Finite-time and fixed-time synchronization criteria for discontinuous fuzzy neural networks of neutral-type in Hale’s form

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ABSTRACT This paper aims to study the impact of discontinuous activations, neutral-type operators and mixed time delays on the finite-time and fixed-time robust synchronization of fuzzy neural networks. By using functional differential inclusions theory, inequality technique and the non-smooth analysis of Lyapunov-Krasovskii functional, a simple switching adaptive controller and a switching state-feedback controller are designed, some new criteria are obtained to achieve the finite-time and fixed-time synchronization of the proposed drive-response systems. Besides, the upper bound of the settling time of finite-time synchronization is estimated, and the settling time of fixed-time synchronization can be given in advance. In spite of many previous results on the synchronization of fuzzy neutral-type neural network and stability analysis of neutral networks of neutral-type in Hale’s form with continuous activation functions, few references on the stability and synchronization control analysis of the neural network like the form of the neural network model addressed in this paper can be cited. From this point of view, the neural network model considered and the theoretical results of this paper are more generalized and inclusive. Finally, simulation examples and remarks have been shown to verify the correctness and advantages of our main results.

INDEX TERMS Fuzzy neutral-type neural networks; Discrete and distributed time-delays; Discontinuous activations; Stability; Finite-time synchronization; Fixed-time synchronization.

I. INTRODUCTION

SYNCHRONIZATION, which implies that the signals of chaotic coupled system fulfill an identical behavior at last. In many real practical applications, such as biological systems, intelligent control, secure communication, and image protection, it is significant to consider the dynamical behavior of synchronization. During the past several years, based on the pioneer work of Yang and Yang [23], who put forward the fuzzy cellular neural networks (FCNNs) in order to cope with the uncertainty or vagueness in human cognitive processes in 1996, various kinds of synchronization control of fuzzy neural networks have been proposed, such as finite-time synchronization, adaptive synchronization control, lag synchronization, exponential lag synchronization. See, to name a few, [1], [6], [15], [20], [21], [24]. For example, Ratnave et al. [20] investigated the synchronization for fuzzy BAM neural networks with various time delays by using LKF and LMI method. Taken the jump discontinuities of the neuron activation functions into account, Duan et al. [6] considered the finite-time synchronization of delayed fuzzy cellular neural networks with discontinuous activations; and Tang et al. [21] further studied the finite-time cluster synchronization for a class of fuzzy cellular neural networks via non-chattering quantized controllers by constructing new Lyapunov-Krasovskii functionals and utilizing \( M \)-matrix methods.
It is worth considering that the settling time of finite-time synchronization is dependent on the initial conditions. But, for many real practical applications, the initial conditions of the models are hardly be given. In 2012, Polyakov [13] carried out a pioneer work concerning on the fixed-time stabilization. In the linear system, the fixed-time synchronization time has nothing to do with the initial synchronization errors. Besides, the settling time can be estimated in advance. From this point of view, the fixed time synchronization takes more advantages. In contrast to lots of finite-time synchronization problems on the fuzzy neural networks, the study on the fixed-time synchronization is not sufficient based on the existing literature, and few research has been investigated on the fixed-time synchronization of fuzzy neural networks, see [26].

Due to the complicated dynamic properties of the neural cells in the real world, it is natural to consider describing these complicated dynamic properties of neural cells by neutral-type neural networks. Neutral neural networks contain some information about the derivative of the past state. Due to this, neutral neural networks can be employed to characterise the properties of a neural reaction process more precisely. From the previous literature review, in spite of many synchronization results of fuzzy neural networks have been established, the synchronization control analysis of fuzzy neural networks of neutral-type is at this moment very incomplete, and few references can be cited if compared with the large number of references, see [2], [13], [17]. From the references above, we can find that the neutral terms in the their systems are \( x'_j(t - \tau_{ij}(t)) \), \( x'_j(t - \tau_{ij}(t)) \) and \( x'_j(t - u) \). For example, Muralisankar et al. [14] as was pointed out by Hale [8] that the properties of neutral operator are more important for studying neutral-type functional differential equation. In recent years, many efforts have devoted into the dynamic behavior analysis of neutral-type neural networks with delays, see to name a few, [4], [5], [13], [16], [22]. For example, by using fixed point theorem, Lyapunov functional method, comparison theorem and assuming that the activation functions are Lipschitz, Wang and Zhu [22] studied the stability of almost periodic solution for a class of generalized neutral-type neural networks with delays. Using coincidence degree theory and assuming that the activation functions are Lipschitz, Du et al. [4] further established some new results on the global exponential stability of periodic solutions of discrete-time neutral-type neural networks with time-varying delays.

Inspired by the above discussions, the main goal of this paper is to investigate the finite-time and fixed-time robust synchronization control of discontinuous fuzzy neutral-type neural networks with discrete and distributed time-delays. The main contributions of this paper are in the following key aspects:

- (1) Almost all works in [4], [5], [13], [16], [22], and [1], [10], [21], [25] and the references related therein studying the neural networks of neutral-type and fuzzy neural networks have still assumed that the activation functions are continuous, Lipschitz continuous or even smooth. In this paper, we focus on the study of discontinuous fuzzy neutral-type neural networks with discrete and distributed time-delays. Easily, one can see that the neural network models considered not only extend the previous results on neural networks of neutral-type to the discontinuous cases, but also complement the existing results on fuzzy neural networks of neutral-type to the more generalized cases in Hale’s form. Hence, a lot of previous neural network models can be included by our system.

- (2) We first attempt to investigate the finite-time and fixed-time robust synchronization control issues for the proposed neural network models. By designing a simple switching adaptive controller and a switching state-feedback control law for the response neural system, some new verifiable algebraic criteria are given to guarantee that the response system can robustly synchronize with the drive system. Moreover, the upper bounds of the settling time of finite-time robust synchronization can be estimated and settling time of fixed-time robust synchronization can be given in advance.

- (3) Finally, numerical examples and corresponding simulations have been investigated to verify the correctness of the main theorems. Several remarks are provided to show the novelty of the theory results.

The remainder part of this paper is organized as follows. System description and some preliminaries are presented in Section II. Two control laws are designed and employed to ensure the finite-time and fixed-time robust synchronization in Section III. In Section IV, numerical examples and illustrative remarks are given to illustrate the effectiveness and advantages of the obtained results. Finally, conclusion is provided in Section V.

II. SYSTEM DESCRIPTION AND PRELIMINARIES

A. SYSTEM DESCRIPTION

In this paper, we consider the following discontinuous fuzzy neutral-type neural networks with discrete and distributed time-delays as follows:

\[
\begin{align*}
(\alpha_f x_i)(t) &= -d_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)f(x_j(t)) + \sum_{j=1}^{n} b_{ij}v_j + \sum_{k=1}^{n} c_{j}(t)x_i(t - \sigma_j(t)) + \sum_{j=1}^{n} h_{ij}(t)\int_{-\xi_{ij}(t)}^{0} f_j(x_j(s))ds \\
&+ \sum_{i=1}^{n} \beta_{ij}(t)\int_{-\delta_{ij}(t)}^{0} f_j(x_j(s))ds + \sum_{j=1}^{n} \gamma_{ij}(t)\int_{-\xi_{ij}(t)}^{0} f_j(x_j(s))ds + \sum_{j=1}^{n} \zeta_{ij}(t)\int_{-\delta_{ij}(t)}^{0} f_j(x_j(s))ds + \sum_{j=1}^{n} \eta_{ij}(t)\int_{-\xi_{ij}(t)}^{0} f_j(x_j(s))ds + \sum_{j=1}^{n} \chi_{ij}(t)\int_{-\delta_{ij}(t)}^{0} f_j(x_j(s))ds + \sum_{j=1}^{n} \theta_{ij}(t)\int_{-\xi_{ij}(t)}^{0} f_j(x_j(s))ds + \sum_{j=1}^{n} \bar{\delta}_{ij}(t)\int_{-\delta_{ij}(t)}^{0} f_j(x_j(s))ds + \sum_{j=1}^{n} \bar{\chi}_{ij}(t)\int_{-\xi_{ij}(t)}^{0} f_j(x_j(s))ds + \sum_{j=1}^{n} \bar{\theta}_{ij}(t)\int_{-\delta_{ij}(t)}^{0} f_j(x_j(s))ds
\end{align*}
\]

where \( \alpha_f \) is a difference operator defined by

\[
(\alpha_f x_i)(t) = x_i(t) - \sum_{j=1}^{n} c_{j}(t)x_i(t - \sigma_j(t)), \quad i = 1, 2, ..., n,
\]
Let (2.1) be the drive system, and propose the following response system:

\[
\begin{align*}
\dot{y}_i(t) &= -d_i(t)y_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j(y_j(t)) \\
&+ \sum_{j=1}^{n} b_{ij}v_j + \sum_{j}^{n} A_j(t) f_j(y_j(t) - \tau_j(t)) \\
&+ \sum_{j}^{n} b_{ij}^\nu v_j + \sum_{j}^{n} \int_{-\delta_j}^{0} f_j(y_j(s)) ds \\
&+ \sum_{j}^{n} b_{ij}^\nu v_j + \sum_{j}^{n} S_i(t) v_j + I_i(t) + u_i(t) \\
y_0(\theta) &= \phi_i(\theta), \quad \theta \in [-\xi, 0], \quad i = 1, 2, ..., n,
\end{align*}
\]

where \(A_i\) is a difference operator defined by

\[
(A_i y_i)(t) = y_i(t) - \sum_{j=1}^{n} c_{ij}(t)(y_i(t) - \sigma_j(t)), \quad i = 1, 2, ..., n,
\]

and \(u_i(t)\) is the control law which will be designed later.

**B. PRELIMINARIES**

**Lemma 1:** If \(\sum_{j=1}^{M} c_{ij} < 1\), then the inverse of difference operator \(A_i\), denoted by \(A_i^{-1}\), exists and

\[
\sup_{t \in [1, \frac{n}{1+c_{ij}}]}|A_i^{-1}(t)| \leq 1 \left(1 - \sum_{j=1}^{M} c_{ij}\right), \quad i = 1, 2, ..., n.
\]

**Proof:** \((B_i x_i)(t) = \sum_{j=1}^{n} c_{ij}(t)x_j(t) - \sigma_j(t)\), then we can see that

\[
\sup_{t \in [1, \frac{n}{1+c_{ij}}]}|B_i(t)| \leq 1. \quad \text{Thus, } A_i^{-1} = (I - B_i)^{-1}\text{ exists and}
\]

\[
\sup_{t \in [1, \frac{n}{1+c_{ij}}]}|B_i^{-1}(t)| = \sup_{t \in [1, \frac{n}{1+c_{ij}}]}|(I - B_i)^{-1}(t)| \leq 1 \left(1 - \sum_{j=1}^{M} c_{ij}\right), \quad i = 1, 2, ..., n.
\]

From Lemma 1, we can see that the inverse of difference operator \(A_i\), denoted by \(A_i^{-1}\), exits. Let \((A_i y_i)(t) = z_i(t)\) and \((A_i y_i)(t) = w_i(t)\), then \(x_i(t) = (A_i^{-1} z_i)(t)\) and \(y_i(t) = (A_i^{-1} w_i)(t)\). Thus, systems (2.1) and (2.2) transform to the following system.

\[
\begin{align*}
\dot{z}_i(t) &= -d_i(t)z_i(t) \\
&-d_i(t)\sum_{j=1}^{n} c_{ij}(t)(A_i^{-1} z_j)(t - \sigma_j(t)) \\
&+ \sum_{j=1}^{n} a_{ij}(t)f_j(z_j(t)) \\
&+ \sum_{j=1}^{n} b_{ij} v_j + \sum_{j}^{n} A_j(t) f_j(z_j(t) - \tau_j(t)) \\
&+ \sum_{j}^{n} b_{ij}^\nu v_j + \sum_{j}^{n} \int_{-\delta_j}^{0} f_j(z_j(s)) ds \\
&+ \sum_{j}^{n} b_{ij}^\nu v_j + \sum_{j}^{n} S_i(t) v_j + I_i(t) \quad (z_0(\theta) = (A_i^{-1} \phi_i)(\theta), \quad \theta \in [-\xi, 0], \quad i = 1, 2, ..., n,)
\end{align*}
\]
and

\begin{equation}
\begin{cases}
\dot{w}_i(t) = -d_i(t)w_i(t) \\
\quad -d_i(t)\sum_{j=1}^n c_{ij}(t)(w_j^{-1}w_i)(t-\sigma_j(t)) \\
\quad + \sum_{j=1}^n a_{ij}(t)\frac{f((w_j^{-1}w_i)(t-\sigma_j(t)))}{\sigma_j(t)} \\
\quad + \sum_{j=1}^n b_{ij}v_j + \sum_{j=1}^n T_jv_j \\
\quad + \sum_{j=1}^n k_{ij}\int_{t-\delta_i(t)}^t \frac{f_j((w_j^{-1}w_i)(s-\tau_j(s)))}{\tau_j(s)} ds \\
\quad + \sum_{j=1}^n \int_{t-\delta_i(t)}^t \frac{f_j((w_j^{-1}w_i)(s-\tau_j(s)))}{\tau_j(s)} ds \\
\quad + \sum_{j=1}^n h_{ij}\int_{t-\delta_i(t)}^t \frac{f_j((w_j^{-1}w_i)(s-\tau_j(s)))}{\tau_j(s)} ds \\
\quad + \sum_{j=1}^n S_jv_j + I_i(t), \quad \text{for a.e. } t \in [0,T], i = 1,2,\ldots,n,
\end{cases}
\end{equation}

Hence, the problem of achieving the fixed-time synchronization between (2.1) and (2.2) reduces to achieving the fixed-time synchronization between (2.3) and (2.4).

For the sake of defining the solution of the discontinuous systems (2.3)-(2.4), in the following we consider solutions of systems (2.3)-(2.4) in Filippov’s sense [7].

Suppose that \( z(t) = (z_1(t), z_2(t), \ldots, z_n(t))^T \) and \( w(t) = (w_1(t), w_2(t), \ldots, w_n(t))^T \) are the solutions of initial value problems (2.3) and (2.4) on \([0,T]\), \( T \in (0, +\infty) \), if \( z_i(t)(i = 1,2,\ldots,n) \) and \( w_i(t)(i = 1,2,\ldots,n) \) are absolutely continuous on any compact subinterval of \([0,T]\) and satisfy the following inclusion respectively:

\( \dot{z}_i(t) \in -d_i(t)z_i(t) - d_i(t)\sum_{j=1}^n c_{ij}(t)(w_j^{-1}z_i)(t-\sigma_j(t)) \)

\( + \sum_{j=1}^n a_{ij}(t)\frac{f((w_j^{-1}z_i)(t-\sigma_j(t)))}{\sigma_j(t)} \)

\( + \sum_{j=1}^n b_{ij}v_j + \sum_{j=1}^n T_jv_j \)

\( + \sum_{j=1}^n k_{ij}\int_{t-\delta_i(t)}^t \frac{f_j((w_j^{-1}z_i)(s-\tau_j(s)))}{\tau_j(s)} ds \)

\( + \sum_{j=1}^n \int_{t-\delta_i(t)}^t \frac{f_j((w_j^{-1}z_i)(s-\tau_j(s)))}{\tau_j(s)} ds \)

\( + \sum_{j=1}^n h_{ij}\int_{t-\delta_i(t)}^t \frac{f_j((w_j^{-1}z_i)(s-\tau_j(s)))}{\tau_j(s)} ds \)

\( + \sum_{j=1}^n S_jv_j + I_i(t) \),

and

\( \dot{w}_i(t) \in -d_i(t)w_i(t) - d_i(t)\sum_{j=1}^n c_{ij}(t)(w_j^{-1}w_i)(t-\sigma_j(t)) \)

\( + \sum_{j=1}^n a_{ij}(t)\frac{f((w_j^{-1}w_i)(t-\sigma_j(t)))}{\sigma_j(t)} \)

\( + \sum_{j=1}^n b_{ij}v_j + \sum_{j=1}^n T_jv_j \)

\( + \sum_{j=1}^n k_{ij}\int_{t-\delta_i(t)}^t \frac{f_j((w_j^{-1}w_i)(s-\tau_j(s)))}{\tau_j(s)} ds \)

\( + \sum_{j=1}^n \int_{t-\delta_i(t)}^t \frac{f_j((w_j^{-1}w_i)(s-\tau_j(s)))}{\tau_j(s)} ds \)

\( + \sum_{j=1}^n h_{ij}\int_{t-\delta_i(t)}^t \frac{f_j((w_j^{-1}w_i)(s-\tau_j(s)))}{\tau_j(s)} ds \)

\( + \sum_{j=1}^n S_jv_j + I_i(t) + u_i(t) \)

have nonempty compact convex values. Thus, they are upper semi-continuous and measurable. By the measurable selection theorem, if \( z_i(t) \) and \( w_i(t) \) are the solution of systems (2.3) and (2.4), there exist measurable function
$\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)^T : [-\xi, T] \to \mathbb{R}^n$ and $\eta = (\eta_1, \eta_2, \ldots, \eta_n)^T : [-\xi, T] \to \mathbb{R}^n$, where $\gamma_j(t) \in \mathcal{C}[f((\alpha f)^{-1}z_j(t))]$ or $\gamma_j(t) \in \mathcal{C}[f(\alpha f^{-1}w_j(t))]$ or $\eta_j(t) \in \mathcal{C}[f(y_j(t))]$ for a.e. $t \in [-\xi, T]$, such that

$$\dot{z}_i(t) = -d_i(t)z_i(t) - d_i(t) \sum_{j=1}^{n} c_{ij}(t)(\alpha f^{-1}z_j)(t - \sigma_j(t)) + \sum_{j=1}^{n} a_{ij}(t)\gamma_j(t) + \sum_{j=1}^{n} b_{ij} v_j + \sum_{j=1}^{n} T_{ij} v_j$$

$$+ \sum_{j=1}^{n} \alpha_{ij}(t)\gamma_j(t - \tau_j(t)) + \sum_{j=1}^{n} k_{ij}(t) \int_{t-\delta(t)}^{t} \gamma_j(s)ds$$

$$+ \sum_{j=1}^{n} \beta_{ij}(t)\gamma_j(t - \tau_j(t)) + \sum_{j=1}^{n} h_{ij}(t) \int_{t-\delta(t)}^{t} \gamma_j(s)ds$$

$$+ \sum_{j=1}^{n} S_{ij} v_j + I_{i}(t), \text{ for a.e. } t \geq 0, i = 1, 2, \ldots, n,$$

(2.5)

Denote the error

$$e_i(t) = w_i(t) - z_i(t),$$

then, from (2.5) and (2.6), we can obtain the error dynamics:

$$\dot{e}_i(t) = -d_i(t)[w_i(t) - z_i(t)] - d_i(t) \sum_{j=1}^{n} c_{ij}(t)(\alpha f^{-1}w_i(t) - \sigma_j(t))$$

$$+ \sum_{j=1}^{n} a_{ij}(t)[\eta_j(t) - \gamma_j(t)]$$

$$+ \sum_{j=1}^{n} \alpha_{ij}(t)\eta_j(t - \tau_j(t)) - \sum_{j=1}^{n} \alpha_{ij}(t)\gamma_j(t - \tau_j(t))$$

$$+ \sum_{j=1}^{n} k_{ij}(t) \int_{t-\delta(t)}^{t} \eta_j(s)ds - \sum_{j=1}^{n} k_{ij}(t) \int_{t-\delta(t)}^{t} \gamma_j(s)ds$$

$$+ \sum_{j=1}^{n} \beta_{ij}(t)\eta_j(t - \tau_j(t)) - \sum_{j=1}^{n} \beta_{ij}(t)\gamma_j(t - \tau_j(t))$$

$$+ \sum_{j=1}^{n} h_{ij}(t) \int_{t-\delta(t)}^{t} \eta_j(s)ds - \sum_{j=1}^{n} h_{ij}(t) \int_{t-\delta(t)}^{t} \gamma_j(s)ds$$

$$+ u_i(t),$$

$$e_{\theta}(t) = (\alpha f^{-1}\phi_i(t) - \sigma_j(t)) \in [-\xi, 0].$$

(2.7)

For convenience, let $e_{\theta}(t) = (e_{\theta 1}(t), e_{\theta 2}(t), \ldots, e_{\theta n}(t))^T, \theta \in [-\xi, 0].$

Definition 2.1: The drive system (2.3) and response system (2.4) are said to be finite-time robustly synchronized if, for a suitable controller, there exists a time $t^*$ such that $|e(t^*)| = 0$ and $|e(t)| = ||w(t) - z(t)|| = 0$ for $t > t^*$, where $z(t)$ and $w(t)$ are the solutions of drive system (2.3) and response system (2.4) with initial conditions $h_i(\phi_i(t))$ and $h_i(\phi_i(t))$, respectively.

Definition 2.2: (See [19]) The origin of error system (2.7) is said to be globally fixed-time stable if it is globally uniformly finite-time stable and the settling time $T$ is globally bounded, i.e., $\exists T_{max} \in \mathbb{R}^+$ such that $T(e_0) \leq T_{max}, \forall e_0 \in \mathbb{R}^n$.

Definition 2.3: The drive-response systems (2.3)-(2.4) are said to achieve robust fixed-time synchronization if there exist a fixed time $T_{max}$ and a settling time function $T(e_0(\theta))$ such that

$$\lim_{t \to T(e_0(\theta))} ||e(t)|| = 0,$$

$$e(t) = 0, \forall t \geq T(e_0(\theta)),$$

$$T(e_0(\theta)) \leq T_{max}, e_0(\theta) \in \mathbb{R}^n \setminus 0,$$

where $|| \cdot ||$ represents the Euclidean norm.

Lemma 2: (See [9]) If there exists a regular, positive definite and radially unbounded function $V(e(t)) : \mathbb{R} \to \mathbb{R}$ and constants $a, b, \delta, k > 0$ and $\delta k > 1$ meet

$$V(e(t)) \leq -(aV^{\delta}(e(t)) + b)^k, \quad e(t) \in \mathbb{R}^n \setminus 0,$$

then the origin is fixed-time stable, and the settling time $T_{max}$ is estimated by

$$T(e_0) \leq T_{max} = \frac{1}{b} \left( \frac{a}{b} \right)^{\frac{1}{k}} \left( 1 + \frac{1}{\delta k - 1} \right).$$

Definition 2.4: (See [3]) A function $V(x) : \mathbb{R}^n \to \mathbb{R}$ is C-regular if $V(x)$ is:

(i) regular in $\mathbb{R}^n$;

(ii) positive definite, i.e., $V(x) > 0$ for $x \neq 0$ and $V(0) = 0$;

(iii) radially unbounded, i.e., $V(x) \to +\infty$ as $|x| \to +\infty$.

Note that a C-regular Lyapunov function $V(x)$ is not necessarily differentiable.

Remark 2: Let $e(t) = (e_1(t), e_2(t), \ldots, e_n(t))^T, t \in [0, T), T \in (0, +\infty), be a solution of the error system (2.7). Then, we can have $|e_i(t)|$ is a locally Lipschitz continuous function in $e_i$ on \( \mathbb{R} \). Moreover, according to the definition of Clarke’s generalized gradient, we have

$$\partial(|e_i(t)|) = \mathcal{C}[\text{sign}(e_i(t))] = \begin{cases} (-1), & \text{if } e_i(t) < 0, \\ [-1, 1], & \text{if } e_i(t) = 0, \\ (1), & \text{if } e_i(t) > 0, \end{cases}$$

which means that, for any $v_i(t) \in \partial(|e_i(t)|)$, we can see that $v_i(t) = \text{sign}(e_i(t))$ if $e_i(t) \neq 0$, and $v_i(t)$ can be arbitrarily chosen in $[-1, 1]$ if $e_i(t) = 0$. In particular, for any $i = 1, 2, \ldots, n$, we choose $v_i(t) = \text{sign}(e_i(t))$, then we can see that

$$v_i(t)e_i(t) = |e_i(t)|, \quad \text{and } v_i(t)\text{sign}(e_i(t)) = 1.$$
Lemma 3: (See [11]) Suppose x and y are two states of system (2.1), then the following inequalities hold
\[
\left| \sum_{j=1}^{n} \alpha_{ij} f_j(x_j) - \sum_{j=1}^{n} \alpha_{ij} f_j(y_j) \right| \leq \sum_{j=1}^{n} |\alpha_{ij}||f_j(x_j) - f_j(y_j)|, \\
\left| \sum_{j=1}^{n} \beta_{ij} f_j(x_j) - \sum_{j=1}^{n} \beta_{ij} f_j(y_j) \right| \leq \sum_{j=1}^{n} |\beta_{ij}||f_j(x_j) - f_j(y_j)|.
\]

Lemma 4: (See [12]) Let \(x_1,x_2,\ldots,x_n \geq 0\), \(0 < p \leq 1\), \(q > 1\), the following two inequalities hold
\[
\sum_{i=1}^{n} x_i^p \geq \left( \sum_{i=1}^{n} x_i \right)^p, \quad \sum_{i=1}^{n} x_i^q \geq n^{-1-q} \left( \sum_{i=1}^{n} x_i \right)^q.
\]

III. MAIN RESULTS OF SYNCHRONIZATION CONTROL

A. FINITE-TIME SYNCHRONIZATION UNDER SWITCHING ADAPTIVE LAW

Defining the following switching adaptive control law for response system (2.4) of the form:
\[
u_i(t) = \text{sign}(e_i(t)) \left( \rho_i - \lambda_i(t)e_i(t) \right) - \mu_i|e_i(t - \tau(t))| \\
- \omega_i|e_i(t - \sigma_i(t))| - \zeta_i \int_{-\delta_i(t)}^{0} e_i(s)ds,
\]
(3.1)
where \(e_i(t) = y_i(t) - x_i(t), \rho_i, \mu_i, \omega_i, \zeta_i\) are the parameters to be designed later, \(i = 1,2,\ldots,n\). For \(e_i(t) \neq 0\), the feedback gains \(\lambda_i(t)\) are adapted and satisfying:
\[\hat{\lambda}_i(t) = \omega_i |e_i(t)|,\]
(3.2)
where \(\omega_i(i = 1,2,\ldots,n)\) are positive constants. For \(e_i(t) \equiv 0\), we set \(\lambda_i(t) \equiv \lambda_i^*\), where \(\lambda_i^*\) are determined to be sufficiently large constants.

Theorem 3.1: Suppose that the conditions (H1) and (H2) hold, then the drive-response systems (2.3) and (2.4) can achieve finite-time robust synchronization based on the controller (3.1) if the design parameters are appropriately selected as follows
\[
\Gamma_i = \liminf_{t \to +\infty} \left\{ \rho_i - \sum_{j=1}^{n} |a_{ij}(t)| + |\alpha_{ij}(t)| + |\beta_{ij}(t)| \right. \\
+ |kJ_i(t)| |\delta_i(t)+ |h_j(t)(t-\delta_i(t))| \right\} \geq 0,
\]
\[
\limsup_{t \to +\infty} \left\{ -d_i(t) - \frac{\lambda_i^*}{1 - \sum_{j=1}^{n} c_{ij}(t)} + \sum_{j=1}^{n} \frac{|a_{ij}(t)|}{\epsilon_j} \right\} \leq 0,
\]
\[
\limsup_{t \to +\infty} \left\{ -\mu_i + \sum_{j=1}^{n} |a_{ij}(t)| + |\beta_{ij}(t)| |\epsilon_j| \right\} \leq 0,
\]
\[
\limsup_{t \to +\infty} \left\{ -\omega_i + \sum_{j=1}^{n} |d_i(t)||c_{ij}(t)| \right\} \leq 0,
\]
\[
\limsup_{t \to +\infty} \left\{ -\zeta_i + \sum_{j=1}^{n} |kJ_i(t)| |h_j(t)| |\epsilon_j| \right\} \leq 0.
\]

Proof. Define the following Lyapunov-Krasovskii functional:
\[
V(e(t)) = \sum_{i=1}^{n} |e_i(t)|^p + \frac{1}{2} \sum_{i=1}^{n} \frac{1}{\omega_i} (\lambda_i(t) - \lambda_i^*)^2.
\]
Then, in view of Definition (2.4) (2.7) and (3.1), we get the derivative of \(V(e(t))\) that
\[
\frac{dV(e(t))}{dt} = \sum_{i=1}^{n} \text{sign}(e_i(t)) \left\{ -d_i(t)e_i(t) \\
- d_i(t) \sum_{j=1}^{n} c_{ij}(t)(|\omega_i^{-1}w_i(t-\sigma_i(t)) - (\omega_i^{-1}z_i)(t-\sigma_i(t))| \\
+ n \sum_{j=1}^{n} a_{ij}(t)|\eta_j(t) - \gamma_j(t)| \\
+ \sum_{j=1}^{n} \alpha_{ij}(t)|\eta_j(t) - \tau_j(t)) \\
- \sum_{j=1}^{n} \alpha_{ij}(t)|\eta_j(t) - \tau_j(t)) \\
+ \sum_{j=1}^{n} k_{ij}(t) \int_{-\delta_i(t)}^{0} \eta_j(s)ds - \sum_{j=1}^{n} k_{ij}(t) \int_{-\delta_i(t)}^{0} \gamma_j(s)ds \\
+ \sum_{j=1}^{n} \beta_{ij}(t)|\eta_j(t) - \tau_j(t)) - \sum_{j=1}^{n} \beta_{ij}(t)|\gamma_j(t) - \tau_j(t)) \\
+ \sum_{j=1}^{n} h_{ij}(t) \int_{-\delta_i(t)}^{0} \eta_j(s)ds - \sum_{j=1}^{n} h_{ij}(t) \int_{-\delta_i(t)}^{0} \gamma_j(s)ds \\
+ u_i(t) \right\} + \sum_{i=1}^{n} \lambda_i(t)|\epsilon_i(t)| - \sum_{i=1}^{n} \lambda_i^*|\epsilon_i(t)|.
\]
(3.3)

It follows that
\[
- \sum_{i=1}^{n} \text{sign}(e_i(t))d_i(t) \sum_{j=1}^{n} c_{ij}(t)(|\omega_i^{-1}w_i(t-\sigma_i(t)) \\
- (\omega_i^{-1}z_i)(t-\sigma_i(t))| \\
\leq \sum_{i=1}^{n} |d_i(t)| \sum_{j=1}^{n} |c_{ij}(t)||\epsilon_i(t) - \sigma_i(t)|.
\]
(3.4)

According to (H2), we can have
\[
\sum_{i=1}^{n} \text{sign}(e_i(t)) \sum_{j=1}^{n} a_{ij}(t)|\eta_j(t) - \gamma_j(t)| \\
\leq \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}(t)||\epsilon_i(t)| + \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}(t)||\epsilon_j|.
\]
(3.5)

From Lemma 3 and (H2), it follows that
\[
\sum_{i=1}^{n} \text{sign}(e_i(t)) \left[ \sum_{j=1}^{n} \alpha_{ij}(t)|\eta_j(t) - \tau_j(t) \right. \\
\left. - \alpha_{ij}(t)|\gamma_j(t) - \tau_j(t)) \right] \\
\leq \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}(t)||\epsilon_i(t)| - |\tau_j(t))| + \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}(t)||\epsilon_j|.
\]
(3.6)
Furthermore, it is easy to see that $\varepsilon_i(t) = (\sigma_i^{-1}\varepsilon_i)(t)$, and the difference operator $\sigma_i^{-1}$ is bounded, then we can see that $\text{sign}(\varepsilon_i(t)) = \text{sign}(\varepsilon_i(t))$. By (3.1), we can have

$$
\sum_{i=1}^{n} \text{sign}(\varepsilon_i(t)) \left[ \sum_{j=1}^{n} k_{ij}(t) \int_{t-\delta_i(t)}^{t} \eta_j(s) ds \right]
- \sum_{i=1}^{n} k_{ij}(t) \int_{t-\delta_i(t)}^{t} \gamma_j(s) ds 
\leq \sum_{i=1}^{n} \sum_{j=1}^{n} |k_{ij}(t)| |\eta_j(t)| + |\varepsilon_j(t)| ds 
+ \sum_{i=1}^{n} \sum_{j=1}^{n} |k_{ij}(t)||\delta_j(t)||m_j.
$$

By using the same way, we also get

$$
\sum_{i=1}^{n} \text{sign}(\varepsilon_i(t)) \left[ \sum_{j=1}^{n} h_{ij}(t) \int_{t-\delta_i(t)}^{t} \eta_j(s) ds \right]
- \sum_{i=1}^{n} h_{ij}(t) \int_{t-\delta_i(t)}^{t} \gamma_j(s) ds 
\leq \sum_{i=1}^{n} \sum_{j=1}^{n} |h_{ij}(t)| |\eta_j(t)| + |\varepsilon_j(t)| ds 
+ \sum_{i=1}^{n} \sum_{j=1}^{n} |h_{ij}(t)||\delta_j(t)||m_j.
$$

Substituting (3.4)-(3.10) into (3.3) and by using Lemma [1], we obtain

$$
\frac{dV(e(t))}{dt} \leq \sum_{i=1}^{n} \left[ -d_i(t) - \frac{\lambda_i^*}{1-\sum_{j=1}^{m} \beta_i} \right] |e_i(t)| 
+ \sum_{i=1}^{n} \left[ -\mu_i + \sum_{j=1}^{m} \left( |\alpha_{ij}(t)| + |\beta_{ij}(t)| \right) |m_j| \right] |e_i(t) - \sigma_i(t)| 
+ \sum_{i=1}^{n} \left[ -\omega_i + \sum_{j=1}^{m} |d_i(t)||e_j(t)|| \right] |e_i(t) - \sigma_i(t)| 
+ \sum_{i=1}^{n} \left[ -\zeta_i + \sum_{j=1}^{m} |k_{ij}(t)||h_j(t)|| |\delta_j(t)||m_j| \right],
$$

which leads to

$$
\frac{dV(e(t))}{dt} \leq -\sum_{i=1}^{n} \Gamma_i(t) \leq 0, \text{ for a.e. } t \geq 0,
$$

where

$$
\Gamma_i(t) = \liminf_{t \to 0^+} \left\{ \rho_i - \sum_{j=1}^{m} (|a_{ij}(t)| + |\alpha_{ij}(t)| + |\beta_{ij}(t)|) 
+ |k_{ij}(t)||\delta_j(t)||m_j| \right\}.
$$

Now, we show that there is a time instant $t^*$ satisfying $0 \leq t^* < +\infty$ such that $V(e(t^*)) = 0$. By using the method of reduction to absurdity, i.e., assume that $V(e(t)) > 0$ for each $t \geq 0$. Then,

$$
\frac{dV(e(t))}{dt} \leq -\sum_{i=1}^{n} \Gamma_i(t) \leq 0, \text{ for a.e. } t \geq 0.
$$

Integrating (3.12) on the interval $[0,t]$, we have

$$
V(e(t)) \leq V(0) - \sum_{i=1}^{n} \Gamma_i(t), \text{ for each } t \geq 0.
$$

Then,

$$
V(e(t)) < 0, \text{ for } t > \bar{t} = \frac{V(0)}{-\sum_{i=1}^{n} \Gamma_i},
$$

which leads to a contradiction. By using a same argument, we can also get that $t^* \leq \bar{t}$. In the following, we will prove that

$$
V(e(t)) \equiv 0, \text{ for each } t \geq t^*.
$$

Indeed, if $V(e(t^*)) > 0$ for some $t' > t^*$, then we can find that some non-degenerate interval $(t_1,t_2) \subset (t',t^*)$ such that $\frac{dV(e(t))}{dt} > 0$ for all $t \in (t_1,t_2)$. This is contradicted with (3.12).

Thus, we can conclude that $V(e(t)) = 0$, for each $t \geq \bar{t}$. Therefore, according to Definition [1], we can conclude that the drive-response systems (2.3) and (2.4) can achieve finite-
time robust synchronization in a finite time $t \geq \frac{V(0)}{\sum_{i=1}^{n} \Gamma_i}$ based on the controller (3.1).

Defining the following switching adaptive control law for response system (2.4) of the form:

$$u_i(t) = \text{sign}(\xi_i(t)) \left( \rho_i - \lambda_i(t) |\xi_i(t)| - \mu_i |\xi_i(t) - \tau_i(t)| - \zeta_i \int_{t-\delta_i(t)}^{t} |\xi_i(s)| ds \right),$$

(3.13)

where $\xi_i(t) = y_i(t) - x_i(t)$, $\rho_i$, $\mu_i$ and $\zeta_i$ are the parameters to be designed later, $i = 1, 2, ..., n$, $j = 1, 2, ..., n$. For $\xi_i(t) \neq 0$, the feedback gains $\lambda_i(t)$ are adapted and satisfying:

$$\dot{\lambda}_i(t) = \Theta_i |\xi_i(t)|,$$

where $\Theta_i(i = 1, 2, ..., n)$ are positive constants. For $\xi_i(t) \equiv 0$, we set $\lambda_i(t) \equiv \lambda^*_i$, where $\lambda^*_i$ are determined to be sufficiently large constants.

**Corollary 1:** Suppose that the conditions (H1) and (H2) hold and $c_{ij}(\cdot) \equiv 0$, then the drive-response systems (2.3) and (2.4) can achieve finite-time robust synchronization based on the controller (3.1) if the design parameters are appropriately selected as follows

$$\Gamma_i = \liminf_{t \to +\infty} \left\{ \rho_i - \sum_{j=1}^{n} (|a_{ij}(t)| + |c_{ij}(t)| + |b_{ij}(t)|) 
+ |k_{ij}(t)||d_i(t)| + |h_{ij}(t)||\delta_i(t)|\right\} \geq 0,$$

$$\limsup_{t \to +\infty} \left\{ -d_i(t) - \lambda^*_i + \sum_{j=1}^{n} |a_{ij}(t)| \right\} \leq 0,$$

$$\limsup_{t \to +\infty} \left\{ -\mu_i + \sum_{j=1}^{n} (|a_{ij}(t)| + |b_{ij}(t)|) \right\} \leq 0,$$

$$\limsup_{t \to +\infty} \left\{ -\zeta_i + \sum_{j=1}^{n} (|k_{ij}(t)| + |h_{ij}(t)|) \right\} \leq 0.$$

Moreover, the settling time can be estimated as

$$t^* \leq \tilde{t} = \frac{V(0)}{\sum_{i=1}^{n} \Gamma_i}.$$

**Remark 3:** Since there is no result concerning with the finite-time synchronization of discontinuous fuzzy neutral-type neural networks with discrete and distributed time-delays, the switching adaptive controller proposed in this paper can effectively cope with the finite-time synchronization of the considered neural system. This implies that this paper originally proposed a new method to study the finite-time synchronization of the fuzzy neural networks.

**B. FIXED-TIME SYNCHRONIZATION UNDER SWITCHING STATE-FEEDBACK CONTROLLER**

Designing the following switching state-feedback controller:

$$u_i(t) = -\text{sign}(\xi_i(t)) \left( \bar{\rho}_i + \bar{\lambda}_i |\xi_i(t)| + \bar{\mu}_i |\xi_i(t) - \tau_i(t)| + \bar{\zeta}_i \int_{t-\delta_i(t)}^{t} |\xi_i(s)| ds + \bar{\gamma}_i |\xi_i(t)| \theta \right),$$

(3.14)

where $\xi_i(t) = y_i(t) - x_i(t)$, $\theta > 1$, $\bar{\rho}_i$, $\bar{\lambda}_i$, $\bar{\mu}_i$, $\bar{\zeta}_i$, $\bar{\gamma}_i$ are the parameters to be designed later, $i = 1, 2, ..., n$, $j = 1, 2, ..., n$.

**Theorem 3.2:** Suppose that the conditions (H1) and (H2) hold, then the response system (2.4) can robustly synchronize with the drive system (2.3) in a fixed time based on the controller (3.14) if the design parameters are appropriately selected as follows

$$\bar{\Gamma}_i = \liminf_{t \to +\infty} \left\{ \bar{\rho}_i - \sum_{j=1}^{n} (|a_{ij}(t)| + |c_{ij}(t)| + |b_{ij}(t)|) 
+ |k_{ij}(t)||d_i(t)| + |h_{ij}(t)||\delta_i(t)|\right\} > 0,$$

$$\limsup_{t \to +\infty} \left\{ -d_i(t) - \frac{\bar{\lambda}_i}{1 - \sum_{j=1}^{n} c_{ij}} + \sum_{j=1}^{n} |a_{ij}(t)| \right\} \leq 0,$$

$$\limsup_{t \to +\infty} \left\{ -\bar{\mu}_i + \sum_{j=1}^{n} (|a_{ij}(t)| + |b_{ij}(t)|) \right\} \leq 0,$$

$$\limsup_{t \to +\infty} \left\{ -\bar{\zeta}_i + \sum_{j=1}^{n} (|k_{ij}(t)| + |h_{ij}(t)|) \right\} \leq 0,$$

$$\bar{\zeta}_i > 0, i = 1, 2, ..., n.$$

Moreover, the settling time $T_{\max}$ can be estimated as

$$T_{\max} = \frac{1}{\bar{\Gamma}} \left( \frac{\bar{\Gamma}}{\min\{\bar{\zeta}_i\}} \right)^{\frac{1}{\theta}} \left( 1 + \frac{1}{\theta - 1} \right),$$

where $\bar{\Gamma} = \sum_{i=1}^{n} \bar{\Gamma}_i$.

**Proof:** Define the following Lyapunov-Krasovskii functional:

$$\bar{V}(e(t)) = \sum_{i=1}^{n} |\xi_i(t)|.$$

(3.15)
It is easy to verify that $\tilde{V}(e(t))$ is $C$-regular. Calculating the derivative of $\tilde{V}(e(t))$, in view of (3.7), we get
\[
\frac{d\tilde{V}(e(t))}{dt} = \sum_{i=1}^{n} \frac{d\varepsilon_i(t)}{dt} = \sum_{i=1}^{n} \varepsilon_i(t) \frac{d\varepsilon_i(t)}{dt}
\]
\[
= \sum_{i=1}^{n} \text{sign}(\varepsilon_i(t)) \left\{ -d_i(t) [w_i(t) - z_i(t)] - d_i(t) \sum_{j=1}^{n} c_{ij}(t) \right\} \\
\cdot [\alpha_{i}^{-1} w_i(t) (t - \sigma_i(t)) - (\alpha_{i}^{-1} z_i(t) - \sigma_i(t))]
\]
\[
+ \sum_{j=1}^{n} a_{ij}(t) \left[ \eta_j(t) - \gamma_j(t) \right] \\
+ \sum_{j=1}^{n} \alpha_{ij}(t) \left[ \eta_j(t) - \tau_j(t) \right] - \sum_{j=1}^{n} \alpha_{ij}(t) \left[ \gamma_j(t) - \tau_j(t) \right]
\]
\[
+ \sum_{j=1}^{n} k_{ij}(t) \left[ \eta_j(t) - \tau_j(t) \right] - \sum_{j=1}^{n} k_{ij}(t) \left[ \gamma_j(t) - \tau_j(t) \right]
\]
\[
\sum_{j=1}^{n} b_{ij}(t) \left[ \eta_j(t) - \tau_j(t) \right] - \sum_{j=1}^{n} b_{ij}(t) \left[ \gamma_j(t) - \tau_j(t) \right]
\]
\[
\sum_{j=1}^{n} h_{ij}(t) \left[ \eta_j(t) - \tau_j(t) \right] - \sum_{j=1}^{n} h_{ij}(t) \left[ \gamma_j(t) - \tau_j(t) \right]
\]
\[
+ u_i(t) \right\}, \tag{3.16}
\]

Noting the fact that $\varepsilon_i(t) = (\alpha_{i}^{-1} \varepsilon_i(t))$, and the difference operator $\alpha_{i}^{-1}$ is bounded, then we can see that $\text{sign}(\varepsilon_i(t)) = \text{sign}(\varepsilon_i(t))$. By (3.14), we can have
\[
\sum_{i=1}^{n} \text{sign}(\varepsilon_i(t)) u_i(t) = \sum_{i=1}^{n} \text{sign}(\varepsilon_i(t)) \cdot \left[ -\text{sign}(\varepsilon_i(t)) \right]
\]
\[
= \left\{ \bar{\rho}_i + \lambda_i \varepsilon_i(t) + \bar{\mu}_i |\varepsilon_i(t) - \tau_i(t)| + \omega_i |\varepsilon_i(t) - \sigma_i(t)| \\
+ \tilde{\zeta}_i \int_{1 - \delta_i(t)}^{t} |\varepsilon_i(s)| ds + \tilde{\omega}_i |\varepsilon_i(t)|^\theta \right\}
\]
\[
\leq - \sum_{i=1}^{n} \bar{\rho}_i - \sum_{i=1}^{n} \lambda_i |\varepsilon_i(t)| - \sum_{i=1}^{n} \bar{\mu}_i |\varepsilon_i(t) - \tau_i(t)| \\
- \omega_i |\varepsilon_i(t) - \sigma_i(t)| - \sum_{i=1}^{n} \tilde{\zeta}_i \int_{1 - \delta_i(t)}^{t} |\varepsilon_i(s)| ds - \sum_{i=1}^{n} \tilde{\omega}_i |\varepsilon_i(t)|^\theta. \tag{3.17}
\]

Substituting (3.17) into (3.16), and based on the proof in Theorem 3.1, we directly have
\[
\frac{d\tilde{V}(e(t))}{dt} \leq \sum_{i=1}^{n} \left[ -d_i(t) - \tilde{\lambda}_i \right]
\]
\[
+ \sum_{j=1}^{n} \left[ \sum_{i=1}^{n} \left( |a_{ij}(t)| l_i + |b_{ij}(t)| l_i + |c_{ij}(t)| \right) l_j \right] |\varepsilon_i(t) - \tau_i(t)|
\]
\[
+ \sum_{i=1}^{n} \left[ -\bar{\mu}_i + \frac{n}{1 - \sum_{j=1}^{n} c_{ij}^M} |\varepsilon_i(t)| \right] \left[ -\bar{\omega}_i + \frac{n}{1 - \sum_{j=1}^{n} c_{ij}^{M}} |\varepsilon_i(t)| \right].
\]

Corollary 2: Suppose that the conditions (H1) and (H2) hold and $c_{ij}(\cdot) \equiv 0$, then the drive-response systems (2.3) and (2.4) can achieve finite-time robust synchronization based on the following controller
\[
u_i(t) = -\text{sign}(\varepsilon_i(t)) \left( \bar{\rho}_i + \lambda_i \varepsilon_i(t) + \bar{\mu}_i |\varepsilon_i(t) - \tau_i(t)| \right)
\]
\[
+ \tilde{\zeta}_i \int_{1 - \delta_i(t)}^{t} |\varepsilon_i(s)| ds + \tilde{\omega}_i |\varepsilon_i(t)|^\theta, \quad i = 1, 2, \ldots, n,
\]
if the design parameters are appropriately selected as follows
\[
\tilde{\lambda}_i = \lim_{t \to +\infty} \left\{ \bar{\rho}_i - \sum_{j=1}^{n} |a_{ij}(t)| + |b_{ij}(t)| + |c_{ij}(t)| \right\} l_j > 0, \quad i = 1, 2, \ldots, n,
\]
\[
\limsup_{t \to +\infty} \left\{ -d_i(t) - \bar{\lambda}_i + \sum_{j=1}^{n} |a_{ij}(t)| l_i \right\} \leq 0, \quad i = 1, 2, \ldots, n,
\]
\[
\limsup_{t \to +\infty} \left\{ -\bar{\mu}_i + \sum_{j=1}^{n} |a_{ij}(t)| + |b_{ij}(t)| l_i \right\} \leq 0, \quad i = 1, 2, \ldots, n,
\]
\[
\limsup_{t \to +\infty} \left\{ -\zeta_i^2 + \sum_{j=1}^{n} (|k_{ji}(t)| + |h_{ji}(t)|) \right\} \leq 0, \quad i = 1, 2, \ldots, n,
\]
\[
\zeta_i > 0, \quad i = 1, 2, \ldots, n.
\]

Moreover, the settling time \(T_{\text{max}}\) can be estimated as
\[
T_{\text{max}} = \frac{1}{\Gamma} \left( \frac{\Gamma}{\min\{\zeta_i\} n^{1-\theta}} \right)^{\frac{1}{\theta}} \left( 1 + \frac{1}{\theta - 1} \right), \quad \Gamma = \sum_{i=1}^{n} \Gamma_i,
\]
where \(\Theta > 0\) and \(i = 1, 2, \ldots, n\).

**Remark 4:** Since the control laws (3.14) contain the discontinuous sign function, it may lead to some undesirable chattering. In order to avoid this situation, a continuous function \(\tanh(\cdot)\) can be applied to replace the sign function. Thus, the control law (3.14) can be re-designed as
\[
u_i(t) = -\tanh(\vartheta_i \varepsilon_i(t)) \left( \tilde{\theta}_i + \tilde{\alpha}_i \varepsilon_i(t) \right) + \tilde{\alpha}_i \varepsilon_i(t) \left( \frac{1}{\min\{\zeta_i\} n^{1-\theta}} \right)^{\frac{1}{\theta}} \left( 1 + \frac{1}{\theta - 1} \right), \quad \Gamma = \sum_{i=1}^{n} \Gamma_i,
\]
where \(\vartheta_i > 0\) and \(i = 1, 2, \ldots, n\).

**Corollary 3:** Suppose that the conditions (H1) and (H2) hold and \(c_{i1}(\cdot) = 0\), then the drive-response systems (2.3) and (2.4) can achieve finite-time robust synchronization based on the controller (3.18) if the design parameters are appropriately selected as those in Corollary 2. Moreover, the settling time \(T_{\text{max}}\) can be estimated as
\[
T_{\text{max}} = \frac{1}{\Gamma} \left( \frac{\Gamma}{\min\{\zeta_i\} n^{1-\theta}} \right)^{\frac{1}{\theta}} \left( 1 + \frac{1}{\theta - 1} \right), \quad \Gamma = \sum_{i=1}^{n} \Gamma_i,
\]
where \(\Gamma \geq \sum_{i=1}^{n} \Gamma_i\).

**IV. NUMERICAL EXAMPLES AND SIMULATIONS**

In this section, numerical examples are dedicated to showing the effectiveness of the proposed criteria.

**Example 4.1:** Consider the three-dimensional discontinuous fuzzy neutral-type neural networks with discrete and distributed time-delays (2.1) with
\[
\begin{align*}
0.01 + 0.01 \cos(t) & \quad 0.01 + 0.01 \sin(t) & \quad 0.01 \cos(t) \\
0.01 \sin(t) & \quad 0.02 - 0.01 \cos(t) & \quad 0.03 - 0.01 \cos(t) \\
0.01 + 0.01 \sin(t) & \quad 0.01 + 0.01 \cos(t) & \quad 0.03 + 0.01 \sin(t)
\end{align*}
\]
\[
d_1(t) = 12.5 + 4.5 \cos(t), \quad d_2(t) = 12.5 + 5 \sin(t), \quad d_3(t) = 12.5 + 5 \cos(t),
\]
\[
(a_{ij}(t))_{3 \times 3} = \begin{pmatrix}
2 + \cos(t) & -10 + \sin(t) & -2 + \sin(t) \\
-7 + 2 \sin(t) & 1 + \cos(t) & 4 + \sin(t) \\
-9 + \cos(t) & -5 + 2 \sin(t) & 4 + 2 \cos(t)
\end{pmatrix},
\]
\[
(\alpha_{ij}(t))_{3 \times 3} = \begin{pmatrix}
-10 + 2 \sin(t) & -12 + 4 \sin(t) & -9 + \cos(t) \\
-6 + 2 \sin(t) & -7 + \cos(t) & -3 + 2 \sin(t)
\end{pmatrix},
\]
\[
(k_{ij}(t))_{3 \times 3} = \begin{pmatrix}
3 + \sin(t) & -6 + 3 \cos(t) & -11 + \sin(t) \\
2 + \cos(t) & 6 + 2 \cos(t) & -9 + 2 \cos(t) \\
-9 + \cos(t) & -2 + \sin(t) & -7 + 2 \sin(t)
\end{pmatrix},
\]
\[
(\beta_{ij}(t))_{3 \times 3} = \begin{pmatrix}
-3 + \sin(t) & -7 + 2 \cos(t) & -4 + 3 \sin(t) \\
-11 + \cos(t) & 6 + \sin(t) & 9 + 4 \sin(t) \\
3 + \sin(t) & -6 + 3 \cos(t) & -11 + \sin(t)
\end{pmatrix},
\]
\[
(b_{ij}(t))_{2 \times 2} = \begin{pmatrix}
5 + 2 \cos(t) & 3 + \sin(t) & 3 + 2 \sin(t) \\
-11 + \cos(t) & 7 + \sin(t) & -8 + 2 \sin(t) \\
\end{pmatrix},
\]
\[
(b_{ij})_{3 \times 3} = (T_{ij})_{3 \times 3} = (S_{ij})_{3 \times 3} = \begin{pmatrix}
0.1 & 0 & 0 \\
0 & 0.2 & 0 \\
0 & 0 & 0.3
\end{pmatrix},
\]
\[
I_1(t) = 0.7 - 0.2 \cos(t), I_2(t) = 0.9 + 0.2 \sin(t),
\]
\[
I_3(t) = 0.5 + 0.2 \cos(t), \quad v_1 = v_2 = v_3 = 1,
\]
\[
\tau_j(t) = 0.5, \quad \sigma_j(t) = 0.5, \quad \delta_j(t) = 0.5, \quad i, j = 1, 2, 3.
\]

Then, \(\varsigma = 0.5\). Moreover, let
\[
f_1(x) = f_2(x) = f_3(x) = 0.3 \tan(x) + 0.2 \cos(x) + 0.3, x \geq 0;
\]
\[
0.3 \tan(x) + 0.2 \sin(x) - 0.2, x < 0.
\]

Obviously, the activation function \(f_j(x)\) is discontinuous and non-monotonic. The activation function \(f_j(x)\) has a discontinuous point \(x = 0\) and \(\overline{f_1(0)} = [f_1^+(0), f_1^-(0)] = [-0.2, 0.5], i = 1, 2, 3, L_{t+1} = 2.5\) and \(\tau_{t+1} = 0.3\), then (H1) and (H2) are satisfied. This fact can be seen in Figure 1.

**FIGURE 1:** Discontinuous activation functions \(f_j(j = 1, 2)\) for drive system.

Let take (2.1) as the drive system, and design three-dimensional (2.2) as the response system.

When there is no control append to the response system, the dynamic behavior of drive-response system is presented in Figures 2.

Let us select the control gains as following
\[
\rho_1 = 19, \quad \rho_2 = 19, \quad \rho_3 = 19, \quad \sigma_1 = \sigma_2 = \sigma_3 = 1,
\]
\[
\lambda^* = 0.38, \quad \lambda^*_2 = 0.71, \quad \lambda^*_3 = 1.44,
\]
\[
\mu_1 = 18, \quad \mu_2 = 14.5, \quad \mu_3 = 9.5,
\]
\[
\omega_1 = 1.22, \quad \omega_2 = 1.43, \quad \omega_3 = 1.01,
\]
\[
\zeta_1 = 11, \quad \zeta_2 = 20, \quad \zeta = 25.
\]

By simple computation, we can further have
\[
\Gamma_1 = \liminf_{t \to +\infty} \left\{ \rho_1 \left( |a_{11}(t)| + |a_{11}(t)| + |B_{11}(t)| + |k_{11}(t)| \right) + |a_{12}(t)| + |a_{12}(t)| + |B_{12}(t)| + |k_{12}(t)| \right) = 11(11) \right) + |a_{12}(t)| + |a_{12}(t)| + |B_{12}(t)| + |k_{12}(t)| \right) + |a_{13}(t)| + |a_{13}(t)| + |B_{13}(t)| + |k_{13}(t)|
\]
\[
\end{align*}
\]

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Consider the three-dimensional discontinuous fuzzy neutral-type neural networks with discrete and distributed time-delays (2.1) with
\[
\begin{align*}
\rho_1 &= 9, \quad \rho_2 = 7, \quad \rho_3 = 9, \quad \lambda_1 = 0.9, \quad \lambda_2 = 8.6, \quad \lambda_3 = 3.3, \\
\tilde{\mu}_1 &= 19, \quad \tilde{\mu}_2 = 14, \quad \tilde{\mu}_3 = 12, \quad \tilde{\omega}_1 = 1.1, \quad \tilde{\omega}_2 = 0.7, \quad \tilde{\omega}_3 = 0.5, \\
\tilde{\zeta}_1 &= 8.5, \quad \tilde{\zeta}_2 = 14.5, \quad \tilde{\zeta}_3 = 12.5, \quad \tilde{\xi}_1 = 7, \quad \tilde{\xi}_2 = 4, \quad \tilde{\xi}_3 = 7.
\end{align*}
\]
and initial values \(\phi(x) = (0.1, 0.5, 0.2)^T, \varphi(s) = (0.4, 0.3, 0.1)^T, s \in [-0.5, 0].\)

\[
\begin{align*}
\left[ \begin{array}{c} x_1(t) \\ x_2(t) \\ x_3(t) \\ y_1(t) \\ y_2(t) \\ y_3(t) 
\end{array} \right] &= \left[ \begin{array}{c} \sin(t) + \cos(t) \\ \sin(t) + \cos(t) \\ \sin(t) + \cos(t) \\ -\sin(t) - \cos(t) \\ -\sin(t) - \cos(t) \\ -\sin(t) - \cos(t) \n\end{array} \right].
\end{align*}
\]

![Figure 2: Time evolution of variables](image)

![Figure 3: The synchronization errors between the drive-response systems with initial values](image)
then by simple computation, we can have
\[
\begin{align*}
\bar{\Gamma}_1 &= \liminf_{t \to +\infty} \left\{ \tilde{\rho}_1 - (|a_{11}(t)| + |\alpha_{11}(t)| + |\beta_{11}(t)|) \\
&\quad + |k_{11}(t)||\delta_1(t)| + |h_{11}(t)||\beta_1(t)||\Pi_1 - (|a_{12}(t)| + |\alpha_{12}(t)| + |\beta_{12}(t)| + |k_{12}(t)||\delta_2(t)| + |h_{12}(t)||\beta_2(t)||\Pi_2 \\
&\quad - (|a_{13}(t)| + |\alpha_{13}(t)| + |\beta_{13}(t)| + |k_{13}(t)||\delta_3(t)|) + |h_{13}(t)||\beta_3(t)||\Pi_3 \right\} = 1.6 > 0,
\end{align*}
\]
\[
\begin{align*}
\bar{\Gamma}_2 &= \liminf_{t \to +\infty} \left\{ \tilde{\rho}_2 - (|a_{21}(t)| + |\alpha_{21}(t)| + |\beta_{21}(t)|) \\
&\quad + |k_{21}(t)||\delta_1(t)| + |h_{21}(t)||\beta_1(t)||\Pi_1 - (|a_{22}(t)| + |\alpha_{22}(t)| + |\beta_{22}(t)| + |k_{22}(t)||\delta_2(t)| + |h_{22}(t)||\beta_2(t)||\Pi_2 \\
&\quad - (|a_{23}(t)| + |\alpha_{23}(t)| + |\beta_{23}(t)| + |k_{23}(t)||\delta_3(t)|) + |h_{23}(t)||\beta_3(t)||\Pi_3 \right\} = 1.3 > 0,
\end{align*}
\]
\[
\begin{align*}
\bar{\Gamma}_3 &= \liminf_{t \to +\infty} \left\{ \tilde{\rho}_3 - (|a_{31}(t)| + |\alpha_{31}(t)| + |\beta_{31}(t)|) \\
&\quad + |k_{31}(t)||\delta_1(t)| + |h_{31}(t)||\beta_1(t)||\Pi_1 - (|a_{32}(t)| + |\alpha_{32}(t)| + |\beta_{32}(t)| + |k_{32}(t)||\delta_2(t)| + |h_{32}(t)||\beta_2(t)||\Pi_2 \\
&\quad - (|a_{33}(t)| + |\alpha_{33}(t)| + |\beta_{33}(t)| + |k_{33}(t)||\delta_3(t)|) + |h_{33}(t)||\beta_3(t)||\Pi_3 \right\} = 0.55 > 0,
\end{align*}
\]
and the other inequalities are also satisfied. Therefore, all the conditions in Theorem 3.2 are satisfied. Namely, under the following discontinuous state-feedback controller:
\[
\begin{align*}
u_1(t) &= -\text{sign}(\varepsilon_1(t))\left(9 + 0.9|\varepsilon_1(t)| + 19|\varepsilon_1(t) - \tau_1(t)| \right. \\
&\quad + 1.1|\varepsilon_1(t) - \sigma_1(t)| + 8.5 \int_{t - \delta_1(t)}^{t} |\varepsilon_1(s)|ds \\
&\quad + 7|\varepsilon_1(t)|^{5.3},
\end{align*}
\]
\[
\begin{align*}
u_2(t) &= -\text{sign}(\varepsilon_2(t))\left(7 + 8.6|\varepsilon_2(t)| + 14|\varepsilon_2(t) - \tau_2(t)| \right. \\
&\quad + 0.7|\varepsilon_2(t) - \sigma_2(t)| + 14.5 \int_{t - \delta_2(t)}^{t} |\varepsilon_2(s)|ds \\
&\quad + 4|\varepsilon_2(t)|^{5.3},
\end{align*}
\]
\[
\begin{align*}
u_3(t) &= -\text{sign}(\varepsilon_3(t))\left(9 + 3.3|\varepsilon_3(t)| + 12|\varepsilon_3(t) - \tau_3(t)| \\
&\quad + 0.5|\varepsilon_3(t) - \sigma_3(t)| + 12.5 \int_{t - \delta_3(t)}^{t} |\varepsilon_3(s)|ds \\
&\quad + 7|\varepsilon_3(t)|^{5.3},
\end{align*}
\]
where \(\varepsilon_i(t) = y_i(t) - x_i(t), \) \(i = 1, 2, 3,\) the response system converges to the drive system with in a finite time. Furthermore, the settling time is
\[
T_{\text{max}} = \frac{1}{\Gamma} \left(\frac{\bar{\Gamma}}{\min_i \{\theta_i - 1\}}\right)^{\frac{1}{\theta}} \left(1 + \frac{1}{\theta - 1}\right) \approx 4.85(s),
\]
where \(\bar{\Gamma} = \sum_{i=1}^{n} \bar{\Gamma}_i.\) This fact can be illustrated by Figure 5.
The authors declare that they have no competing interests.

COMPETING INTERESTS

The authors declare that they have no competing interests.

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