Interval-Valued \(q\)-rung Orthopair 2-Tuple Linguistic Aggregation Operators and their applications to Decision Making Process

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ABSTRACT The objective of this manuscript is to present the concept of interval-valued \(q\)-rung orthopair 2-tuple linguistic set (IVq-RO2TLSs) and their basic operations. As a generalization of the sets, interval-valued \(q\)-rung orthopair fuzzy set is a more profitable way to express the uncertainties in the data while 2-tuple linguistic set deals with the qualitative aspects of the information. By utilizing both these features, we propose a concept of IVq-RO2TLSs and studies their properties. For it, firstly, we define the basic operational laws between the pairs of the sets and define the score, accuracy and basic operations. Secondly, based on the operational laws, we develop some weighted averaging and geometric aggregation operators for IVq-RO2TLSs and characterize their desirable properties. Thirdly, a novel approach is developed to solve multi-attribute decision making (MADM) problem with interval-valued \(q\)-rung orthopair 2-tuple linguistic information. Finally, several numerical examples are provided with some comparative study to validate the approach.

INDEX TERMS interval-valued \(q\)-rung orthopair set; 2-tuple linguistic number; aggregation operator; Multi attribute decision making

I. INTRODUCTION

MULTIPLE attribute decision making (MADM) is one of the processes for selecting the optimal alternative from a set of given alternatives according to some attributes. Due to the complexity of the socio-economic environment, the decision maker’s insufficient knowledge and judgments, it is difficult for decision-makers to give accurate information about alternatives. For this, Atanassov [1] proposed the intuitionistic fuzzy set (IFS), IFS is an extension of fuzzy set (FS) [2], which is characterized by a membership degree and a nonmembership degree satisfying the condition that the sum of these two degrees is equal to or less than 1, therefore, IFS is a very useful tool in processing fuzziness and uncertainty information. To aggregate the intuitionistic fuzzy data and make a decision, many intuitionistic fuzzy aggregation operators have been suggested for aggregating the different alternatives [3–9]. However, in some particular conditions, the sum of the membership degree (MD) and the nonmembership degree (NMD) provided by the decision-maker may be greater than 1, for instances, if decision-maker gives the membership as 0.3, whereas the nonmembership degree maybe 0.9, it is easily seen that \(0.3 + 0.9 > 1\). Thus, this condition cannot be managed by IFS. To succeed it, Yager [10, 11] revealed the concept of Pythagorean fuzzy set (PFS) with the requirements on MD and NMD that their square sum is equal to or less than 1. In some cases, the PFS can solve plenty of problems that the IFS cannot. In other words, PFS is a generalization of IFS, therefore, PFS is more competent than IFS to reduce the imprecise and imperfect information in useful MADM problems. Since PFS features, numerous aggregation operators have been proposed for aggregating Pythagorean fuzzy information [10–18]. Further, Zhang [19] extended the concept of PFS to interval-valued Pythagorean fuzzy set (IVPFS). Peng and Yang [20] proposed interval-valued Pythagorean
fuzzy weighted average (IVPFWA) operator, interval-valued Pythagorean fuzzy weighted geometric (IVPFWG) operator, and interval-valued Pythagorean fuzzy point weighted averaging (IVPFPPWA) operator. Based on fundamental laws of IVPFSs [19], Liang et al. [21] and Garg [22] introduced another kind of IVPFWA operator and IVPFWG operator, respectively. Some continuous interval-valued Pythagorean fuzzy operators were developed by Wang and Li [23]. Garg [24] describe the aggregation operators based on the immediate probabilities in PFS environment.

With high difficulty of environment and the hurried development of hypothesis, a new concept was manifested by Yager [25, 26], the \( q \)-rung orthopair fuzzy sets (\( q \)-ROFSs), whose prominent feature is that the sum of the \( q \)th power of the MD and \( q \)th power of the NMD is equal or less than 1. IFS and PFS are all their special cases, so the \( q \)-ROFSs are more general. Under that environment, Liu and Wang [27] proposed the \( q \)-rung orthopair fuzzy weighted averaging operator and the \( q \)-rung orthopair fuzzy weighted geometric operator. Considering the interactive relationship among input \( q \)-ROFSs, Liu and Liu [28] introduced some \( q \)-rung orthopair fuzzy Bonferroni mean (BM) operators to aggregate \( q \)-ROFSs. Further, Liu and Wang [29] revealed the \( q \)-rung orthopair fuzzy Archimedean BM (\( q \)-ROFABM) operator and the \( q \)-rung orthopair fuzzy weighted Archimedean BM (\( q \)-ROFWABM) operator and explained their desirable properties. Yang and Pang [30] represented \( q \)-rung orthopair fuzzy partitioned Bonferroni mean operators and employed them to solve MADM problems. Liu et al. [31] and Wei et al. [32] introduced the \( q \)-rung orthopair fuzzy Heronion mean operators, respectively. Some \( q \)-rung orthopair fuzzy Maclaurin symmetric mean operators were explained by Wei et al. [33]. Liu et al. [34] developed \( q \)-rung orthopair fuzzy power Maclaurin symmetric mean operators. Peng et al. [35] proposed a \( q \)-rung orthopair fuzzy weighted exponential aggregation (\( q \)-ROFWEA) operator to process evaluation information for \( q \)-ROFSs. More recently, Joshi et al.[36] extended the concept of \( q \)-ROFSs to interval-valued \( q \)-rung orthopair fuzzy set (IV\( q \)-ROFSs), whose membership degrees and nonmembership degrees can be represented by the subset of the closed interval \([0, 1]\) instead of a crisp value, meanwhile, some aggregation operators were proposed and corresponding to desirable properties of these operators are also discussed. Wang et al. [37] presented some \( q \)-rung interval-valued orthopair fuzzy aggregation operators for IV\( q \)-ROFSs.

In the real world, plenty of problems are too complex to be used for general quantitative description. Usually, decision-makers employ linguistic terms to describe and evaluate decision information. 2-tuple linguistic representation models were introduced by Herrera and Martinez [38, 39], which is a useful tool for handling MADM problems with qualitative information. Following the pioneering work of Herrera and Martinez, many aggregation operators have been proposed. For instance, Herrera and Martinez [39] introduced the 2-tuple ordered weighted average (OWA) operator. Xu and Wang [40] explained the 2-tuple linguistic power aggregation operators. Wei [41] proposed extended 2-tuple weighted geometric operator and the extended 2-tuple ordered weighted geometric operator. 2-tuple linguistic hybrid arithmetic aggregation operators have been proposed by Wan [42]. Merigo and Gil-Lafuente [43] proposed the induced 2-tuple linguistic generalized aggregation operators to handle MADM problem. Based on Archimedean t-norm and s-norm, Tao et al. [44] presented the 2-tuple weighted averaging operator and the 2-tuple weighted geometric operator. Jiang and Wei [45] defined some Bonferroni mean operators for aggregating 2-tuple linguistic information. 2-tuple linguistic Muirhead mean operators were introduced by Qin and Liu [46]. Additionally, under Pythagorean fuzzy environment, Wei et al. [47] presented the Pythagorean 2-tuple linguistic aggregation operators based on the operational laws of Pythagorean 2-tuple linguistic number (P2TLN).

From above examination, although, IV\( q \)-ROFSs can be used to express a wider range of fuzzy information than IVPFSs and IFSs, it is not suitable to describe the interval membership degree and the interval degree of nonmembership an element to a 2-tuple linguistic label, which can reflect the decision maker’s confidence degree when they are making decision [47–53]. Furthermore, P2TLSs [47, 52] can be used to describe the membership degree and the nonmembership degree of an element to a 2-tuple linguistic label, however, the scope of applications of P2TLSs is narrow because it must satisfy the condition that the square sum of the membership degrees and the nonmembership degrees is no more than 1. Motivated by IV\( q \)-ROFSs and P2TLSs, we shall present the notions of interval-valued \( q \)-rung orthopair 2-tuple linguistic number (IV\( q \)-RO2TLN). The main feature of IV\( q \)-RO2TLN is that MD and NMD of an element to a 2-tuple linguistic label are described by IV\( q \)-ROFN. The IV\( q \)-RO2TLN can tackle some issues that P2TLN can’t, which is a successful generalization of P2TLN. So far, there have been no studies on IV\( q \)-RO2TLN. It requires more attention to the IV\( q \)-RO2TLN. Motivated by this view, the main contribution of this paper is as follows:

1) to develop the notion of interval-valued \( q \)-rung orthopair 2-tuple linguistic number (IV\( q \)-RO2TLN), and define some new operation laws and comparison rules of IV\( q \)-RO2TLNs which satisfies closeness, also introduces some comparison rules;
2) to present some interval-valued \( q \)-rung orthopair 2-tuple linguistic information aggregation operators, including the weighted average, ordered weighted average, hybrid average, weighted geometric, ordered weighted geometric and hybrid geometric operators, labeled as IV\( q \)-RO2TLWA, IV\( q \)-RO2TLWAWA, IV\( q \)-RO2TLHA, IV\( q \)-RO2TLWG, IV\( q \)-RO2TLWGW and IV\( q \)-RO2TLHG operator, also discuss some desirable properties and special cases of these operators;
3) to establish an efficient MADM method to solve the problems based on the proposed IV\( q \)-RO2TLWA and
IV\textsubscript{q}-RO2TLWG operators;
4) to illustrate the mentioned method with a numerical example while the effectiveness and advantages of it are explored by comparing the results with the several existing studies.

To do so, the rest of this paper is organized as follows: In Section 2, we briefly review some basic concepts of IV\textsubscript{q}-ROFSs and 2-tuple linguistic term sets. Meanwhile, we develop the basic concepts of the interval-valued q-rung orthopair 2-tuple linguistic sets (IV\textsubscript{q}-RO2TTLs) and the fundamental operational rules of IV\textsubscript{q}-RO2TLNs. In Section 3, we propose some interval-valued q-rung orthopair 2-tuple linguistic arithmetic and geometric aggregation operators, and investigate some desirable properties of these operators. In Section 4, we develop a method for solving the MADM problems using the IV\textsubscript{q}-RO2TLWA operator and the IV\textsubscript{q}-RO2TLWG operator. In Section 5, an illustrative example is provided to demonstrate the applicability and effectiveness of the proposed method. Finally, Section 6 concludes the paper.

II. PRELIMINARIES
A. INTERVAL-VALUED Q-RUNG ORTHOPAIR FUZZY SETS
In this subsection, we briefly review basic concepts of q-ROFS [25, 26] and IV\textsubscript{q}-ROFS [36]. Afterwards, new score function and accuracy function for IV\textsubscript{q}-ROFSs are developed. Furthermore, a novel comparison method for IV\textsubscript{q}-ROFSs is presented.

Definition 1: [25, 26] Let \( X \) be a fixed set. A q-ROFS \( A \) is defined as:
\[
A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}
\]
where the functions \( \mu_A(x), \nu_A(x) : X \to [0, 1] \) satisfy the condition: \( 0 \leq (\mu_A(x))^q + (\nu_A(x))^q \leq 1, \ (q \geq 1) \), \( \mu_A(x), \nu_A(x) \) denote, respectively, the degree of membership and the degree of non-membership of the element \( x \in X \) to the set \( A \). The degree of indeterminacy \( \pi_A(x) = (\sqrt[1-q]{1-(\mu_A(x))^q} - (\nu_A(x))^q) \).

For convenience, Liu and Wang [27] defined \( \alpha = (\mu_\alpha, \nu_\alpha) \) as the q-rung orthopair fuzzy number (q-ROFN), where \( \mu_\alpha \in [0, 1], \nu_\alpha \in [0, 1] \) and \( 0 \leq \mu_\alpha^q + \nu_\alpha^q \leq 1 \).

Definition 2: [36] Let \( X \) be a fixed set. An IV\textsubscript{q}-ROFS \( \tilde{A} \) can be defined as:
\[
\tilde{A} = \{(x, (\mu_A(x), \nu_A(x))) \mid x \in X\}
\]
where the functions \( \mu_A(x), \nu_A(x) : X \to [0, 1], \mu_A(x) = [\mu_A^L(x), \mu_A^U(x)], \nu_A(x) = [\nu_A^L(x), \nu_A^U(x)], \) satisfy \( (\mu_A^U(x))^q + (\nu_A^U(x))^q \leq 1, \ (q \geq 1) \). The degree of indeterminacy, \( \pi_A(x) = \left[\pi_A^L(x), \pi_A^U(x)\right] \), where \( \pi_A^L(x) = \sqrt[1-q]{1-(\mu_A(x))^q} - (\nu_A(x))^q \) and \( \pi_A^U(x) = \sqrt[1-q]{1-(\mu_A(x))^q} - (\nu_A(x))^q \).

If \( \mu_A^L(x) = \mu_A^U(x) \) and \( \nu_A^L(x) = \nu_A^U(x) \), then the IV\textsubscript{q}-ROFS reduces to the q-ROFS. For simplicity, we call \( \tilde{\alpha} = (\mu_\tilde{\alpha}, \nu_\tilde{\alpha}) = ([\mu_\alpha^L, \nu_\alpha^L], [\mu_\alpha^U, \nu_\alpha^U]) \) an interval-valued q-rung orthopair fuzzy number (IV\textsubscript{q}-ROFN) denoted by \( \tilde{\alpha} = ([a, b], [c, d]) \), where \( [a, b] \subseteq [0, 1], [c, d] \subseteq [0, 1] \), and \( 0 \leq b^q + d^q \leq 1 \).

Based on the operational rules of q-ROFNs [28, 29], we can define some basic operations on IV\textsubscript{q}-ROFNs as follows:

Definition 3: [25, 26] Let \( \tilde{\alpha} = ([a, b], [c, d]), \tilde{\alpha}_1 = ([a_1, b_1], [c_1, d_1]) \) and \( \tilde{\alpha}_2 = ([a_2, b_2], [c_2, d_2]) \) be three IV\textsubscript{q}-ROFNs, then:
1) \( \tilde{\alpha}_1 \oplus \tilde{\alpha}_2 = \left(\sqrt[1-q]{(a_1)^q + (a_2)^q} - (a_1)^q, (b_1)^q + (b_2)^q - (b_1)^q(b_2)^q, [c_1c_2, d_1d_2]\right) \);
2) \( \tilde{\alpha}_1 \odot \tilde{\alpha}_2 = \left(\sqrt[1-q]{(c_1)^q + (c_2)^q} - (c_1)^q(c_2)^q, \sqrt[1-q]{(d_1)^q + (d_2)^q} - (d_1)^q(d_2)^q, \right) \);
3) \( \lambda \tilde{\alpha}_1 = \left(\sqrt[1-q]{1-(a^q)}, \sqrt[1-q]{1-(b^q)}, [c^q, d^q]\right), \lambda > 0 \);
4) \( \tilde{\alpha}_1^\lambda = \left(\lambda^{a_1}, \lambda^{b_1}, \right) \) \( \left(\sqrt[1-q]{1-(c^q)}, \sqrt[1-q]{1-(d^q)}, \lambda > 0 \right) \);
5) \( \tilde{\alpha}_1^\lambda = ([c, d], [b, a]) \).

From Definition 3, we can obtain the following properties easily.

Theorem 1: Let \( \tilde{\alpha}_1 = ([a, b], [c, d]), \tilde{\alpha}_1 = ([a_1, b_1], [c_1, d_1]) \) and \( \tilde{\alpha}_2 = ([a_2, b_2], [c_2, d_2]) \) be three IV\textsubscript{q}-ROFNs, and \( \lambda > 0, \lambda_1 > 0, \lambda_2 > 0 \), then:
(1) \( \tilde{\alpha}_1 \odot \tilde{\alpha}_2 = \tilde{\alpha}_2 \odot \tilde{\alpha}_1 \);
(2) \( \tilde{\alpha}_1 \oplus \tilde{\alpha}_2 = \tilde{\alpha}_2 \oplus \tilde{\alpha}_1 \);
(3) \( \lambda \tilde{\alpha}_1 \odot \tilde{\alpha}_2 = \lambda \tilde{\alpha}_1 \odot \lambda \tilde{\alpha}_2 \);
(4) \( \tilde{\alpha}_1 \odot \tilde{\alpha}_2^\lambda = (\tilde{\alpha}_1^\lambda \odot \tilde{\alpha}_2) \);
(5) \( \lambda \tilde{\alpha}_1 \odot \tilde{\alpha}_2^\lambda = (\lambda \tilde{\alpha}_1 \odot \lambda \tilde{\alpha}_2) \);
(6) \( \tilde{\alpha}_1 \odot \tilde{\alpha}_2^\lambda = \tilde{\alpha}_1 \odot \tilde{\alpha}_2 \);
(7) \( \tilde{\alpha}_1 \odot \tilde{\alpha}_2^\lambda = \tilde{\alpha}_1 \odot \tilde{\alpha}_2 \).

We now propose a score function and an accuracy function for IV\textsubscript{q}-ROFN \( \tilde{\alpha} \).

Definition 4: Let \( \tilde{\alpha} = ([a, b], [c, d]) \) be an IV\textsubscript{q}-ROFN, then the score function of \( \tilde{\alpha} \) is defined as follows:
\[
s(\tilde{\alpha}) = (2 + a^q + b^q - c^q - d^q)/4, \quad s(\tilde{\alpha}) \in [0, 1]
\]
Definition 5: Let \( \tilde{\alpha} = ([a, b], [c, d]) \) be an IV\textsubscript{q}-ROFN, then the accuracy function of \( \tilde{\alpha} \) is defined as follows:
\[
h(\tilde{\alpha}) = (a^q + b^q + c^q + d^q)/2, \quad h(\tilde{\alpha}) \in [0, 1]
\]
Based on the score function \( s \) and the accuracy function \( h \), a comparison method for IV\textsubscript{q}-ROFNs is defined as follows:

Definition 6: Let \( \tilde{\alpha}_1, \tilde{\alpha}_2 \) be two IV\textsubscript{q}-ROFNs, then:
(1) If \( s(\tilde{\alpha}_1) > s(\tilde{\alpha}_2) \), then \( \tilde{\alpha}_1 > \tilde{\alpha}_2 \);
(2) If \( s(\tilde{\alpha}_1) = s(\tilde{\alpha}_2) \), then:
(a) If \( h(\tilde{\alpha}_1) > h(\tilde{\alpha}_2) \), then \( \tilde{\alpha}_1 > \tilde{\alpha}_2 \);
(b) If \( h(\tilde{\alpha}_1) = h(\tilde{\alpha}_2) \), then \( \tilde{\alpha}_1 = \tilde{\alpha}_2 \).

B. 2-TUPLE LINGUISTIC TERM SETS
Let \( S = \{s_0, s_1, \ldots, s_t\} \) be a linguistic term set with odd cardinality. The term \( s_i (i = 0, 2, \ldots, t) \) represents a possible
value for linguistic variable. For example, a set of seven terms $S$ can be defined as follows:

$$S = \{ s_0 = \text{"extremely poor"}, s_1 = \text{"very poor"}, s_2 = \text{"poor"}, s_3 = \text{"medium"}, s_4 = \text{"good"}, s_5 = \text{"very good"}, s_6 = \text{"extremely good"}\}.$$ Usually, the linguistic term set $S$ should satisfy the following characteristics:

1. The set is ordered: $s_i > s_j$, if $i > j$.
2. There is a negation operator: $\text{Neg}(s_i) = s_j$ such that $j = t - i$.
3. Max operator: $\max(s_i, s_j) = s_i$, if $s_i \geq s_j$.
4. Min operator: $\min(s_i, s_j) = s_i$, if $s_i \leq s_j$.

Based on the concept of symbolic translation, Herrera and Martinez [38, 39] initially presented a 2-tuple linguistic representation model for handling with linguistic information.

**Definition 7:** [38, 39]. Let $\beta$ be the result of an aggregation of the indices of a set of labels assessed in a linguistic term set $S$, i.e., the result of a symbolic aggregation operation, being $t + 1$ the cardinality of $S$. Let $i = \text{round}(\beta)$ and $\alpha = \beta - i$ be two values, such that, $i \in [0, t]$ and $\alpha \in [-0.5, 0.5)$, then $\alpha$ is called a symbolic translation.

**Definition 8:** [38, 39]. Let $S = \{ s_i | i = 0, 1, \ldots, t \}$ be a linguistic term set and $\beta \in [0, t]$, a value representing the result of symbolic aggregation operation, then the 2-tuple linguistic information that expresses the equivalent information to $\beta$ is defined as follows:

$$\Delta : [0, t] \rightarrow S \times [-0.5, 0.5)$$

$$\Delta(\beta) = \begin{cases} s_i, & i = \text{round}(\beta), \\ \alpha = \beta - i, & \alpha \in [-0.5, 0.5) \end{cases}$$

Where $\text{round}(\cdot)$ is the usual round operation, $s_i$ has the closest index label to $\beta$ and $\alpha$ is the value of the symbolic translation.

**Definition 9:** [38, 39]. Let $S = \{ s_i | i = 0, 1, \ldots, t \}$ be a linguistic term set and $(s_i, \alpha_i)$ be a linguistic 2-tuple. There is always a function $\Delta^{-1}$, such that it returns its equivalent numerical value $\beta \in [0, t] \subset \mathbb{R}$ from a 2-tuple $(s_i, \alpha_i)$, where

$$\Delta^{-1} : S \times [-0.5, 0.5) \rightarrow [0, t]$$

$$\Delta^{-1}(s_i, \alpha_i) = i + \alpha = \beta$$

From Definitions 7 and 8, a linguistic term can be converted into a linguistic 2-tuple by adding a value zero as symbolic translation

$$\Delta(s_i) = (s_i, 0)$$

**Definition 10:** [38, 39]. Let $(s_k, \alpha_k)$ and $(s_l, \alpha_l)$ be two linguistic 2-tuples, then

1. If $k < l$, then $(s_k, \alpha_k)$ is smaller than $(s_l, \alpha_l)$.
2. If $k = l$, then
   a. If $\alpha_k = \alpha_l$, then $(s_k, \alpha_k)$ and $(s_l, \alpha_l)$ represent the same information denoted by $(s_k, \alpha_k) = (s_l, \alpha_l)$.
   b. If $\alpha_k < \alpha_l$, then $(s_k, \alpha_k)$ is smaller than $(s_l, \alpha_l)$ denoted by $(s_k, \alpha_k) < (s_l, \alpha_l)$.

### III. INTERVAL-VALUED Q-RUNG ORTHOPAIR 2-TUPLE LINGUISTIC AGGREGATION OPERATORS

Based on the rules of the operation of Interval-valued q-rung orthopair fuzzy set and 2-tuple linguistic model. In what follows, we will introduce the concepts and basic operations of the interval-valued q-rung orthopair 2-tuple linguistic sets.

#### A. INTERVAL-VALUED Q-RUNG ORTHOPAIR 2-TUPLE LINGUISTIC SETS

**Definition 11:** An interval-valued q-rung orthopair 2-tuple linguistic sets (IVq-RO2TLS) $\tilde{A}$ is defined as:

$$\tilde{A} = \{ (s_{\theta(x)}, \rho), (\mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)) | x \in X \}$$

where $s_{\theta(x)} \in S, \rho \in [-0.5, 0.5)$, the functions $\mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x) \subseteq [0, 1]$ satisfy the condition $\mu_{\tilde{A}}(x) = [\mu^L_{\tilde{A}}(x), \mu^U_{\tilde{A}}(x)], \nu_{\tilde{A}}(x) = [\nu^L_{\tilde{A}}(x), \nu^U_{\tilde{A}}(x)]$, where $\mu^L_{\tilde{A}}(x) = \frac{1}{q} - (\mu^U_{\tilde{A}}(x))^q - (\nu^U_{\tilde{A}}(x))^q$ and $\mu^U_{\tilde{A}}(x) = \frac{1}{q} - (\mu^L_{\tilde{A}}(x))^q - (\nu^L_{\tilde{A}}(x))^q$.

From Equation (10), an IVq-RO2TLS $\tilde{A}$ can be expressed as $\tilde{A} = \{ (s_{\theta(x)}, \rho), (\mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)) | x \in X \}$. For convenience, we call $\tilde{A} = (\langle s_{\alpha(x)} , \rho \rangle, \langle [a, b], [c, d] \rangle)$ an interval-valued q-rung orthopair 2-tuple linguistic number (IVq-RO2TLS), where, $s_{\alpha(x)} \in S, \rho \in [-0.5, 0.5), [a, b], [c, d] \subseteq [0, 1]$ and $b^q + d^q \leq 1, \ (q \geq 1)$. Let $\Gamma$ be the set of all IVq-RO2TLSs.

**Definition 12:** Let $\tilde{\alpha} = (\langle s_{\alpha(x)} , \rho \rangle, \langle [a, b], [c, d] \rangle)$ be an IVq-RO2TLS, then the score function of $\tilde{\alpha}$ is given as:

$$S(\tilde{\alpha}) = \Delta \left( \left( \Delta^{-1}(s_{\alpha(x)}, \rho) \left( \frac{2 + a^q + b^q - c^q - d^q}{2} \right) \right) \right)$$

**Definition 13:** Let $\tilde{\alpha} = (\langle s_{\alpha(x)} , \rho \rangle, \langle [a, b], [c, d] \rangle)$ be an IVq-RO2TLS, then the accuracy function of $\tilde{\alpha}$ is defined as:

$$H(\tilde{\alpha}) = \Delta \left( \left( \Delta^{-1}(s_{\alpha(x)}, \rho) \left( \frac{\alpha^q + b^q + c^q + d^q}{2} \right) \right) \right)$$

Clearly $\Delta^{-1}(H(\tilde{\alpha})) \in [0, t]$. Based on the above score function $S$ and the accuracy function $H$, a comparison method of IVq-RO2TLSs is developed, which is defined as follows:

**Definition 14:** For two IVq-RO2TLSs $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, a comparison law between them, denoted by $\tilde{\alpha}_1 \geq \tilde{\alpha}_2$ holds, if either of the following inequality satisfy

1. $S(\tilde{\alpha}_1) > S(\tilde{\alpha}_2)$
2. $S(\tilde{\alpha}_1) = S(\tilde{\alpha}_2)$ and $H(\tilde{\alpha}_1) > H(\tilde{\alpha}_2)$

Motivated by the operations of the 2-tuple linguistic numbers [44] and Definition 3, we introduce some operational laws of IVq-RO2TLSs as follows.
Definition 15: Let \( \tilde{\alpha} = \{(s_\alpha, \rho), ([a, b], [c, d])\} \), \( \tilde{\alpha}_1 = \{(s_{\alpha_1}, \rho_1), ([a_1, b_1], [c_1, d_1])\} \) and \( \tilde{\alpha}_2 = \{(s_{\alpha_2}, \rho_2), ([a_2, b_2], [c_2, d_2])\} \) be three IV\(Q\)-RO2TLNs and \( S = \{s_0, s_1, \ldots, s_t\} \), \( \lambda \geq 0 \), then

\[
\begin{align*}
(1) & \quad \tilde{\alpha}_1 \oplus \tilde{\alpha}_2 = \begin{bmatrix}
\Delta \left( \frac{\Delta^{-1}(s_{\alpha_1}, \rho_1) + \Delta^{-1}(s_{\alpha_2}, \rho_2) - \Delta^{-1}(s_{\alpha_1}, \rho_1) \Delta^{-1}(s_{\alpha_2}, \rho_2)}{t} \right) & \left( \sqrt[\lambda]{(a_1)^q + (a_2)^p - (a_1)^q (a_2)^p}, \right. \\
\left. \sqrt[\lambda]{(b_1)^q + (b_2)^p - (b_1)^q (b_2)^p}, \right. \\
(c_1, d_1), (c_2, d_2) & \left. \right)
\end{bmatrix}, \\
(2) & \quad \tilde{\alpha}_1 \otimes \tilde{\alpha}_2 = \begin{bmatrix}
\Delta \left( \frac{\Delta^{-1}(s_{\alpha_1}, \rho_1) \Delta^{-1}(s_{\alpha_2}, \rho_2)}{t} \right) & \left( \sqrt[\lambda]{(c_1)^q + (c_2)^p - (c_1)^q (c_2)^p}, \\
\sqrt[\lambda]{(d_1)^q + (d_2)^p - (d_1)^q (d_2)^p}, \right. \\
(c_1, d_1), (c_2, d_2) & \left. \right)
\end{bmatrix}; \\
(3) & \quad \lambda \tilde{\alpha}_1 = \begin{bmatrix}
\Delta \left( t - \left( 1 - \frac{\Delta^{-1}(s_{\alpha}, \rho)}{t} \right)^\lambda \right) & \left( \sqrt[\lambda]{1 - (1 - a)^q t^{\lambda}}, \right. \\
\left. \sqrt[\lambda]{1 - (1 - b)^q t^{\lambda}}, \right. \\
1) & \left. \right)
\end{bmatrix}; \\
(4) & \quad \tilde{\alpha}_1^\lambda = \begin{bmatrix}
\Delta \left( \frac{\Delta^{-1}(s_{\alpha}, \rho) t^{1-\lambda}}{t} \right) & \left( \sqrt[\lambda]{1 - (1 - c)^q t^{\lambda}}, \right. \\
\left. \sqrt[\lambda]{1 - (1 - d)^q t^{\lambda}}, \right. \\
1) & \left. \right)
\end{bmatrix}
\end{align*}
\]

Remark 1: Based on the results of Ref. [44] and Definition 3, we can obtain the basic operation in Definition 15 are closed, that is \( \tilde{\alpha}_1 \oplus \tilde{\alpha}_2, \tilde{\alpha}_1 \otimes \tilde{\alpha}_2, \lambda \tilde{\alpha}_1, \tilde{\alpha}_1^\lambda \in \Gamma \).

Example 1: Let \( \tilde{\alpha}_1 = \{(s_{0.3}, 0.4), ([0.4, 0.6], [0.1, 0.4])\} \) and \( \tilde{\alpha}_2 = \{(s_{0.4}, 0.2), ([0.3, 0.5], [0.2, 0.4])\} \) be two IV\(Q\)-RO2TLNs, and \( S = \{s_0, s_1, \ldots, s_6\}, \lambda = 2, q = 3 \), we have

\[
\begin{align*}
(1) & \quad \tilde{\alpha}_1 \oplus \tilde{\alpha}_2 = \begin{bmatrix}
\Delta \left( \frac{\Delta^{-1}(s_{0.3}, 0.4) + \Delta^{-1}(s_{0.4}, 0.2) - \Delta^{-1}(s_{0.3}, 0.4) \Delta^{-1}(s_{0.4}, 0.2)}{6} \right) & \left( \sqrt[3]{0.4^3 + 0.3^3 - 0.4^3 0.3^3}, \\
\sqrt[3]{0.6^3 + 0.5^3 - 0.6^3 0.5^3}, \right. \\
0.1 \times 0.2, 0.4 \times 0.4) & \left. \right)
\end{bmatrix}, \\
(2) & \quad \tilde{\alpha}_1 \otimes \tilde{\alpha}_2 = \begin{bmatrix}
\Delta \left( \frac{\Delta^{-1}(s_{0.3}, 0.4) \Delta^{-1}(s_{0.4}, 0.2)}{6} \right) & \left( \sqrt[3]{0.4^3 + 0.3^3 - 0.4^3 0.3^3}, \\
\sqrt[3]{0.6^3 + 0.5^3 - 0.6^3 0.5^3}, \right. \\
0.1 \times 0.2, 0.4 \times 0.4) & \left. \right)
\end{bmatrix}.
\end{align*}
\]

3) \( \lambda_1 \tilde{\alpha}_1 + \lambda_2 \tilde{\alpha}_2 = (\lambda_1 + \lambda_2) \tilde{\alpha}_1 \).
4) \( \tilde{\alpha}_1 \otimes \tilde{\alpha}_2 = \tilde{\alpha}_2 \otimes \tilde{\alpha}_1 \).
5) \( \tilde{\alpha}_1 \lambda \tilde{\alpha}_2 = (\tilde{\alpha}_1)^\lambda \otimes (\tilde{\alpha}_2)^\lambda \).
6) \( \tilde{\alpha}_1^\lambda \otimes \tilde{\alpha}_2^\lambda = (\tilde{\alpha}_1^\lambda)^\lambda \otimes (\tilde{\alpha}_2^\lambda)^\lambda \).
7) \( (\tilde{\alpha}_1^\lambda)^\lambda = \tilde{\alpha}_1^{\lambda 1 2} \).

IV\(Q\)-RO2TLWA \(A(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \oplus \omega \tilde{\alpha}_j \) (13)

where \( \omega = (\omega_1, \omega_2, \cdots, \omega_n)^T \) is the weight vector of \( \tilde{\alpha}_j (j = 1, 2, \cdots, n) \), with \( \omega_j \in [0, 1] \) and \( \sum_{j=1}^n \omega_j = 1 \).

Based on the Definitions 15 and 16, we can obtain the following result:

Theorem 3: Let \( \tilde{\alpha}_j = \{(s_{\alpha_j}, \rho_j), ([a_j, b_j], [c_j, d_j])\} \), \( (j = 1, 2, \cdots, n) \) be a collection of IV\(Q\)-RO2TLNs, then the aggregated value with IV\(Q\)-RO2TLWA operator is an IV\(Q\)-RO2TLN, and

\[
IV\(Q\)-RO2TLWA \left( \Delta \left( t \left( 1 - \prod_{j=1}^n \left( 1 - \frac{\Delta^{-1}(s_{\alpha_j}, \rho_j)}{t} \right)^{\omega_j} \right) \right) \right) = \left( 1 - \prod_{j=1}^n \left( 1 - (a_j)^{\omega_j} \right), \left( 1 - \prod_{j=1}^n \left( 1 - (b_j)^{\omega_j} \right) \right) \right)
\]

where \( \omega = (\omega_1, \omega_2, \cdots, \omega_n)^T \) is the weight vector of \( \tilde{\alpha}_j (j = 1, 2, \cdots, n) \), with \( \omega_j \in [0, 1] \) and \( \sum_{j=1}^n \omega_j = 1 \).

Remark 2: If \( q = 1 \), then the IV\(Q\)-RO2TLWA operator reduces to the interval-valued intuitionistic fuzzy 2-tuple linguistic weighted average (IVIF2TLWA) operator:
The Proof of Theorem 3: We prove Equation (14) by using mathematical induction on $n$.

(1) When $n = 2$,

$$IV q - RO2TLWA(\tilde{\alpha}_1, \tilde{\alpha}_2) = \omega_1 \tilde{\alpha}_1 \oplus \omega_2 \tilde{\alpha}_2$$

From Definition 15, we can see that both $\omega_1 \tilde{\alpha}_1$ and $\omega_2 \tilde{\alpha}_2$ are IV$q$-RO2TLNs, and the value of $\omega_1 \tilde{\alpha}_1 \oplus \omega_2 \tilde{\alpha}_2$ is an IV$q$-RO2TLN. By the operational law (3) in Definition 15, we obtain

$$\begin{align*}
\omega_1 \tilde{\alpha}_1 &= \Delta \left( t \left( 1 - \frac{\Delta^{-1}(s_{n_1}, p_{n_1})}{t} \right)^{\omega_1} \right), \\
\omega_2 \tilde{\alpha}_2 &= \Delta \left( t \left( 1 - \frac{\Delta^{-1}(s_{n_2}, p_{n_2})}{t} \right)^{\omega_2} \right), \\
\left[ \sqrt{1 - (1 - (a_1)^{\omega_1})^{\omega_1}}, \sqrt{1 - (1 - (b_1)^{\omega_1})^{\omega_1}} \right], \\
&\left[ (c_1)^{\omega_1}, (d_1)^{\omega_1} \right]
\end{align*}$$

Then, based on the Definition 15, we have

$$\begin{align*}
IV q - RO2TLWA(\tilde{\alpha}_1, \tilde{\alpha}_2) &= \omega_1 \tilde{\alpha}_1 \oplus \omega_2 \tilde{\alpha}_2 = \Delta \left( t \left( 1 - \frac{\Delta^{-1}(s_{n_1}, p_{n_1})}{t} \right)^{\omega_j} \right), \\
&\left( \sqrt{1 - (1 - (a_1)^{\omega_1})^{\omega_2}}, \sqrt{1 - (1 - (b_1)^{\omega_1})^{\omega_2}} \right), \\
&\left[ (c_1)^{\omega_2}, (d_1)^{\omega_2} \right]
\end{align*}$$

(2) Assume that result is true for $n = k$, Equation (14) holds, i.e.,

$$\begin{align*}
IV q - RO2TLWA(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_k) &= \omega_1 \tilde{\alpha}_1 \oplus \omega_2 \tilde{\alpha}_2 \oplus \cdots \oplus \omega_k \tilde{\alpha}_k = \\
\Delta \left( t \left( 1 - \prod_{j=1}^{k} \left( 1 - \frac{\Delta^{-1}(s_{n_j}, p_{n_j})}{t} \right)^{\omega_j} \right) \right), \\
&\left( \sqrt{1 - \prod_{j=1}^{k} (1 - (a_j)^{\omega_j})^{\omega_j}}, \sqrt{1 - \prod_{j=1}^{k} (1 - (b_j)^{\omega_j})^{\omega_j}} \right), \\
&\left[ \prod_{j=1}^{k} (c_j)^{\omega_j}, \prod_{j=1}^{k} (d_j)^{\omega_j} \right]
\end{align*}$$

Then when $n = k + 1$, by laws (1) and (3) in Definition 15, we obtain

$$\begin{align*}
IV q - RO2TLWA(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_{k+1}) &= \omega_1 \tilde{\alpha}_1 \oplus \omega_2 \tilde{\alpha}_2 \oplus \cdots \oplus \omega_k \tilde{\alpha}_k \oplus \omega_{k+1} \tilde{\alpha}_{k+1} = \\
&\Delta \left( t \left( 1 - \prod_{j=1}^{k+1} \left( 1 - \frac{\Delta^{-1}(s_{n_j}, p_{n_j})}{t} \right)^{\omega_j} \right) \right), \\
&\left( \sqrt{1 - \prod_{j=1}^{k+1} (1 - (a_j)^{\omega_j})^{\omega_j}}, \sqrt{1 - \prod_{j=1}^{k+1} (1 - (b_j)^{\omega_j})^{\omega_j}} \right), \\
&\left[ \prod_{j=1}^{k+1} (c_j)^{\omega_j}, \prod_{j=1}^{k+1} (d_j)^{\omega_j} \right]
\end{align*}$$

That is, when $n = k + 1$, Equation (14) holds.

Hence, from (1) and (2), Equation (14) holds for all $n$. The proof is completed.
Similarly to the Theorem 3, we have the following theorem.

**Theorem 7:** Let 
\[
\tilde{\alpha}_j = \langle (s_{\alpha_j}, \rho_j), ([a_j, b_j], [c_j, d_j]) \rangle (j = 1, 2, \ldots, n)
\]
be a collection of IVq-RO2TLNs, then the aggregated value derived from IVq-RO2TLOWA operator is an IVq-RO2TLN, and

\[
IVq - RO2TLOWA(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \left\langle \Delta \left( t \left( 1 - \prod_{j=1}^{n} \left( 1 - \frac{\Delta^{-1}(s_{\alpha_j}, \rho_j)}{t} \right)^{\omega_j} \right) \right) \right\rangle
\]

where \( \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T \) is the weight vector of \( \tilde{\alpha}_j (j = 1, 2, \ldots, n) \), with \( \omega_j \in [0, 1] \) and \( \sum_{j=1}^{n} \omega_j = 1 \).

It is easy to prove that the IVq-RO2TLOWA operator has the following desirable properties.

**Theorem 8:** (Idempotency). Let \( \tilde{\alpha}_j (j = 1, 2, \ldots, n) \) be a collection of IVq-RO2TLNs. If \( \tilde{\alpha}_j = \tilde{\alpha} \) for all \( j \), then

\[
IVq - RO2TLOWA(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \tilde{\alpha}
\]

**Theorem 9:** (Boundedness). Let \( \tilde{\alpha}_j (j = 1, 2, \ldots, n) \) be a collection of IVq-RO2TLNs, then

\[
\tilde{\alpha}^- \leq IVq - RO2TLOWA(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) \leq \tilde{\alpha}^+
\]

where \( \tilde{\alpha}^- = \min \{ \tilde{\alpha}_j \} \), \( \tilde{\alpha}^+ = \max \{ \tilde{\alpha}_j \} \).

**Theorem 10:** (Monotonicity). Let \( \tilde{\alpha}_j (j = 1, 2, \ldots, n) \) and \( \tilde{\alpha}_j' (j = 1, 2, \ldots, n) \) be two collection of IVq-RO2TLNs. If \( \tilde{\alpha}_j \geq \tilde{\alpha}_j' \) for all \( j \), then

\[
IVq - RO2TLOWA(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) \geq IVq - RO2TLOWA(\tilde{\alpha}_1', \tilde{\alpha}_2', \ldots, \tilde{\alpha}_n')
\]

Further, we propose an IVq-RO2TLOWA operator as follow.

**Definition 17:** Let \( \tilde{\alpha}_j = \langle (s_{\alpha_j}, \rho_j), ([a_j, b_j], [c_j, d_j]) \rangle \) be a collection of IVq-RO2TLNs, an IVq-RO2TLOWA operator of dimension \( n \) is a mapping of \( IVq - RO2TLOWA: \mathbb{F}^n \rightarrow \Gamma \), such that

\[
IVq - RO2TLOWA(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \sum_{j=1}^{n} \omega_j \tilde{\alpha}_{\sigma(j)}
\]

where \( \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T \) is the weight vector of \( \tilde{\alpha}_j (j = 1, 2, \ldots, n) \), with \( \omega_j \in [0, 1] \) and \( \sum_{j=1}^{n} \omega_j = 1 \). \((\sigma(1), \sigma(2), \ldots, \sigma(n))\) is a permutation of \((1, 2, \ldots, n)\) such that \( \tilde{\alpha}_{\sigma(j-1)} \geq \tilde{\alpha}_{\sigma(j)} \) for all \( j = 1, 2, \ldots, n \).
Definition 18: Let $\tilde{\alpha}_j = \{(s_{\alpha_j}, \rho_j), ([a_j, b_j], [c_j, d_j])\} (j = 1, 2, \ldots, n)$ be a collection of IV$_q$-RO2TLNs, an IV$_q$-RO2TLHA operator of dimension $n$ is a mapping: IV$_q$ – RO2TLHA : $\Gamma^n \rightarrow \Gamma$, such that

$$\text{IV}_q - \text{RO2TLHA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \bigoplus_{j=1}^{n} \omega_j \tilde{\alpha}_{\sigma(j)}$$

(24)

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weight vector of $\tilde{\alpha}_j$ ($j = 1, 2, \cdots, n$), with $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$. $\sigma$ is a permutation of $(1, 2, \cdots, n)$ such that $\sigma_{\alpha(j-1)} \geq \sigma_{\alpha(j)}$ for all $j$. $\tilde{\alpha}_{\sigma(j)}$ is the $j$th largest value of the weighted IV$_q$-RO2TLNs $\{\omega_1 \alpha_1, \omega_2 \alpha_2, \cdots, \omega_n \alpha_n\}$, $n$ is the balancing coefficient, and $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weight vector of $\tilde{\alpha}_j$ ($j = 1, 2, \cdots, n$) with $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$. Special, if the weight vector $\omega = (1/n, 1/n, \cdots, 1/n)$, then the IV$_q$-RO2TLHA operator is reduced to the IV$_q$-RO2TLA operator. If the weight vector $\omega = (1/n, 1/n, \cdots, 1/n)$, then the IV$_q$-RO2TLHA operator is reduced to the IV$_q$-RO2TLOWA operator.

Theorem 12: Let $\tilde{\alpha}_j = \{(s_{\alpha_j}, \rho_j), ([a_j, b_j], [c_j, d_j])\} (j = 1, 2, \cdots, n)$ be a collection of IV$_q$-RO2TLNs, then

$$\text{IV}_q - \text{RO2TLHA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \left\langle \Delta \left( t \left( 1 - \prod_{j=1}^{n} \left( 1 - \Delta^{-1}(s_{\alpha(j)}, \rho_{\sigma(j)}) \right) \omega_j \right) \right) \right\rangle$$

(25)

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weight vector of $\tilde{\alpha}_j$ ($j = 1, 2, \cdots, n$), with $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

Proof. The proof is similar to Theorem 3, so it is omitted here.

From Equation (25), we know that the aggregated value by using IV$_q$-RO2TLHA operator is also an IV$_q$-RO2TLN.

C. INTERVAL-VALUED Q-RUNG ORTHOPAIR 2-TUPLE LINGUISTIC GEOMETRIC AGGREGATION OPERATORS

In this subsection, we will develop interval-valued $q$-rung orthopair 2-tuple linguistic geometric aggregation operators such as interval-valued $q$-rung orthopair 2-tuple linguistic weighted geometric (IV$_q$-RO2TLWG) operator, interval-valued $q$-rung orthopair 2-tuple linguistic ordered weighted geometric (IV$_q$-RO2TLWOG) operator, interval-valued $q$-rung orthopair 2-tuple linguistic hybrid geometric (IV$_q$-RO2TLHGW) operator.

Definition 19: Let $\tilde{\alpha}_j = \{(s_{\alpha_j}, \rho_j), ([a_j, b_j], [c_j, d_j])\} (j = 1, 2, \cdots, n)$ be a collection of IV$_q$-RO2TLNs, an interval-valued $q$-rung orthopair 2-tuple linguistic weighted geometric (IV$_q$-RO2TLWG) operator of dimension $n$ is a mapping

$$\text{IV}_q - \text{RO2TLWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \bigoplus_{j=1}^{n} \omega_j \tilde{\alpha}_{\sigma(j)}$$

(26)

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weight vector of $\tilde{\alpha}_j$ ($j = 1, 2, \cdots, n$), with $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

Based on the Definitions 15 and 19, we can obtain the following result:

Theorem 13: Let $\tilde{\alpha}_j = \{(s_{\alpha_j}, \rho_j), ([a_j, b_j], [c_j, d_j])\} (j = 1, 2, \cdots, n)$ be a collection of IV$_q$-RO2TLNs, then the aggregated value derived from IV$_q$-RO2TLWG operator is an IV$_q$-RO2TLN, and

$$\text{IV}_q - \text{RO2TLWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \left\langle \Delta \left( t \left( 1 - \prod_{j=1}^{n} \left( 1 - \Delta^{-1}(s_{\alpha_j}, \rho_j) \right) \omega_j \right) \right) \right\rangle$$

(27)

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weight vector of $\tilde{\alpha}_j$ ($j = 1, 2, \cdots, n$), with $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

Remark 3. If $q = 1$, then the IV$_q$-RO2TLWOG operator reduces to the interval-valued intuitionistic fuzzy 2-tuple linguistic weighted geometric (IVIF2TLWG) operator:

$$\text{IVIF2TLWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \left\langle \Delta \left( t \left( 1 - \prod_{j=1}^{n} \left( 1 - \Delta^{-1}(s_{\alpha_j}, \rho_j) \right) \omega_j \right) \right) \right\rangle$$

If $q = 2$, then the IV$_q$-RO2TLWOG operator reduces to the interval-valued Pythagorean fuzzy 2-tuple linguistic weighted geometric (IVPF2TLWG) operator:

$$\text{IVPF2TLWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \left\langle \Delta \left( t \left( 1 - \prod_{j=1}^{n} \left( 1 - \Delta^{-1}(s_{\alpha_j}, \rho_j) \right) \omega_j \right) \right) \right\rangle$$
The Proof of Theorem 13: We prove Equation (27) by using mathematical induction on \( n \).

(1) When \( n = 2 \),

\[
IVq - RO2TLWG(\tilde{\alpha}_1, \tilde{\alpha}_2) = (\tilde{\alpha}_1)^{\omega_1} \otimes (\tilde{\alpha}_2)^{\omega_2}
\]

From Definition 15, we can see that both \((\tilde{\alpha}_1)^{\omega_1}\) and \((\tilde{\alpha}_2)^{\omega_2}\) are IVq-RO2TLNs, and the value of \((\tilde{\alpha}_1)^{\omega_1} \otimes (\tilde{\alpha}_2)^{\omega_2}\) is an IVq-RO2TLN. By the operational law (4) in Definition 15, we obtain

\[
(\tilde{\alpha}_1)^{\omega_1} = \Delta \left( \left( \Delta^{-1}(s_{\alpha_1}, \rho_1) t^{1-\omega_1} \right), \left( (a_1)^{\omega_1}, (b_1)^{\omega_1} \right), \left[ \sqrt{1 - (1 - (c_1)^{\omega_1})(1 - (d_1)^{\omega_1})}, \sqrt{1 - (1 - (c_2)^{\omega_1})(1 - (d_2)^{\omega_1})} \right] \right)
\]

\[
(\tilde{\alpha}_2)^{\omega_2} = \Delta \left( \left( \Delta^{-1}(s_{\alpha_2}, \rho_2) t^{1-\omega_2} \right), \left( (a_2)^{\omega_2}, (b_2)^{\omega_2} \right), \left[ \sqrt{1 - (1 - (c_1)^{\omega_2})(1 - (d_1)^{\omega_2})}, \sqrt{1 - (1 - (c_2)^{\omega_2})(1 - (d_2)^{\omega_2})} \right] \right)
\]

Then, based on the Definition 15, we have

\[
IVq - RO2TLWG(\tilde{\alpha}_1, \tilde{\alpha}_2) = (\tilde{\alpha}_1)^{\omega_1} \otimes (\tilde{\alpha}_2)^{\omega_2} = \Delta \left( \left( \prod_{j=1}^{2} \left( \Delta^{-1}(s_{\alpha_j}, \rho_j) t^{1-\omega_j} \right)^{\omega_j} \right), \left( (a_1)^{\omega_1}, (b_1)^{\omega_1}, (a_2)^{\omega_2}, (b_2)^{\omega_2} \right), \left[ \sqrt{1 - (1 - (c_1)^{\omega_1})(1 - (c_2)^{\omega_2})}, \sqrt{1 - (1 - (d_1)^{\omega_1})(1 - (d_2)^{\omega_2})} \right] \right)
\]

(2) Assume that result is true for \( n = k \), Equation (27) holds, i.e.,

\[
IVq - RO2TLWG(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_k) = \tilde{\alpha}_1^{\omega_1} \otimes \tilde{\alpha}_2^{\omega_2} \otimes \cdots \otimes \tilde{\alpha}_k^{\omega_k} = \Delta \left( \left( \prod_{j=1}^{k} \left( \Delta^{-1}(s_{\alpha_j}, \rho_j) t^{1-\omega_j} \right)^{\omega_j} \right), \left( \prod_{j=1}^{k} (a_j)^{\omega_j}, \prod_{j=1}^{k} (b_j)^{\omega_j} \right), \left[ \sqrt{1 - \prod_{j=1}^{k} (1 - (c_j)^{\omega_j})}, \sqrt{1 - \prod_{j=1}^{k} (1 - (d_j)^{\omega_j})} \right] \right)
\]

Then when \( n = k + 1 \), by operational laws (2) and (4) in Definition 15, we obtain

\[
IVq - RO2TLWG(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_{k+1}) = \omega_1 \tilde{\alpha}_1^{\omega_1} \otimes \tilde{\alpha}_2^{\omega_2} \otimes \cdots \otimes \tilde{\alpha}_k^{\omega_k} \otimes \tilde{\alpha}_{k+1}^{\omega_{k+1}} = \Delta \left( \left( \prod_{j=1}^{k+1} \left( \Delta^{-1}(s_{\alpha_j}, \rho_j) t^{1-\omega_j} \right)^{\omega_j} \right), \left( \prod_{j=1}^{k+1} (a_j)^{\omega_j}, \prod_{j=1}^{k+1} (b_j)^{\omega_j} \right), \left[ \sqrt{1 - \prod_{j=1}^{k+1} (1 - (c_j)^{\omega_j})}, \sqrt{1 - \prod_{j=1}^{k+1} (1 - (d_j)^{\omega_j})} \right] \right)
\]

That is, when \( n = k + 1 \), Equation (27) holds.

Hence, from (1) and (2), Equation (27) holds for all \( n \). The proof is completed.

Theorem 16: (Monotonicity). Let \( \tilde{\alpha}_j(j = 1, 2, \cdots, n) \) and \( \tilde{\alpha}'_j(j = 1, 2, \cdots, n) \) be two collection of IVq-RO2TLNs. If \( \tilde{\alpha}_j \geq \tilde{\alpha}'_j \) for all \( j \), then

\[
IVq - RO2TLWG(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) \geq IVq - RO2TLWG(\tilde{\alpha}'_1, \tilde{\alpha}'_2, \cdots, \tilde{\alpha}'_n)
\]

Further, we propose an interval-valued \( q \)-rung orthopair 2-tuple linguistic ordered weighted geometric (IVq-RO2TLowG) operator as follow:

Definition 20: Let \( \tilde{\alpha}_j = ((s_{\alpha_j}, \rho_j), ([a_j, b_j], [c_j, d_j]))(j = 1, 2, \cdots, n) \) be a collection of IVq-RO2TLNs, an IVq-
RO2TLOWG operator of dimension $n$ is a mapping $IVq - RO2TLOWG : \Gamma^n \rightarrow \Gamma$, such that

$$IVq - RO2TLOWG(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \frac{1}{n} \sum_{j=1}^{n} (\tilde{\alpha}_{\sigma(j)})^{\omega_j}$$

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weight vector of $\tilde{\alpha}_j (j = 1, 2, \cdots, n)$, with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$. $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ is a permutation of $(1, 2, \cdots, n)$ such that $\tilde{\alpha}_{\sigma(j-1)} \geq \tilde{\alpha}_{\sigma(j)}$ for all $j = 1, 2, \cdots, n$.

Similarly to the Theorem 12, we have the following theorem.

Theorem 17: Let $\tilde{\alpha}_j = ((s_{\alpha_j}, \rho_j), ([a_j, b_j], [c_j, d_j])) (j = 1, 2, \cdots, n)$ be a collection of IVq-RO2TLNs, then the aggregated value derived from IVq-RO2TLOWG operator is an IVq-RO2TLN, and

$$IVq - RO2TLOWG(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \left\langle \Delta \left( t \prod_{j=1}^{n} \left( \frac{\Delta^{-1}(s_{\alpha_{\sigma(j)}}, \rho_{\sigma(j)})}{t} \right)^{\omega_j} \right) \right\rangle$$

$$= \left\langle \frac{\prod_{j=1}^{n} (a_{\sigma(j)})^{\omega_j}, \prod_{j=1}^{n} (b_{\sigma(j)})^{\omega_j}}{\prod_{j=1}^{n} (1 - (c_{\sigma(j)})^{q})^{\omega_j}, \prod_{j=1}^{n} (1 - (d_{\sigma(j)})^{q})^{\omega_j}} \right\rangle$$

(32)

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weight vector of $\tilde{\alpha}_j (j = 1, 2, \cdots, n)$, with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

It is easy to prove that the IVq-RO2TLOWG operator has the following desirable properties.

Theorem 18: (Idempotency). Let $\tilde{\alpha}_j (j = 1, 2, \cdots, n)$ be a collection of IVq-RO2TLNs. If $\tilde{\alpha}_j = \hat{\alpha}$ for all $j$, then

$$IVq - RO2TLOWG(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \hat{\alpha}$$

(33)

Theorem 19: (Boundedness). Let $\tilde{\alpha}_j (j = 1, 2, \cdots, n)$ be a collection of IVq-RO2TLNs, then

$$\hat{\alpha}^- \leq IVq - RO2TLOWG(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) \leq \hat{\alpha}^+$$

(34)

where $\hat{\alpha}^- = \min\{\tilde{\alpha}_j\}$, $\hat{\alpha}^+ = \max\{\tilde{\alpha}_j\}$.

Theorem 20: (Monotonicity). Let $\tilde{\alpha}_j (j = 1, 2, \cdots, n)$ and $\tilde{\alpha}_j' (j = 1, 2, \cdots, n)$ be two collection of IVq-RO2TLNs. If $\tilde{\alpha}_j \geq \tilde{\alpha}_j'$ for all $j$, then

$$IVq - RO2TLOWG(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) \geq IVq - RO2TLOWG(\tilde{\alpha}_1', \tilde{\alpha}_2', \cdots, \tilde{\alpha}_n')$$

(35)

Theorem 21: (Commutativity). Let $\tilde{\alpha}_j (j = 1, 2, \cdots, n)$ be a collection of and IVq-RO2TLNs, $\tilde{\beta}_j (j = 1, 2, \cdots, n)$ is any permutation of $\tilde{\alpha}_j (j = 1, 2, \cdots, n)$, then

$$IVq - RO2TLOWG(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = IVq - RO2TLOWG(\tilde{\beta}_1, \tilde{\beta}_2, \cdots, \tilde{\beta}_n)$$

(36)

Since IVq-RO2TLOWG operator weighs only the importance of IVq-RO2TLNs themselves, while the IVq-RO2TLOWG operator weighs only the ordered positions of the IVq-RO2TLNs instead of weighting the IVq-RO2TLNs themselves. In order to weigh both the given IVq-RO2TLNs and their ordered positions, in the following we introduce the interval-valued $q$-rung orthopair 2-tuple linguistic hybrid geometric (IVq-RO2TLLHG) operator.

Definition 21: Let $\tilde{\alpha}_j = ((s_{\alpha_j}, \rho_j), ([a_j, b_j], [c_j, d_j])) (j = 1, 2, \cdots, n)$ be a collection of IVq-RO2TLNs, an IVq-RO2TLLHG operator of dimension $n$ is a mapping $IVq - RO2TLLHG : \Gamma^n \rightarrow \Gamma$, such that

$$IVq - RO2TLLHG(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \frac{1}{n} \sum_{j=1}^{n} (\hat{\alpha}_{\sigma(j)})^{\omega_j}$$

(37)

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weight vector of $\tilde{\alpha}_j (j = 1, 2, \cdots, n)$, with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$. $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ is a permutation of $(1, 2, \cdots, n)$ such that $\hat{\alpha}_{\sigma(j-1)} \geq \hat{\alpha}_{\sigma(j)}$ for all $j$. $\hat{\alpha}_{\sigma(j)}$ is the jth largest value of the weighted IVq-RO2TLNs $\{(\tilde{\alpha}_1)^{\omega_1}, (\tilde{\alpha}_2)^{\omega_2}, \cdots, (\tilde{\alpha}_n)^{\omega_n}\}$, $n$ is the balancing coefficient, and $w = (w_1, w_2, \cdots, w_n)^T$ is the weight vector of $\tilde{\alpha}_j (j = 1, 2, \cdots, n)$ with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Special, if the weight vector $w = (1/n, 1/n, \cdots, 1/n)$, then the IVq-RO2TLLHG operator is reduced to the IVq-RO2TLOWG operator. If the weight vector $w = (1/n, 1/n, \cdots, 1/n)$, then the IVq-RO2TLLHG operator is reduced to the IVq-RO2TLOWG operator.

Theorem 22: Let $\tilde{\alpha}_j = ((s_{\alpha_j}, \rho_j), ([a_j, b_j], [c_j, d_j])) (j = 1, 2, \cdots, n)$ be a collection of IVq-RO2TLNs, then

$$IVq - RO2TLLHG(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \left\langle \Delta \left( t \prod_{j=1}^{n} \left( \frac{\Delta^{-1}(s_{\alpha_{\sigma(j)}}, \rho_{\sigma(j)})}{t} \right)^{\omega_j} \right) \right\rangle$$

$$= \left\langle \frac{\prod_{j=1}^{n} (\tilde{a}_{\sigma(j)})^{\omega_j}, \prod_{j=1}^{n} (\tilde{b}_{\sigma(j)})^{\omega_j}}{\prod_{j=1}^{n} (1 - (\tilde{c}_{\sigma(j)})^{q})^{\omega_j}, \prod_{j=1}^{n} (1 - (\tilde{d}_{\sigma(j)})^{q})^{\omega_j}} \right\rangle$$

(38)

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weight vector of $\tilde{\alpha}_j (j = 1, 2, \cdots, n)$, with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

From Equation (38), we know that the aggregated value by using IVq-RO2TLLHG operator is also an IVq-RO2TLN.

IV. LINGUISTIC MADM METHOD WITH INTERVAL-VALUED $Q$-RUNG ORTHOPAIR 2-TUPLE LINGUISTIC INFORMATION

In this section, the IVq-RO2TWA and IVq-RO2TLOWG operators are applied to MADM problems based on the interval-valued $q$-rung orthopair 2-tuple linguistic information. On the basis of the given linguistic term set $S = \{s_0, s_1, \cdots, s_t\}$,
let $A = \{A_1, A_2, \ldots, A_m\}$ be a discrete set of alternatives, and $C = \{G_1, G_2, \ldots, G_n\}$ be the set of attributes, and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of attribute $G_j (j = 1, 2, \ldots, n)$ with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

Assume that $\tilde{R} = (\tilde{r}_{ij})_{m \times n}$ be an interval-valued $q$-rung orthopair 2-tuple linguistic decision matrix, where $\tilde{r}_{ij} = (s_{r_{ij}}, \rho_{ij}), [(a_{ij}, b_{ij}), [c_{ij}, d_{ij}]]$ is attribute value given by decision maker as the alternative $A_i \in A$ with respect to the attribute $G_j \in C$, where $[a_{ij}, b_{ij}]$ and $[c_{ij}, d_{ij}]$ indicate the degree that the alternative $A_i$ satisfies the attribute $G_j$ and does not satisfy the attribute $G_j$ given by the decision maker respectively, such that $a_{ij} \in [0, 1], b_{ij} \in [0, 1], c_{ij} \in [0, 1], d_{ij} \in [0, 1], 0 \leq (b_{ij})^q + (d_{ij})^q \leq 1, q \geq 1, s_{r_{ij}} \in S, \rho_{ij} \in [-0.5, 0.5], i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$.

In what follows, with the aid of the IV$q$-RO2TLWA operator and the IV$q$-RO2TLWG operator, we develop a new MADM approach, and the detailed steps of the proposed approach are described as follows:

**Step 1.** Aggregate the collective information $\tilde{R} = (\tilde{r}_{ij})_{m \times n}$ into the aggregated one $r_i, i = 1, 2, \ldots, m$ of the alternative $A_i$ by using either IV$q$-RO2TLWA or IV$q$-RO2TLWG operator. For instance, by utilizing IV$q$-RO2TLWA operator, then value of $r_i = \langle(s_{r_i}, \rho_i), ([a_i, b_i], [c_i, d_i]) \rangle$ are obtained as

$$r_i = \text{IV$q$-RO2TLWA}(\tilde{r}_{i1}, \tilde{r}_{i2}, \ldots, \tilde{r}_{in}) = \bigoplus_{j=1}^{n} \omega_j \tilde{r}_{ij}$$

$$= \Delta \left( t \left( \prod_{j=1}^{n} \left( 1 - \frac{\Delta^{-1}(s_{r_{ij}}, \rho_{ij})}{t} \right) \omega_j \right) \right),$$

$$= \begin{pmatrix} \prod_{j=1}^{n} (1 - (a_{ij})^q)^{\omega_j}, & \prod_{j=1}^{n} (c_{ij})^{\omega_j} \\ \prod_{j=1}^{n} (1 - (b_{ij})^q)^{\omega_j}, & \prod_{j=1}^{n} (d_{ij})^{\omega_j} \end{pmatrix} \left( \frac{1}{t} \prod_{j=1}^{n} \Delta^{-1}(s_{r_{ij}}, \rho_{ij}) \omega_j \right).$$

(39)

On the other hand, by utilizing IV$q$-RO2TLWG operator, the value of $r_i$ can be found as

$$r_i = \text{IV$q$-RO2TLWG}(\tilde{r}_{i1}, \tilde{r}_{i2}, \ldots, \tilde{r}_{in}) = \bigotimes_{j=1}^{n} \tilde{r}_{ij}$$

$$= \Delta \left( t \left( \prod_{j=1}^{n} \frac{\Delta^{-1}(s_{r_{ij}}, \rho_{ij})}{t} \right) \omega_j \right),$$

$$= \begin{pmatrix} \prod_{j=1}^{n} (a_{ij})^{\omega_j}, & \prod_{j=1}^{n} (1 - (b_{ij})^q)^{\omega_j} \\ \prod_{j=1}^{n} (c_{ij})^{\omega_j}, & \prod_{j=1}^{n} (d_{ij})^{\omega_j} \end{pmatrix} \left( \frac{1}{t} \prod_{j=1}^{n} \Delta^{-1}(s_{r_{ij}}, \rho_{ij}) \omega_j \right).$$

(40)

**Step 2:** Calculate the score values of the aggregated IV$q$-RO2TLNAs $r_i$ as

$$S(r_i) = \Delta \left( \frac{1}{4} \Delta^{-1}(s_i, \rho_i) \left( 2 + a_i^q + b_i^q - c_i^q - d_i^q \right) \right).$$

(41)

If there is no difference between two scores for two indices $r_i$ and $r_j$, then compute the accuracy function values for such indices as

$$H(r_i) = \Delta \left( \frac{1}{4} \Delta^{-1}(s_i, \rho_i) \left( a_i^q + b_i^q + c_i^q + d_i^q \right) \right).$$

(42)

**Step 3:** Rank all the alternatives $A_i (i = 1, 2, \ldots, m)$ according to score values $S(r_i)$, and hence select the best alternative(s).

**Step 4:** End.

V. ILLUSTRATIVE EXAMPLE

The above mentioned approach has been demonstrated with a numerical example as below.

A. A CASE STUDY

With the continuous progress of Chinese society, the rapid development of national economy and science and technology has been constantly updated, the company has achieved leap-forward development. Companies continue to grow larger and stronger, to the expansion of many corporate groups. The development of the investment company has gained new opportunities and challenges. Through expanding its business and exploring new fields, investment company can obtain more production funds, which become an important role in promoting their development. How to invest is related to the survival and development of an investment company. This selection process can be regarded as a multiple attribute decision-making problem. There are many factors affect the investment decision making, the focus of the research is on the following focus aspects:

1) The capability of win profit: As stakeholders of enterprises, investment company very concern about the profitability of enterprises. The capability of win profit refers to the enterprise’s ability to make profits, also known as the enterprise’s capital or capital appreciation capacity. How to judge whether a company is “making money” principal depends on its profitability. Therefore, the profitability of an enterprise company is shown by the amount and level of its earnings in a certain period.

2) Competitive power on the market: Competitive power is that under the environment of market competition and with aid of effective utilization of enterprise resources, enterprises can better and faster meet consumers’ needs than their competitors in product design, production, sales and other aspects of business activities, as well as in product price, quality and service, so as to obtain more profits to enterprises and further elevate the competitive power enterprises.
other words, during the process of market competition, through their optimization and interaction with the external environment, enterprises have a comparative superiority in the finite allocation of market resources, hence the ability of sustainable development in a virtuous circle.

3) **The management ability:** The level of enterprise manager’s ability will have a different effect on alleviating the financing constraints faced by companies. The stronger the manager’s ability, the stronger the manager’s ability to grasp the information related to the company’s internal and external investment, and the lower the sensitivity of the company’s investment cash flow are. The company makes investment decisions in light of this can valid avoid or decrease the company into financing constraints.

4) **The organizational culture:** The organizational culture refers to the basic belief, behavior standard and value standard formed in the longtime survival and development of an organization and followed by most of the members of the organization. It can guide employees to work hard for the enterprise without swerving and accelerate the formation of the sense of mission, pride, and perception of affiliation of employees so that employees form a strong cohesive force to the enterprise instinctively.

Investment company makes a comprehensive assessment on the alternative by evaluating the four aspects.

Suppose an investment company wants to invest a sum of money (revised from Ref. [45]). After the careful analysis of the market, the investment company considers the five money (revised from Ref. [45]). After the careful analysis of the alternative by evaluating the four aspects.

TABLE 1: Input preferences of the given alternatives in terms of IVq-RO2TLNs

<table>
<thead>
<tr>
<th>Alternative</th>
<th>IVq-RO2TLWG operators are</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1 = (s2, 0), (0.1, 0.2), (0.4, 0.5)</td>
<td>(s5, 0), (0.2, 0.4), (0.5, 0.6)</td>
</tr>
<tr>
<td>A2 = (s3, 0), (0.4, 0.5), (0.2, 0.3)</td>
<td>(s2, 0), (0.3, 0.4), (0.2, 0.3)</td>
</tr>
<tr>
<td>A3 = (s4, 0), (0.4, 0.5), (0.3, 0.5)</td>
<td>(s3, 0), (0.2, 0.4), (0.5, 0.3)</td>
</tr>
<tr>
<td>A4 = (s2, 0), (0.2, 0.4), (0.5, 0.6)</td>
<td>(s1, 0), (0.1, 0.3), (0.3, 0.5)</td>
</tr>
<tr>
<td>A5 = (s2, 0), (0.1, 0.4), (0.4, 0.5)</td>
<td>(s2, 0), (0.4, 0.6), (0.1, 0.3)</td>
</tr>
</tbody>
</table>

obtained as

r1 = (s3, −0.3186), (0.1627, 0.3043), (0.4472, 0.5477),
ri = (s4, 0.0104), (0.3930, 0.5312), (0.1712, 0.2781),
ri = (s3, −0.0157), (0.2766, 0.3860), (0.4134, 0.5741),
ri = (s4, 0.1869), (0.3527, 0.4637), (0.1552, 0.3393),
and
ri = (s4, 0.3492), (0.3226, 0.5118), (0.2107, 0.4134).

On the other hand, if we utilize IVq-RO2TLWG operator to aggregate the information then we get the collective values as

r1 = (s2, 0.0939), (0.1320, 0.2847), (0.4586, 0.5576),
ri = (s4, 0.4661), (0.3887, 0.5165), (0.2332, 0.3224),
ri = (s3, 0.1863), (0.1741, 0.2973), (0.5210, 0.6247),
ri = (s4, 0.3007), (0.2297, 0.4443), (0.2398, 0.4125),
and
ri = (s3, 0.1783), (0.2144, 0.4704), (0.3179, 0.4766).

**Step 2.** The score values of these aggregated numbers are S(r1) = (s1, 0.2598), S(r2) = (s2, 0.1021), S(r3) = (s1, 0.4102), S(r4) = (s2, 0.1435) and S(r5) = (s2, −0.1306). On the other hand, the score values of the alternatives corresponding to the numbers obtained through IVq-RO2TLWG operators are S(r1) = (s1, −0.0232), S(r2) = (s2, −0.1622), S(r3) = (s1, 0.2541), S(r4) = (s2, −0.1415) and S(r5) = (s1, 0.4003).

**Step 3.** Based on these values, the ordering of the alternatives are obtained as A4 ≻ A2 ≻ A5 ≻ A3 ≻ A1 by both the operators. Hence, we conclude that A4 is the best alternative.

**B. Sensitivity analysis**

To analyze the flexibility and sensitivity of the parameter q, we set the different value q to sort the above MADM example. The ranking results are shown in Tables 2 and 3. From Table 2, we easily find that the aggregation results are different, S(A1), S(A4) increasing and S(A2), S(A4) decreasing with parameter q increasing in the IVq-RO2TLWG operator and the final ordering of the alternative...
changes. The best alternative varies from \( A_2 \) to \( A_4 \) according to values of \( q \). On the other hand, from Table 3, we can observe that the final optimal values of the alternatives are changes with the change of the parameter \( q \). For example, with the increase of the parameter \( q \), the score value of the alternatives \( A_1, A_3 \) and \( A_5 \) are increasing while for others decreasing. Further, it has been seen that when \( q \) changes from 1 to 4, the best alternative is \( A_2 \) while for some bigger values of \( q \) such as 4, 7, 11, 16, 22, etc., the optimal alternative change to \( A_4 \). Thus, based on the optimal decision-maker choice, the person can select suitable operators as well as the desired values of \( q \) to handle the more uncertainties during the analysis. However, the influence of the parameter \( q \) can be selected from the fact that it should be the smallest integer which satisfies the inequality \( 0 \leq b^q + d^q \leq 1 \). For example, if the evaluating information is \( ((s_2, 0.2), ([0.3, 0.7], [0.5, 0.8])) \), then clearly it is seen from them that \( 0.7^2 + 0.8^2 > 1 \) and \( 0.7^2 + 0.8^3 < 1 \), so the selected values of the parameter \( q \) is 3, i.e., the smallest integer \( q \) is 3. Further, to demonstrate the feasibility of the proposed approach with some of the existing approaches, an analysis has been done which are described in the next section.

**C. COMPARATIVE ANALYSIS AND DISCUSSION**

In the following, we will demonstrate the effectiveness and advantages of proposed operators by comparing with the existing methods [47–53]. Since Pythagorean 2-tuple information is a special case of the \( q \)-RO2TLNs with \( q = 2 \) and hence we compare our approach with the Pythagorean 2-tuple weighted averaging operator \( (P2TLWA) \) and the Pythagorean 2-tuple weighted geometric operator \( (P2TLWG) \) operator proposed by [47]. As the considered data is given in the interval form, so we first convert into a crisp form by taking the averaging of the interval membership degrees and hence the IVq-RO2TLNs information are converted into q-RO2TLNs, given in Table 4.

**TABLE 4: The q-rung orthopair 2-tuple linguistic decision matrix**

<table>
<thead>
<tr>
<th>( A_1 )</th>
<th>( q_1 )</th>
<th>( q_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( ((s_2, 0), (0.15, 0.45)) )</td>
<td>( ((s_5, 0), (0.30, 0.56)) )</td>
<td></td>
</tr>
<tr>
<td>( ((s_3, 0), (0.45, 0.25)) )</td>
<td>( ((s_2, 0), (0.35, 0.25)) )</td>
<td></td>
</tr>
<tr>
<td>( ((s_4, 0), (0.45, 0.40)) )</td>
<td>( ((s_2, 0), (0.30, 0.20)) )</td>
<td></td>
</tr>
<tr>
<td>( ((s_4, 0), (0.45, 0.15)) )</td>
<td>( ((s_3, 0), (0.20, 0.40)) )</td>
<td></td>
</tr>
<tr>
<td>( ((s_4, 0), (0.30, 0.45)) )</td>
<td>( ((s_1, 0), (0.30, 0.35)) )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( A_1 )</th>
<th>( q_3 )</th>
<th>( q_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( ((s_1, 0), (0.25, 0.45)) )</td>
<td>( ((s_3, 0), (0.20, 0.55)) )</td>
<td></td>
</tr>
<tr>
<td>( ((s_2, 0), (0.50, 0.35)) )</td>
<td>( ((s_5, 0), (0.45, 0.15)) )</td>
<td></td>
</tr>
<tr>
<td>( ((s_2, 0), (0.25, 0.55)) )</td>
<td>( ((s_3, 0), (0.15, 0.65)) )</td>
<td></td>
</tr>
<tr>
<td>( ((s_2, 0), (0.25, 0.40)) )</td>
<td>( ((s_2, 0), (0.45, 0.20)) )</td>
<td></td>
</tr>
<tr>
<td>( ((s_2, 0), (0.25, 0.45)) )</td>
<td>( ((s_2, 0), (0.50, 0.20)) )</td>
<td></td>
</tr>
</tbody>
</table>

On this collective information, we have implemented the P2TLWA and P2TLWG operator along with the proposed method to rank the numbers. The final aggregated numbers, as well as the ranking order corresponding to them, are summarized in Table 5. From this table, it is concluded that the ranking order of the given alternative is \( A_4 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \) and get \( A_4 \) is the best alternative. Since the obtained ordering of the alternative by the proposed method and the existing method [47] coincides which explains that the developed approach is valid and reasonable. However, the methods presented in this paper describe more widely fuzzy information, and they can permit the sum and the square sum of membership degree and non-membership degree exceed one, respectively. Therefore, their scope of application is wider than the P2TLWA [47] and P2TLWG [47] operators.

**TABLE 5: Comparative study with existing methods**

<table>
<thead>
<tr>
<th>P2TLWA [47]</th>
<th>P2TLWG [47]</th>
<th>IVq-RO2TLWA ( q = 2 )</th>
<th>IVq-RO2TLWG ( q = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 )</td>
<td>( (s_1, 0.2000) )</td>
<td>( (s_2, 0.2042) )</td>
<td>( (s_2, 0.2049) )</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>( (s_2, 0.2770) )</td>
<td>( (s_2, 0.3922) )</td>
<td>( (s_2, 0.3244) )</td>
</tr>
<tr>
<td>( A_3 )</td>
<td>( (s_2, -0.3728) )</td>
<td>( (s_2, 0.4373) )</td>
<td>( (s_2, 0.2452) )</td>
</tr>
<tr>
<td>( A_4 )</td>
<td>( (s_2, -0.3415) )</td>
<td>( (s_2, 0.4560) )</td>
<td>( (s_2, 0.2716) )</td>
</tr>
<tr>
<td>( A_5 )</td>
<td>( (s_2, 0.2040) )</td>
<td>( (s_2, -0.0584) )</td>
<td>( (s_2, -0.0842) )</td>
</tr>
</tbody>
</table>

However, apart from that the basic differences between the presented method over the existing methods [47–53] are demonstrated as below.

1) The basic operation of IVq-RO2TLNs proposed in this paper is closed, but the operational laws of P2LTSs [47, 52] is not closed in some situations. Moreover, the methods presented in this paper describe more widely fuzzy information, and they can permit the sum and the square sum of membership degree and non-membership degree exceed one, respectively. Therefore, their scope of application is wider than Pythagorean 2-tuple linguistic (P2TL) aggregation operators [47]. Although, Picture 2-tuple linguistic (P2TL) aggregation operators [48], Bipolar 2-tuple linguistic (B2TL) aggregation operators [49], hesitant 2-tuple linguistic (H2TL) aggregation operators [50], single-valued neutrosophic 2-tuple linguistic Muirhead mean (SVN2TLMM) operators [51], generalized Pythagorean 2-tuple linguistic Bonferroni mean (GP2TLBM) operators [52] and hesitant picture 2-tuple linguistic (HP2TL) aggregation operators [53] can express the membership degree and the degree of non-membership an element to a linguistic label, they are failure to process IVq-RO2TLNs.

2) When \( q = 1 \), the IVq-RO2TLWA operator and the IVq-RO2TLWG operator in our methods reduce to the 1VF2TLWA operator and the 1VF2TLWG operator. When \( q = 2 \), the IVq-RO2TLWA operator and the IVq-RO2TLWG operator in our methods reduce to the 1VPF2TLWA operator and the 1VPF2TLWG operator. When \( q \geq 3 \), our proposed operators can aggregate IVq-RO2TLNs. Additionally, according to the decision maker’s preference, he/she can select different values of \( q \).

Therefore, our presented methods are more general, they can more effectively deal with fuzzy decision-making information.

**D. CHARACTERISTICS COMPARISON**

To further explain the superiorities of the novel method under the fuzzy information, we present the characteristics of the
proposed operators concerning the existing operators [47–53]. Results of them can be found in Table 6.

In this table, the symbol “✓” represents that the desired operator satisfy the corresponding properties while the symbol “×” does not satisfy. It is evident from this table that the proposed operator has taken the confidence degree of the decision-makers in the analysis. Further, the proposed operators have considered a flexible parameter which helps the decision-makers to choose their optimal alternatives according to their desired goals. However, on the other hand, the existing approaches [47–51, 53] does not involve any such parameters and hence their corresponding approaches are limited to one variable. Finally, the presented approach has considered the IVq-ROFNs to describe the information related to the evaluation of the alternatives. Hence, based on the above analysis and its characteristic comparisons, we can conclude that the proposed approach based on the aggregation operators is more general than the existing approaches [47–53] under the different environment to solve the MADM problems. Thus, there is a wider scope to handle the fuzzy information by using a parameter q than IFS and PFS information. Therefore, the novel proposed method is more flexible and suitable to solve the decision-making problems and hence the proposed method outperforms over the existing methods.

VI. CONCLUSIONS

In this paper, based on the IVq-ROFNs and 2-tuple linguistic representation model, we have studied interval-valued q-rung orthopair 2-tuple linguistic information aggregation operators. To aggregate the IVq-ROFNs, IVq-ROF2TLWA operator, IVq-ROF2TLLOG operator, IVq-ROF2TLHA operator, IVq-ROF2TLWG operator, IVq-ROF2TLNW operator and IVq-ROF2TLHG operator have been proposed. Later, the desirable characteristics corresponding to each operator has been investigated. Additionally, a new method for MADM problems with IVq-ROFNs is also developed based on the proposed operators and an illustrative example is provided to show the feasibility and advantages of the new method. Moreover, we have analyzed the influence of parameter values q on the ranking result. From comparison analysis, it has been concluded that since IVq-ROFNs have parameter values q, so they can describe decision information more flexible and can express the range of the space larger. In the future study, we will further extend the proposed aggregating operators to some other uncertain environment [54–58].

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