Distributed Finite-Time Rotating Encirclement Control of Multiagent Systems with Nonconvex Input Constraints

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This work was supported by the National Key Research and Development Plan (2017YFB0503300), the Beijing Educational Committee Foundation (No. KM201910011007, PXM2019_014213_000007) and the Beijing Natural Science Foundation (No. Z180005).

ABSTRACT This paper is devoted to the distributed finite-time rotating encirclement control problem of multiagent systems with nonconvex input constraints. Three distributed finite-time estimators are designed to estimate the targets’ geometric center, the reference polar radius and the reference polar angular. Then, the estimated values are employed to get the reference trajectory for each agent, which guarantees the references can uniformly distribute on a circle and have same angular velocity. The distributed finite-time control laws are proposed for all agents to drive their positions tracking reference trajectories by introducing a constraint operator to cope with the nonconvex input constraints, which guarantees the control inputs can invariably lie in the corresponding nonconvex constraint sets. A numerical simulation is performed to demonstrate the correctness and effectiveness of the proposed control scheme.

INDEX TERMS Encirclement control, nonconvex input constraints, finite-time tracking, distributed estimator, multiagent systems

I. INTRODUCTION

In military and civilian, when a group of autonomous vehicles (including robots, unmanned ground vehicles, unmanned aerial vehicles, and so on) are dispatched to protect/rescue our targets or detect/attack enemy targets, these vehicles are often required to provide a coverage or encircle the targets. For a single vehicle, many commendable control methods have been proposed such as robust control, adaptive control, sliding mode control, and so forth [1]–[6]. In [3], a new result about robust control by considering decoupling performance for mobile robots with noise and disturbance was first obtained. Adaptive control methods were presented in [4], [5] for collision avoidance and vehicle active suspension system. And an adaptive scheme that could be used to tolerate parameter variations and multiple-output-delay in the plants was first proposed in [6].

With the rapid development of computation and communication capabilities, it is possible to coordinate a number of autonomous vehicles to perform various challenging tasks, and a great deal of favorable results have been achieved in [7]–[12]. Particularly, the encirclement control problem of multiple vehicles has attracted tremendous attention with lots of relative results [13]–[21]. In [15], Zhang et al. presented a distributed encirclement control method and achieved the circular formation for multiagent systems with multiple dynamic targets, where a target can be tracked by only one agent. Mo et al. proposed a finite-time distributed rotating encirclement control scheme for multiagent systems in [16], where the agents corresponds to the targets one by one. For the leader-follower case, a finite-time rotating target-encirclement control was proposed in [20], where the target’s information can be only obtained by the leader. And [21] has extended the result to fractional-order multi-agent systems.

Although these studies greatly boost the development of the encirclement control, there still exist some gaps. Firstly, most of aforementioned control methods are not distributed. Secondly, the topological structure between agents and targets of these results are relatively special since the targets’
information can be obtained by only one agent or there is only one target.

Besides, most reported results for multiagent systems assume that the states and inputs of all agent are constrained free or in convex sets [22]–[24]. However, in practical systems, the control input or states of each agent might be constrained in the nonconvex set, such as quadrotors [25]. To cope with nonconvex constraints, a constraint operator is introduced in [25] and lots of results with respect to nonconvex constraints are proposed subsequently [26]–[29]. Though these articles studied the nonconvex constraints problem for multiagent systems, the encirclement control scheme are distributed, while relative information can be obtained by only one agent or there is only one target. Moreover, each target can be tracked by at least one agent, rather than only one.

1) For the first time, the nonconvex input constraints of the encirclement control problem is considered.

2) In our work, estimators of the proposed rotating encirclement control scheme are distributed, while relative results in [15]–[21] are centralized.

3) Compared with [13]–[21], the general situation of multiagent systems is considered, where a group of targets should be encircled by another group of agents. Moreover, each target can be tracked by at least one agent, rather than only one.

II. PRELIMINARIES AND PROBLEM DESCRIPTION

A. GRAPH THEORY

Let $\mathcal{G}(\mathcal{V}, \mathcal{E})$ be an undirected graph consisting of a vertex set $\mathcal{V} = \{1, \cdots, n\}$ and an edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. An edge $(i, j)$ in $\mathcal{G}$ represents the $i$-th agent and the $j$-th agent can share information with each other. The neighbor set of vertex $i$ is defined as $\mathcal{N}_i = \{ j \in \mathcal{V} : (i, j) \in \mathcal{E} \}$. The weighted adjacency matrix is denoted by $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{n \times n}$ satisfying $a_{ij} = a_{ji} > 0$ if $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. Accordingly, the Laplacian matrix of the undirected weighted graph is defined as $\mathcal{L}(\mathcal{A}) = [l_{ij}]$, where $l_{ii} = \sum_{j=1}^{n} a_{ij}$ and $l_{ij} = -a_{ij}$ for all $i \neq j$, which is symmetric. A path connected $i$ and $j$ is a sequence of edges as $(v_1, v_2), (v_2, v_3), \cdots, (v_{s-1}, v_s)$, where $v_1 \in \mathcal{V}, i = 1, \cdots, s$ and $v_1 = i, v_s = j$. The graph $\mathcal{G}$ is connected if there exists at least a path from any vertex to any other one. If the graph $\mathcal{G}$ is connected, the distance between vertices $i$ and $j$ is denoted by $d(i, j)$, which represents the shortest path from $i$ to $j$, for example, $d(i, j) = 1$ if $(i, j) \in \mathcal{E}$.

In this paper, the multiagent system under consideration is composed of $n$ agents (Index set $\mathcal{V} = \{1, \cdots, n\}$) and $m$ targets (Index set $\mathcal{T} = \{n+1, \cdots, n+m\}$), and every agent can be considered as a vertex of an undirected graph $\mathcal{G}$. Furthermore, $\mathcal{N}_i^T \subseteq \mathcal{T}$ denotes the set of targets whose position information can be obtained by the $i$-th agent, $\mathcal{N}_T^T \subseteq \mathcal{V}$ represents the set of agents which can obtain the position information of the $j$-th target, and $|\mathcal{N}_T^T|$ denotes the number of elements in the set $\mathcal{N}_T^T$.

\textbf{Notations:} $\mathbb{R}^{n \times m}$ and $\mathbb{R}^n$ denote the sets of $n \times m$ real matrices and $n$-dimensional real vectors, respectively. $\mathbb{R}_+$ represents the set of positive real numbers. $\mathbf{1}_n = [1, 1, \cdots, 1]^T \in \mathbb{R}^n$. For a real vector $x \in \mathbb{R}^n$, $\|x\|$ denotes the Euclidean norm, and $\frac{1}{\|x\|} = 0$ if $x = 0$.

\textbf{Assumption 1:} The graph corresponding to agents is connected and $|\mathcal{N}_T^T| \geq 1, \forall j \in \mathcal{T}$, i.e., each target can be tracked by at least one agent.

\textbf{Lemma 1} [9]: For a connected undirected graph $\mathcal{G}$, the second smallest eigenvalue of Laplacian matrix $\mathcal{L}(\mathcal{A})$ is called the algebraic connectivity and denoted by $\lambda_2(\mathcal{L}(\mathcal{A}))$, which is larger than zero and satisfies

\begin{equation}
\label{eq1}
x^T \mathcal{L}(\mathcal{A}) x \geq \lambda_2(\mathcal{L}(\mathcal{A})) x^T x, \forall x \neq 0
\end{equation}

\textbf{B. PROBLEM DESCRIPTION}

Consider the multiagent system consisting of $n$ agents and $m$ targets. The dynamic equation of the $i$-th agent is given by

\begin{equation}
\label{eq2}
\dot{x}_i(t) = u_i(t), i \in \mathcal{V}
\end{equation}

where $x_i(t) \in \mathbb{R}^2, u_i(t) \in \mathbb{R}^2$ denote the position and the control input of the $i$-th agent. In practical applications, since different constraints of each agent’s driving forces are always in different directions, the input of each agent may be constrained in a nonempty bounded nonconvex set, such as robots, quadrotors. To cope with nonconvex constraints, we need the following assumption.

\textbf{Assumption 2} [25]: Let $\mathcal{U}_i \subseteq \mathbb{R}^2, i \in \mathcal{V}$ be nonempty bounded closed sets and $0 \in \mathcal{U}_i$, $S_{\mathcal{U}_i}(\cdot)$ is a constraint operator such that $\sup_{x \in \mathcal{U}_i} \|S_{\mathcal{U}_i}(x)\| = \tilde{\gamma}_i > 0$ and $\inf_{x \notin \mathcal{U}_i} \|S_{\mathcal{U}_i}(x)\| = \gamma_i > 0$, where

\begin{equation}
\label{eq3}
S_{\mathcal{U}_i}(x) = \begin{cases}
\frac{x}{\|x\|} \sup_{0 \leq \beta \leq \|x\|} \left\{ \beta \frac{\mathbf{a}_\beta x}{\|x\|} \in \mathcal{U}_i, \forall 0 \leq \alpha \leq 1 \right\}, & x \neq 0 \\
0, & x = 0
\end{cases}
\end{equation}

\textbf{Remark 1:} In view of the existence of nonconvex input constraints, it is a essential requirement that $\alpha u_i(t)$ invariably lies in the corresponding nonconvex set for all $\alpha \in [0, 1]$. By means of the operator $S_{\mathcal{U}_i}(x)$, the feedback information $u_i(t)$ is transformed into the vector $S_{\mathcal{U}_i}(u_i)$, which has the maximum magnitude with the same direction as $u_i$ such that $\alpha S_{\mathcal{U}_i}(u_i) \in \mathcal{U}_i$, $\forall 0 \leq \alpha \leq 1$, i.e., the control input will stay in $\mathcal{U}_i$ without any convexity assumption. The supremum $\tilde{\gamma}_i$ implies the distance between any point in $\mathcal{U}_i$ and the origin is upper bounded, which indicates the control input is bounded. The infimum $\gamma_i$ implies the distance between any point outside $\mathcal{U}_i$ and the origin is lower bounded, and accurately the origin is a interior point of $\mathcal{U}_i$, which indicates the control input can vary in any direction.
The objective of this paper is to design a distributed finite-time rotating encirclement control scheme for the multiagent system (2) with nonconvex input constraints, which ensures all agents achieve circular formation and all targets are encircled by the circle in finite time. Moreover, all agents will uniformly distribute on the circle and have the common angular velocity.

To this end, by introducing the following polar coordinate transformation, the definition of the finite-time rotating encirclement control problem is given in Definition 1.

\[
x_i(t) = \bar{r}(t) + [\bar{e}_i(t) \cos(\vartheta_i(t)), \bar{e}_i(t) \sin(\vartheta_i(t))]^T
\]

where \( \bar{r}(t) = 1/m \sum_{k=1}^{m} r_k(t) \) and \( r_k(t) \) denote the targets’ geometric center and the position of the \( k \)-th target, respectively. The polar radius \( \bar{e}_i(t) \) represents the distance from the \( i \)-th agent to the center, and the polar angle \( \vartheta_i(t) \) represents the angle formed by the horizontal positive direction and the polar radius with the origin at the targets’ geometry center. Since \( \bar{r}(t), \bar{e}_i(t), \vartheta_i(t) \) are unavailable, we need to estimate these values for each agent only using the neighbors’ information.

**Definition 1 (Finite-Time Rotating Encirclement Control Problem)** [16]: For all \( i \in V \), if there exists the distributed control law \( u_i(t) \) and a finite time \( T > 0 \) such that system (2) and (4) satisfy conditions (5), then the distributed finite-time rotating encirclement control problem is solved.

\[
\begin{align*}
\lim_{t \to T} [\bar{e}_i(t) - \kappa z(t)] &= 0 \\
\lim_{t \to T} [\vartheta_i(t) - \vartheta_j(t) - \frac{2\pi(i-j)}{n}] &= 0 \\
\lim_{t \to T} [\vartheta_i(t) - \omega(t)] &= 0
\end{align*}
\]

where \( i, j \in V, z(t) = \max_{k \in T} \{ \|r_k(t) - \bar{r}(t)\| \} \), \( \kappa > 1 \) is a positive design parameter, \( \omega(t) \) represents the desired angular velocity.

**Remark 2**: In Definition 1, the encirclement problem and the rotating formation problem are formulated simultaneously. The first condition determines the range of the formation, i.e., the size of polar radius, where \( z(t) \) denotes the maximum distance between all targets and their geometry center. The second condition guarantees all agents will uniformly distribute on the circle. The third condition means all agents will have the common angular velocity ultimately. To illustrate the definition more clearly, an example is given in Fig.1, where the multiagent system has six agents and four targets.

To facilitate our analysis and design, we need some useful assumptions and lemmas.

**Assumption 3**: The speeds of targets have a common upper bound, i.e., \( \|r_{j}(t)\| \leq r^*_d, j \in T \), where \( r^*_d \) is a positive constant. The desired angular velocity of all agents satisfies \( \|\omega(t)\| \leq \omega^* \) with \( \omega^* > 0 \). The maximum distance between all targets and their geometry center is upper bounded with the positive constant \( z^* \), i.e., \( z(t) \leq z^* \).

**Lemma 2** [31]: Consider the system in Definition 2, suppose that there exists a continuous radially unbounded positive definite function \( V: \mathbb{R}^n \to \mathbb{R} \cup \{0\} \) such that \( V'(x) \leq -\alpha V(x) - \beta V^n(x) \), where \( \alpha > 0, \beta > 0, p > 1, 0 < q < 1 \). Then, the equilibrium is finite-time stable and the settling time \( T(x_0) \) satisfies \( T(x_0) \leq \frac{\left( \frac{1}{\alpha + \beta} \right) + \frac{1}{\beta(1-q)}}{\alpha} \forall x_0 \in \mathbb{R}^n \).

**Lemma 3** [32]: Let \( x_1, \ldots, x_n \geq 0 \) and \( 0 < q \leq 1, p \geq 1 \), then

\[
\begin{align*}
\frac{1}{n} \sum_{i=1}^{n} x_i^q &\geq \left( \frac{1}{n} \sum_{i=1}^{n} x_i^p \right)^{\frac{q}{p}} \\
\frac{1}{n} \sum_{i=1}^{n} x_i &\leq \left( \frac{1}{n} \sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}}
\end{align*}
\]

**III. MAIN RESULT**

In this section, the distributed finite-time rotating encirclement control scheme is presented, which is mainly composed of two parts. Firstly, a reference trajectory of \( x_i(t) \) is constructed by using reference estimations of the targets’ geometric center \( \bar{r}(t) \), the polar radius \( \bar{e}_i(t) \) and the polar angle \( \vartheta_i(t) \), which satisfies the finite-time rotating encirclement conditions. Secondly, the distributed finite-time constrained control input \( u_i(t) \) is designed for system (2) to drive \( x_i(t) \) tracking the reference trajectory. And the process of stability analysis is given in detail.

**A. CONTROL DESIGN**

Firstly, we will construct the distributed estimator (7) for each agent \( i \in V \) to obtain the estimation \( p_i(t) \) of the targets’ geometric center \( \bar{r}(t) \), since the \( i \)-th agent may not receive the position information of all targets, let alone the center.

\[
\begin{align*}
\dot{\varphi}_i(t) &= -\alpha_{i1} \sum_{j \in N_i} \xi_{ij}(t) - \alpha_{i2} \sum_{j \in N_i} \xi_{ij}(t) ||\xi_{ij}(t)|| \\
\xi_{ij}(t) &= p_i(t) - p_j(t) \\
p_i(t) &= \varphi_i(t) + \frac{1}{m} \sum_{j \in N_i} \frac{1}{|N_j^i|} r_j(t)
\end{align*}
\]

where \( \varphi_i(t) \in \mathbb{R}^2 \) is the intermediate vector with \( \frac{1}{n} \sum_{i=1}^{n} \varphi_i(t) = 0 \), without loss of generality, we choose \( \varphi_i(0) = 0 \). And \( \alpha_{i1}, \alpha_{i2} \) are positive design parameters.
Then, the distributed estimator (8) is established to obtain the reference polar radius \( l_i(t) \).

\[
\begin{align*}
\dot{\rho}_{11}(t) &= -\beta_{11} \frac{\varepsilon_{11}(t)}{|\varepsilon_{11}(t)|} - \beta_{12} \varepsilon_{11}(t) |\varepsilon_{11}(t)|, \\
\varepsilon_{11}(t) &= \rho_{11}(t) - \max_{j \in \mathcal{N}_i} (z_{ij}(t)) \\
\dot{\rho}_{12}(t) &= -\beta_{11} \frac{\varepsilon_{12}(t)}{|\varepsilon_{12}(t)|} - \beta_{12} \varepsilon_{12}(t) |\varepsilon_{12}(t)|, \\
\varepsilon_{12}(t) &= \rho_{12}(t) - \max_{j \in \mathcal{N}_i \cup \{i\}} (\rho_{j1}(t)) \\
\dot{\rho}_{iM}(t) &= -\beta_{11} \frac{\varepsilon_{iM}(t)}{|\varepsilon_{iM}(t)|} - \beta_{12} \varepsilon_{iM}(t) |\varepsilon_{iM}(t)|, \\
\varepsilon_{iM}(t) &= \rho_{iM}(t) - \max_{j \in \mathcal{N}_i \cup \{i\}} (\rho_{j(M-1)}(t)) \\
l_i(t) &= \kappa \rho_{iM}(t)
\end{align*}
\]

where \( z_{ij}(t) = \|r_j(t) - p_i(t)\|, M = \max_{i,j \in \mathcal{V}} \{d(i,j)\} \), and \( \beta_{11}, \beta_{12} \) are positive design parameters. If \( M \) is not the prior information, we can choose \( M = n - 1 \).

Next, the following distributed estimator (9) is proposed to obtain the reference polar angle \( \theta_i(t) \) and guarantee all reference trajectories have a common angular velocity.

\[
\begin{align*}
\dot{\theta}_i(t) &= -\delta_{11} \sum_{j \in \mathcal{N}_i} a_{ij} \frac{s_{ij}(t)}{|s_{ij}(t)|} - \delta_{12} \sum_{j \in \mathcal{N}_i} a_{ij} s_{ij}(t) |s_{ij}(t)| + \omega(t) \\
s_{ij}(t) &= \theta_i(t) - \theta_j(t) - \frac{2\pi(i-j)}{n}
\end{align*}
\]

where \( \delta_{11}, \delta_{12} \) are positive design parameters.

Accordingly, we can obtain the following reference trajectory for each agent by the polar coordinate transformation, which satisfies the finite-time rotating encirclement conditions in Definition 1.

\[
\bar{x}_i(t) = p_i(t) + [l_i(t) \cos(\theta_i(t)), l_i(t) \sin(\theta_i(t))]^T
\]

**Remark 3:** In [16], [20], [21], the multiagent system under consideration either has a one-to-one correspondence between agents and targets, or contains only one target, which can be regarded as the special cases of the problem herein. Compared with [15], [18], [19], where estimators of the maximum distance \( z(t) \) are centralized, in this paper, estimators (7), (8), (9) are fully distributed with only using the information of their neighbor targets and agents, which reduces the requirements of system communication.

**Remark 4:** Compared with estimators in [15]–[21], all of the above estimators have better convergence performance since their settling time is independent of initial states.

Finally, the nonconvex constrained control law of the system (2) is designed as

\[
\begin{align*}
u_i(t) &= S_{U_i}(-\sigma_{11} x_i(t) - \bar{x}_i(t)) \\
&\quad - \sigma_{12} (x_i(t) - \bar{x}_i(t)) \|x_i(t) - \bar{x}_i(t)\|
\end{align*}
\]

where \( S_{U_i}() \) is the constraint operator elucidated in (3). And \( \sigma_{11}, \sigma_{12} \) are positive design parameters.

**Remark 5:** In [15], [18], [19], the polar coordinate transformation is utilized to construct the final control law containing derivatives of the polar radius and polar angle, which is unpractical in physical systems since the derivative terms may amplify noises. In this paper, by designing a reference trajectory for each agent, the control law (11) overcomes this shortcoming, which drives the position of each agent to track the reference trajectory.

**Remark 6:** For the encirclement problem, the constraint operator \( S_{U_i}() \) is firstly introduced to the control law (11), which can restrict the input \( u_i(t) \) lying in the corresponding nonconvex constraint set \( U_i \).

### B. STABILITY ANALYSIS

Employing estimators (7), (8), (9), and control law (11), we obtain following lemmas to analyze the finite-time rotating encirclement control problem of the multiagent system (2).

**Lemma 4:** Consider the multiagent system (2) under Assumptions 1 and 3. For each agent \( i \in \mathcal{V} \), choose design parameters \( \alpha_{11} > (n-1)r_{a1}^2, \alpha_{12} > 0 \) in estimator (7), then there exists a finite-time \( T_1 > 0 \) such that \( \lim_{t \to T_1} \{p_i(t) - \bar{r}(t)\} = 0 \), i.e., the estimation \( p_i(t) \) can converge to the targets’ geometric center \( \bar{r}(t) \) in finite-time. Specifically, the settling time \( T_1 \) satisfies \( T_1 \leq \frac{\sqrt{2n}}{(n-1)}r_{a1}^2 + \frac{n}{2\alpha_{12}} \).

**Proof:** Define \( \bar{p}(t) = \frac{1}{n} \sum_{k=1}^{n} p_k(t) \), then we choose the following Lyapunov function candidate.

\[
V_1(t) = \frac{1}{2} \sum_{i=1}^{n} \left\{ |p_i(t) - \bar{p}(t)|^T [p_i(t) - \bar{p}(t)] \right\} \leq \frac{n}{2} s(t)^2
\]

where \( s(t) = \max_{i,j \in \mathcal{V}} \{ \|p_i(t) - p_j(t)\| \} \).

Calculate the time derivative of \( V_1(t) \) along (7), yields

\[
\dot{V}_1(t) = \sum_{i=1}^{n} \left\{ [p_i(t) - \bar{p}(t)]^T \left[ -\alpha_{11} \sum_{j \in \mathcal{N}_i} \xi_{ij}(t) \frac{\xi_{ij}(t)}{\|\xi_{ij}(t)\|} \right. \right.
\]

\[
- \alpha_{12} \sum_{j \in \mathcal{N}_i} \xi_{ij}(t) \|\xi_{ij}(t)\| \\
+ \frac{n}{m} \sum_{j \in \mathcal{N}_i} \frac{1}{|\mathcal{N}_j|} \hat{r}_j(t) - \bar{p}(t) \left\} \right.
\]

In line with Assumption 1 and Lemma 3, we obtain

\[
- \alpha_{11} \sum_{i=1}^{n} \left\{ [p_i(t) - \bar{p}(t)]^T \sum_{j \in \mathcal{N}_i} \frac{\xi_{ij}(t)}{\|\xi_{ij}(t)\|} \right\}
\]

\[
= - \frac{\alpha_{11}}{2} \sum_{i=1}^{n} \sum_{j \in \mathcal{N}_i} \frac{\xi_{ij}(t)^T \xi_{ij}(t)}{\|\xi_{ij}(t)\|} \leq -\alpha_{11} s(t)
\]
\[
- \alpha_{i2} \sum_{i=1}^{n} \left\{ [p_i(t) - \bar{p}(t)]T \sum_{j \in N_i} \xi_{ij}(t) \| \xi_{ij}(t) \| \right\} \\
= -\frac{\alpha_{i2}}{2} \sum_{i=1}^{n} \sum_{j \in N_i} \| \xi_{ij}(t) \|^3 \\
\leq -\frac{\alpha_{i2}}{2n^4} \left( \sum_{i=1}^{n} \sum_{j \in N_i} \| \xi_{ij}(t) \| \right)^3 \\
\leq -\frac{4\alpha_{i2}}{n^4} s(t)^3
\]

Then, from Assumption 3, we have
\[
\frac{n}{m} \sum_{i=1}^{n} \left\{ \left[ p_i(t) - \bar{p}(t) \right]T \sum_{j \in N_i} \frac{1}{|N_j^\alpha|} \| \bar{r}_j(t) \| \right\} \\
\leq \frac{1}{m} \sum_{i=1}^{n} \left\{ \left( \| n \bar{p}_i(t) - \sum_{k=1}^{n} p_k(t) \| \right) \sum_{j \in N_i} \frac{r_d^*}{|N_j^\alpha|} \right\} \\
\leq \frac{n-1}{m} \sum_{i=1}^{n} \left\{ \max_{j \neq i} (\| \xi_{ij}(t) \|) \sum_{j \in N_i} \frac{r_d^*}{|N_j^\alpha|} \right\} \\
\leq \frac{n-1}{m} \sum_{i=1}^{n} \sum_{j \in N_i} \frac{r_d^*}{|N_j^\alpha|} \\
= (n-1)r_d^* s(t)
\]

Besides,
\[
\sum_{i=1}^{n} \left[ p_i(t) - \bar{p}(t) \right]T \bar{p}(t) = \left[ \sum_{i=1}^{n} p_i(t) - n \bar{p}(t) \right]T \bar{p}(t) = 0
\]

Hence,
\[
\dot{V}_1(t) \leq -[\alpha_{i1} - (n-1) r_d^*] s(t) - \frac{4\alpha_{i2}}{n^4} s(t)^3 \\
\leq -\frac{\sqrt{2}(\alpha_{i1} - (n-1) r_d^*)}{\sqrt{n}} V_1^{1/2} - \frac{2\sqrt{2} \alpha_{i2}}{n^5 \sqrt{n}} V_1^{3/2}
\]

From Lemma 2, there exists a finite-time \( T_1 > 0 \) satisfying
\[
\lim_{t \to T_1} \{ p_i(t) - \bar{p}(t) \} = 0 \\
\lim_{t \to T_1} \{ p_i(t) - p_j(t) \} = 0 \\
T_1 = \frac{2n}{\alpha_{i1} - (n-1) r_d^*} + \frac{n^5 \sqrt{n}}{2 \alpha_{i2}}
\]

Furthermore, since \( \sum_{i=1}^{n} \dot{\varphi}_i(t) = 0 \), we have \( \dot{\bar{p}}(t) = \bar{r}(t) \) and \( \lim_{t \to T_1} \{ p_i(t) - \bar{r}(t) \} = 0 \).

**Lemma 5:** Consider the multiagent system (2) under Assumptions 1 and 3. For each agent \( i \in \mathcal{V} \), choose design parameters \( \delta_{i1} > 0, \delta_{i2} > 0 \) in estimator (8), then there exists a finite-time \( T_2 > 0 \) such that \( \lim_{t \to T_2} \{ l_i(t) - \kappa z(t) \} = 0 \), which implies the reference polar radius \( l_i(t) \) can satisfy the finite-time rotating encirclement conditions in Definition 2. Specifically, the settling time \( T_2 \) satisfies \( T_2 \leq T_1 + \frac{\sqrt{2} M}{\delta_{i1} - 2r_d^*} + \frac{\sqrt{2} M}{\delta_{i2} - 2r_d^*} \).

**Proof:** For the first equation in estimator (8), the Lyapunov function candidate is chosen as \( V_2(t) = \frac{1}{2} \varepsilon_{i1}(t)^2 \), whose time derivative is
\[
\dot{V}_2(t) = \varepsilon_{i1}(t) \left[ -\beta_{i1} \varepsilon_{i1}(t) - \beta_{i2} \varepsilon_{i1}(t) \varepsilon_{i1}(t) \right] \\
\leq -\max_{j \in N_i^\alpha} \{ \dot{\varepsilon}_{ij}(t) \}
\]

According to Lemma 4, \( p_i(t) = \bar{r}(t), \forall t > T_1. \) Then, from Assumption 3, we have
\[
\max_{j \in N_i^\alpha} \{ \dot{\varepsilon}_{ij}(t) \} \leq \max_{j \in N_i^\alpha} (\| \bar{r}_j(t) \| + \| \bar{p}_i(t) \|) \leq 2r_d^*
\]

Therefore,
\[
V_2(t) \leq -(\beta_{i1} - 2r_d^*) \varepsilon_{i1}(t) - \beta_{i2} \varepsilon_{i1}(t) \varepsilon_{i1}(t)^3 \\
= -\sqrt{2}(\beta_{i1} - 2r_d^*) V_2^{1/2} - 2\sqrt{2} \beta_{i2} V_2^{3/2}
\]

As a matter of course, the first equation of (8) is finite-time stable from Lemma 2, and there exists a finite-time \( T_{21} > 0 \) such that
\[
\lim_{t \to T_{21}} \left[ \rho_{i1}(t) - \max_{j \in N_i^\alpha} \{ \| \bar{r}_j(t) - \bar{r}(t) \| \} \right] = 0
\]
\[
T_{21} \leq T_1 + \frac{\sqrt{2}}{2 \beta_{i2}} + \frac{\sqrt{2}}{\beta_{i1} - 2r_d^*}
\]

Due to \( \rho_{i1}(t) = \max_{j \in N_i^\alpha} \{ \| \bar{r}_j(t) - \bar{r}(t) \| \}, \forall t > T_{21}, \) we have \( |\rho_{i1}(t)| \leq 2r_d^* \).

Proceeding similarly, the \( k \)-th (\( k = 2, \cdots, M \)) Lyapunov function candidate is chosen as \( V_{2k}(t) = \frac{1}{2} \varepsilon_{i1}(t)^2 \). Then, the \( k \)-th equation of (8) can be proved to be finite-time stable and satisfy
\[
\lim_{t \to T_{2k}} \left[ \rho_{ik}(t) - \max_{j \in N_i^\alpha \cup \{ k \}} \{ \rho_{j(k-1)}(t) \} \right] = 0
\]
\[
T_{2k} \leq T_{2(k-1)} + \frac{\sqrt{2}}{2 \beta_{i2}} + \frac{\sqrt{2}}{\beta_{i1} - 2r_d^*}
\]

Since the graph \( \mathcal{G} \) corresponding to agents is connected by Assumption 1, there exists at least one index \( q \) such that \( \rho_{i1}(t) = z(t) = \max_{j \in \mathcal{E}} \{ \| \bar{r}_j(t) - \bar{r}(t) \| \}, \forall t > T_{21}. \) Moreover, it is clear that
\[
\lim_{t \to T_2} \left[ \rho_M(t) - d(t) \right] = 0 \\
T_2 \leq T_1 + \frac{\sqrt{2} M}{2 \beta_{i2}} + \frac{\sqrt{2} M}{\beta_{i1} - 2r_d^*}
\]

where \( T_2 = T_{2N}. \) Since \( l_i(t) = \kappa \rho_M(t), \) we have \( \lim_{t \to T_2} \{ l_i(t) - \kappa z(t) \} = 0. \)

**Lemma 6:** Consider the multiagent system (2) under Assumptions 1 and 3. For each agent \( i \in \mathcal{V}, \) choose design parameters \( \delta_{i1} > 0, \delta_{i2} > 0 \) in estimator (9), then the reference polar angle \( \theta_i(t) \) satisfies the finite-time rotating encirclement conditions in Definition 1, i.e., there exists a finite time \( T_3 > 0 \) such that
\[
\lim_{t \to T_3} \left[ \theta_i(t) - \theta_j(t) - \frac{2\pi (i-j)}{n} \right] = 0 \\
\lim_{t \to T_3} \left[ \dot{\theta}_i(t) - \omega(t) \right] = 0
\]
Specifically, the settling time $T_3$ satisfies

$$
T_3 \leq \frac{2\sqrt{2}}{\delta_{11} \sqrt{\lambda_2(\mathcal{L}(A_1))}} + \frac{\sqrt{2n}}{\delta_{12} \lambda_2(\mathcal{L}(A_2))^2}
$$

(26)

where $A_1 = [a^n_{ij}] \in \mathbb{R}^{n \times n}$, $A_2 = [\frac{\lambda}{2i}] \in \mathbb{R}^{n \times n}$, and $\mathcal{L}(A_1), \mathcal{L}(A_2)$ can be defined as the Laplacian matrix of undirected graph $G(A_1), G(A_2)$.

**Proof.** Define $\hat{\theta}_i(t) = \theta_i(t) - \frac{1}{n} \int_0^t \omega(\tau) d\tau - \frac{2\pi}{n} i$.

Then the dynamic equation of the reference polar angle in (9) can be rewritten as

$$
\dot{\theta}_i(t) = -\delta_{11} \sum_{j \in N_i} a_{ij} \frac{s_{ij}(t)}{|s_{ij}(t)|} - \delta_{12} \sum_{j \in N_i} a_{ij} s_{ij}(t) |s_{ij}(t)|
$$

$$
\dot{s}_{ij}(t) = \hat{\theta}_i(t) - \hat{\theta}_j(t)
$$

(27)

Let $\tilde{\theta}(t) = \frac{1}{n} \sum_{i=1}^n \hat{\theta}_i(t), e_i = \hat{\theta}_i(t) - \tilde{\theta}(t), e(t) = [e_1(t), \cdots, e_n(t)]^T$. Thus, we have $\dot{s}_{ij}(t) = e_i(t) - e_j(t)$, and the following equation holds under Assumption 1.

$$
\dot{\theta}(t) = \frac{1}{n} \sum_{i=1}^n \dot{\theta}_i(t) = 0, \ 1^T e(t) = 0
$$

(28)

Consider the Lyapunov function candidate $V_3(t) = \frac{1}{2} e(t)^T e(t)$. Then, the time derivative of $V_3$ can be obtained with the aid of Lemma 3.

$$
\dot{V}_3(t) = \sum_{i=1}^n e_i \left[ -\delta_{11} \sum_{j \in N_i} a_{ij} \frac{s_{ij}(t)}{|s_{ij}(t)|} \right.
$$

$$
-\delta_{12} \sum_{j \in N_i} a_{ij} s_{ij}(t) |s_{ij}(t)|
$$

$$
= -\frac{\delta_{11}}{2} \sum_{i=1}^n \sum_{j \in N_i} a_{ij} \left| s_{ij}(t) \right|^2
$$

$$
-\frac{\delta_{12}}{2} \sum_{i=1}^n \sum_{j \in N_i} a_{ij} \left| s_{ij}(t) \right|^3
$$

$$
= -\frac{\delta_{11}}{2} \sum_{i=1}^n \sum_{j \in N_i} a_{ij} \left| s_{ij}(t) \right|^2
$$

$$
-\frac{\delta_{12}}{2} \sum_{i=1}^n \sum_{j \in N_i} \left| a_{ij} \frac{s_{ij}(t)}{|s_{ij}(t)|} \right|^2
$$

$$
\leq -\frac{\delta_{11}}{2} \left( \sum_{i=1}^n \sum_{j \in N_i} a_{ij}^2 |s_{ij}(t)|^2 \right)^{\frac{1}{2}}
$$

$$
-\frac{\delta_{12}}{2n} \left( \sum_{i=1}^n \sum_{j \in N_i} a_{ij}^2 |s_{ij}(t)|^2 \right)^{\frac{1}{2}}
$$

$$
\leq -\frac{\delta_{11}}{2} \left[ e(t)^T \mathcal{L}(A_1) e(t) \right]^{\frac{1}{2}}
$$

$$
-\frac{\delta_{12}}{2n} \left[ e(t)^T \mathcal{L}(A_2) e(t) \right]^{\frac{1}{2}}
$$

(29)

In view of Lemma 1, we have

$$
e(t)^T L(A_1) e(t) \geq \lambda_2(\mathcal{L}(A_1)) e(t)^T e(t)
$$

$$
= 2\lambda_2(\mathcal{L}(A_1)) V_3(t)
$$

$$
e(t)^T L(A_2) e(t) \geq \lambda_2(\mathcal{L}(A_2)) e(t)^T e(t)
$$

$$
= 2\lambda_2(\mathcal{L}(A_2)) V_3(t)
$$

(30)

Thus, we obtain that

$$
\dot{V}_3(t) \leq -\frac{\delta_{11} \sqrt{2\lambda_2(\mathcal{L}(A_1))}}{2} V_3(t) - \frac{\sqrt{2\lambda_2(\mathcal{L}(A_2))}}{n} V_3(t)^{\frac{3}{2}}
$$

(31)

Then, according to Lemma 2, there exists a finite-time $T_3 > 0$ such that

$$
\lim_{t \to T_3} \hat{\theta}_i(t) - \hat{\theta}_j(t) = 0
$$

$$
\lim_{t \to T_3} \hat{\theta}_i(t) - \hat{\theta}_j(t) = 0, i, j \in \mathcal{V}
$$

(32)

Furthermore, from the definition of $\hat{\theta}_i(t)$, we have

$$
\lim_{t \to T_3} \hat{\theta}_i(t) - \omega(t) = 0
$$

(33)

From above lemmas, it is clear that the reference trajectory $\bar{x}_i(t)$ satisfies the finite-time rotating encirclement conditions within the settling time $T_0 = \max\{T_0, T_3\}$. Furthermore, from Assumption 3, we can obtain $\|\hat{p}_i(t)\| = \|\hat{\omega}(t)\| \leq \kappa \omega^*$, $\|\hat{l}_i(t)\| = \|\kappa \omega\| \leq \kappa \omega^*$, $\|\hat{\theta}_i(t)\| = \|\omega(t)\| \leq \omega^*, \forall t > T_0$, which leads to

$$
\|\hat{x}_i(t)\| \leq \|\hat{p}_i(t)\| + \sqrt{(1 + \lambda_2(t)) (\hat{l}_i(t)^2 + \hat{\theta}_i(t)^2)}
$$

$$
\leq r_d^* + \sqrt{(1 + \kappa \omega)(4\kappa^2 r_d^2 + \omega^2)} \leq G
$$

(34)

Then, we will analyze the stability of system (2) with the control law (11) and present the main result of this paper.

**Theorem 1:** Consider the multiagent system (2) under Assumptions 1, 2 and 3. For each agent $i \in \mathcal{V}$, suppose that $\gamma_i > G$, design the control law (11) and estimators (7), (8), (9), choose design parameters satisfy

$$
\alpha_{i1} > (n - 1) r_d^*, \alpha_{i2} > 0
$$

$$\beta_{i1} > 2 r_d^*, \beta_{i2} > 0
$$

$$\delta_{i1} > 0, \delta_{i2} > 0
$$

$$\sigma_{i1} > \frac{2G \gamma_i}{G + \gamma_i}
$$

$$\sigma_{i2}(t) = \frac{\sigma_{i1}(\gamma_i - G)}{2GQ_1(t)^2} > 0
$$

(35)

where $Q_1(t) = \|x_i(t) - \bar{x}_i(t)\|$. Then, the finite-time rotating encirclement problem is solved and the nonconvex input
satisfies
\[ T = \max \{ T_0, T_{4i} \}, \quad i \in \mathcal{V} \]
\[ T_{4i} \leq \frac{2(\gamma_i + G)}{G(\gamma_i - G)} \sqrt{V_4i(0)} \]  
(36)

**Proof:** Define
\[ h_i(t) = \begin{cases} \frac{\|S_{U_i}(\mu_i(t))\|}{\|\mu_i(t)\|}, & \mu_i(t) \neq 0, \\ 1, & \mu_i(t) = 0 \end{cases} \]  
(37)
where \( \mu_i(t) = -\sigma_{i1} \frac{x_i - \bar{x}_i}{\|x_i - \bar{x}_i\|} - \sigma_{i2} (x_i - \bar{x}_i) \|x_i - \bar{x}_i\|, \quad i \in \mathcal{V} \). Since the constraint operator \( S_{U_i}(x) \) has same direction with the vector \( x \), the control law can be represented as \( u_i(t) = h_i(t) \mu_i(t) \).

It is clear that \( \|\mu_i(t)\| \leq \sigma_{i1} + \sigma_{i2} Q_i(t)^2 \), \( \forall t > 0 \). Note that \( S_{U_i}(\mu_i(t)) \geq \gamma_i > 0 \) in Assumption 1, then we have \( h_i(t) \geq \frac{\gamma_i}{\sigma_{i1} + \sigma_{i2} Q_i(t)^2} \). Since \( \sigma_{i1} > \frac{2G}{\gamma_i + G} \), it is easily to see that \( \frac{\gamma_i}{\sigma_{i1} + \sigma_{i2} Q_i(t)^2} < 1 \).

Choose the Lyapunov function candidate as
\[ V_4(t) = \frac{1}{2} \|x_i(t) - \bar{x}_i(t)\|^T [x_i(t) - \bar{x}_i(t)] \]  
(38)
Then, we have the time derivative of \( V_4(t) \) along the system 2.
\[ \dot{V}_4 = [x_i(t) - \bar{x}_i(t)]^T \left[ -\sigma_{i1} h_i(t) \frac{x_i(t) - \bar{x}_i(t)}{\|x_i(t) - \bar{x}_i(t)\|} - \sigma_{i2} h_i(t) (x_i(t) - \bar{x}_i(t)) \|x_i(t) - \bar{x}_i(t)\| - \dot{x}_i(t) \right] \]
\[ \leq \left[ \frac{\sigma_{i1} \gamma_i}{\sigma_{i1} + \sigma_{i2} Q_i(t)^2} - G \right] V_4^\frac{1}{2} - \frac{\sigma_{i2} \gamma_i}{\sigma_{i1} + \sigma_{i2} Q_i(t)^2} V_4^\frac{1}{2} \]
\[ \leq - \frac{G(\gamma_i - G)}{\gamma_i + G} V_4^\frac{1}{2} \]  
(39)
Thus, from the Lyapunov finite-time stability theorem [16], there exists a finite time \( T_{4i} > 0 \) such that
\[ \lim_{t \to T_{4i}} (x_i(t) - \bar{x}_i(t)) = 0 \]
\[ T_{4i} \leq \frac{2(\gamma_i + G)}{G(\gamma_i - G)} \sqrt{V_4i(0)} \]  
(40)
Furthermore, the reference trajectory \( \bar{x}_i(t) \) could achieve the finite-time rotating encirclement motion within finite-time \( T_0 \). Then, the state \( x_i(t) \) will realize the same motion within finite-time \( T = \max \{ T_0, T_{4i} \} \). □

**Remark 7:** If \( \gamma_i \leq G \), the i-agent may not be able to track the corresponding reference trajectory, since the maximum input, i.e., the maximum velocity is smaller than the velocity of the reference trajectory. Thus, it is reasonable to suppose that \( \gamma_i > G \).

**Remark 8:** Since the function \( \frac{x_i}{\|x_i\|} \) is discontinuous, the chattering phenomenon may be inevitable in practical applications. To solve this problem, the following continuous control law can be used to replace control law (11).
\[ u_i(t) = S_{U_i}(-\sigma_{i1} \text{tanh}(\frac{x_i(t) - \bar{x}_i(t)}{\tau_i}) - \sigma_{i2} (x_i(t) - \bar{x}_i(t)) \|x_i(t) - \bar{x}_i(t)\|) \]  
(41)
where \( \tau_i > 0 \). And the closed-loop system will be practical finite-time stable [33].

**IV. SIMULATION**

In this section, a numerical simulation example is given to illustrate the effectiveness of the above control scheme.

Consider a multiagent system consisting of 6 agents (Called A1 – A6) and 4 targets (Called T1 – T4), and Fig.2 shows the communication topology. The control input constraint sets and the initial positions of all agents and targets are chosen as
\[ U_i = \left\{ x | -10 \leq [1,0]^T x \leq 10, 8 \leq [1,0]^T x \leq 12 \right\} \cup \left\{ x | \|x\| \leq 10 \right\}, \quad i \in \mathcal{V} \]
\[ x_1(0) = [-5,-2]^T, x_2(0) = [7,5]^T, x_3(0) = [1,5]^T, x_4(0) = [-2,-2]^T, x_5(0) = [5,-3]^T, x_6(0) = [-6,-8]^T, \]
\[ r_1(0) = [-2,-1]^T, r_2(0) = [0,-3]^T, r_3(0) = [1,2]^T, r_4(0) = [2,1]^T \]
where the first coordinate and the second one can be regarded as \( x = \text{axial} \) and \( y = \text{axial} \), respectively.

The dynamical equations of targets are defined as
\[ \dot{r}_1(t) = [0.5e^{-t} + 0.3 \sin(t), 0.5e^{-t} - 0.3 \cos(t)]^T \]
\[ \dot{r}_2(t) = [0.5e^{-t} - 0.3 \cos(t), 0.5e^{-t} + 0.3 \sin(t)]^T \]
\[ \dot{r}_3(t) = [0.5e^{-t} + 0.3 \sin(t), 0.5e^{-t} + 0.3 \cos(t)]^T \]
\[ \dot{r}_4(t) = [0.5e^{-t} - 0.3 \cos(t), 0.5e^{-t} - 0.3 \sin(t)]^T \]
Choose the desired angular velocity \( \omega(t) = 1 \) and take design parameters as
\[ \alpha_{i1} = 1.5, \alpha_{i2} = 1; \beta_{i1} = 1, \beta_{i2} = 1; \delta_{i1} = 1, \delta_{i2} = 1; \sigma_{i1} = 15, \sigma_{i2} = 0.05; \kappa = 2 \]
By using the proposed distributed finite-time rotating encirclement control scheme in Section III, the simulation results are displayed in Fig.3-8. From Fig.3, it is easily to see that the error of estimator (7) converges to zero in finite-time about 1s. Similarly, Fig.4 shows that estimators (8) and (9) can guarantee the reference trajectories of all agents have the same polar radiuses and uniformly distribute on the circle in finite-time about 1.5s. What’s more, this implies all targets rotate at the same angular velocity. Fig.5 indicates the reference trajectories $\bar{x}_i(t)$ are tracked well by agents’ positions $x_i(t)$ in finite-time about 1s with the control law (11). Furthermore, it is clear that the finite-time rotating encirclement problem is solved from Fig.6. Fig.7 shows that the encirclement motion in phase diagram at different times, which implies the rotating encirclement motion will be achieved in finite-time about 4.5s. Finally, Fig.8 manifests the control inputs invariably lie in the corresponding nonconvex constraint sets $U_i$, which is in keeping with Theorem 1.
In this paper, the distributed finite-time rotating encirclement control problem is investigated for the multiagent system with nonconvex input constraints and multiple dynamic targets. To solve this problem, three finite-time estimators, which only use local information, are designed to calculate the targets’ geometric center, the reference polar radius and the reference polar angular. Then, the estimated values are used to construct the reference trajectory for each agent, which satisfies the finite-time rotating encirclement conditions. Finally, the distributed finite-time control laws are proposed for the multiagent system by introducing a constraint operator.

V. CONCLUSION

The distributed finite-time rotating encirclement control problem is investigated for the multiagent system with nonconvex input constraints and multiple dynamic targets. To solve this problem, three finite-time estimators, which only use local information, are designed to calculate the targets’ geometric center, the reference polar radius and the reference polar angular. Then, the estimated values are used to construct the reference trajectory for each agent, which satisfies the finite-time rotating encirclement conditions. Finally, the distributed finite-time control laws are proposed for the multiagent system by introducing a constraint operator.

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