hedging strategy should be adjusted by liquidity when imperfect hedging occurs.

VI. CONCLUSION

Imperfect liquidity occurs when a shortage or surplus of assets exists in the market. Many studies have shown that stock liquidity is a significant effect factor in financial asset pricing and risk management. However, existing literatures
TABLE 1. Posterior estimations of model parameters using Metropolis-Hastings algorithm.

<table>
<thead>
<tr>
<th>Liquidity</th>
<th>Parameter</th>
<th>Mean</th>
<th>Std.Dev.</th>
<th>NSE</th>
<th>CD</th>
<th>95%HPDI</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\lambda$</td>
<td>0.0934</td>
<td>0.0608</td>
<td>0.0004</td>
<td>-0.2549</td>
<td>[0.0079, 0.2021]</td>
</tr>
<tr>
<td>RDV</td>
<td>$\xi$</td>
<td>10.2703</td>
<td>6.3885</td>
<td>0.0452</td>
<td>-0.1026</td>
<td>[1.2624, 21.9762]</td>
</tr>
<tr>
<td>PDV</td>
<td>$\lambda$</td>
<td>0.0882</td>
<td>0.0606</td>
<td>0.0004</td>
<td>-0.1193</td>
<td>[0.0078, 0.2022]</td>
</tr>
<tr>
<td>PDV</td>
<td>$\xi$</td>
<td>4.1189</td>
<td>2.5627</td>
<td>0.0181</td>
<td>0.4601</td>
<td>[0.4445, 8.7270]</td>
</tr>
</tbody>
</table>

TABLE 2. Posterior estimations of the optimal strike price with stock liquidity.

<table>
<thead>
<tr>
<th>Liquidity</th>
<th>Optimal Strike Price Mean</th>
<th>Std.Dev.</th>
<th>95%HPDI</th>
</tr>
</thead>
<tbody>
<tr>
<td>RDV</td>
<td>$X^*_{\lambda,\xi,\omega_t}$</td>
<td>2370.1</td>
<td>59.2656</td>
</tr>
<tr>
<td>PDV</td>
<td>$X^*_{\lambda,\xi,\omega_t}$</td>
<td>2370.3</td>
<td>59.2088</td>
</tr>
</tbody>
</table>

FIGURE 3. Posterior histogram and posterior kernel density for $X^*_{\lambda,\xi,\omega_t}$ under different liquidity measures: RDV (a) and PDV (b), respectively.

TABLE 3. Posterior estimations of the minimal VaR with stock liquidity.

<table>
<thead>
<tr>
<th>Liquidity</th>
<th>Minimal VaR Mean</th>
<th>Std.Dev.</th>
<th>95%HPDI</th>
</tr>
</thead>
<tbody>
<tr>
<td>RDV</td>
<td>VaR$^*<em>t+\tau</em>{\lambda,\xi,\omega_t}$</td>
<td>232.5945</td>
<td>64.5541</td>
</tr>
<tr>
<td>PDV</td>
<td>VaR$^*<em>t+\tau</em>{\lambda,\xi,\omega_t}$</td>
<td>234.5875</td>
<td>64.4301</td>
</tr>
</tbody>
</table>

FIGURE 4. Posterior histogram and posterior kernel density for VaR$^*_t+\tau_{\lambda,\xi,\omega_t}$ under different liquidity measures: RDV (a) and PDV (b), respectively.

study the problem of hedging the VaR of risky asset by buying option contracts under the assumption of a perfectly liquid market. Therefore, we consider the problem of hedging the VaR of a position in stock by using put options in an imperfectly liquid market. We aim to find an optimal hedging strategy which minimizes the VaR of the hedged portfolio by determining an optimal strike price for the put option in an incomplete market. By using Esscher transforms, we firstly obtain a liquidity-adjusted European put option pricing formula for hedging. Then, an analytical expression for the optimal risk hedging strategy with liquidity-adjustment is obtained. The effect of stock liquidity on the risk hedging has been considered.

Although an analytical form of the optimal hedging strategy is obtained, parameter estimations are still required. We propose a new perspective to estimate parameters entering the minimal VaR. A Bayesian statistical method is used to perform posterior inference on the minimal VaR and the optimal strike price. The proposed method allows for the influence of parameter uncertainty on risk hedging. The ran-
dom walk chain Metropolis-Hastings algorithm is conducted to generate samples from posterior kernels for statistical inference on the minimal VaR, including posterior mean, standard deviation, confidence interval, etc. These results can provide more decision information for financial institutions and investors with different risk preferences to avoid risk. An empirical application to S&P 500 index is illustrated. We find that the risk hedging strategy with liquidity-adjustment differs from the hedging strategy based on Black-Scholes model. Empirical results show that the effect of stock liquidity on hedging strategy is significant, supporting the idea of incorporating stock liquidity into risk management.

This paper expands the application of Bayesian statistical method in risk management. In the future research, several related extensions can be made under Bayesian framework. For instance, one can consider the problem of hedging the exposures to multiple assets under the liquidity-adjusted asset pricing model.

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