Bipartite Tracking Consensus Control of Nonlinear High-Order Multi-Agent Systems Subject to Exogenous Disturbances

QIANG WANG¹, WEIMIN ZHONG¹,², JIAPENG XU¹, WANGLI HE¹, AND DAYU TAN¹

¹Key Laboratory of Advanced Control and Optimization for Chemical Process, Ministry of Education, East China University of Science and Technology, Shanghai, 200237, China
²Shanghai Institute of Intelligent Science and Technology, Tongji University, Shanghai, 201804, China

Corresponding author: Weimin Zhong (e-mail: wmpz@ecust.edu.cn).

This work is supported by National Key R & D Program of China (2016YFB0303403), National Natural Science Foundation of China (61890930-3, 61720106008), Programme of Introducing Talents of Discipline to Universities (the 111 Project) under Grant B17017.

ABSTRACT This study concentrates on the bipartite tracking consensus for nonlinear high-order multi-agent systems (MASs) with exogenous disturbances over a signed directed graph. First, under the condition of nonlinear dynamics, two new control protocols are proposed to guarantee high-order bipartite tracking consensus under two cases with and without exogenous disturbances, respectively. Second, a disturbance observer is designed to estimate the external disturbances which originated from an external system. Third, to provide some efficient criteria for high-order bipartite consensus of MASs, we propose a novel pinning nodes selected scheme and get the lower bound of the pinning gains. Furthermore, by virtue of Lyapunov stability theory and graph theory, sufficient conditions for bipartite tracking consensus control of nonlinear high-order MASs subject to external disturbances are presented. Finally, simulations are performed for demonstration.

INDEX TERMS MASs, bipartite consensus, high-order, exogenous disturbances.

I. INTRODUCTION

Up to now, MASs have attracted a great devotion due to its widespread applications in consensus [1], [2], sensor networks [3], unmanned air vehicles (UAV) [4], flocking/swarming control [5], and synchronization [6]. As a consequence, the consensus problem of MASs has been a hot research topic in the filed of cooperative control [7]. Inspired by diverse consensus control problems of MASs, such as consensus control with linear dynamics [8], nonlinear dynamics [9], [10], time-delay consensus [11], and high-order consensus [12], [13], which generalizes the existing first-order [14] and second-order [15] consensus results, has received an increasing research interest. Based on cooperative interactions between neighbors, the main idea of cooperative consensus control is addressed by developing an appropriate consensus protocol, and then to attain collective behaviors, such as tracking consensus problem in MASs. Note that tracking consensus [16], [17] is a special case of consensus in which only part of agents has access to the leader’s information, and the objective is that each agent follows the leader asymptotically. For the MASs with fixed interaction topology, pinning control approaches [18], [19] are more available to control MASs for the benefit of reducing computational burden and equipment resource. In addition, the MASs with semi-Lipschitz nonlinear dynamics [20], [21] has been a significant application in cooperative consensus control.

Generally speaking, most of the foregoing research focused on MASs with cooperative communication link [22]. However, the agents may also interact in an antagonistic way [23]. Actually, the interests or objectives of two agents usually are contradictory and hence the agents regard one-another as opponents in practical scenarios. Such situations generally happen in social networks and biological systems, where the interactions described by signed graphs coexist within a team of agents are either collaborative or antagonistic. By taking advantage of signed graph, Altafini [24] introduced a concept of bipartite consensus where all agents reach an identical modulus value except for the sign. Zhao et al. [25] adopted distributed adaptive protocols to deal with the prob-
lem of finite-time bipartite tracking consensus. Zhang et al. [26] investigated the effect of event-triggered control on the bipartite consensus problem. Zhu et al. [27] solved bipartite consensus of MASs subject to quantization. What’s more, in [28], the distributed model reference adaptive control protocols were proposed for MASs to resolve the bipartite consensus problem. The bipartite consensus for MASs based on directed communication topologies was studied in [29].

Finite-time [30], time-delay [31], linear [32] and nonlinear adaptive bipartite consensus control [33], [34] of MASs also have been investigated, which pointed out the direction for the combination of bipartite consensus and other practical nonlinearity applications in the future.

As far as we know, most of the existing works take bipartite consensus of MASs without disturbances into consideration. However, the unknown exogenous disturbances are inevitably generated in practical processes. In recent years, the disturbance rejection has been a particularly significant problem in the controller design of MASs. Wang et al. [35] proposed a disturbance observer-based control (DOBC) for high-order MASs with Lipschitz dynamic and input delay. Du et al. [36] presented a non-smooth algorithm to reach consensus with disturbance by back-stepping control and finite-time control. Motivated by the foregoing research about bipartite consensus subject to the matched disturbances [37] and the bounded disturbances [38], a disturbance observer is designed to estimate the external disturbance generated by a linear exogenous system. However, until now, it should notice that there are very few results on nonlinear high-order dynamic model which is ubiquitous in practical applications. Furthermore, it is still difficult to determine the disturbances generated from the external environment, especially if the disturbances are assumed to be unmatched, which is challenging from practical viewpoint.

Inspired by the related works on this topic, the contributions of this paper can be divided into threefold:

- We pose and address the bipartite tracking consensus for the MASs with high-order and nonlinear dynamics, two novel distributed protocols under two cases with and without exogenous disturbances are presented respectively, which can better improve the robustness and reliability of the systems. However, there are very few works about high-order nonlinear bipartite consensus control algorithm and the disturbances were not considered in [11], [12], [16].
- By utilizing the approaches of linear matrix inequality and Lyapunov stability analysis, the disturbance observer is designed to estimate the external disturbance generated by a linear exogenous system. Compare with the exiting works [36]–[38], the unavailable upper bounds of external disturbances make the underlying problem rather challenging, and it is relatively difficult for the estimation of system states owing to the unknown disturbances.

- To amend the drawbacks of the commonly leader-follower method [2], [29], [32], [38], the pinning control is developed to apply local feedback injects to part agents, which not only reduces the number of controllers but also achieves bipartite tracking consensus efficiently.

The remainder of this paper is composed of five sections. Section II gives the preliminary knowledge, some useful definitions and lemmas, and problem statement. Section III develops the main results. The simulation results are given in Section IV. Section V concludes the paper.

The following notations are utilized throughout this paper. \( R^n \) stands for \( n \)-dimensional Euclidean space. \( I_N \) denotes the \( N \times N \) identity matrix. \( 1_N \) and \( 0_N \) denote the \( N \times 1 \) column vector of all ones and all zeros, respectively. \( 0_{m \times n} \) denotes the \( m \times n \) matrix with all zero entries. \( \lambda_{\max} (\cdot) \) and \( \lambda_{\min} (\cdot) \) represent the maximum and minimum eigenvalues of a matrix. Let \( \text{sign} (\cdot) \) be sign function with the definition

\[
\text{sign} (x) = \begin{cases} 
1, & x > 0 \\
0, & 1 = 0 \\
-1, & x < 0 
\end{cases}
\]

II. PROBLEM STATEMENT

A. GRAPH THEORY

In this section, the communication network topology of the system is represented by \( G = (V, E, A) \), where \( V = \{1, 2, 3, \cdots, N\} \) represents a vertex set, \( E \subseteq V \times V \) represents the set of communication edges, and each communication edge is represented by a pair of ordered nodes \( \{j, i\} \). If \( \{j, i\} \subseteq E \), then we say that \( i \) and \( j \) are adjacent. The neighbors of agent \( i \) is denoted by \( N_i = \{j \in V : (j, i) \in E\} \). \( A = [a_{ij}] \in R^{N \times N} \) is a symmetric matrix defined as graph adjacency matrix, \( a_{ij} > 0 \) if \( (j, i) \in E, i \neq j \), \( a_{ij} = 0 \) if \( (j, i) \notin E \). Throughout the paper, suppose that the signed graph \( G \) is structurally balanced. The set of nodes \( V \) are divided as disjoint subsets \( V_1 \) and \( V_2 \). Generally, \( \exists D = \{D = \text{diag} \{d_1, d_2, \cdots, d_N\}, d_i = \{\pm 1\}\} \), in which \( D \) represents a diagonal matrix with elements \( d_1, d_2, \cdots, d_N \), and \( D \) exists two partitions such that \( V_1 = \{i | d_i > 0\} \) and \( V_2 = \{i | d_i < 0\} \). If for every vertex, the in-degree and out-degree are equal, we called a digraph is balanced. The Laplacian matrix \( L \) for structurally balanced graph \( G \) is given in the form of

\[
l_{ii} = \sum_{j=1, j \neq i}^{N} |a_{ij}|; l_{ij} = -a_{ij}, i \neq j,
\]

where \( | | \) is the absolute value symbol.

B. MODEL DESCRIPTION

In this paper, the dynamics of \( N \) followers subject to exogenous disturbances in the MASs are given as below

\[
\begin{align*}
\dot{x}_{11} &= x_{21}, \\
\vdots \\
\dot{x}_{(n-1)1} &= x_{n1}, \\
\dot{x}_{n1} &= f (x_{11}, x_{21}, \cdots, x_{n1}) + u_1 + w_1,
\end{align*}
\]

This work is licensed under a Creative Commons Attribution 4.0 License. For more information, see https://creativecommons.org/licenses/by/4.0/.
where $x_{ki} \in \mathbb{R}^m$, $k = 1, 2, \cdots, n$ is the state of follower, $u_i \in \mathbb{R}^m$ is the control input, $w_i$ represents external disturbance, and $x_{ki}$ denotes the $k$th order derivative of $x_i$. Moreover, $f : \mathbb{R}^{nxm} \rightarrow \mathbb{R}^m$ is a convex-valued mapping, nonlinear continuous and differentiable in $t$, which describes the intrinsic dynamic of agent $i$. Similarly, the model of the leader agent can be expressed as

\[
\begin{align*}
\begin{cases}
  \dot{x}_0^1 = x_0^1, \\
  \vdots \\
  \dot{x}_0^{(n-1)} = x_n^0,
\end{cases} \\
  \dot{x}_0^n = f(x_1^0, x_2^0, \cdots, x_n^0),
\end{align*}
\]  

where $x_0^k \in \mathbb{R}^m$, $k = 1, 2, \cdots, n$ is the state of leader. In this case, the following distributed bipartite consensus control protocol is proposed as

\[
\begin{align*}
u_i &= -\beta \sum_{j=1}^{N} |a_{ij}| [(x_{1i} - sgn(a_{ij})x_{1j})] \\
&+ \cdots + [(x_{ni} - sgn(a_{ij})x_{nj})] \\
&- \beta_0 \{[(x_{1i} - d_{1i}x_0^1) \\
&+ \cdots + (x_{ni} - d_{ni}x_0^n)] + w_i,\}
\end{align*}
\]  

where $\beta > 0$, and the pinning feedback gain $\alpha_i$ satisfies $\alpha_i > 0$, $i = 1, 2, \cdots, l$; $\alpha_i = 0$, $i = l + 1, l + 2, \cdots, N$.

**Definition 1:** (bipartite tracking consensus) Consider the networks are modeled by a single leader agent and $N$ follower agents, bipartite tracking consensus for MASs can be achieved if there exists either

\[
\lim_{t \to \infty} \|x_{ki} - x_{ki}^0\| = 0,
\]

or

\[
\lim_{t \to \infty} \|x_{ki} + x_{ki}^0\| = 0.
\]

Evidently, the equalities can be further written as

\[
\lim_{t \to \infty} \|x_{ki} - d_{ki}x_0\| = 0, \quad i = 1, 2, \cdots, N.
\]  

Furthermore, let $x_0 = \left( (x_0^1)^T, (x_0^2)^T, \cdots, (x_0^n)^T \right)^T$, $x_1 = \left( (x_{10}^T, x_{12}^T, \cdots, x_{1n}^T) \right)^T$, $x_n = \left( (x_{n1}^T, x_{n2}^T, \cdots, x_{nn}^T) \right)^T$, $i = 1, \cdots, N$, $x_n$ is the column stack vector of $x_0^k, k = 1, 2, \cdots, n$, $x_1$ is the column stack vectors of $x_{ki}, i = 1, \cdots, N$. Similarly, $x_n$ is the column vector of $s_{ki}$, $i = 1, \cdots, N$.

**Definition 2:** A signed graph $G$ is called structurally balanced if it has a node bipartition with $\{V_1, V_2\}$ satisfying $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$, such that $a_{ij} \geq 0$ if $\forall v_i, v_j \in V_k (k \in \{1, 2\})$, and $a_{ij} \leq 0$ if $\forall v_i \in V_k, v_j \in V_k, v \neq k \notin l (k, l \in \{1, 2\})$.

Some assumptions are listed as follows

**Assumption 1:** The signed digraph $G$ is structurally balanced and contains a directed spanning tree, where the leader is located at the root node.

**Assumption 2:** The nonlinear dynamics $f(\cdot)$ is a nonlinear function, and there is a positive constant $\rho$ such that

\[
\| f(a_1, t) - f(a_2, t) \| \leq \rho \| a_1 - a_2 \|, \forall a_1, a_2 \in \mathbb{R}^m.
\]  

**Assumption 3:** Suppose that there exists non-negative constants $h_1, \cdots, h_{n-2},$ and the continuously differentiable function $f$ satisfies the following semi-Lipschitz condition such that

\[
\begin{align*}
(x_1 - x_0^0)^T f(x_1, \cdots, x_n) - f(x_0^1, \cdots, x_0^n) \\
\leq h_1(x_1 - x_0^0)^T (x_1 - x_0^1) \\
+ \cdots + h_n(x_n(t) - x_n^0)^T (x_n - x_0^n) \\
+ \cdots + h_{n-1} 1(x_1 - x_0^0)^T (x_1 - x_0^1) \\
+ \cdots + h_{n(n-1)+1} 1(x_1 - x_0^0)^T (x_1 - x_0^1) \\
+ \cdots + h_{n(n-1)+1} 1(x_1 - x_0^0)^T (x_1 - x_0^1).
\end{align*}
\]  

**Remark 1:** As shown in [20], the semi-Lipschitz condition yields an effective method for rendering the proceedings which present to some relevant applications of complex network system. The distributed control strategies with semi-Lipschitz dynamics were investigated in [21]. Gao et al. [21] have proved that the uniform semi-Lipschitz condition is greater openness than the uniform Lipschitz condition. Note that Assumption 3 is a semi-Lipschitz condition, which is satisfied by Lorenz system, Chua oscillator, Chen system and so on.

Two useful lemmas are given as follows:

**Lemma 1:** [6] Given linear matrix inequality

\[
\begin{pmatrix}
q_{11} & q_{12} \\
q_{21} & q_{22}
\end{pmatrix} > 0,
\]

where $q_{11}, q_{12}, q_{21}, q_{22} \in \mathbb{R}^{n \times n}$, are equivalent to either of following conditions

\[
q_{11} > 0, q_{22} - q_{21}q_{11}^{-1}q_{12} > 0,
\]

\[
q_{22} > 0, q_{11} - q_{12}q_{22}^{-1}q_{21} > 0.
\]

**Lemma 2:** [18] For a symmetric matrix $M \in \mathbb{R}^{N \times N}$ and diagonal matrix $\alpha = diag\{(\alpha_1, \cdots, \alpha_1, 0, \cdots, 0)\}_{N \times N}$ with $\alpha_i > 0, i = 1, \cdots, l$ ($1 \leq l \leq N$). Let

\[
M - \alpha = \begin{bmatrix}
G - \hat{a} & \hat{Q} \\
\hat{Q}^T & M_l
\end{bmatrix},
\]

where $M_l$ is the minor matrix of $M$ by removing its first $p$ row-column pairs, $G$ and $\hat{Q}$ are matrices with appropriate dimensions, $\hat{a} = diag\{(\alpha_1, \cdots, \alpha_1)\}$. If $\alpha_i > \lambda_{max}\left(G - QM_l^{-1}\hat{Q}\right), i = 1, 2, \cdots, l$, then $M - \alpha < 0$ is equivalent to $M_l < 0$.

**III. MAIN RESULTS**

In this section, a disturbance observer respect to unknown external disturbances is constructed. Then based on the distributed consensus algorithm (5), we will derive sufficient conditions for bipartite tracking consensus of the high-order MASs with and without disturbances, respectively.
A. BIPARTITE TRACKING CONSENSUS WITH EXOGENOUS DISTURBANCES

In this subsection, assuming that the exogenous control disturbance $w_i(t)$ is expressed as a linear control system, and let $w_i^*$ be the estimate of $w_i$. Then we can propose the following form

$$\begin{cases}
\dot{\delta}_i = A\delta_i, \\
w_i = C\delta_i,
\end{cases}$$

(10)

where $\delta_i \in \mathbb{R}^{m_o}$ indicates the internal state value of the exogenous disturbance, $A \in \mathbb{R}^{m_o \times m_o}$ and $C \in \mathbb{R}^{m \times m_o}$ indicate the control matrix for exogenous disturbances. In order to determine unknown exogenous disturbance, an observer with respect to the disturbance can be expressed as

$$\dot{\tilde{\delta}}_i = (A + KG_iC)\dot{\delta}_i - KG_iw_i,$$

(11)

where $\tilde{\delta}_i$ and $w_i^*$ represent the estimates of $\delta_i$ and $w_i$, respectively. $G_i \in \mathbb{R}^{m \times m}$ is the coefficient matrix and $K \in \mathbb{R}^{m \times m_o}$ is the adjustable feedback control gain. Then we can get the error of state estimation as follows

$$e_i = \delta_i - \tilde{\delta}_i.$$

(12)

According to (11) and (12), one obtains

$$\dot{e}_i = (A + KG_iC)e_i.$$

(13)

Therefore, we conclude that the disturbance observer (11) can exponentially track the disturbance if and only if $A + KG_iC < 0$ [18]. Based on the previous analysis, one could construct the following theorem. And the high-order bipartite tracking consensus problem of MASs can be solved under the control protocol (5) together with the disturbance-based observer (11).

Theorem 1: Consider the high-order MASs (3) and (4) with the exogenous disturbances. Under Assumptions 1, 2, and 3, the high-order bipartite tracking consensus control can be realized asymptotically, if there exists appropriate matrix $P > 0$ such that

$$\gamma I_N - \frac{\beta (L + L^T)}{2} - \beta\alpha < 0,$$

(14)

and

$$Q = \begin{bmatrix} Z & \Psi \\ \Psi^T & I_n \otimes F \end{bmatrix} < 0,$$

(15)

where

$$\gamma = \max \left\{ \left( h_1 + \cdots + h_{n(n-1)+1} \right), \cdots, \left( h_n + \cdots + h_{n^2+1} \right) \right\},$$

(16)

$$\Psi = \begin{bmatrix}
\beta \left( L + L^T \right) + 2\beta\alpha & I_N & \cdots & I_N \\
I_N & I_N & \cdots & I_N \\
\vdots & \vdots & \ddots & \vdots \\
I_N & I_N & \cdots & I_N
\end{bmatrix},$$

(17)

$$Z = \begin{bmatrix}
\gamma I_N + \Sigma & 0 & \cdots & 0 \\
0 & \gamma I_N + \Sigma & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \gamma I_N + \Sigma
\end{bmatrix},$$

(18)

$$\Sigma = -\beta (L + \alpha) \otimes I_n,$$

(19)

$$F = \hat{A}^TP + P\hat{A},$$

(20)

$$\hat{A} = A + KG_iC.$$

Proof: Let $w_i^*$ instead of $w_i$, and the composite controller is rewritten as

$$u_i = -\beta \sum_{j=1}^N |a_{ij}| \left[ (x_{1i} - sgn(a_{ij})x_{1j}) + \cdots + (x_{ni} - sgn(a_{ij})x_{nj}) \right] + \beta\alpha \left[ (x_{1i} - d_ix_{1i}^0) + \cdots + (x_{ni} - d_ix_{ni}^0) \right] - w_i^*.$$  

(21)

Due to $sgn(a_{ij})d_i = d_j$, one has

$$u_i = -\beta \sum_{j=1}^N |a_{ij}| \left[ (x_{1i} - d_ix_{1i}^0) - sgn(a_{ij}) (x_{1j} - d_ix_{1i}^0) + \cdots + (x_{nj} - d_nx_{nj}^0) \right] + \beta\alpha \left[ (x_{1i} - d_ix_{1i}^0) + \cdots + (x_{ni} - d_nx_{ni}^0) \right] - w_i^*.$$  

(22)

Let

$$\begin{cases}
\ddot{x}_{1i} = x_{1i} - d_ix_{1i}^0, \\
\vdots \\
\ddot{x}_{ni} = x_{ni} - d_nx_{ni}^0,
\end{cases}$$

(23)

with (22) and (23), one can obtain

$$\begin{cases}
\dot{x}_{1i} = \ddot{x}_{1i}, \\
\vdots \\
\dot{x}_{(n-1)i} = \ddot{x}_{(n-1)i}, \\
\dot{x}_{ni} = -\beta \sum_{j=1}^N |a_{ij}| \left[ (\ddot{x}_{1i} - sgn(a_{ij})\ddot{x}_{1j}) + \cdots + (\ddot{x}_{nj} - sgn(a_{ij})\ddot{x}_{nj}) \right] + \beta\alpha \left[ (\ddot{x}_{1i} - d_i\ddot{x}_{1i}^0) + \cdots + (\ddot{x}_{ni} - d_n\ddot{x}_{ni}^0) \right] + f(\ddot{x}_{1i}, \cdots, \ddot{x}_{ni}) - d_if(\ddot{x}_{1i}, \cdots, \ddot{x}_{ni})^T + Ce_i.
\end{cases}$$

(24)

The compact form of (24) is expressed by

$$\begin{cases}
\dot{\ddot{x}}_{1i} = \ddot{x}_{2i}, \\
\vdots \\
\dot{\ddot{x}}_{(n-1)i} = \ddot{x}_{ni}, \\
\ddot{x}_{ni} = - \left( (L + S) \otimes I_n \right) (\ddot{x}_{1i} + \cdots + \ddot{x}_{n}) + F(x_{1i}, \cdots, x_{ni}) + \left( I_N \otimes f(\ddot{x}_{1i}, \cdots, \ddot{x}_{ni}) + (C \otimes I_n)e, \right.
\end{cases}$$

(25)
where
\[
\begin{align*}
\dot{x}_1 &= d_i x_i, \\
\vdots \\
\dot{x}_n &= d_i x_n.
\end{align*}
\] (26)

Moreover, \(\dot{x}_1\) is the column stack vectors of \(\dot{x}_{i}, i = 1, \ldots, N\). Similarly, \(\dot{x}_n\) is the column stack vector of \(\dot{x}_{ni}, i = 1, \ldots, N\). Let \(\dot{x} = (\dot{x}_1^T, \ldots, \dot{x}_n^T)^T\), then
\[
\dot{V} = F (x_1, \ldots, x_n, x_0, \ldots, x_0) + (B \otimes I_n) \dot{x} + (C \otimes I_n) e,
\] (27)

where
\[
\begin{align*}
F (x_1, \ldots, x_n, x_0, \ldots, x_0) &= \begin{bmatrix} F (x_1, \ldots, x_n) \\
-1_N \otimes f (x_1, \ldots, x_n) \end{bmatrix}, \\
B &= \begin{bmatrix} 0_N & I_N & 0_N & \cdots & 0_N \\
0_N & 0_N & I_N & \cdots & 0_N \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\Sigma & \Sigma & \Sigma & \cdots & I_N
\end{bmatrix},
\end{align*}
\] (28)

in which
\[
\Sigma = -\beta (L + \alpha) \otimes I_n.
\] (30)

The Lyapunov function candidate \(V\) is further constructed as
\[
V = \frac{1}{2} \dot{x}^T (\Psi \otimes I_n) \dot{x} + e^T P e,
\] (31)

where
\[
\Psi = \begin{bmatrix} \beta (L + L^T) + 2\beta \alpha & I_N & \cdots & I_N \\
I_N & I_N & \cdots & I_N \\
\vdots & \vdots & \ddots & \vdots \\
I_N & I_N & \cdots & I_N
\end{bmatrix}.
\] (32)

In view of condition (14), afterwards, it is not difficult to obtain that
\[
\beta \frac{L + L^T}{2} + \beta \alpha - \gamma I_N > 0.
\]

Invoking the above inequality, it follows that
\[
\beta (L + L^T) + 2\beta \alpha - I_N > 0.
\]

From Lemma 1, and the above derivation proves that
\[
\begin{bmatrix} \beta (L + L^T) + 2\beta \alpha & I_N & \cdots & I_N \\
I_N & I_N & \cdots & I_N \\
\vdots & \vdots & \ddots & \vdots \\
I_N & I_N & \cdots & I_N
\end{bmatrix} > 0,
\]
which indicates that
\[
V \geq 0.
\]

The derivative of \(V\) is given as
\[
\dot{V} = \dot{V}_1 + \dot{V}_2,
\] (33)

where
\[
\dot{V}_1 = \dot{x}^T (\Psi \otimes I_n) (F (x_1, \ldots, x_n, x_0, \ldots, x_0)) + \dot{x}^T (B \otimes I_n) \dot{x} + (C \otimes I_n) e,
\] (34)

\[
\dot{V}_2 = \dot{x}^T (\Psi \otimes I_n) e + e^T (A^T P + P A) e.
\] (35)

Differentiating \(V_1\), we can get
\[
\dot{V}_1 = \dot{x}^T \left[ \beta (L + L^T + 2\alpha) \otimes I_n \right] \dot{x}_1 + \cdots + \dot{x}^T \left[ \beta (L + L^T + 2\alpha) \otimes I_n \right] \dot{x}_N
\]
\[
- \dot{x}^T \left[ (\beta (L + \alpha) \otimes I_n) \left( \dot{x}_1 + \dot{x}_2 + \cdots + \dot{x}_N \right) \right]
\]
\[
- \cdots - \dot{x}^T \left[ (\beta (L + \alpha) \otimes I_n) \left( \dot{x}_1 + \dot{x}_2 + \cdots + \dot{x}_N \right) \right]
\]
\[
+ \dot{x}^T \left( (F (x_1, \ldots, x_n) - 1_N f (x_1, \ldots, x_0)) \right) - 1_N f (x_1, \ldots, x_0)
\]
\[
+ \cdots + \dot{x}^T \left( (F (x_1, \ldots, x_n) - 1_N f (x_1, \ldots, x_0)) \right) - 1_N f (x_1, \ldots, x_0)
\]
\[
= -\dot{x}^T \left[ \beta (L + \alpha) \otimes I_n \right] \dot{x}_1 + \cdots + \dot{x}^T \left[ (k_1 I_N - k_\beta (L + \alpha) \otimes I_n) \right] \dot{x}_N
\]
\[
+ \cdots + \dot{x}^T \left( (F (x_1, \ldots, x_n) - 1_N f (x_1, \ldots, x_0)) \right) - 1_N f (x_1, \ldots, x_0)
\]
\[
+ \cdots + \dot{x}^T \left( (F (x_1, \ldots, x_n) - 1_N f (x_1, \ldots, x_0)) \right) - 1_N f (x_1, \ldots, x_0)
\].

By applying Assumption 3, we have that (34) should satisfy the following inequality
\[
\dot{V}_1 \leq \dot{x}^T \left[ (\dot{h}_1 + \cdots + \dot{h}_n) - \beta (L + \alpha) \otimes I_n \right] \dot{x}_1 + \cdots + \dot{x}^T \left[ (k_1 h_{n+1} + \cdots + h_{2n}) \right] - k_\beta (L + \alpha) \otimes I_n
\]
\[
+ \cdots + \dot{x}^T \left[ k_{n-1} (h_{n(n-1)+1} + \cdots + h_{2n}) \right] - k_{n-1} (L + \alpha) \otimes I_n \dot{x}_N
\]
\[
\leq \dot{x}^T \left[ (h_1 I_N - \beta (L + \alpha) \otimes I_n) \dot{x}_1 + \cdots + \dot{x}^T \left[ (k_1 \gamma I_N - k_{n-1} \beta (L + \alpha) \otimes I_n) \right] \dot{x}_N
\]
\[
\vdots
\]
\[
+ \dot{x}^T \left[ (k_{n-1} \gamma I_N - k_{n-1} \beta (L + \alpha) \otimes I_n) \right] \dot{x}_N
\]
\[
= \dot{x}^T Z \dot{x}.
\] (36)

Therefore, the following holds
\[
\dot{V} \leq \dot{x}^T Z \dot{x} + \dot{x}^T \Psi e + e^T (A^T P + P A) e
\]
\[
= (\dot{x}^T, e^T) Q \begin{bmatrix} \dot{x} \\ e \end{bmatrix}.
\] (37)

By Lyapunov stability theory, one has \(V > 0, \dot{V} < 0\), if and only if \(\gamma I_N - \beta \frac{L + L^T}{2} - \beta \alpha < 0\) and under the condition (15). That is to say, if the above conditions are satisfied can the bipartite tracking consensus with external disturbances be completed. This completes the proof.

Remark 2: Different from [35]–[38], which calculated the bounds of disturbance by a series of complex derivation, we propose a disturbance observer to estimate the external disturbance generated by linear exogenous system, which can ensure the agents to accommodate the changes of environments and distributed cooperative tasks.

Remark 3: Recently, bipartite consensus problems with general linear dynamics have been extensively considered in the related works [29], [31], [32]. Moreover, with the widely usage of digital sensors and wireless networks, MASs...
usually work in the nonlinear environment, and there are limited storage and communication bandwidth when the information is transmitted through digital channels, which result the agents depend on digital communication to acquire key information in its nonlinear quantized form rather than receive the precise information from neighboring agents via real-time sensing. However, it is notable that the bipartite consensus protocol design for nonlinear dynamics is more complex than for the linear dynamics, due to its limitation to local stability, network communication and so on. In view of the above facts [11, 8], based on our prior work on bipartite consensus, a bipartite tracking consensus problem is formulated for MASs with nonlinear dynamics, which can better improve the stability and robustness when facing to external disturbances.

Remark 4: Different from [17, 15, 20], in which the bipartite consensus control problems based on first-order dynamic and second-order dynamic were settled for MASs, respectively. Nevertheless, high-order dynamic is unavoidable in the practical environment. Hence, it is of significance to study the bipartite tracking consensus for high-order MASs. This paper aims to solve the bipartite tracking consensus for a flock of N follower agents and a leader agent under the condition of high-order dynamic.

### B. BIPARTITE TRACKING CONSENSUS WITHOUT EXOGENOUS DISTURBANCES

In this subsection, we analyze the stability of bipartite tracking consensus for high-order MASs without exogenous disturbances. Then the system of the followers is expressed as

$$
\begin{align*}
\dot{x}_{11} &= x_{21}, \\
\vdots \\
\dot{x}_{(n-1)i} &= x_{ni}, \\
x_{ni} &= f(x_{11}, x_{2i}, \ldots, x_{ni}) + u_i,
\end{align*}
$$

and the form of leader

$$
\begin{align*}
\dot{x}_1^0 &= x_2^0, \\
\vdots \\
\dot{x}_{(n-1)}^0 &= x_n^0, \\
x_0^0 &= f(x_1^0, x_2^0, \ldots, x_n^0).
\end{align*}
$$

It becomes obvious that the protocol of system (5) is reconstructed as

$$
\begin{align*}
u_i &= -\beta \sum_{j=1}^{N} |a_{ij}| [(x_{i1} - sgn(a_{ij}) x_{1j}) \\
&\quad + \cdots + (x_{ni} - sgn(a_{ij}) x_{nj})] \\
&\quad - \beta \alpha_i [(x_{i1} - d_i x_1^0) + \cdots + (x_{ni} - d_i x_n^0)].
\end{align*}
$$

And make use of Lemma 3, let $\gamma I_N - \beta \frac{L + L^T}{2} = M$, then

$$
\gamma I_N - \beta \frac{L + L^T}{2} - \beta \alpha = \begin{bmatrix} G & -\beta \alpha \\ Q & M \end{bmatrix}.
$$

**Theorem 2:** Under Assumptions 1-3, the bipartite tracking consensus control for nonlinear high-order multi-agent systems (MASs) with exogenous disturbances can be realized if the following conditions are met.

**Proof:** By taking the controllers (3) and (4), we get

$$
\begin{align*}
\dot{x}_{11} &= \dot{x}_{21}, \\
\vdots \\
\dot{x}_{(n-1)i} &= \dot{x}_{ni}, \\
x_{ni} &= f(x_{11}, x_{2i}, \ldots, x_{ni}) + u_i,
\end{align*}
$$

Construct Lyapunov functional candidate as

$$
V = \frac{1}{2} \dot{x}^T (\Psi \otimes I_N) \dot{x},
$$

where

$$
\dot{x} = (\dot{x}_1^T, \ldots, \dot{x}_n^T),
$$

and

$$
\Psi = \begin{bmatrix} \beta (L + L^T) + 2\beta \alpha & I_N & \cdots & I_N \\ I_N & I_N & \cdots & I_N \\
\vdots & \vdots & \ddots & \vdots \\ I_N & I_N & \cdots & I_N \end{bmatrix}.
$$

The compact form of Equation (44) is expressed by

$$
\begin{align*}
\dot{x}_1^0 &= \dot{x}_2^0, \\
\vdots \\
\dot{x}_{(n-1)}^0 &= \dot{x}_n^0, \\
x_0^0 &= f(x_1^0, x_2^0, \ldots, x_n^0).
\end{align*}
$$

It follows that

$$
\beta \lambda_{max} \left( \frac{-L + L^T}{2} \right) + \gamma < 0.
$$

Since $\gamma \geq 1$, one has

$$
\lambda_{max} \left( \left( \gamma I_N - \beta \frac{L + L^T}{2} \right) \right) \leq \beta \lambda_{max} \left( \frac{-L + L^T}{2} \right) + \gamma < 0.
$$
It follows from Lemma 2 and (49) that
\[ \gamma I_N - \beta \frac{L + LT}{2} - \beta \alpha < 0. \]  
(51)

Then, we can get
\[ \beta (L + LT) + 2\alpha \beta - I_N > 0, \]  
(52)

based on Lemma 1, we yield
\[
\begin{bmatrix}
\beta (L + LT) + 2\beta \alpha & I_N & \cdots & I_N \\
I_N & I_N & \cdots & I_N \\
\vdots & \vdots & \ddots & \vdots \\
I_N & I_N & \cdots & I_N
\end{bmatrix} > 0. 
\]
(53)

According to Lemma 3, one gets that \( V > 0, \) which, in turn, implies that \( V > 0. \) Rewrite (45) as
\[
V = \frac{\beta}{2} x_1^T \left( (L + LT + 2\alpha) \otimes I_n \right) x_1 + \cdots + \frac{\beta}{2} x_n^T \left( (L + LT + 2\alpha) \otimes I_n \right) x_n \\
- \frac{\beta}{2} x_1^T \left[ (L + L\alpha) \otimes I_n \right] (x_1 + x_2 + \cdots + x_n) \\
- \cdots - \frac{\beta}{2} x_n^T \left[ (L + L\alpha) \otimes I_n \right] (x_1 + x_2 + \cdots + x_n) \\
+ \frac{\beta}{2} x_1^T x_2 + \cdots + \frac{\beta}{2} x_n^T x_n \\
+ \frac{\beta}{2} x_1^T \left[ F \left( x_1, \cdots, x_n \right) - 1_N f \left( x_1^0, \cdots, x_n^0 \right) \right] \\
+ \cdots + \frac{\beta}{2} x_n^T \left[ F \left( x_1, \cdots, x_n \right) - 1_N f \left( x_1^0, \cdots, x_n^0 \right) \right] \\
= -\frac{\beta}{2} x_1^T \left( (L + L\alpha) \otimes I_n \right) x_1 \\
+ \frac{\beta}{2} x_2^T \left( k_1 I_N - k_1 \beta (L + \alpha) \otimes I_n \right) x_2 \\
+ \cdots + \frac{\beta}{2} x_n^T \left( k_{n-1} I_N - k_{n-1} \beta (L + \alpha) \otimes I_n \right) x_n \\
+ \frac{\beta}{2} x_1^T \left[ F \left( x_1, \cdots, x_n \right) - 1_N f \left( x_1^0, \cdots, x_n^0 \right) \right] \\
+ \cdots + \frac{\beta}{2} x_n^T \left[ F \left( x_1, \cdots, x_n \right) - 1_N f \left( x_1^0, \cdots, x_n^0 \right) \right].
\]
(54)

The derivative of \( V \) is given by
\[
\dot{V} \leq \frac{\beta}{2} \max_i \left[ \left( h_i + \cdots + h_{n_i} \right) - \beta (L + \alpha) \otimes I_n \right] x_1 \\
+ \frac{\beta}{2} x_2^T \left[ k_1 h_{n+i} \cdots + h_{n_2} \right] \\
- k_1 \beta (L + \alpha) \otimes I_n \\
\vdots \\
+ \frac{\beta}{2} x_{n-1}^T \left[ k_{n-1} h_{n(n-1)+1} \cdots + h_{n_2} \right] \\
- k_{n-1} \beta (L + \alpha) \otimes I_n \\
\leq \frac{\beta}{2} \gamma I_N - \beta (L + \alpha) \otimes I_n \left[ I_N \right] x_1 \\
+ \frac{\beta}{2} x_2^T \left[ k_1 \gamma I_N - k_1 \beta (L + \alpha) \otimes I_n \right] x_1 \\
\vdots \\
+ \frac{\beta}{2} x_{n-1}^T \left[ k_{n-1} \gamma I_N - k_{n-1} \beta (L + \alpha) \otimes I_n \right] x_n.
\]
(55)

Based on the conditions (52) and (53), one can conclude that
\[ \gamma I_N - \beta \frac{L + LT}{2} - \beta \alpha < 0. \]  
Together with above analysis (54) and (55), which means that \( V < 0. \) This completes the proof.

Remark 5: Up to now, as is known to all of us, how to select the pinning control nodes effectively for MASs with fixed topology is a difficult subject due to the complex network environment. Particularly, it is very complicated to select the pinning nodes under a directed network. Based on the previous results of selecting pinning control nodes \([18], [19]\), we present the pinning control nodes selection strategies as follows: (i) The agents with zero in-degrees must be pinned because their states are not influenced by any other agent. (ii) The signed digraph \( G \) is structurally balanced and contains a directed spanning tree, where the leader is located at the root node. And the states of \( N \) follower agents will be influenced by the state of the leader.

Remark 6: Consider reorganizing the sequence of pinning control nodes depended on both in-degrees and out-degrees, Song et al \([18]\) discovered that if the value of \( l \) increases, then it will reduce \( \lambda_{\max} \left( \frac{L + LT}{2} \right) \). Therefore, we should propose a comparatively lower \( \beta \) in a real application. Since \( \lambda_{\max} \left( \frac{L + LT}{2} \right) \) becomes larger as more and more nodes are selected, then \( \beta^* = -\gamma / \lambda_{\max} \left( \frac{L + LT}{2} \right) \) decreases with the increases of \( l \), where \( \beta^* \) denotes the lower bound of \( \beta \). And it is notable that (42) holds if \( \beta \) is designed appropriately, let \( \beta = \beta^* + \epsilon_1, \epsilon_1 > 0 \), then by increasing the value of \( l \), we can get the suitable \( \beta \). In addition, the pinning feedback gains \( \alpha_i, i = 1, \cdots, l \) should have their lower bounds \( \alpha_i > \alpha^* = \frac{\lambda_{\max}(G - QM_i^{-1}Q^T)}{\beta} \), and we present a scheme between the parameters \( \beta \) and \( \alpha_i, i = 1, \cdots, l \) such that \( l \) is as little as possible, \( \beta \) and \( \alpha_i \) are suitable for practical use.

Furthermore, we will give the scheme to guarantee bipartite tracking consensus:

(a) Let \( \deg_{\text{invar}}(i) = \deg_{\text{out}}(i) - \deg_{\text{in}}(i), i = 1, \cdots, N, \) in which \( \deg_{\text{var}}(i) \) represents a degree variance vector.

(b) Select the agents with zero in-degrees and reorganize the sequence of pinning control nodes based on degree-variations.

(c) Let \( l = l_0 \), and \( l_0 \) denotes the smallest number of agents which satisfies \( \lambda_{\max} \left( \frac{L + LT}{2} \right)_{l_0 - 1} \geq 0 \) and \( \lambda_{\max} \left( \frac{L + LT}{2} \right)_{l_0} < 0. \)

(d) Calculate the lower bound \( \beta^* = -\gamma / \lambda_{\max} \left( \frac{L + LT}{2} \right) \).

(e) Let \( \alpha_i = \alpha^* + \epsilon_2, i = 1, \cdots, l, \) in which \( \alpha^* \) represents the lower bound of pinning feedback gains, and \( \epsilon_2 \) denotes a positive definite parameter.

To summarize, by utilizing the scheme and based on Remark 5 and Remark 6, the pinning control nodes can be selected and correlation parameters can be confirmed to guarantee bipartite tracking consensus.

IV. SIMULATION RESULTS

In this section, suppose that the signed network is comprised of six agents shown in Figure 1, whose initial conditions are set randomly. Assume that a group of agents consisting of one leader and five followers are involved in the networked MAS, where the five followers indexed by 1, 2, \cdots, 5, and one leader labeled by 0. The interaction topology involving the leader has spanning tree and one can easily verify that the topology corresponding to the digraph is connected. Obviously, the communication topology among the agents in Figure 1 satisfies Assumption 1. And the solid and dashed
FIGURE 1: The communication diagraph $G$ with six agents.

FIGURE 2: The trajectories of states without disturbances.

FIGURE 3: The trajectories of states with disturbances.

FIGURE 4: The tracking errors for the states of MASs without external disturbances by pinning control.

FIGURE 5: The tracking errors for the states of MASs with external disturbances by pinning control.

FIGURE 6: The tracking errors for the velocities of MASs without external disturbances by pinning control.

lines represent the collaborative and antagonistic relationships between agents, respectively. The followers are split into two groups: $V_1 = \{1, 2\}$ and $V_2 = \{3, 4, 5\}$ according to the property of structural balance. In Figure 2, the blue line denotes the state of the leader. Similarly, in Figure 3, the blue line represents the leader’s state. Then one can easily conclude from Figures 2-3 that the agents referring subset $V_1$ asymptotically approach the leader’s state $r_0$, while the agents referring subset $V_2$ asymptotically track the leader’s opposite state $-r_0$, which is in accordance with the partition $V_1$ and $V_2$. Furthermore, from the results shown in Figure 3, it is straightforward to see that the third-order MAS subject to exogenous disturbance can achieve bipartite tracking consensus with the disturbance observer-based control protocol (5), which conforms to the result obtained in Theorem 1. And in Figure 2, under the case without exogenous disturbances, the third-order MAS also can achieve bipartite tracking consensus, which similarly conforms to the result obtained in Theorem 2. Without any loss of generality, the corresponding kinematics of the followers with disturbances are given by

$$\begin{cases}
\dot{r}_i = v_i, \\
\dot{v}_i = a_i, \\
\dot{a}_i = f(r_i, v_i, a_i) + \tilde{u}_i + \tilde{w}_i,
\end{cases}$$  

(56)
and the leader in the MASs is as follows

\[
\begin{aligned}
\dot{r}_i &= v_i, \\
\dot{v}_i &= a_i, \\
\dot{a}_i &= f(r_0, v_0, a_0),
\end{aligned}
\]  

(57)

where \( r_i \in R^m \), \( v_i \in R^m \), and \( a_i \in R^m \) denote the position, velocity, and acceleration of followers, respectively. Similarly, \( r_0 \in R^m \), \( v_0 \in R^m \), and \( a_0 \in R^m \) denote the position, velocity, and acceleration of the leader. The corresponding control protocol is

\[
\dot{a}_i = -\beta \alpha_i \sum_{j=1}^{N} |a_{ij}| [(r_i - sgn(a_{ij}) r_j) + (v_i - sgn(a_{ij}) v_j) + (a_i - sgn(a_{ij}) a_j)] + (v_i - d_i v_0) + (a_i - d_i a_0) + w_i.
\]  

(58)

The laplacian matrix \( L \) is written as

\[
L = \begin{bmatrix}
1 & 0 & -1 & 0 & 0 \\
-2 & 4 & 0 & -2 & 0 \\
0 & 0 & 1 & 0 & -1 \\
0 & 0 & -2 & 2 & 0 \\
0 & 0 & 0 & -1 & 1
\end{bmatrix}.
\]  

(59)

By solving the matrix in main results, it is found that the eigenvalues of \( L \) are \( 4.7873, 1.646 \pm 0.6930i, 1.646 - 0.6930i, 2.9598 + 1.2043i, 2.9598 - 1.2043i \). Then we present the nonlinear function to show the organized methods could achieve control requirements. And the nonlinear function can be constructed as

\[
f(x_i, v_i, a_i) = \begin{pmatrix}
s_1(a_{i2} - a_{i1} - \varepsilon a_{i1}) \\
-2s_2a_{i2} - s_3a_{i3}
\end{pmatrix},
\]

(60)

in which \( s_1 = 8, s_2 = 17.351, s_3 = 1.756, \varepsilon = -0.8259a_{i1}^2 - 0.4174(|a_{i1} + 1| - |a_{i1} - 1|) \). According to Assumption 3, we can get \( h_1 + h_4 + h_7 = 13.2689, h_2 + h_5 + h_8 = 19.76, h_3 + h_6 + h_9 = 33.8, \gamma = 34.8 \). Let \( \beta = 130 \), we yield \( \lambda_{max} \left( \frac{L + L^T}{2} \right) = -0.8935 < -0.8917 \). Based on (42) and (44), it then follows \( \alpha_i > 5.06 \). In FIGURES 4-9, the followers’ states move into the leader’s state, which implies the tracking errors of the velocities and acceleration for the close-loop MAS asymptotically converge to zero, and the tracking errors of the position converge to a constant value. The above reflect the control protocol (58) shows better convergence performance for the MASs with and without exogenous disturbances by pinning control.

V. CONCLUSION

In this paper, we have investigated the bipartite tracking consensus for nonlinear high-order MASs with and without external disturbances, respectively. Compared with some related works on this area, two new distributed control protocols for a group of anonymous agents with nonlinear dynamics are proposed to fulfill bipartite tracking consensus via pinning control. Moreover, the disturbance observer is designed to estimate the disturbances yield by an external system. By utilizing algebraic graph theory, linear matrix inequality and Lyapunov inequality analysis, the sufficient conditions for the realization bipartite tracking consensus are given. Finally, simulations are performed for demonstration. Furthermore, it would be challenging to combine the problem of leader-follower bipartite consensus with practical application. And the future work will further study bipartite consensus with event-trigger.

REFERENCES


