A Class of Linear Codes and Their Complete Weight Enumerators

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ABSTRACT Linear codes may have a few weights if their defining sets are chosen properly. Let $s$ and $t$ be positive integers. For an odd prime $p$ and an even integer $m$, let $q = p^m$, $m = 2^s$ and $Tr_m$ (resp. $Tr_s$) be the absolute trace function from $\mathbb{F}_q$ (resp. $\mathbb{F}_{p^s}$) to $\mathbb{F}_p$. In this paper, we define

$$D_b = \{(x_1, \cdots, x_t) \in \mathbb{F}_q^t \setminus \{(0, \cdots, 0)\} : Tr_m(x_1 + \cdots + x_t) = b\},$$

where $b \in \mathbb{F}_p$. By employing exponential sums, we demonstrate the complete weight enumerators of a class of $p$-ary linear codes given by

$$C_{D_b} = \{c(a_1, \cdots, a_t) : a_1, \cdots, a_t \in \mathbb{F}_{p^s}\},$$

where

$$c(a_1, \cdots, a_t) = (Tr_s(a_1x_1^{p^s+1} + \cdots + a_tx_t^{p^s+1}))(x_1, \cdots, x_t) \in D_b.$$ 

Then we get their weight enumerators explicitly, which will give us several linear codes with a few weights. The presented codes are suitable with applications in secret sharing schemes.

INDEX TERMS Linear code, complete weight enumerator, Gauss sum, cyclotomic number.

I. INTRODUCTION

Throughout this paper, let $q = p^m$ for an odd prime $p$ and a positive integer $m$. An $[n, k, d]$ linear code $C$ over $\mathbb{F}_p$ is a $k$-dimensional subspace of $\mathbb{F}_p^n$ with minimum distance $d$. For a codeword $c \in C$ the (Hamming) weight $wt(c)$ is the number of nonzero coordinates in $c$. We use $A_i$ to denote the number of codewords of weight $i$ in $C$. Then $(1, A_1, \cdots, A_n)$ is called the weight distribution of $C$ and its weight enumerator is denoted by the polynomial $1 + A_1x + \cdots + A_nx^n$.

The complete weight enumerator of a code $C$ over $\mathbb{F}_p$ enumerates the codewords according to the number of symbols of each kind contained in each codeword (see [13], [21]). Denote $\mathbb{F}_p = \{z_0, z_1, \cdots, z_{p-1}\}$, where $z_0 = 0$. If $c \in \mathbb{F}_p^n$ and $z_j \in \mathbb{F}_p$, let $s_j(c)$ be the number of components of $c$ that equal $z_j$. The polynomial

$$\text{CWE}(C) = \sum_{c \in C} z_0^{s_0(c)}z_1^{s_1(c)}\cdots z_{p-1}^{s_{p-1}(c)}$$

is called the complete weight enumerator of $C$.

The complete weight enumerators of linear codes contain important information because they not only give the weight enumerators but also show the frequency of each symbol appearing in each codeword. In [2], [14], Blake and Kith showed the complete weight enumerators of Reed–Solomon codes and concluded that they could be helpful in soft decision decoding. The study was extended to the generalized Kerdock code and related linear codes over Galois rings in [15] and [16]. It was found in [12] that monomial and quadratic bent functions have close relation with the complete weight enumerators of linear codes. Recently, a lot of progress had been made on this subject. Ding et al. [5], [6] indicated that complete weight enumerators can be applied to the calculation of the deception probabilities of certain authentication codes. In [3], [7], [8], the authors studied the complete weight enumerators of some constant composition codes and presented some families of optimal constant composition codes. Weight distributions were also determined in [22], [28], [32], [34], [37], [38] for different types of codes.

Ding et al. [9]–[11] developed a generic construction of linear codes. Denote $D = \{d_1, d_2, \cdots, d_n\} \subseteq \mathbb{F}_q$, where
$q = p^m$. A linear code associated with $D$ is defined by

$$C_D = \{(\text{Tr}_m(ax))_{x \in D} : a \in \mathbb{F}_q \},$$

where $\text{Tr}_m$ is the absolute trace function of $\mathbb{F}_q$. The set $D$ is called the defining set of $C_D$. This construction is generic since lots of codes were constructed by choosing proper defining sets. In [11, 17–19, 29–31, 33], some specific codes were defined and their complete weight enumerators and weight enumerators were also determined. Note that by choosing different defining sets, the authors constructed many different kinds of optimal codes from trace codes over different finite rings in [23–26]. Moreover, most of these constructed codes over finite fields or rings can be used to construct secret sharing schemes.

In this paper, we assume that $s$ and $t$ are positive integers, and $m = 2s$. For $b \in \mathbb{F}_p$, we define

$$D_b = \{(x_1, \ldots , x_t) \in \mathbb{F}_p^t \backslash \{(0, \ldots , 0)\} : \text{Tr}_m(x_1 + \cdots + x_t) = b\}.$$

Then a class of linear codes are defined by

$$C_{D_b} = \{c(a_1, \ldots , a_t) : a_1, \ldots , a_t \in \mathbb{F}_p^t \}, \tag{1}$$

where the codeword $c(a_1, \ldots , a_t)$ is given by

$$(\text{Tr}_s(a_1x_1^{p^s+1} + \cdots + a_tx_t^{p^s+1}))_{(x_1, \ldots , x_t) \in D_b}, \tag{2}$$

and $\text{Tr}_s$ is the absolute trace function of $\mathbb{F}_p^s$. Using exponential sums, we investigate the complete weight enumerators and weight enumerators of $C_{D_b}$ for each $b \in \mathbb{F}_p$. This work is strongly inspired by the above construction and [35]. Actually, we generalize the main result of [35] where the case $t = 1$ was settled explicitly. In addition, some examples are included to illustrate our results.

II. MATHEMATICAL FOUNDATIONS

We begin with some basic concept of cyclotomic numbers and Gauss sums over finite fields. The reader is referred to [20, 27] for more information. Recall that $q = p^m$. Let $\alpha$ be a primitive element of $\mathbb{F}_q$ and $q = Nh + 1$ for two positive integers $N > 1$ and $h > 1$. The cyclotomic classes of order $N$ in $\mathbb{F}_q$ are the cosets $C_{i}\mathbb{F}_q(\alpha) = \alpha^i\mathbb{F}_q(\alpha)$ for $i = 0, 1, \ldots , N - 1$, where $\mathbb{F}_q(\alpha)$ denotes the subgroup of $\mathbb{F}_q$ generated by $\alpha^N$.

For fixed $i$ and $j$, the cyclotomic number $(i,j)(\mathbb{F}_q)$ is the number of ordered pairs $(s,t)$ such that

$$\alpha^{Ns+t} + 1 = \alpha^{Nt+j} \quad (0 \leq s, t \leq h - 1),$$

where $1 = \alpha^0$ is the multiplicative unit of $\mathbb{F}_q$.

If $\lambda$ is a multiplicative character of $\mathbb{F}_q^*$, then the Gauss sum $G(\lambda)$ over $\mathbb{F}_q$ is defined by

$$G(\lambda) = \sum_{x \in \mathbb{F}_q^*} \lambda(x)\zeta_p \text{Tr}_m(x),$$

where $\text{Tr}_m$ is the absolute trace function of $\mathbb{F}_q$ and $\zeta_p$ is a $p$-th primitive root of unity in the complex number field.

Let $\eta_m$ denote the quadratic character of $\mathbb{F}_q$ by defining $\eta_m(0) = 0$. The quadratic Gauss sum $G(\eta_m)$ is denoted by $G_m$ for simplicity. Especially when $m = 1$, we briefly write $G(\eta)$ as $G_1$, where $\eta := \eta_1$ is the quadratic character over $\mathbb{F}_p$.

Next, let us review some results on cyclotomic numbers and Gauss sums.

**Lemma 1** ([27]). When $N = 2$, the cyclotomic numbers are given by

1. $h$ even: $(0,0)(2^s) = \frac{h-2}{2}$, $(0,1)(2^s) = (1,0)(2^s) = (1,1)(2^s) = \frac{h}{2}$.
2. $h$ odd: $(0,0)(2^s) = (1,0)(2^s) = (1,1)(2^s) = \frac{h-1}{2}$.

**Lemma 2** ([20]). Let $\eta_m$ be the quadratic character of $\mathbb{F}_q$, where $q = p^m$, $m \geq 1$. Then

$$G_m = (-1)^{m-1}(-1)\left(\frac{p-1}{m}\right) \zeta_p.$$ 

In particular, $G_1 = (-1)^{\frac{p-1}{2}} \zeta_p$ and $G_2^2 = \eta(-1)p$.

**Lemma 3** ([20]). Let $q = p^m$ and $f(x) = a_2x^2 + a_1x + a_0 \in \mathbb{F}_q[x]$ with $a_2 \neq 0$. Then

$$\sum_{x \in \mathbb{F}_q} \zeta_p \text{Tr}_m(f(x)) = \zeta_p \text{Tr}_m(a_0-a_2^2(4a_2)^{-1}) \eta_m(a_2)G(\eta_m),$$

where $\eta_m$ is the quadratic character of $\mathbb{F}_q^*$.

**Lemma 4** ([4]). Let $m = 2s$, $q = p^m$, $a \in \mathbb{F}_p^*$ and $b \in \mathbb{F}_p$. Then

$$\sum_{x \in \mathbb{F}_q} \zeta_p \text{Tr}_s(ax^{p^s+1} + \text{Tr}_m(bx)) = -p \zeta_p \text{Tr}_s(a^{-1}b^{p^s+1}).$$

III. MAIN RESULTS

In this section, we will focus our attention on the complete weight enumerators and weight enumerators of $C_{D_b}$ defined by (1) and (2), where

$$D_b = \{(x_1, \ldots , x_t) \in \mathbb{F}_q^t \backslash \{(0, \ldots , 0)\} : \text{Tr}_m(x_1 + \cdots + x_t) = b\}.$$

The length of $C_{D_b}$ is determined in the following lemma by noting that the trace function is balanced.

**Lemma 5.** Denote

$$n_b = \#\{(x_1, \ldots , x_t) \in \mathbb{F}_q^t \backslash \{(0, \ldots , 0)\} : \text{Tr}_m(x_1 + \cdots + x_t) = b\},$$

for each $b \in \mathbb{F}_p$. Then

$$n_b = \begin{cases} \frac{p^m}{2} - 1 & \text{if } b = 0, \\ \frac{p^m}{2} & \text{if } b \neq 0. \end{cases}$$

For a codeword $c(a_1, \ldots , a_t)$ of $C_{D_b}$, since $(a_1, \ldots , a_t) = (0, \ldots , 0)$ gives the zero codeword, we only consider the complete weight and weight of $c(a_1, \ldots , a_t)$ with $(a_1, \ldots , a_t) \neq (0, \ldots , 0)$ in the sequel. Let $p \in \mathbb{F}_p$, we denote $N_p := N_p(a_1, \ldots , a_t)$ to be the number of components $\text{Tr}_s(a_1x_1^{p^s+1} + \cdots + a_tx_t^{p^s+1})$ of $c(a_1, \ldots , a_t)$.

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that are equal to \( \rho \). Then \( N_0 = n_0 - \sum_{\rho \in \mathbb{F}_p^*} N_\rho \). So it suffices to study \( N_\rho \) for all \( \rho \in \mathbb{F}_p^* \). It follows that
\[
N_\rho = \sum_{x_1, \ldots, x_t \in \mathbb{F}_p} \left( \frac{1}{p} \sum_{y \in \mathbb{F}_p} \zeta_p^{y \text{Tr}_p(x_1 + \cdots + x_t - yb)} \right)
\times \left( \frac{1}{p} \sum_{z \in \mathbb{F}_p} \zeta_p^{z \text{Tr}_p(a_1 x_1^{p^\alpha} + \cdots + a_t x_t^{p^\alpha} - zp)} \right)
= \frac{1}{p^2} \sum_{x_1, \ldots, x_t \in \mathbb{F}_p} \left( 1 + \sum_{y \in \mathbb{F}_p} \zeta_p^{y \text{Tr}_p(x_1 + \cdots + x_t - yb)} \right)
\times \left( 1 + \sum_{z \in \mathbb{F}_p} \zeta_p^{z \text{Tr}_p(a_1 x_1^{p^\alpha} + \cdots + a_t x_t^{p^\alpha} - zp)} \right)
= p^{tm-2} + p^{-2}(\Omega_1 + \Omega_2 + \Omega_3),
\]
where
\[
\begin{align*}
\Omega_1 &= \sum_{y \in \mathbb{F}_p} \zeta_p^{-yb} \prod_{i=1}^{t} \zeta_p^{\text{Tr}_p(y x_i)} = 0, \\
\Omega_2 &= \sum_{z \in \mathbb{F}_p} \zeta_p^{-zp} \prod_{i=1}^{t} \zeta_p^{\text{Tr}_p(z a_i x_i^{p^\alpha} + 1)}, \\
\Omega_3 &= \sum_{y \in \mathbb{F}_p} \zeta_p^{-yb} \sum_{z \in \mathbb{F}_p} \zeta_p^{-zp} \prod_{i=1}^{t} \zeta_p^{\text{Tr}_p(z a_i x_i^{p^\alpha} + \text{Tr}_p(y x_i))}.
\end{align*}
\]
Now let us determine the values of \( \Omega_2 \) and \( \Omega_3 \).

**Lemma 6.** Suppose that there are exactly \( k \) nonzero elements, say \( a_1, \ldots, a_k \), among \( a_1, a_2, \ldots, a_t \), for \( 1 \leq k \leq t \), then
\[
\Omega_2 = (-1)^k + 1 p^{tm - sk}.
\]

**Proof.** From Lemma 4, we have
\[
\Omega_2 = \sum_{z \in \mathbb{F}_p} \zeta_p^{-zp} \prod_{i=1}^{t} \zeta_p^{\text{Tr}_p(z a_i x_i^{p^\alpha} + 1)}
= q^t - k \sum_{z \in \mathbb{F}_p} \zeta_p^{-zp} \prod_{j=1}^{k} \zeta_p^{\text{Tr}_p(z a_j x_j^{p^\alpha} + 1)}
= (-1)^k + 1 q^{t-k} p^{sk}.
\]
This completes the proof. \( \Box \)

**Lemma 7 ([1]).** For \( c \in \mathbb{F}_p \), let
\[
B_c = \{(a_1, \ldots, a_t) \in (\mathbb{F}_p^*)^t : \text{Tr}_p(a_1^{-1} + a_2^{-1} + \cdots + a_t^{-1} - c) \}
\]
and let \( n_c = \#B_c \). Then we have
\[
n_c^\prime = \begin{cases} \frac{1}{p} (p^t - 1)^t & \text{if } c = 0, \\ \frac{1}{p} (p^t - 1)^t - (-1)^t & \text{if } c \neq 0. \end{cases}
\]

In the following, we denote \( A = a_1 a_2 \cdots a_t \) and \( \lambda = \text{Tr}_p(a_1^{-1} + a_2^{-1} + \cdots + a_t^{-1}) \) for the simplicity of formulae. To determine the complete weight enumerators of \( C_{D_t} \), we need to do some preparations.

**Lemma 8.** If \( A = 0 \), then \( \Omega_3 = 0 \). Assume that \( A \neq 0 \) and \( \rho \in \mathbb{F}_p^* \).

1. If \( b = 0 \), then
\[
\Omega_3 = \begin{cases} (-1)^{t+1}(p - 1)p^{st} & \text{if } \lambda = 0, \\ (-1)^{t} p^{st} \left( \eta (\rho \lambda) + 1 \right) & \text{if } \lambda \neq 0. \end{cases}
\]
2. If \( b \neq 0 \), then
\[
\Omega_3 = \begin{cases} (-1)^{t} p^{st} & \text{if } \lambda = 0, \\ (-1)^{t} p^{st} \left( \eta (b^2 - 4\rho \lambda) + 1 \right) & \text{if } \lambda \neq 0. \end{cases}
\]

**Proof.** If \( A = 0 \), then clearly \( \Omega_3 = 0 \). Let us suppose that \( A \neq 0 \) and \( \rho \in \mathbb{F}_p^* \) for the rest of the proof. It follows from Lemma 4 that
\[
\Omega_3 = \sum_{y \in \mathbb{F}_p} \zeta_p^{-yb} \sum_{z \in \mathbb{F}_p} \zeta_p^{-zp} \prod_{i=1}^{t} \left( -p^{s} \zeta_p^{-\text{Tr}_p(z^{-1} a_i y^{p^\alpha} + 1)} \right)
= (-p)^t \sum_{y \in \mathbb{F}_p} \zeta_p^{-yb} \sum_{z \in \mathbb{F}_p} \zeta_p^{-z^{-1} \lambda y^2 - by}. \tag{3}
\]
To calculate \( \Omega_3 \), we will divide \( b \) into two cases. We first consider the case that \( b = 0 \). Then it follows from (3) that
\[
\Omega_3 = (-p)^t \sum_{z \in \mathbb{F}_p} \zeta_p^{-z^{-1} \lambda y^2}.
\]
If \( \lambda = 0 \), then
\[
\Omega_3 = (-p)^t (p - 1) \sum_{z \in \mathbb{F}_p} \zeta_p^{-z^{-1} \lambda y^2}
= (-1)^{t+1} p^{st} (p - 1).
\]
Otherwise, if \( \lambda \neq 0 \), then
\[
\Omega_3 = (-p)^t \sum_{z \in \mathbb{F}_p} \zeta_p^{-z^{-1} \lambda y^2} \left( \eta (\zeta_p^{-1}) G_1 - 1 \right)
= (-p)^t \left( G_1^2 \eta (\rho \lambda) + 1 \right).
\]
The desired conclusion follows from the fact that \( G_1^2 = \eta (\zeta_p^{-1}) \).

Now suppose that \( b \neq 0 \). By Lemma 3 and (3), we have
\[
\Omega_3 = (-p)^t \sum_{z \in \mathbb{F}_p} \zeta_p^{-z^{-1} \lambda y^2} \left( \sum_{y \in \mathbb{F}_p} \zeta_p^{-z^{-1} \lambda y^2 - by} - 1 \right).
\]
If \( \lambda = 0 \), then
\[
\Omega_3 = (-p)^t.
\]
Otherwise, if \( \lambda \neq 0 \), then
\[
\Omega_3 = (-p)^t \sum_{z \in \mathbb{F}_p} \zeta_p^{-z^{-1} \lambda y^2} \left( \eta (\zeta_p^{-1}) G_1 - 1 \right)
= (-p)^t \left( G_1 \eta (\zeta_p^{-1}) \sum_{z \in \mathbb{F}_p} \zeta_p^{-z^{-1} \lambda y^2 - by} + 1 \right)
= \begin{cases} (-p)^t & \text{if } \lambda = 0, \\ (-p)^t \left( G_1 \eta (\zeta_p^{-1}) \left( \eta (\zeta_p^{-1}) (\frac{b^2}{4\lambda} - \rho) + 1 \right) \right) & \text{if } \lambda \neq 0, \end{cases}
\]
By noticing that $C_2^2 = \eta(-1)p$, we get the desired results. The proof is completed.

With the above preparations, we are ready to determine the complete weight enumerators of $C_{D_b}$ for each $b \in \mathbb{F}_p$.

**Theorem 1.** Let $C_{D_b}$ be the linear code defined by (1) and (2) for $b \in \mathbb{F}_p$, where

$$D_b = \{(x_1, \ldots, x_t) \in \mathbb{F}_p^t \setminus \{(0, \ldots, 0)\} : \text{Tr}_m(x_1 + \cdots + x_t) = b\}.$$  

Assume that $\rho \in \mathbb{F}_p^*$ and $1 \leq k < t$.

(1) If $b = 0$, the code $C_{D_0}$ has parameters $[p^{tm-1} - 1, ts]$ and its complete weight enumerator is given as follows:

$$N_{\rho} = p^{tm-2} - (-1)^k p^{tm-sk-2}$$

occurs $\binom{t}{k} (p^s - 1)^k$ times for every $k$, 

$$N_{\rho} = p^{tm-2} - (-1)^t p^{st-1}$$

occurs $\frac{1}{p} \binom{t}{t} (p^s - 1)^t + (-1)^t (p-1)$ times, 

$$N_{\rho} = p^{tm-2} + (-1)^t p^{st-1} \eta(\rho)$$

occurs $\frac{p-1}{2p} \binom{t}{t} (p^s - 1)^t - (-1)^t$ times, 

$$N_{\rho} = p^{tm-2} + (-1)^t p^{st-1} \eta(\rho)$$

occurs $\frac{p-1}{2p} \binom{t}{t} (p^s - 1)^t - (-1)^t$ times.

(2) If $b \neq 0$, the code $C_{D_b}$ has parameters $[p^{tm-1}, ts]$ and its complete weight enumerator is given as follows:

$$N_{\rho} = p^{tm-2} - (-1)^k p^{tm-sk-2}$$

occurs $\binom{t}{k} (p^s - 1)^k$ times for every $k$, 

$$N_{\rho} = p^{tm-2} + (-1)^t p^{st-1}$$

occurs $\frac{1}{p} \binom{t}{t} (p^s - 1)^t + (-1)^t (p-1)$ times, 

$$N_{\rho} = \left\{\begin{array}{ll}
p^{tm-2} & \text{if } \rho = \rho_0 \\
p^{tm-2} + (-1)^t p^{st-1} & \text{if } \eta(1 - \rho \rho_0^{-1}) = 1 \\
p^{tm-2} - (-1)^t p^{st-1} & \text{if } \eta(1 - \rho \rho_0^{-1}) = -1 \\
\end{array}\right.$$ 

for all $\rho \in \mathbb{F}_p^*$, 

occurs $\frac{1}{p} \binom{t}{t} (p^s - 1)^t - (-1)^t$ times.

**Proof.** We have from the definition that $C_{D_b}$ has length $n_b$ and dimension $ts$. Recall that

$$N_{\rho} = p^{tm-2} + p^{-2}(\Omega_1 + \Omega_2 + \Omega_3),$$

for $\rho \in \mathbb{F}_p^*$. We will employ Lemmas 6 and 8 to compute $N_{\rho}$. Suppose that there are exactly $k$ nonzero elements $a_{i_1}, \ldots, a_{i_k}$ among $a_1, a_2, \ldots, a_t$ for $1 \leq k \leq t$.

We first consider the case that $b = 0$. If $1 \leq k < t$, then we have from Lemma 6 that

$$N_{\rho} = p^{tm-2} + p^{-2}\Omega_2 = p^{tm-2} + (-1)^{k+1} \eta^{-1} p^{sk-2} = p^{tm-2} - (-1)^{k} p^{tm-sk-2}.$$ 

The frequencies of these values are $\binom{t}{k} (p^s - 1)^k$ for every $1 \leq k < t$.

If $k = t$ and $\lambda = 0$, then it follows from Lemmas 6 and 8 that

$$N_{\rho} = p^{tm-2} + p^{-2}(\Omega_2 + \Omega_3) = p^{tm-2} + p^{-2}\left((-1)^{t+1} p^t + (-1)^{t+1}(p-1)p^t\right) = p^{tm-2} + (-1)^{t+1} p^{st-1}.$$ 

By Lemma 7, the frequency is $\frac{1}{p} \binom{t}{t} (p^s - 1)^t - (-1)^t$ times according to Lemma 7.

In the following, we suppose that $b \neq 0$. If $1 \leq k < t$, then

$$N_{\rho} = p^{tm-2} + p^{-2}\Omega_2 = p^{tm-2} - (-1)^{k} p^{tm-sk-2}.$$ 

The frequencies of these values are $\binom{t}{k} (p^s - 1)^k$ for every $1 \leq k < t$.

If $k = t$ and $\lambda = 0$, then

$$N_{\rho} = p^{tm-2} + p^{-2}(\Omega_2 + \Omega_3) = p^{tm-2} + p^{-2}\left((-1)^{t+1} p^t + (-1)^{t+1}(p-1)p^t\right) = p^{tm-2}.$$ 

This value occurs $\frac{1}{p} \binom{t}{t} (p^s - 1)^t - (-1)^t$ times.

If $k = t$ and $\lambda \neq 0$, then

$$N_{\rho} = p^{tm-2} + p^{-2}(\Omega_2 + \Omega_3) = p^{tm-2} + p^{-2}\left((-1)^{t+1} p^t + (-1)^{t+1}(p-1)p^t\right) = p^{tm-2}.$$ 

This value occurs $\frac{1}{p} \binom{t}{t} (p^s - 1)^t - (-1)^t$ times according to Lemma 7.

This completes the whole proof.

The next result describes the weight distributions of $C_{D_b}$ for each $b \in \mathbb{F}_p$, which directly follows from their complete weight enumerators.
Corollary 1. Let $C_{D_b}$ be the linear code defined by (1) and (2), where

$$D_b = \{(x_1, \cdots, x_t) \in \mathbb{F}_q^t \setminus \{(0, \cdots, 0)\}: \text{Tr}_m(x_1 + \cdots + x_t) = b\},$$

for $b \in \mathbb{F}_p$. Assume that $1 \leq k < t$. The following assertions hold.

1. If $b = 0$, then the weight distribution of $C_{D_0}$ is given in Table 1.
2. If $b \neq 0$, then the weight distribution of $C_{D_b}$ is given in Table 2.

**TABLE 1.** The weight distribution of $C_{D_0}$

<table>
<thead>
<tr>
<th>Weight</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$(p-1)(p^m-2) - (-1)^k 2^m - 2$</td>
</tr>
<tr>
<td>$(p-1)p^m - 1$</td>
<td>$(p^s-1)^k$ for every $k$</td>
</tr>
<tr>
<td>$(p-1)p^m - 2$</td>
<td>$\sum_{j=1}^{p} \frac{(p^s-1)^k + (-1)^k(p-1)}{p}</td>
</tr>
</tbody>
</table>

**TABLE 2.** The weight distribution of $C_{D_b}$ for $b \neq 0$

<table>
<thead>
<tr>
<th>Weight</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$(p-1)(p^m-2) - (-1)^k 2^m - 2$</td>
</tr>
<tr>
<td>$(p-1)p^m - 1$</td>
<td>$(p^s-1)^k$ for every $k$</td>
</tr>
<tr>
<td>$(p-1)p^m - 2$</td>
<td>$\sum_{j=1}^{p} \frac{(p^s-1)^k + (-1)^k(p-1)}{p}</td>
</tr>
</tbody>
</table>

**Proof.** We only prove the results for $b \neq 0$. Let $\rho$ and $\rho_0$ be two distinct elements in $\mathbb{F}_p$. The only thing left is to compute $\sum_{\rho \neq \rho_0} \eta(1 - \rho \rho_0)$. If $\eta(\rho_0) = 1$, we need to compute the number of $\rho$ such that $\eta(\rho_0 - \rho) = 1$. Let $d = \rho_0 - \rho \in \mathbb{F}_0$. Then

$$\rho + 1 = \rho_0 d.$$

From Lemma 1, the number of $\rho$ such that $\eta(\rho_0 - \rho) = 1$ is

$$\frac{(0,0)^{(2,p)} + (1,0)^{(2,p)}}{p} = \frac{p-1}{2} - 1.$$

On the other hand, if $\eta(\rho_0) = 1$, the number of $\rho$ such that $\eta(\rho_0 - \rho) = -1$ is

$$\frac{(0,1)^{(2,p)} + (1,1)^{(2,p)}}{p} = \frac{p-1}{2}.$$

Therefore, if $\eta(\rho_0) = 1$, then

$$\sum_{\rho \neq \rho_0} \eta(1 - \rho \rho_0) = \sum_{\rho \neq \rho_0} \eta(\rho_0) \eta(\rho_0 - \rho) = \left(\frac{p-1}{2} - 1\right) \times 1 + \frac{p-1}{2} \times (-1) = -1.$$

Similarly, if $\eta(\rho_0) = -1$, then

$$\sum_{\rho \neq \rho_0} \eta(1 - \rho \rho_0) = -\sum_{\rho \neq \rho_0} \eta(\rho_0 - \rho) = -\frac{p-1}{2} \times 1 - \left(\frac{p-1}{2} - 1\right) \times (-1) = -1.$$

The desired conclusion then follows from Theorem 1.

**Remark 1.** In fact, when $t = 1$, the authors in [35] studied $C_{D_b}$ for its weight enumerator and complete weight enumerator. Thus the main result presented here can be viewed as a generalization of [35]. Also by Corollary 1, we easily get several linear codes with a few weights. For instance, when $t = 2$, $C_{D_b}$ is a three-weight linear code.

To illustrate our results, we provide some examples for the code $C_{D_b}$.

**Example 1.** Let $(p,m,t) = (3,2,3)$. By Theorem 1 and Corollary 1, the code $C_{D_0}$ is a $[242, 3, 144]$ linear code with complete weight enumerator

$$x_0^{242} + 12x_0^{98}(z_1z_2)^{72} + 3x_0^{80}z_1z_2^2 + 3x_0^{80}z_1z_2^2 \cdot 2 + 2z_0^{12}(z_1z_2)^{90} + 6z_0^{12}(z_1z_2)^{108},$$

and weight enumerator

$$1 + 12x_1^{144} + 6x_1^{162} + 2x_1^{180} + 6x_1^{216}.$$

These experimental results are checked by Magma programs.

**Example 2.** Let $(p,m,t) = (3,4,2)$. By Theorem 1, the code $C_{D_1}$ is a $[2187, 4, 1431]$ linear code. Its complete weight enumerator is

$$x_0^{2187} + 21x_0^{756}z_1z_2^2 + 21x_0^{756}z_1z_2^2 + 21x_0^{756}z_1z_2^2 + 21x_0^{756}z_1z_2^2 + 21z_0^{12}(z_1z_2)^{810},$$

and its weight enumerator is

$$1 + 42x_1^{1431} + 22x_1^{1458} + 16x_1^{1620},$$

which are checked by Magma programs.

**IV. CONCLUDING REMARKS**

In this paper, we studied the complete weight enumerators and weight enumerators of a class of trace codes $C_{D_b}$ with defining set $D_b$ for $m = 2s$. Let $w_{\min}$ (resp. $w_{\max}$) be the minimum (resp. maximum) nonzero weights in $C_{D_b}$. It follows from Corollary 1 that for every integer $t$, the inequality

$$\frac{w_{\min}}{w_{\max}} > \frac{p-1}{p}$$

always holds provided that $s \geq 2$. According to [36], we conclude that the codes $C_{D_b}$ are suitable for constructing secret sharing schemes with interesting access structures.
REFERENCES


