Trajectory Optimization for Asteroid Landing Considering Gravitational Orbit-Attitude Coupling

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ABSTRACT This paper proposes a trajectory optimization method considering gravitational orbit-attitude coupling (GOAC) for asteroid landing. First, by modeling the spacecraft as a rigid body rather than a point mass, and using the polyhedral method to describe the irregular gravitational field of the asteroid, GOAC is fully embedded into the 6 degree-of-freedom (DOF) dynamic model of the spacecraft. Second, attitude control, which is assumed to consume electric energy, is introduced into the optimization, the indirect method is adopted to formulate the 6-DOF fuel-optimal control problem, and the boundary conditions are modified to expand the search space of the problem. Third, a two-phase homotopic approach is proposed to realize a smooth continuation from the 6-DOF energy-optimal problem to the fuel-optimal problem. Finally, a numerical example is presented to demonstrate the feasibility of the proposed method, and the comparison results show that the obtained 6-DOF optimal trajectory can guarantee the fuel-optimality of the landing mission under GOAC.

INDEX TERMS Asteroid, landing mission, orbit-attitude coupling, trajectory optimization.

I. INTRODUCTION

Landing on the surface of an asteroid, by which soil samples and high resolution data can be obtained, has become a widely approved way of asteroid exploration, e.g. Hayabusa2 [1] and OSIRIS-Rex [2]. To save fuel and extend mission time, trajectory planning is usually upgraded to trajectory optimization to minimize the fuel consumption in the landing phase [3]–[5].

To develop an optimal asteroid landing trajectory, describing the gravitational field of the asteroid is usually the first concern, due to the irregular shape of the asteroid. Two commonly used methods are spherical harmonics method [6] and polyhedral method [7], which root their theoretical basis in series approximation and geometric approximation, respectively. The spherical harmonics method has a faster computation speed than the polyhedral method because of its analytic nature. However, the spherical harmonics method suffers severe divergence inside the Brillouin sphere of the asteroid [8], while the polyhedral method can calculate the gravitational potential precisely at any distance from the asteroid.

After modeling the gravitational field of the asteroid, an optimization method is picked to design the optimal trajectory. Various trajectory optimization methods have been proposed, e.g. direct method [9], indirect method [10], evolutionary algorithms [11], convex optimization [12], etc. Among them, the indirect method has its distinct advantages on high accuracy and guarantee of optimality [13]. However, two-point-boundary-value problem (TPBVP) formulated by the indirect method suffers from small convergence radius and sensitivity of the initial guess, due to the fact that fuel-optimal control is generally bang-bang control according to Pontryagin’s minimum principle (PMP). To help release the difficulty of the indirect method, a homotopic method has been proposed by Betrand and Epenoy [14]. The homotopic approach links the original fuel-optimal problem to a related, but easier one—energy-optimal problem—by the parameter ε. The fuel optimal problem can be solved by solving the easier problem first and decrease ε iteratively.
What is noteworthy is that, the spacecraft is traditionally modeled as a point mass in the studies of trajectory optimization for asteroid landing [3]–[5]. This approximation excludes attitude motion from the optimization by assuming it is controlled ideally, neglecting an important perturbation, known as gravitational orbit-attitude coupling (GOAC).

GOAC is a phenomenon that, for a rigid-body spacecraft, its orbital motion and attitude motion affects each other under the influence of gravitational force and gravity gradient torque of the central celestial body. The researches concerning GOAC around an asteroid mainly focus on dynamical behavior analysis [15]–[17] and control law design [18]–[20]. The necessity of taking GOAC into landing trajectory optimization originates from the unnegligible magnitude of GOAC near an asteroid. To describe the coupling effect, Sincarsin and Hughes [21] defined the parameter $\psi = \rho / r$, where $\rho$ is the characteristic of spacecraft size and $r$ is the orbit radius. For Earth missions, the orbit radius is much larger than the spacecraft, $\psi$ is generally in an order of magnitude $10^{-6}$ (e.g. $\rho = 10\,\text{m}$, $r = 1500\,\text{km}$, $\psi = 6.7 \times 10^{-6}$), the coupling effect is inherently weak and can be easily overcome by the control system of the spacecraft. But for an asteroid landing mission, there are two distinct features: first, as the spacecraft descends towards the landing site, $r$ is decreasing and may lead $\psi$ to an order of magnitude $10^{-2}$ (e.g. $\rho = 4\,\text{m}$, $r = 100\,\text{m}$, $\psi = 4 \times 10^{-2}$) at the end of the landing trajectory; second, the highly irregular gravity field of the asteroid will also intensify the coupling effect. Therefore, the GOAC is getting gradually stronger during the landing phase, exerting a significant influence on spacecraft’s motion. Modeling the spacecraft as a point mass—rather than a rigid body—in trajectory optimization, which means neglecting this important coupling effect, will cause extra fuel consumption for spacecraft’s controller to overcome the coupling perturbation during trajectory tracking, thus destroying the fuel-optimality of the landing mission (here we refer fuel-optimality of the landing mission as a situation that the fuel consumed after trajectory tracking is equal to the fuel consumed derived from trajectory optimization). To avoid extra fuel consumption, Li et al. [19] proposed a control method using only attitude control. This paper intends to solve the problem in the domain of trajectory optimization.

To develop an optimal landing trajectory under GOAC, the spacecraft should be modeled as a rigid body and its dynamic model is therefore of 6 degree-of-freedom (DOF), containing both orbital and attitude motion. In this paper, the polyhedral method is chosen to reflect GOAC precisely in dynamics of the spacecraft due to its modeling accuracy, and a distributed point-mass model is established to serve as a rigid-body approximation for the spacecraft, to ensure computation efficiency in simulation.

To optimize a trajectory with 6-DOF coupled dynamics, introducing attitude control into the problem and optimizing orbit and attitude of the spacecraft simultaneously is a commonly adopted way. The idea can generally be seen in trajectory optimization of rockets, whose coupled dynamics originates from the structure rather than gravity—the thrust direction of a rocket is controlled by its attitude [22]–[24]. The thought of 6-DOF trajectory optimization is adopted in this paper to deal with GOAC, and the indirect method is used to do the 6-DOF optimization for it can guarantee the optimality of a trajectory. What is noteworthy is that, the dimensionality of the optimal control problem is doubled compared with a 3-DOF point-mass trajectory optimization, thus higher dimensionality brings higher nonlinearity and discontinuity. To solve this difficulty, boundary conditions are modified to expand the search space for TPBVP, and an improved homotopic method is proposed to ease the discontinuity by introducing a rotational energy consumption term and an extra attitude homotopic parameter into the homotopic process. Based on the constraint analysis of the problem, the orbit homotopic parameter (the original homotopic parameter) and the attitude homotopic parameter are iterated separately. After the continuation, the 6-DOF fuel-optimal trajectory is obtained. By tracking the obtained trajectory, extra fuel consumption can be avoided, which means the fuel-optimality of the landing mission can be guaranteed.

The contributions of this paper mainly include the following points:

- This paper considers GOAC in trajectory optimization for asteroid landing for the first time, optimizing the trajectory in a more realistic situation.
- An improved trajectory optimization method for asteroid landing considering GOAC is proposed by optimizing orbit and attitude of the spacecraft simultaneously.
- In 6-DOF indirect method, the terminal constraint of attitude is relieved to expand the search space and reduce the difficulty of solving TPBVP.
- A two-phase homotopic approach based on constraint analysis is proposed to achieve a smooth continuation from the 6-DOF energy-optimal problem to the fuel-optimal problem.

The rest of this paper is organized as follows: In Section II, the reference frames, and the polyhedral method are introduced, a distributed point-mass model of the spacecraft is developed, based on which, the 6-DOF dynamic model of the spacecraft is derived. Section III formulates the 6-DOF fuel-optimal control problem by the indirect method with modified boundary conditions, and a two-phase homotopic method is proposed to solve the optimal problem. Section IV shows simulation results of the proposed method. Section V summarizes the research results and provides some future research interests.

II. ESTABLISHMENT OF 6-DOF DYNAMIC MODEL

A. REFERENCE FRAMES

A rigid-body spacecraft is going to execute a landing maneuver from its parking orbit or hovering position at some given initial time. Three reference frames are established to describe the orbital and attitude motion of the spacecraft, as shown in Fig. 1.
Inertial Reference Frame O-XYZ: centered on the asteroid’s center of mass (CM), with axes $X$, $Y$ and $Z$ fixed to the principal axes of the asteroid at the initial time.

Asteroid Body-Fixed Frame $O$-xyz: centered on the asteroid’s CM, with axes $x$, $y$ and $z$ coincides with the principal axes of the asteroid. The asteroid is assumed to rotate around axis $z$ with a constant angular rate $\omega_z$.

Spacecraft Body-Fixed Frame $b$-ijk: centered on the spacecraft’s CM, with axes $i$, $j$ and $k$ coincides with the spacecraft’s principal axes of inertia.

**B. GRAVITY POTENTIAL AND ITS DERIVATIVES OF THE ASTEROID**

An accurate characterization of GOAC needs a gravity model which can reflect the irregularity of the potential field in close proximity to the asteroid. Considering the spherical harmonics expansion does not converge inside the Brillouin sphere, the polyhedral method is adopted. The gravitational potential of the constant-density polyhedron in frame $O$-xyz is derived with

$$
\varrho U = -\frac{1}{2} G \rho \sum_{e \in \text{edges}} r_e \cdot E_e \cdot r_e \cdot L_e + \frac{1}{2} G \rho \sum_{f \in \text{faces}} r_f \cdot F_f \cdot r_f \cdot \omega_f,
$$

where $G$ is the universal gravitational constant, $\rho$ is the density of the polyhedron, and $r_e$ and $r_f$ are vectors pointing from the spacecraft to any point on the edge and face of the polyhedron, respectively.

The constants edge dyad $E_e$ and face dyad $F_f$ are defined as

$$
E_e = \hat{n}_A \left( \hat{n}_{12}^A \right)^T + \hat{n}_B \left( \hat{n}_{21}^B \right)^T,
$$

$$
F_f = \hat{n}_f \left( \hat{n}_f \right)^T,
$$

where $\hat{n}_{ij}$ is the edge-normal vector on face $f$, $\hat{n}_f$ is the face-normal vector, and $1$ and $2$ represent two vertices of an edge shared by faces $A$ and $B$.

The dimensionless variable $L_e$ and $\omega_f$ are defined as

$$
L_e = \ln \frac{a + b + e}{a + b - e},
$$

$$
\omega_f = 2 \arctan \frac{\hat{r}_1 \cdot (\hat{r}_2 \times \hat{r}_3)}{1 + \hat{r}_1 \cdot \hat{r}_2 + \hat{r}_2 \cdot \hat{r}_3 + \hat{r}_3 \cdot \hat{r}_1},
$$

where $a$ and $b$ are distances from the spacecraft to the two vertices of the common edge, $e$ is the length of this edge. $\hat{r}_1$, $\hat{r}_2$ and $\hat{r}_3$ are vectors from the spacecraft to three consecutive vertices of face $f$.

The gravitational gradient and gravitational gradient matrix can be expressed as

$$
\nabla \varrho U = G \rho \sum_{e \in \text{edges}} E_e \cdot r_e \cdot L_e - G \rho \sum_{f \in \text{faces}} F_f \cdot r_f \cdot \omega_f,
$$

$$
\nabla^2 \varrho U = -G \rho \sum_{e \in \text{edges}} E_e \cdot L_e + G \rho \sum_{f \in \text{faces}} F_f \cdot \omega_f.
$$

**C. DISTRIBUTED POINT-MASS MODEL OF THE SPACECRAFT**

The gravitational force and torque acting on the spacecraft derived from the polyhedral method cannot be analytically expressed, due to the numerical nature of the method. To calculate the exact gravitational force and torque exerted on the spacecraft and increase computational efficiency, a distributed point-mass model with a cuboid shape is proposed to approximate the rigid-body spacecraft in real world, as shown in Fig. 2, where $l_1$, $l_2$, and $l_3$ represent the length, width, and height of the imaginary cuboid, respectively.

The model consists of 9 point masses connected with rigid massless rods, which means the positions of the point masses are fixed in the spacecraft’s body-fixed frame $b$-ijk. Point mass $m_i$ is located at the geometric center of the imaginary cuboid, which is also the origin of the frame $b$-ijk, the other eight equal-weighted point masses named as $m_i$ are located at the vertices of the cuboid. Considering the convenience for dynamic modeling, number the 9 point masses as $m_i$, $i = 1 \sim 9$, with $m_e$ numbered as $m_1$. Thus the total mass of the spacecraft can be expressed as

$$
m = \sum_{i=1}^{9} m_i.
$$

To better observe the effect of the GOAC on the spacecraft, the mass loss of the system is assumed to be caused only by the fuel consumption, and the mass loss is only reflected in the center point mass $m_e$. 

![FIGURE 1. Reference frames definition.](image1)

![FIGURE 2. Distributed point-mass model of the spacecraft.](image2)
D. 6-DOF DYNAMIC MODEL OF SPACECRAFT

In this paper, the orbital motion is assumed to be conducted by thrusters, which consume fuel, and the attitude motion is assumed to be conducted by reaction wheels or similar actuators, which consume electric energy.

1) EQUATIONS OF ORBITAL MOTION

The orbital motion of the spacecraft in the frame $O$-$xyz$ is given by

\[ \dot{r} = v, \]

\[ \dot{v} = -2\omega_d \times v - \omega_d \times (\omega_d \times r) - \nabla^\omega U + \frac{F_{\text{max}}u_0}{m} \alpha_o + F_s, \]

where $r$ and $v$ are the position vector and the velocity vector of the spacecraft’s CM, respectively. $\omega_d$ is the angular velocity of the rotation of the asteroid, which equals $[0 \ 0 \ 0]^T$. $F_s$ is the perturbation force acted on the spacecraft, e.g., solar radiation pressure and the Sun’s gravitational perturbation.

Denote

\[ F = F_{\text{max}}u_0, \quad \alpha_o = \begin{bmatrix} F_x & F_y & F_z \end{bmatrix}^T, \]

where $F = \sqrt{F_x^2 + F_y^2 + F_z^2}$ is the magnitude of the thrust, $F_{\text{max}}$ is the maximal magnitude of the thrust, $u_0$ is the engine thrust ratio within domain $[0, 1]$, and $\alpha_o$ is the unit vector of the thrust direction.

In (8), gravitational acceleration $\nabla^\omega U$ of the rigid-body spacecraft can be expressed as

\[ \nabla^\omega U = \sum_{i=1}^{9} m_i \nabla^\omega U_i, \]

where $m$ is the total mass of the spacecraft, $m_i (i = 1 \sim 9)$ represents each point mass, and $\nabla^\omega U_i$ is the gravitational acceleration of $m_i$. Equation (10) is derived based on the fact that the gravitational acceleration of a point-mass system equals to all the gravity forces acting on it divided by its total mass.

For point masses on the vertices of the cuboid, their positions in $O$-$xyz$ can be expressed by the vector

\[ r_i = r + bR \cdot b_{r_i}, \quad i = 1 \sim 8, \]

where $b_{r_i}$ is the position vector of the $i$-th point mass in spacecraft’s body-fixed frame $b$-$ijk$, and $bR$ is the attitude matrix of the frame $b$-$ijk$ with respect to the frame $O$-$xyz$.

2) EQUATIONS OF ATTITUDE MOTION

Based on the Euler’s equations of rigid body dynamics, the attitude motion of the spacecraft under control of reaction wheels in the spacecraft body-fixed frame $b$-$ijk$ is given by

\[ \dot{\sigma} = \frac{1}{4} B(\sigma) \omega, \]

\[ I\dot{\omega} + \omega^\times I\omega = T_{\text{max}}u_a\alpha_a + M(\sigma, r) + T_s, \]

where $T_s$ is the perturbation torque acted on the spacecraft, i.e. Sun’s gravity gradient torque. $\omega$ is the angular velocity of the spacecraft, $\omega^\times$ is a cross-product operator of the vector $\omega = [\omega_1 \ \omega_2 \ \omega_3]^T$, which is

\[ \omega^\times = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}, \]

and $B(\sigma)$ is a matrix of attitude $\sigma$, which is given by

\[ B(\sigma) = \begin{bmatrix} 1 - \sigma_2^2 + 2\sigma_1^2 & 2(\sigma_1\sigma_2 - \sigma_3) & 2(\sigma_1\sigma_3 + \sigma_2) \\ 2(\sigma_2\sigma_1 + \sigma_3) & 1 - \sigma_1^2 + 2\sigma_2^2 & 2(\sigma_2\sigma_3 - \sigma_1) \\ 2(\sigma_3\sigma_1 - \sigma_2) & 2(\sigma_3\sigma_2 + \sigma_1) & 1 - \sigma_1^2 + 2\sigma_3^2 \end{bmatrix}, \]

$I$ is the inertia matrix of the point-mass system. $T_{\text{max}}, u_a$ and $\alpha_a$ are of the same form as in (9), which can be expressed as

\[ T = T_{\text{max}}u_a, \quad \alpha_a = \begin{bmatrix} T_x & T_y & T_z \end{bmatrix}^T, \]

where $T = \sqrt{T_x^2 + T_y^2 + T_z^2}$ is the magnitude of the control torque, $T_{\text{max}}$ is the maximal magnitude of the torque exerted by reaction wheels, $u_a$ is the output ratio of control torque within domain $[0, 1]$, and $\alpha_a$ is the unit vector of the control torque direction in the frame $b$-$ijk$.\]
The gravitational torque acting on the spacecraft in frame \( b-ijk \) is defined as
\[
M(\sigma, r) = \sum_{i=1}^{8} b_r_i \times bG_i = \sum_{i=1}^{8} b_r_i \times (bR^{-1} m_i \nabla^\alpha U_i(\sigma, r)) = \sum_{i=1}^{8} b_r_i \times (m_i \nabla^b U_i(\sigma, r)),
\]
where \( bG_i \) is the gravitational force acting on the \( i \)-th point mass in \( b-ijk \).

From (10) and (18) it can be seen that the gravitational force and torque acting on the spacecraft both are functions of spacecraft’s position and attitude, therefore the orbital and attitude motion of the spacecraft are coupled.

3) FUEL CONSUMPTION

By assuming the reduction of spacecraft’s mass is due solely to the fuel consumption, the fuel consumption rate is given by
\[
\dot{m} = -\frac{F}{I_{sp} g_0},
\]
where \( I_{sp} \) is the thruster specific impulse; and \( g_0 \) is the standard acceleration of gravity at sea level.

Thus, the dynamic model considering GOAC is established by (7), (8), (13), (14), and (19). It can be seen that there is a mutual influence between the orbital motion and attitude motion of the spacecraft, therefore GOAC is fully embedded into the dynamics of the spacecraft, which lays a foundation for the 6-DOF trajectory optimization.

III. 6-DOF FUEL-OPTIMAL TRAJECTORY DESIGN

In this section, the fuel-optimal trajectory is obtained by introducing attitude control into the indirect method, and a two-phase homotopic approach is proposed to achieve the continuation from the 6-DOF energy-optimal problem to the fuel-optimal problem.

A. FORMULATION OF FUEL-OPTIMAL CONTROL PROBLEM AND BOUNDARY MODIFICATIONS

Suppose the spacecraft has been moving around the asteroid in the circular parking orbit at initial time \( t_0 \), it ought to reach the pre-designed landing site at the fixed terminal time \( t_f \). The initial and terminal state constraints are given by
\[
\begin{align*}
&\{ r(t_0) = r_0, \ v(t_0) = v_0, \ m(t_0) = m_0, \ \\
&\{ \sigma(t_0) = \sigma_0, \ \omega(t_0) = \omega_0, \ \\
&\{ r(t_f) = r_f, \ v(t_f) = [0 \ 0 \ 0]^T, \ \omega(t_f) = [0 \ 0 \ 0]^T, \ 
\end{align*}
\]

With attitude control introduced into the optimization, the dimension of the optimal control problem is expanded to 6 DOFs, which means the attitude and angular velocity are included in the state constraints, increasing the difficulty of solving TPBVP. For an asteroid-landing spacecraft, its orbit control is relatively more important than its attitude control, for the prior target of the mission is to land at the desired landing position. Based on this consideration, the orbit-related boundary conditions are fixed, and the terminal attitude of the spacecraft, \( \sigma_f \), is claimed to be free at the terminal time \( t_f \). By doing so, the search space of TPBVP is expanded and the solving difficulty is reduced, a better attitude trajectory can be found to minimize the fuel consumption. To cope with the modification, the target position \( r_f \) is assumed to be several meters (e.g. 50 m) above the exact landing site, leaving margin for attitude adjustment before landing on the surface of the asteroid.

To design the minimum-fuel trajectory with optimal control theory, the optimal index is chosen as
\[
J = \frac{F_{max}}{I_{sp} g_0} \int_{t_0}^{t_f} u_o \ dt.
\]

To transform the optimal control problem into a TPBVP by applying the theory of the PMP, a Hamiltonian \( H \) is defined as
\[
H = \lambda_r \cdot \dot{r} + \lambda_v \cdot \dot{v} + \lambda_\sigma \cdot \dot{\sigma} + \lambda_\omega \cdot \dot{\omega} + \lambda_m \cdot \dot{m} + \frac{F_{max} u_o}{I_{sp} g_0} = \lambda_r \cdot \dot{r} + \lambda_v \cdot \left(-2\omega_\sigma \times v - \omega_\sigma \times (\omega_\sigma \times v) - \nabla U(\sigma, r) \right) + \lambda_\omega \cdot \left[I^{-1} \left[T_{max} u_o a_o + M(\sigma, r) + T_3 - \omega^T I_\omega \right]\right] + \lambda_m \cdot \left(\frac{F_{max} u_o}{I_{sp} g_0} + \frac{F_{max} u_o}{I_{sp} g_0} \right),
\]

where \( \lambda_r, \lambda_v, \lambda_\sigma, \lambda_\omega, \) and \( \lambda_m \) are the costate variables. According to PMP, to obtain the optimal control law, the Hamiltonian needs to be minimized. Considering there are only two of the terms \( F_{max} u_o / m \lambda_\omega a_o \) and \( T_{max} u_o \lambda_\omega I^{-1} a_o \) containing the direction vector of control in the Hamiltonian, the direction vectors \( a_o \) and \( a_o \) are to be parallel and opposite to \( \lambda_\omega \) and \( \lambda_\omega I^{-1} \), respectively, which are
\[
\begin{align*}
\alpha_o &= -\frac{\lambda_v}{\|\lambda_v\|}, \\
\alpha_o &= -\frac{\lambda_\omega I^{-1}}{\|\lambda_\omega I^{-1}\|}.
\end{align*}
\]

Substituting (24) into (23), the Hamiltonian can be rewritten as
\[
H = \lambda_r \cdot \dot{r} + \lambda_v \cdot \left(-2\omega_\sigma \times v - \omega_\sigma \times (\omega_\sigma \times v) - \nabla U(\sigma, r) + F_r \right) + \lambda_\sigma \cdot \frac{1}{4} \left[ B(\sigma) \omega + \lambda_\omega \cdot \left[I^{-1} \left[M(\sigma, r) + T_3 - \omega^T I_\omega \right]\right] \right] + B_{max} u_o a_o + B_{max} u_o a_o,
\]

where \( B_o \) and \( B_o \) are called switch functions and expressed as
\[
\begin{align*}
B_o &= 1 - \frac{I_{sp} g_0}{\|\lambda_v\|}, \\
B_o &= -\frac{\lambda_\omega I^{-1}}{\|\lambda_\omega I^{-1}\|}.
\end{align*}
\]
To minimize the Hamiltonian, the output ratio of control force and control torque should be

\[
\begin{align*}
    u_o &= 0, \quad \rho_o > 0 \\
    u_o &= 1, \quad \rho_o < 0 \\
    u_o \in [0,1], \quad \rho_o = 0, \\
    u_o &= 0, \quad \rho_o > 0 \\
    u_o &= 1, \quad \rho_o < 0 \\
    u_o \in [0,1], \quad \rho_o = 0.
\end{align*}
\]

(28)

Generally, the switch functions \(\rho_o\) and \(\rho_a\) are zero only at the finite isolated points, so the control magnitude \(u_o\) and \(u_a\) take only a value of 0 or 1 at any interval, which means the optimal control is a bang-bang control. However, as can be seen in (27), the value of \(\rho_a\) is non-positive, so \(u_a\) can not be 0 according to (29). Then the optimal control law for \(u_a\) can be simplified to

\[
\begin{align*}
    u_a &= 1, \quad \rho_o < 0 \\
    u_a \in [0,1], \quad \rho_o = 0.
\end{align*}
\]

(30)

According to the Euler-Lagrange conditions, the ordinary differential equations (ODEs) of the costates are given by

\[
\begin{align*}
    \dot{\lambda}_r &= -\frac{\partial H}{\partial r}, \quad \dot{\lambda}_v = -\frac{\partial H}{\partial v}, \quad \dot{\lambda}_m = -\frac{\partial H}{\partial m},
\end{align*}
\]

(31)

and they are specified as follows.

1) ODE OF THE POSITION COSTATE \(\lambda_r\)

\[
\dot{\lambda}_r = \lambda_v \left( \frac{\partial}{\partial r} (\omega_o \times (\omega_o \times v)) + \nabla^2 \omega U - \frac{\partial F_s}{\partial r} \right) - \lambda_\omega I^{-1} \frac{\partial}{\partial r} M(\sigma, r),
\]

(32)

where \(\nabla^2 \omega U\) is given in the same form as in (10), and the last differential term \(\frac{\partial}{\partial r} M(\sigma, r)\) is calculated by

\[
\frac{\partial}{\partial r} M(\sigma, r) = \sum_{i=1}^{n} b_{ri} \times \left( \frac{b_i R^{-1}}{m_i} \cdot m_i \nabla^2 \omega U_i(\sigma, r) \right)
= \sum_{i=1}^{n} (b_{ri})^\times \cdot \left( m_i \nabla^2 b_i U_i(\sigma, r) \right),
\]

(33)

where \(\nabla^2 b_i U_i(\sigma, r)\) represents the gravitational gradient matrix in the spacecraft body-fixed frame \(b-ijk\).

2) ODE OF THE VELOCITY COSTATE \(\lambda_v\)

\[
\dot{\lambda}_v = -\frac{\partial H}{\partial v} = -\lambda_r - 2(\omega_o \times \lambda_v).
\]

(34)

3) ODE OF THE ATTITUDE COSTATE \(\lambda_\sigma\)

\[
\dot{\lambda}_\sigma = -\frac{1}{4} \omega^T \cdot \frac{\partial}{\partial \sigma} [\lambda_\sigma B(\sigma)] + \lambda_v \frac{\partial}{\partial \sigma} \nabla^\omega U(\sigma, r) - \lambda_\omega I^{-1} \frac{\partial}{\partial \sigma} M(\sigma, r),
\]

(35)

where \(\frac{\partial}{\partial \sigma} \nabla^\omega U(\sigma, r)\) is given by

\[
\frac{\partial}{\partial \sigma} \nabla^\omega U(\sigma, r) = \sum_{i=1}^{9} m_i \frac{\partial}{\partial \sigma} \nabla^\omega U_i(\sigma, r),
\]

(36)

where \(\nabla^\omega U_i(\sigma, r)\) denotes the gravitational acceleration of the \(i\)-th point mass. It is noteworthy that \(\nabla^\omega U_1(\sigma, r)\) of the central point mass \(m_c\) is unrelated to the attitude \(\sigma\), so there’s only eight non-zero terms in (36). The differential term in numerator of (36) can be given by

\[
\frac{\partial}{\partial \sigma} \nabla^\omega U_i(\sigma, r) = G_\rho \sum_{e} E_{e} \cdot \frac{\partial r_{ei}}{\partial \sigma} \cdot L_e
\]

\[
- G_\rho \sum_{f} F_{f} \cdot \frac{\partial r_{fj}}{\partial \sigma} \cdot \omega_f, \quad i = 1 \sim 8,
\]

(37)

where the first partial derivation term can be expressed as

\[
\frac{\partial}{\partial \sigma} r_{ei} = \frac{\partial}{\partial \sigma} \left( r_e - r - b_i R^b r_i \right)
= -\frac{\partial}{\partial \sigma} \left( b_i R^b r_i \right), \quad i = 1 \sim 8,
\]

(38)

where \(r_e\) is the position vector of a random point on an edge of the polyhedron in the frame \(O-xyz\). The result in (38) is obtained based on the fact that only the rotation matrix \(b_i R^b\) is a function of the attitude \(\sigma\). Similarly, the second partial derivation term in (38) can be expressed as \(\frac{\partial^2 r_{ei}}{\partial \sigma^2} = -\frac{\partial}{\partial \sigma} (b_i R^b r_i)\). Let \(a_i J_i = \frac{\partial}{\partial \sigma} (b_i R^b r_i)\) be the partial derivative matrix, (37) can be renewed as

\[
\frac{\partial}{\partial \sigma} \nabla U_i(\sigma, r) = -G_\rho \sum_{e} E_{e} \cdot a_i J_i \cdot L_e
+ G_\rho \sum_{f} F_{f} \cdot a_i J_i \cdot \omega_f, \quad i = 1 \sim 8,
\]

(39)

where the explicit formulation of \(a_i J_i\) is obtained with the help of Mathematica, which is presented in the Appendix. The explicit formulations given in the Appendix are tedious but essential for the numerical integration of the ODE of \(\lambda_\sigma\), which is indispensable for simulations. The differential term \(\frac{\partial}{\partial \sigma} M(\sigma, r)\) in (35) can be given by

\[
\frac{\partial}{\partial \sigma} M(\sigma, r) = \sum_{i=1}^{8} b_{ri} \cdot m_i \frac{\partial b_i R^{-1} \nabla^\omega U_i(\sigma, r)}{\partial \sigma}
= \sum_{i=1}^{8} m_i (b_{ri})^\times \cdot \frac{\partial b_i R^{-1} \nabla^\omega U_i(\sigma, r)}{\partial \sigma},
\]

(40)
where
\[
\frac{\partial}{\partial \sigma} \nabla^b U_i (\sigma, r) = G \rho \sum_e E_e \cdot \frac{\partial (b^R_i - r_{\sigma, i})}{\partial \sigma}, L_e
\]
\[
- G \rho \sum_f F_f \cdot \frac{\partial (b^R_i - r_{\sigma, i})}{\partial \sigma}, \omega_f, \quad i = 1 \sim 8,
\]
from (38), we have
\[
\frac{\partial (b^R_i - r_{\sigma, i})}{\partial \sigma} = \frac{\partial}{\partial \sigma} b^R_i \left(r_{\sigma} - r - b^R_i r_{\omega} \right)
\]
\[
= \frac{\partial}{\partial \sigma} b^R_i r_{\sigma, e},
\]
where \(r_{\sigma, e} = r_{\sigma} - r\) denotes a vector pointing from the CM of the spacecraft to any point on an edge of the asteroid. Similarly, \(\frac{\partial (b^R_i - r_{\sigma, i})}{\partial \sigma}\) is the partial derivative matrix, which is given in the Appendix in detail, and (41) is renewed as
\[
\frac{\partial}{\partial \sigma} \nabla^b U_i (\sigma, r) = G \rho \sum_e E_e \cdot b^R_i \cdot J \cdot L_e
\]
\[
- G \rho \sum_f F_f \cdot b^R_i \cdot J \cdot \omega_f, \quad i = 1 \sim 8,
\]
then \(\frac{\partial}{\partial \sigma} M (\sigma, r)\) can be obtained by substituting (43) into (40).

4) ODE OF THE ANGULAR VELOCITY COSTATE \(\lambda_\omega\)
\[
\dot{\lambda}_\omega = - \frac{\partial H}{\partial \omega} = - \frac{1}{4} \lambda_\sigma B (\sigma) + \lambda_\omega I^{-1} \left[ \frac{\partial}{\partial \omega} (\omega^\times I \omega) \right]
\]
\[
= - \frac{1}{4} \lambda_\sigma B (\sigma) + \lambda_\omega I^{-1} \left[ \omega^\times I - (I \omega)^\times \right].
\]

5) ODE OF THE MASS COSTATE \(\lambda_m\)
\[
\dot{\lambda}_m = \frac{\partial H}{\partial m} = - \frac{F_{\max} u_o \| \lambda_\sigma \|}{m^2}.
\]

No terminal attitude and mass constraint is claimed at the terminal time \(t_f\), so the transversality conditions are
\[
\lambda_\sigma (t_f) = 0,
\]
\[
\lambda_m (t_f) = 0.
\]

So far, the 6-DOF fuel-optimal control problem has been converted into a TPBVP described by dynamic ODEs: equations (7), (8), (13), (14), and (19); costate ODEs: equations (32), (34), (35), (44), and (45); and boundary conditions: equations (20), (21), (46) and (47). The integrated vector is a 26-dimensional variable which is expressed as
\[
\chi = [r \dot{r} \sigma \omega \lambda_\sigma \lambda_\nu \lambda_\lambda \lambda_\omega \lambda_\mu].
\]

The TPBVP can be solved by single shooting methods such as Newton’s, theoretically. The shooting function of TPBVP can be built as
\[
\Phi (\lambda (t_0)) = \left[ r (t_f) - r_f, \dot{r} (t_f) - \dot{r}_f, \sigma (t_f) - \sigma_f, \omega (t_f) - \omega_f, \lambda_\sigma (t_f) - \lambda_\sigma_f, \lambda_\nu (t_f) - \lambda_\nu_f, \lambda_\lambda (t_f) - \lambda_\lambda_f, \lambda_\omega (t_f) - \lambda_\omega_f, \lambda_\mu (t_f) - \lambda_\mu_f \right] = 0.
\]

The shooting methods are gradient-based algorithms [4], meaning it is extremely sensitive to the smoothness of the shooting function. However, according to (28) and (30), the optimal control law for \(u_o\) is of a bang-bang control type, and the optimal control law for \(u_a\) cannot guarantee the smoothness due to its “\(\rho_a = 0\)” case, either. The right hand side of the ODEs thus appears nonsmooth and discontinuous, therefore, challenge arises when solving (49) by the shooting methods.

### B. Two Phase Homotopic Continuation

To solve the difficulties caused by the nonsmooth optimal control of both orbit control \(u_o\) and attitude control \(u_a\), the homotopic approach is applied. The original homotopic approach builds a bridge between the original difficult problem with an easier one, in fuel-optimal control problem specifically, it connects the fuel-optimal control problem with the energy-optimal control problem. By decreasing the continuation parameter iteratively, the optimal solution can be obtained.

However, according to the above analysis, the ODEs related with \(\dot{r}, \dot{\sigma}, \dot{\lambda}_\sigma, \text{ and } \dot{\lambda}_\nu\) and the ODEs related with attitude \((\dot{\sigma}, \omega, \dot{\lambda}_\lambda, \text{ and } \dot{\lambda}_\omega)\) are tightly coupled, which inevitably leads to the coupling between the two control variables \(u_o\) and \(u_a\) during optimization. It means that except...
from the nonsmoothness $u_a$ owns itself, the discontinuous feature of $u_a$ will also bring side-effects to the optimization of $u_o$, causing extra troubles to the homotopic method. Therefore, to fully smoothen the solving process, we decide to include the attitude control in the homotopic continuation procedure by adding a rotational energy consumption term $T_{\max} \int_0^t \epsilon_a u_a^2 dt$ to the original homotopic index, extending the original 3-DOF energy-optimal problem to a 6-DOF energy-optimal problem, which is shown in the equation below:

$$J = \lambda_0 \frac{F_{\text{max}}}{I_{sp\text{g}0}} \int_{t_0}^{t_f} \left[ \epsilon_o u_o^2 + \left( 1 - \epsilon_o \right) u_o + \frac{I_{sp\text{g}0}}{F_{\text{max}}} T_{\max} \cdot \epsilon_a u_a^2 \right] dt,$$

(50)

where the positive numerical factor $\lambda_0$ [10] is used to reduce the search space of the initial costates.

As can be seen in the above equation, there are two homotopic parameters: $\epsilon_o$—the orbit homotopic parameter, and $\epsilon_a$—the attitude homotopic parameter. When $\epsilon_o = \epsilon_a = 1$, the index refers to the 6-DOF energy-optimal problem, where the energy includes both translational and rotational energy; when $\epsilon_o = \epsilon_a = 0$, the two energy terms disappear, and the index refers to the fuel-optimal problem.

Based on the extended homotopic index (50), the Hamiltonian can be renewed as

$$H = \lambda_r \dot{r} + \lambda_\omega \left[ -2 \omega_a \times v - \omega_i \times (\omega_a \times v) \right]$$

$$- \nabla U(\sigma, r) + \frac{F_{\text{max}} u_o}{m} \alpha_o + F_s$$

$$+ \lambda_\sigma \left[ I^{-1} \left[ T_{\max} \alpha_o M(\sigma, r) + T_r - \omega^x I_\omega \right] \right]$$

$$+ \lambda_m \left[ -\frac{F_{\text{max}} u_o}{I_{sp\text{g}0}} \right]$$

$$- \lambda_0 \frac{F_{\text{max}}}{I_{sp\text{g}0}} \left[ \epsilon_o u_o^2 + \left( 1 - \epsilon_o \right) u_o + \frac{I_{sp\text{g}0}}{F_{\text{max}}} T_{\max} \epsilon_a u_a^2 \right].$$

(51)

To minimize the renewed Hamiltonian, the directions of the control keep the same form as in (24) while the magnitudes of orbit and attitude control are renewed as

$$u_o = 0,$$

$$\rho_o > \epsilon_o$$

$$u_o = \frac{\epsilon_o - \rho_o}{2 \epsilon_o}, \quad |\rho_o| \leq \epsilon_o$$

$$u_o = 1, \quad \rho_o < -\epsilon_o,$$

$$u_a = \frac{-\rho_a}{2 \epsilon_a}, \quad -2 \epsilon_a \leq \rho_a \leq 0$$

$$u_a = 1, \quad \rho_a < -2 \epsilon_a,$$

and the switching functions in the above control laws are changed to

$$\rho_o = 1 - \frac{I_{sp\text{g}0} \lambda \omega}{\lambda_a} - \frac{\lambda_m}{\lambda_0},$$

(54)

$$\rho_a = -\frac{\lambda_a \sigma^{-1}}{\lambda_0}.$$

(55)

The factor $\lambda_0$ is introduced in (50) to normalize the initial costate variables, which will reduce the search space for the algorithm. The original costate vector $\lambda = [\lambda_r \lambda_\omega \lambda_{\sigma} \lambda_{\sigma} \lambda_m \lambda_{\sigma}]$ is expanded to $\lambda_e = [\lambda_r \lambda_\omega \lambda_{\sigma} \lambda_m \lambda_{\sigma} \lambda_0]$, and the normalization condition

$$\|\lambda_e (t_0)\| = 1$$

(56)

needs to be followed. Thus the TPBVP can be renewed as

$$\Phi (\lambda_e (t_0)) = [r (t_f) - r_i, v (t_f) - v_i, \lambda_\omega (t_f) \omega (t_f) - \omega_f, \lambda_m (t_f) \lambda_e (t_0)] - 1 = 0.$$

(57)

To start the homotopic process, the PSO method is adopted and modified to obtain the solution of the energy-optimal problem. The objective function of the energy-optimal problem is constructed as

$$F (x) = \frac{F_{\text{max}}}{I_{sp\text{g}0}} \int_{t_0}^{t_f} u_o^2 dt + T_{\max} \int_{t_0}^{t_f} u_a^2 dt + R_f \cdot \|\Phi (\lambda (t_0))\|,$$

(58)

where $x$ is the search-variable vector and has the same dimension with the integrated vector in (48). The first two terms of the above index refers to the energy consumption index with $\epsilon_o = \epsilon_a = 1$, and the last term is the shooting function multiplied by a random factor $R_f = rand (10^m, 10^m)$, where rand $(a, b)$ is a random number between $a$ and $b$. Detailed derivation can be seen in [10].

To solve the fuel-optimal control problem, there are two homotopic parameters need to be decreased. If decreasing $\epsilon_o$ and $\epsilon_a$ simultaneously, the coupling between the controls $u_o$ and $u_a$, and the discontinuity caused by them in the homotopic process have to be dealt with at the same time, which leads to a higher possibility of failure of the continuation. To reduce the difficulty, we divide the iteration process into two phases—phase 1: iteration of $\epsilon_o$, and phase 2: iteration of $\epsilon_a$. The two controls are treated separately, thus the coupling between them can be weakened, and the discontinuity caused by each control can be handled one at a time. The sequence is settled based on the thought that, at the end of phase 2, the two homotopic parameters are zero or near zero, high discontinuity will emerge both in orbit control $u_o$ and attitude control $u_a$, making the TPBVP more difficult to solve. The boundary conditions of attitude ($\omega (t_f) = \omega_f$ and $\lambda_\sigma (t_f) = 0$) is less than that of orbit ($r (t_f) = r_i, v (t_f) = v_i$, and $\lambda_m (t_f) = 0$), regardless of $\|\lambda_e (t_0)\| = 1$ which is shared in common. Fewer constraints equals less difficulty, which means the $u_a$ can do better than $u_o$ at the end of phase 2. Therefore, the orbit control with more constraint is placed at phase 1, and the attitude control with fewer constraints is placed at phase 2.

The solving process is summarized as follows.

1) Use the PSO algorithm to search the guess of $\lambda_e (t_0)$ to the 6-DOF energy-optimal problem.

2) Take the approximate solution obtained from 1) as initial values to solve the 6-DOF energy-optimal problem. If it does not converge, go to 1.

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3) Phase 1 iteration: take the solution obtained from 2) as the initial value, the factor \( \lambda_0 \) is fixed with the value obtained from 2), \( \varepsilon_a \) is fixed to 1, solve \( \lambda \) \((t_0)\) successively with the decreased \( \varepsilon_o \) until \( \varepsilon_o = 0 \).

4) Phase 2 iteration: take the solution corresponding to \( \varepsilon_o = 0 \) and \( \varepsilon_a = 1 \) from 3) as the initial value, solve \( \lambda \) \((t_0)\) successively with the decreased \( \varepsilon_o \) until \( \varepsilon_o = 0 \). When \( \varepsilon_o = 0 \), the fuel-optimal solution is obtained. With the help of Equations (32), (34), (35), (44), and (45), the solution meets the first-order optimal necessary condition, thus the optimality of the solution can be guaranteed. A simple flow chart of the algorithm is presented as in Fig. 4.

\[
\begin{align*}
\varepsilon_a = 1 & \quad \text{Phase 1} \quad \varepsilon_a = 0 \quad \text{Phase 2} \quad \\
\varepsilon_a = 1 & \quad \quad \quad \varepsilon_a = 1 \quad \quad \quad \varepsilon_a = 0
\end{align*}
\]

**FIGURE 4.** Coupling among the costate variables and its effect on state variables.

### IV. NUMERICAL RESULTS

In this section, a numerical example is presented to verify the proposed method. The 6-DOF fuel-optimal trajectory is first designed, and the fuel consumption is compared between the obtained 6-DOF trajectory and the point-mass trajectory using the same controller. To better observe the effect of GOAC, other perturbation forces and torques are assumed to be zero, and attitude constraints that a landing mission may require are not considered in this paper.

### A. EXAMPLE OF 6-DOF OPTIMAL TRAJECTORY DESIGN

Eros 433 is chosen as the target asteroid for landing, and build its polyhedral model with 1708 faces and 856 vertices, whose shape model is available on the website of the Planetary Data System [27]. The density of Eros is considered to be a constant value of 2.67 g/cm³, and the magnitude of angular velocity of Eros is 3.31E-4 rad/s, based on its rotation period. MATLAB’s Mex-function is used to accelerate the calculation of the asteroid’s gravitational potential, and fsolve is adopted to solve the shooting functions with an accuracy requirement of 1E-6. The integration step size is set as \( h = 1 \) s.

The parameters of the spacecraft and boundary conditions of the landing mission are stated in Table 1 and Table 2, respectively.

Based on the settings above, moment of inertia of the spacecraft is obtained as \( I = \text{diag}(131.3, 92.9, 118.4) \) kg m². \( T_{\text{max}} \) is set to a relatively small value due to the consideration that the optimal attitude control \( u_{\text{at}} \) is turned on along the whole trajectory.

Terminal velocities \( r_f \) and \( \omega_f \) are set to zero as mentioned in Section III. The target position \( r_f \) is chosen near the equator, and it is close to the Brillouin sphere of the asteroid (the Euclidean norm of \( r_f \) is 14.77 km, close to the reference radius of the asteroid \( R_0 = 16 \) km), because we want to prove that the coupling effect still exerts a significant influence even in a region with a less irregular gravitational field.

In PSO searching, the random factor is set to \( R_f = \text{rand} (10^{-1}, 10^{5}) \), maximal iteration number is set to 100, and the swarm size is 10. The initial guess is obtained after the searching, and it is taken as the input to solve the 6-DOF energy-optimal problem, which is corresponding to \( \varepsilon_o = 1 \) and \( \varepsilon_a = 1 \). The solution \( \lambda_e \) \((t_0)\), \( \varepsilon_o=1, \varepsilon_a=1 \) is obtained, which is presented in Table 3.

Phase 1 iteration is started by taking \( \lambda_e \) \((t_0)\), \( \varepsilon_o=0, \varepsilon_a=1 \) as the input. With \( \varepsilon_a \) fixed to 1, and \( \lambda_0 \) fixed to the value in \( \lambda_e \) \((t_0)\), \( \varepsilon_o=1, \varepsilon_a=1 \), the orbit homotopic parameter \( \varepsilon_o \) is decreased along the route \([1.0, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3, 0.2, 0.1, 0.08, 0.008, 0.0] \), the iteration process is illustrated in Fig. 5. The costate variables \( \lambda \) \((t_0)\), \( \varepsilon_o=0, \varepsilon_a=1 \) obtained from the Phase 1 iteration is presented in Table 3, and the fuel consumed is 7.3114 kg.

Phase 2 iteration is started by taking \( \lambda \) \((t_0)\), \( \varepsilon_o=0, \varepsilon_a=1 \) as the input, which is corresponding to \( \varepsilon_o = 0, \varepsilon_a = 1 \). With \( \varepsilon_o \) fixed to 0, the attitude homotopic parameter \( \varepsilon_a \) is decreased along the route \([1.0, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3, 0.2, 0.1, 0.08, 0.04, 0.004, 0.0] \). The iteration process is illustrated in Fig. 6. After the iteration, the fuel-optimal solution \( \lambda \) \((t_0)\), \( \varepsilon_o=0, \varepsilon_a=0 \) is obtained, which is presented in Table 3, the final fuel consumption is reduced to

**TABLE 1.** Parameters of the spacecraft.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_0 )</td>
<td>1000 kg</td>
</tr>
<tr>
<td>( m_s )</td>
<td>5 kg</td>
</tr>
<tr>
<td>( F_{\text{max}} )</td>
<td>30 N</td>
</tr>
<tr>
<td>( \sigma_{\text{max}} )</td>
<td>0.006 Nm</td>
</tr>
<tr>
<td>( I_{sp} )</td>
<td>300 s</td>
</tr>
<tr>
<td>( I/\ell_1/\ell_2 )</td>
<td>2.8/2.3 m</td>
</tr>
</tbody>
</table>

**TABLE 2.** Boundary conditions of the landing mission.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_0 )</td>
<td>0 s</td>
</tr>
<tr>
<td>( t_f )</td>
<td>3000 s</td>
</tr>
<tr>
<td>( r_f )</td>
<td>[17.5, -30.311, 0.0] km</td>
</tr>
<tr>
<td>( v_f )</td>
<td>[-6.496, -4.01, 0.0] m/s</td>
</tr>
<tr>
<td>( \sigma_v )</td>
<td>[0.211, -0.996, -0.211] rad</td>
</tr>
<tr>
<td>( \omega_v )</td>
<td>[0.001, 0.003, 0.002] rad/s</td>
</tr>
<tr>
<td>( r_f )</td>
<td>[12.29, -8.20, 0.05] km</td>
</tr>
</tbody>
</table>
TABLE 3. Results of costate variables at different stages.

<table>
<thead>
<tr>
<th>Costates</th>
<th>Energy-optimal solution</th>
<th>Phase 1 solution</th>
<th>Phase 2 solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_1(t_0) )</td>
<td>[-1.54E-5, -6.51E-5, -6.99E-7]</td>
<td>[-1.75E-5, -7.02E-5, 1.06E-6]</td>
<td>[-1.19E-5, -7.59E-5, -7.17E-7]</td>
</tr>
<tr>
<td>( \lambda_2(t_0) )</td>
<td>[0.0276, -0.0888, 5.19E-4]</td>
<td>[0.0357, -0.0910, 4.57E-5]</td>
<td>[0.0376, 0.0933, -3.83E-4]</td>
</tr>
<tr>
<td>( \lambda_3(t_0) )</td>
<td>[9.00E-10, 7.55E-10, 6.21E-10]</td>
<td>[1.11E-5, 1.11E-5, -1.94E-5]</td>
<td>[-0.0011, -0.0019, 0.0020]</td>
</tr>
<tr>
<td>( \lambda_4(t_0) )</td>
<td>[0.3523, -0.5981, 0.6784]</td>
<td>[0.3358, -0.5964, 0.6774]</td>
<td>[0.2197, -0.5616, 0.7596]</td>
</tr>
<tr>
<td>( \lambda_5 )</td>
<td>0.0018</td>
<td>0.0019</td>
<td>0.0019</td>
</tr>
<tr>
<td>( \lambda_6 )</td>
<td>0.222</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

7.2740 kg, 0.0374 kg less before the Phase 2 iteration. It can be seen that optimization of attitude have a positive effect on the optimization of the fuel consumption.

After the two-phase homotopic continuation, the optimal orbital and attitude trajectories of the spacecraft are obtained, which are shown in Fig. 7. Attitude parameter is changed from MRP to Euler angles to get an intuitive view of the attitude motion.

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B. COMPARISON OF FUEL CONSUMPTIONS

The aim of this section is to verify the advantage of the 6-DOF optimal trajectory obtained from the last section over the 3-DOF point-mass optimal trajectory, through comparing the fuel consumptions derived from tracking the two trajectories with the same controller.

For comparison, the 3-DOF fuel-optimal trajectory modeling the spacecraft as a point mass (which means there is only orbital control \( u_0 \), no attitude control \( u_a \) in optimization) is designed by using the traditional indirect method and homotopic method. The basic parameters of the spacecraft—\( m_0, F_{max} \), and \( I_{sp} \)—are the same as in Table 1 and boundary conditions of the trajectory are the same as in Table 2. After optimization, the optimal control sequence of \( u_0 \) is in the same form as in Fig. 5, which refers to \( [1, 0, 1] \). The fuel consumption of the 3-DOF optimal trajectory is 7.3114 kg, the same as the fuel consumed after the Phase 1 iteration.

Before comparison, there is a necessity to show that if there is GOAC and how deep it influences the orbit of the spacecraft during the landing process. So the rigid body spacecraft is made to follow the 3-DOF fuel-optimal trajectory, and the attitude of the spacecraft is set free. The geometric parameters of the spacecraft are the same as in Table 1. The position error of the spacecraft along the trajectory in the frame \( O-xyz \) is shown in Fig. 8. Under the consumption of no other perturbation is considered, we can draw the conclusion that the position error fully comes from GOAC, which means the attitude motion of the spacecraft has a significant influence on the orbit of the spacecraft. The relatively small position error on \( z \) axis originates from the relatively small distance between \( r_0 \) and \( r_f \) on \( z \) axis. And it can be seen that the position error equals zero for about 392 s due to a relatively large distance between the spacecraft and the asteroid, besides, \( u_0 \) is 1. When \( u_0 \) turns off at 392 s, the position error tends to grow as the spacecraft descending towards the asteroid. The gradients of the position errors grow at about 2676 s, which is because the orbit control \( u_a \) turns from 0 to 1 at the time. Due to the neglect of the coupling effect, the orbit control intensifies the position error.

To verify the advantage of the 6-DOF trajectory over the 3-DOF point-mass trajectory, two simulations for comparison are implemented:

1) The rigid-body spacecraft follows the 3-DOF fuel-optimal trajectory, its attitude motion is planned and the 6-DOF trajectory is tracked by a PD controller to see the fuel consumption under GOAC. The attitude motion is planned to ensure the controller both tracks...
where the polynomial coefficients are calculated according to (13), which are given by polynomial method, and the angular velocity of the spacecraft trajectory. The attitude of the spacecraft is planned by the for a rigid-body spacecraft following the 3-DOF point-mass trajectory. The attitude of the spacecraft is planned and controlled to see the real fuel consumption of the attitude-free spacecraft.

![FIGURE 7. 6-DOF Fuel-optimal trajectory.](image)

![FIGURE 8. Position error of the attitude-free spacecraft.](image)

and the boundary condition $\sigma_f$ is set as the terminal attitude of the 6-DOF optimal trajectory in the previous section.

By planning the attitude of the spacecraft and combining the attitude trajectory with the point-mass trajectory stated above, the DOF of the combined trajectory is 6, which is the same as that of the trajectory generated by the method proposed in this paper. Besides, the boundary conditions of the two trajectories are the same, which makes the comparison more objective.

A PD controller is designed to track the combined 6-DOF trajectory, which is presented as

$$a_{\text{control}} = k_d (v_d - v) + k_p (r_d - r),$$

$$a_{\text{control}} = k_d (\omega_d - \omega) + k_p (\sigma_d - \sigma),$$

where $a_{\text{control}}$ and $a_{\text{control}}$ denote the translational and rotational acceleration of control, $r_d$, $v_d$, $\sigma_d$, and $\omega_d$ denote the reference trajectory with optimized orbit and planned attitude, and the PD parameters are set as $k_p = \text{diag} (5E-4, 5E-4, 5E-4)$ and $k_d = \text{diag} (0.5, 0.5, 0.5)$. The fuel consumption which can be calculated by

$$\dot{m} = -m \frac{\|a_{\text{control}}\|}{I_{\text{SP}} g_0}.$$  

After trajectory tracking, the Euclidean norm of the position error at final time $t_f$ is 15.86 m, and the fuel consumption after trajectory tracking is shown as a red solid line in Fig. 9(a), it is compared with the original fuel consumption derived from 3-DOF trajectory optimization, which is shown as a blue dotted line.

It can be seen that, when $u_o = 1$ (oblique line segments of the dotted line), the fuel consumed is fit to the original one. When $u_o = 0$ (level line segments of the dotted line), the optimal control $u_o$ turns off, to attenuate the position error caused by the coupling effect, extra fuel needs to be consumed, as can be seen in the solid line. The fuel consumption after tracking is 8.2529 kg, 0.9415 kg more than the original consumption, 7.3114 kg.

In simulation 2), the 6-DOF fuel-optimal trajectory obtained from the previous section is tracked by the same
controller as in simulation 1). After trajectory tracking, the Euclidean norm of the position error at final time $t_f$ is 15.93 m, only 0.06 m bigger than that in simulation 1, but the fuel consumption after trajectory tracking is different, which is shown in Fig. 9(b).

It can be seen that, the fuel consumption under control is quite fit to the original one, the final consumption is 7.285 kg, 0.0114 kg more than the original one, 7.274 kg. The extra consumption of the 6-DOF trajectory only accounts for 0.16% of the original fuel consumption, the conclusion can be draw that the 6-DOF optimal trajectory designed by the proposed method can avoid extra fuel consumption under GOAC.

The fuel consumed in simulation 1) and 2) are stated in Table 4, where “Optimization result” refers to the fuel consumption obtained from trajectory optimization, and “Tracking result” refers to one that obtained after trajectory tracking. From Table 4 it can be seen that the controlled consumption in simulation 2) is 0.9675 kg less than that in simulation 1), indicating the obtained 6-DOF optimal trajectory saves more fuel than the traditional 3-DOF optimal trajectory in trajectory tracking.

<table>
<thead>
<tr>
<th>Simulation</th>
<th>Optimization result</th>
<th>Tracking result</th>
<th>Extra consumption</th>
<th>Extra consumption %</th>
</tr>
</thead>
<tbody>
<tr>
<td>1)</td>
<td>7.3114 kg</td>
<td>8.2529 kg</td>
<td>0.9415 kg</td>
<td>12.88%</td>
</tr>
<tr>
<td>2)</td>
<td>7.2740 kg</td>
<td>7.285 kg</td>
<td>0.0114 kg</td>
<td>0.16%</td>
</tr>
</tbody>
</table>

From the simulations above, it can be summarized that:

First, the GOAC has a significant influence on motion of a rigid-body spacecraft when it descends towards an asteroid.

Second, additional fuel with a significant amount will be consumed if a rigid-body spacecraft follows the 3-DOF point-mass trajectory.

Third, the 6-DOF trajectory obtained from the proposed method in this paper can guarantee the fuel-optimality under GOAC to a certain extent, and saves more fuel than the 3-DOF optimal trajectory in trajectory tracking.

V. CONCLUSION

A trajectory optimization method for asteroid landing considering GOAC is proposed in this paper. The coupling effect is embedded into the dynamic of the spacecraft with the help of the polyhedral method and a distributed point-mass model. Attitude control is introduced into the indirect method to formulate a 6-DOF fuel-optimal problem. Terminal attitude constraint is relieved to expand the search space of TPBVP. A two-phase homotopic method is proposed by introducing an extra attitude homotopic parameter to achieve a smooth continuation from 6-DOF energy optimal problem to 6-DOF fuel-optimal problem.

Simulation results show that the proposed optimization method can decrease orbit and attitude homotopic parameters smoothly and generate the 6-DOF fuel-optimal trajectory. Comparison after trajectory tracking shows that the fuel consumption of the 6-DOF trajectory is lower than that of the 3-DOF trajectory. Besides, 12.88% more fuel is additionally consumed after tracking the point-mass trajectory generated by the traditional method, while the proposed method consumes only 0.16% more fuel, indicating the fuel-optimality of the landing mission can be guaranteed to a certain extent by the proposed method.

The contributions of this paper mainly include the following points:

- This paper considers GOAC in trajectory optimization for asteroid landing for the first time, optimizing the trajectory in a more realistic situation.
- An improved trajectory optimization method for asteroid landing considering GOAC is proposed by optimizing orbit and attitude of the spacecraft simultaneously.
• In 6-DOF indirect method, the terminal constraint of attitude is relieved to expand the search space and reduce the difficulty of solving TPBVP.

• A two-phase homotopic approach based on constraint analysis is proposed to achieve a smooth continuation from the 6-DOF energy-optimal problem to the fuel-optimal problem.

Several improvements will be applied in the future work. A more complex geometric model composed of more asymmetricaly distributed point masses can be applied to simulate the mass distribution of a spacecraft in real world. Liquid sloshing can be considered in optimization to deduce a more realistic analysis on a spacecraft under GOAC. Furthermore, with attitude considered into the dynamics, the stability of the optimal trajectory and its influence on feasibility of the method will also be investigated.

APPENDIX

A. EXPLICIT FORMULATIONS OF \( ^oJ_i \)

Let \( ^b\mathbf{r}_i = [x\ y\ z]^T \), the nine elements of the \( 3 \times 3 \) partial derivative matrix \( ^oJ_i \) are as follows.

\[
^oJ_i(1, 1) = \frac{8}{(1 + \sigma^2)^3} \left( \sigma_1^3 (\sigma_3 y - \sigma_2 z) - (3\sigma_1^2 - 1 - \sigma_2^2 - \sigma_3^2) (\sigma_2 y + \sigma_3 z) - \sigma_1 (4\sigma_2 x + 4\sigma_3 x - 3y + 8\sigma_2 \sigma_3 y + 8\sigma_3^2 y - \sigma_2 z) - \sigma_2 z (\sigma_2^2 + \sigma_3^2 - 3) \right)
\]

\[
^oJ_i(1, 2) = \frac{4}{(1 + \sigma^2)^3} \left( 4\sigma_2 \left( -\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 1 \right) x + 2 (\sigma_1 + 1 - 3\sigma_2 \sigma_3 + \sigma_2 (3 + \sigma_3^2)) \sigma_3 + \sigma_1 \sigma_2^2 + \sigma_2 \sigma_3^3 \right) y + \sigma_3^4 + 6\sigma_2 - \sigma_3^2 - 1
\]

\[
^oJ_i(1, 3) = \frac{8}{(1 + \sigma^2)^3} \left( y - (\sigma_1^2 + \sigma_2^2)^2 y + \sigma_3^4 y - 8\sigma_1 \sigma_2 \sigma_3 + 2\sigma_1 \sigma_2^2 + \sigma_2 \sigma_3^2 - 3 \right) z
\]

\[
^oJ_i(2, 1) = \frac{4}{(1 + \sigma^2)^3} \left( 2 (\sigma_2 - 3\sigma_1^2 \sigma_2 + \sigma_3^3 - \sigma_2 \sigma_3^2 - \sigma_1 \sigma_3^2) x - (3 + \sigma_2 + \sigma_3) \sigma_3 x + 2\sigma_1 \sigma_2 y - 6\sigma_2^2 (y + \sigma_1 z) + \sigma_3^3 (4x - 2\sigma_2 z) \right)
\]

\[
^oJ_i(2, 2) = \frac{4}{(1 + \sigma^2)^3} \left( \sigma_1^2 (-\sigma_2 \sigma_3 x + 4\sigma_2 y + \sigma_3 z) + \sigma_1^3 (x + \sigma_2 z) + \sigma_3 (3\sigma_2 x - \sigma_3^2 x - \sigma_2 \sigma_3) + 4\sigma_2 \sigma_3 y + z - 3\sigma_2^2 z + \sigma_3^2 z \right)
\]

\[
^oJ_i(2, 3) = \frac{4}{(1 + \sigma^2)^3} \left( 4\sigma_3 \left( \sigma_1^2 - \sigma_2^2 + \sigma_3^2 - 1 \right) y + \left( 6\sigma_2^2 - \sigma_3^4 + \sigma_1^2 + \sigma_3^2 \right) - 8\sigma_1 \sigma_2 \sigma_3 - 1 \right) x + 2 (\sigma_2^2 + \sigma_1 \sigma_2 \sigma_3 + 8\sigma_2 \sigma_3 (1 + \sigma_1^2 - 3\sigma_3^2) + \sigma_1 \sigma_3 (-3 + \sigma_2^2 + \sigma_3^2) \right) z
\]

\[
^oJ_i(3, 1) = \frac{4}{(1 + \sigma^2)^3} \left( -\sigma_1^4 y + 6\sigma_2^2 (-\sigma_3 x + y)
\right)
\]

\[
^oJ_i(3, 2) = \frac{4}{(1 + \sigma^2)^3} \left( 4\sigma_2 \left( \sigma_1^2 + \sigma_2^2 - \sigma_3^2 - 1 \right) z
\right)
\]

\[
^oJ_i(3, 3) = \frac{8}{(1 + \sigma^2)^3} \left( \sigma_1 (1 + \sigma_2^2 - 3\sigma_3^2) x - \sigma_1 \sigma_3 (-3 + \sigma_2^2 + \sigma_3^2) y + \sigma_2 \left( (-3 + \sigma_2^2 + \sigma_3^2) \sigma_3 x + \sigma_3^3 x + y + \sigma_2^2 y - 3\sigma_2^3 y + 4\sigma_2 \sigma_3 z + \sigma_1^3 (x - \sigma_3 y) + \sigma_1^2 (\sigma_2 \sigma_3 x + y) + 4\sigma_3 z \right) \right)
\]

B. EXPLICIT FORMULATIONS OF \( ^bJ \)

Let \( ^b\mathbf{r}_{ce} = [x\ y\ z]^T \), the nine elements of the \( 3 \times 3 \) partial derivative matrix \( ^bJ \) are as follows.

\[
^bJ(1, 1) = \frac{8}{(1 + \sigma^2)^3} \left( \sigma_1^3 (-\sigma_2 \sigma_3 z - 3\sigma_1^2 \sigma_2 y + \sigma_3 z) + \left( 1 + \sigma_2^2 + \sigma_3^2 \right) \sigma_2 y + \sigma_3 z
\]

\[
^bJ(1, 2) = \frac{8}{(1 + \sigma^2)^3} \left( \sigma_1^3 (\sigma_3 - \sigma_2 x + 4\sigma_2^2 x + 3\sigma_3 y + 3\sigma_2^2 y - \sigma_3^2 y - \sigma_2 \sigma_3 z) \right)
\]
\[ bJ(1, 2) = -\frac{4}{(1 + \sigma^2)^3} \left( \sigma^2 (2 \sigma + 2 \sigma^3 y) - \sigma^2 z + 2 \sigma y \right) + \sigma (3 \sigma^2 + 2 \sigma^3 y + 4 \sigma \sigma^2 \sigma z) + \left( 1 + \sigma^2 + \sigma^3 \right) \left( -2 \sigma y \left( \sigma^2 + \sigma^3 - 1 \right) \right) \]

\[ bJ(1, 3) = \frac{4}{(1 + \sigma^2)^3} \left( \left( 1 + \sigma^2 \right) - 2 \sigma (3 \sigma^2 + 2 \sigma^3 y + 4 \sigma \sigma^2 \sigma z) + \sigma (3 \sigma^2 + 2 \sigma^3 y + 4 \sigma \sigma^2 \sigma z) + \sigma^2 \right) \]

\[ bJ(2, 1) = \frac{4}{(1 + \sigma^2)^3} \left( \left( 1 + \sigma^2 \right) - 2 \sigma (3 \sigma^2 + 2 \sigma^3 y + 4 \sigma \sigma^2 \sigma z) + \sigma (3 \sigma^2 + 2 \sigma^3 y + 4 \sigma \sigma^2 \sigma z) + \sigma^2 \right) \]

\[ bJ(2, 2) = \frac{8}{(1 + \sigma^2)^3} \left( \sigma (2 \sigma^2 \sigma x + 4 \sigma y + \sigma z) + \sigma^3 (x - 2 \sigma^2 \sigma y + 4 \sigma^2 \sigma y) + 2 \sigma^3 (x + 2 \sigma^2 \sigma y + 4 \sigma y + \sigma^2 \sigma z) + \sigma^3 \left( 1 - \sigma^2 - \sigma^3 \right) + 4 \sigma^3 \right) - \sigma^3 \left( 1 - \sigma^2 - \sigma^3 \right) \]

\[ bJ(2, 3) = \frac{4}{(1 + \sigma^2)^3} \left( \left( 1 + \sigma^2 \right) - 2 \sigma x \left( -3 + \sigma^2 + \sigma^3 \right) x + 2 \sigma \left( 1 + \sigma^2 + \sigma^3 \right) x - \sigma \left( -3 \sigma^2 + \sigma^3 \right) x \right) - \sigma^3 \left( 1 - \sigma^2 - \sigma^3 \right) \]

\[ bJ(3, 1) = \frac{4}{(1 + \sigma^2)^3} \left( \sigma \left( 1 - \sigma^2 - \sigma^3 \right) x + 8 \left( \sigma^2 - \sigma \sigma^2 \sigma + 4 \sigma^2 \right) y + \sigma \left( -3 + \sigma^2 + \sigma^3 \right) \right) \]

\[ bJ(3, 2) = -\frac{4}{(1 + \sigma^2)^3} \left( \left( \sigma^4 + 6 \sigma^2 + \sigma^3 - 1 \right) x - 8 \sigma \sigma^2 \sigma + 2 \sigma^2 y \sigma^2 + \sigma^3 \right) x + \sigma^3 \left( 2 \sigma^2 + \sigma^3 - 3 \sigma^2 \sigma \right) + \sigma \left( -3 + \sigma^2 + \sigma^3 \right) y + 4 \sigma^2 \left( -1 + \sigma^2 + \sigma^3 \right) z \]

\[ bJ(3, 3) = \frac{8}{(1 + \sigma^2)^3} \left( \sigma \left( 1 - \sigma^2 - \sigma^3 \right) x + \sigma \left( -3 + \sigma^2 + \sigma^3 \right) y + \sigma \left( 3 \sigma^2 + 2 \sigma^3 \right) + 8 \left( \sigma^2 - \sigma \sigma^2 \sigma + 4 \sigma^2 \right) y + \sigma \left( -3 + \sigma^2 + \sigma^3 \right) + \sigma^2 \right) \]

\[ \sigma (3 \sigma^2 - 3 \sigma^2 + \sigma^3) \left( 2 \sigma^2 + \sigma^3 \right) x + \sigma \left( -3 + \sigma^2 + \sigma^3 \right) y + \sigma \left( 3 \sigma^2 - \sigma^2 \sigma + 3 \sigma^3 \right) \]

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