Abstract—This paper considers the satellites autonomous navigation problem with an extended sliding mode observer method. First, on consideration that the relative dynamic equation of chaser and target spacecrafts refers to a nonlinear time-varying system, a transformation is performed on the original relative dynamic equation to obtain a linear time-invariant system with a nonlinear part. Second, an augmentation strategy is made on the new system to obtain a descriptor system. A new extended sliding mode observer is then designed for the descriptor system which can generate the simultaneous estimation of state and disturbance sliding mode observer method.

Index Terms—satellites autonomous navigation; State estimation; Sliding mode observer

I. INTRODUCTION

Due to the fast development of aerospace technology, more and more space missions have been raised and realized being regarded as advanced space applications, for instance, on-orbit servicing, space materials transport and space emergency rescue, active debris removal, earth gravity field detection, magnetic fields detection [1], [2], [3], etc. In these tasks, the leader spacecraft (chaser) is always required to implement autonomous relative navigation maneuvers (rendezvous and docking, station keeping, and monitoring) when it falls in a close-proximity of the prescribed target. Furthermore, it is desirable for the spacecraft to have the ability to determine relative position and velocity in real time without the assistance of external support, which can then achieve a distinguished reduction in ground operating costs [4], [5]. In this sense, autonomous navigation technique has been developed in spacecraft fields to satisfy increasing realistic engineering requirements [6], [7]. Compared to traditional absolute satellite navigation, autonomous navigation can provide immediate availability for spacecraft position and velocity estimates measurements, and further realize low cost on satellites and less ground-based operations. Due to these advantages, autonomous navigation has become an active area of research which attracted considerable interest over the past few years [8], [9].

In fact, during the design of autonomous navigation system for spacecraft, the kalman filter theory has become the most significant filtering algorithm for the cruise phase since there exists nonlinear pole for the orbital dynamics equation of the spacecraft. In this research forefront, there has been a few popular autonomous navigation methods reported in the existing literature, for instance, extended Kalman filter method (EKF) [10], Unscented Kalman Filter (UKF) [11], [12], and Square root unscented Kalman filter (SRUKF) [13], [14], etc. Among these methods, the main idea of EKF is to perform linearization for the kinematics and dynamics up to first order term of Taylor series expansion, which however may result in unavoidable large errors and further degrade the convergence performance of the algorithm. With consideration on the limitation of EKF, scholars have developed UKF method in [11], which makes approximation for the probability density by evaluating the nonlinear function with sigma points [15], [16]. However, UKF may not achieve ideal performance for weak observability system with large initial error. Due to this fact, the IUKF (Iterated Unscented Kalman Filter) method has been presented which is an extension of UKF and can adjust the state estimation to converge to the true value online by corrections of the measurement [17], [18].

It should be pointed out that, however, in autonomous navigation for realistic spacecraft task, external disturbance, and non-white output measured noise with always exist inevitably [19], [20], which may increase the calculation complexity the aforementioned autonomous navigation filter approaches (EKF, UKF, and SRUKF) [21], [22], [23]. In addition, all these conventional autonomous navigation filtering method may not be effective and cannot achieve desired accuracy in the presence of external disturbance, input disturbance and non-white measure noise [24], [25], [26], [27]. Hence, it is of both theoretical significance and practical aerospace requirement to develop new effective autonomous navigation approaches to cope with this issue. The objective of this note is, therefore, to develop a new extended sliding mode observer method to fully solve this design problem [28], [29].

In this paper, we investigate the autonomous navigation for satellite formation problem with the sliding mode observer method in the presence of external disturbance and non-white output measure noise. An extended sliding mode observer approach is proposed to solve the considered design problem, and the design procedure of the presented observer method is as follows: i) an augmentation scheme is first developed for the relative dynamic model of the leader spacecraft and the follower spacecraft to obtain a descriptor system, where the output measured noise are assembled into the extended vector...
of the new system; ii) a new sliding mode observer is designed for the descriptor system, which can generate simultaneous estimates of the state vector and output measured noise; and iii) based on the estimation, the equivalent output error injection method is used to reconstruct the external disturbance. Finally, a simulation example is given to demonstrate the effectiveness of the developed autonomous navigation observer technique.

II. PROBLEM FORMULATION

Suppose there are $\mathcal{N}$ spacecrafts for the spacecraft formation, the dynamic model of the leader spacecraft is as follows:

$$\ddot{\mathbf{r}}_l = -\frac{G(m_e + m_l)}{r^3_l} \mathbf{r} + f_l + u_l$$  \hspace{0.5cm} (1)

and the dynamic model of the $i$th ($i \in \mathcal{N}$) follower spacecraft is as follows:

$$\ddot{\mathbf{r}}_{fi} = -\frac{G(m_e + m_{fi})}{r^3_{fi}} \mathbf{r} + f_{fi} + u_{fi}$$  \hspace{0.5cm} (2)

where $\mathbf{r}_l$ and $\mathbf{r}_{fi}$ denotes the position vector of the leader and follower spacecrafts in the Earth-centered inertial frame, $m_e$, $m_l$ and $m_{fi}$ represent the masses of earth, the leader spacecraft and the follower spacecraft, respectively, $f_l$ and $f_{fi}$ denote the external disturbance of the leader and follower spacecrafts, $G$ denotes the gravitational constant, $u_l$ and $u_{fi}$ denotes the control input of the the leader spacecraft and the follower spacecraft. Define the Earth’s center as the origin of the inertial coordinate system $S = XYZ$, $\mathbf{r} = (0, r, 0)^T$ is the position vector of the leader spacecraft relative to coordinate system. Suppose the orbital rotational speed of the leader spacecraft is

$$\dot{\theta} = \sqrt{\frac{\mu}{r^3}},$$

$$\ddot{\theta} + \frac{2\dot{\theta}}{r} = 0,$$  \hspace{0.5cm} (3)

where $\mu$ denote the gravitational parameter of the earth.

We denote $\mathbf{p}_i(t) = [x_i \ y_i \ z_i]^T$ as the relative position vector of the follower spacecraft relative to the leader spacecraft, based on (1)-(3) we have the following

$$\dot{x}_i - 2\dot{\theta} y_i - \dot{\theta}^2 x_i - \dot{y}_i + \frac{\mu x_i}{|\mathbf{r} + \mathbf{p}_i|^3} = \frac{u_{ix}}{m_{fi}} + f_{ix}$$

$$\dot{y}_i + 2\dot{\theta} x_i + \dot{\theta}^2 y_i + \frac{\mu(y_i + r)}{|\mathbf{r} + \mathbf{p}_i|^3} - \frac{\mu r}{|\mathbf{r} + \mathbf{p}_i|^3} = \frac{u_{iy}}{m_{fi}} + f_{iy}$$

$$\dot{z}_i + \frac{\mu z_i}{|\mathbf{r} + \mathbf{p}_i|^3} = \frac{u_{iz}}{m_{fi}} + f_{iz}$$  \hspace{0.5cm} (4)

We define the state vector $\mathbf{x}_i(t) = [\mathbf{p}_i \dot{\mathbf{p}}_i]^T$ and $R_i = |\mathbf{r} + \mathbf{p}_i|$, then we can obtain the following state equation

$$\dot{\mathbf{x}}_i(t) = A_i \mathbf{x}_i(t) + B_i u_i(t) + g_i(\mathbf{x}_i(t)) + d_i(t)$$  \hspace{0.5cm} (5)

where $u_i(t) = [u_{ix} \ u_{iy} \ u_{iz}]^T$, and

$$A_i = \begin{bmatrix} 0_{3 \times 3} & I_3 \\ A_{i1} & A_{i2} \end{bmatrix}, \quad B_i = \frac{1}{m_{fi}} \begin{bmatrix} 0_{3 \times 3} \\ I_3 \end{bmatrix},$$

$$A_{i1} = \begin{bmatrix} \dot{\theta}^2 & \dot{\theta} & 0 \\ -\dot{\theta} & \ddot{\theta} & 0 \\ 0 & 0 & -\dot{\theta}^2 \end{bmatrix}, \quad A_{i2} = \begin{bmatrix} 0 & 2\dot{\theta} & 0 \\ -2\dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$h_i(\mathbf{x}_i(t)) = \begin{bmatrix} -\frac{\mu x_i(t)}{r_i^3} - \frac{\mu r}{r_i^3} + \frac{\mu r_i}{r^2} \dot{z}_i(t) \\ \frac{0_{3 \times 1}}{h_i(\mathbf{x}_i(t))} \end{bmatrix},$$

$$g_i(\mathbf{x}_i(t)) = \begin{bmatrix} 0_{3 \times 1} \\ h_i(\mathbf{x}_i(t)) \end{bmatrix},$$

$$f_i(t) = [f_{ix} \ f_{iy} \ f_{iz}]^T,$$

$$d_i(t) = [0_{3 \times 1} \ f_{ix} \ f_{iy} \ f_{iz}]^T.$$  \hspace{0.5cm} (6)

On the other hand, the output measurement is given as follows:

$$y(t) = \hat{\mathbf{x}}_i(t) + Dw(t)$$  \hspace{0.5cm} (7)

where

$$w(t) = [w_1(t) \ w_2(t) \ w_3(t)]^T$$  \hspace{0.5cm} (8)

denotes the unknown non-white measurements noise with $w_j(t) \in \mathbb{R}$ ($i = 1, 2, 3$). As a result, we obtain the following system

$$\begin{cases} \dot{\mathbf{x}}_i(t) = A_i \mathbf{x}_i(t) + B_i u_i(t) + g_i(\mathbf{x}_i(t)) + d_i(t), \\ y(t) = \hat{\mathbf{x}}_i(t) + Dw(t), \end{cases}$$  \hspace{0.5cm} (9)

where $D \in \mathbb{R}^{6 \times 3}$ is system matrix with $D = [0_{3 \times 3} \ D_1^T]^T$, $D_1 \in \mathbb{R}^{3 \times 3}$.

Note that in system (9), the nonlinear function $g_i(\mathbf{x}_i(t))$ and external disturbance $d_i(t)$ exist with a “mismatched” form. In the following discussion, we should consider system (9) to transmit it into a standard form with matched nonlinear function and disturbance to facilitate the observer design work.

We rewritten $A_i$ as

$$\tilde{A}_i = \begin{bmatrix} 0_{3 \times 3} & I_3 \\ 0_{3 \times 3} & -0.5I_3 \end{bmatrix} + \begin{bmatrix} 0_{3 \times 3} & 0_{3 \times 3} \\ A_{i1} & A_{i2} + 0.5I_3 \end{bmatrix}$$  \hspace{0.5cm} (10)

and we then have

$$\tilde{A}_i \tilde{x}_i(t) = \begin{bmatrix} 0_{3 \times 3} & I_3 \\ 0_{3 \times 3} & -0.5I_3 \end{bmatrix} \tilde{x}_i(t) + \begin{bmatrix} 0_{3 \times 3} & 0_{3 \times 3} \\ A_{i1} & A_{i2} + 0.5I_3 \end{bmatrix} \tilde{x}_i(t)$$

$$= \begin{bmatrix} 0_{3 \times 3} & I_3 \\ 0_{3 \times 3} & -0.5I_3 \end{bmatrix} \tilde{x}_i(t) + \begin{bmatrix} 0_{3 \times 1} \\ A_{i1} \tilde{p}_i + (A_{i2} + 0.5I_3)\tilde{p}_i \end{bmatrix}$$  \hspace{0.5cm} (11)

Therefor, the system (9) can be rewritten as

$$\begin{cases} \dot{\mathbf{x}}_i(t) = A \tilde{x}_i(t) + B_i u_i(t) + g_i(\mathbf{x}_i(t)) + d_i(t) + \begin{bmatrix} 0_{3 \times 1} \\ (A_{i1} \tilde{p}_i + (A_{i2} + 0.5I_3)\tilde{p}_i) \end{bmatrix}^T, \\ y(t) = \tilde{x}_i(t) + Dw(t), \end{cases}$$  \hspace{0.5cm} (12)

where

$$A = \begin{bmatrix} 0_{3 \times 3} & I_3 \\ 0_{3 \times 3} & -0.5I_3 \end{bmatrix}.$$  \hspace{0.5cm} (13)
Considering the nonlinearity and disturbance term in system (12), we rewrite it as
\[ \begin{bmatrix} 0^T_{3 \times 1} & \left( \hat{A}_{i1} \hat{p}_i + (\hat{A}_{i2} + 0.5I_3) \hat{p}_i \right)^T \end{bmatrix}^T + g_i(\hat{x}_i(t)) + d_i(t) = \begin{bmatrix} A_{i1} \hat{p}_i + (\hat{A}_{i2} + 0.5I_3) \hat{p}_i + h_i(t) + f_i(t) \end{bmatrix} \]
\[ = \frac{1}{m_i} \begin{bmatrix} 0^T_{3 \times 3} \end{bmatrix} \begin{bmatrix} A_{i1} \hat{p}_i + (\hat{A}_{i2} + 0.5I_3) \hat{p}_i + h_i(t) + f_i(t) \end{bmatrix} \]
\[ = B_1(\psi_i(x_i(t), f_i(t))) \] (14)
with \( \psi_i(x_i(t), f_i(t)) = m_i(\hat{A}_{i1} \hat{p}_i + (\hat{A}_{i2} + 0.5I_3) \hat{p}_i + h_i(t) + f_i(t)) \).

As a result, system (12) is finally transferred into the following form
\[ \begin{cases} \dot{x}_i(t) = A\bar{x}_i(t) + B_1(u_i(t) + \psi_i(x_i(t), f_i(t))) \\ y(t) = \bar{x}_i(t) + Dw(t) \end{cases} \] (15)

We now consider the system (15) where \( A \) is an open loop system matrix which is not Hurwitz. To facilitate the analysis design, we propose the control input term as \( u_i(t) = u_{i1}(t) + u_{i2}(t) \) with
\[ \begin{align*}
u_{i1}(t) &= K_iy(t) = K_i(\hat{x}_i(t) + Dw(t)), \\
u_{i2}(t) &= -m_i(\hat{A}_{i1} \hat{p}_i(t) - m_i(\hat{A}_{i2} + 0.5I_3) \hat{p}_i(t) + D_1w(t)) - m_i h_i(t),
\end{align*} \]
where \( K_i \) is designed such that \( A + B_i K_i \) is Hurwitz for each \( i \in \mathcal{N} \). Note that
\[ y(t) = \bar{x}_i(t) + Dw(t) = \begin{bmatrix} \bar{p}_i \bar{x}_i \end{bmatrix} + \begin{bmatrix} 0 \\ D_1 \end{bmatrix} w(t) \]
is known, thus all the signals in (16) are available.

By substituting the controller (16) into (15), we have the following closed-loop system
\[ \begin{cases} \dot{x}_i(t) = (A + B_i K_i) \bar{x}_i(t) + B_1(\psi_i(x_i(t), f_i(t))) \\ y(t) = C\bar{x}_i(t) + Dw(t), \end{cases} \] (17)
where \( \psi_i(x_i(t), f_i(t)) = [K_i D - m_i(\hat{A}_{i2} + 0.5I_3)D_1]w(t) + f_i(t) \).

The following assumptions are made for system (17) throughout this paper.

**Assumption 1:** The coefficient matrix \( D_1 \) is nonsingular.

**Assumption 2:** For each \( i \in \mathcal{N} \), given \( \forall s \in \mathcal{C}_+ \), the following rank condition holds:
\[ \text{rank} \left( \begin{bmatrix} sI_9 - (A + B_i K_i) & 0_{6 \times 3} \\ C & D \end{bmatrix} \right) = 12. \]
(18)

**Assumption 3:** For each \( i \in \mathcal{N} \), the non-white measurements noise \( w(t) \) and the external disturbance \( f_i(t) \) satisfy the following norm constraints
\[ \|w(t)\| \leq w_{\text{max}}, \quad \|f_i(t)\| \leq f_{\text{max}} \] (19)
where \( w_{\text{max}} > 0 \) and \( f_{\text{max}} > 0 \) are known constants.

The main purpose of this paper is to develop an effective robust observer methodology for system (17) to obtain the asymptotical estimates of \( \dot{x}_i(t) \) and \( d_i(t) \).

### III. EXTENDED SLIDING MODE OBSERVER DESIGN

In this section, we will present a novel sliding mode observer approach for system (17) to reconstruct the state vector and unknown input disturbance \( d_i(t) \) simultaneously.

We define the following augmented vector and matrices:
\[ \begin{bmatrix} \dot{\bar{x}}_i(t) = [\dot{x}_i(t), \dot{w}(t)]^T \end{bmatrix} \in \mathbb{R}^9, \]
\[ E = [I_6 \ 0_{6 \times 3}] \in \mathbb{R}^{6 \times 9}, \]
\[ \bar{A}_i = [A + B_i K_i \ 0_{6 \times 3}] \in \mathbb{R}^{6 \times 9}, \]
\[ \bar{C} = [I_6 \ D] \in \mathbb{R}^{6 \times 9}, \]
and build the following augmented singular system
\[ \begin{cases} E\ddot{x}_i(t) = \bar{A}_i \bar{x}_i(t) + B_1 \psi_i(x_i(t), f_i(t)), \\
y(t) = \bar{C}\bar{x}_i(t). \end{cases} \] (20)

To design a sliding mode observer for system (20), we first introduce two parameter matrices \( T \in \mathbb{R}^{9 \times 6} \) and \( \bar{L} \in \mathbb{R}^{9 \times 6} \) which satisfy
\[ [T \ \bar{L}] \begin{bmatrix} ET & \bar{C}T \end{bmatrix}^T = I_9, \]
and define
\[ \bar{E} \triangleq [ET \ \bar{C}T]^T \in \mathbb{R}^{12 \times 9}, \]
then (21) can be rewritten as
\[ [T \ \bar{L}] \bar{E} = I_9, \]
(23)

Since \( \bar{E} \in \mathbb{R}^{12 \times 9} \), we can always find a solution for the equation (23) with the following form
\[ [T \ \bar{L}] = E^\dagger - G(I_{12} - \bar{E}\bar{E}^\dagger), \]
(24)
where \( G \in \mathbb{R}^{9 \times 12} \) is an arbitrary matrix, and \( E^\dagger \in \mathbb{R}^{9 \times 12} \) represents the generalized inverse of \( E \). In particular, we can set \( E^\dagger = (\bar{E}^T \bar{E})^{-1} \bar{E}^T \), and in this case a special solution of (21) is
\[ [T \ \bar{L}] = (\bar{E}^T \bar{E})^{-1} \bar{E}^T. \]
(25)

In the following analysis, \( T \) and \( \bar{L} \) will be important for the design work.

The next lemma reveals the minimum phase equivalence for the triples \( (A + B_i K_i, \bar{C}, B_i) \) and \( \{T \bar{A}_i, \bar{C}, TB_i\} \).

**Lemma 1:** All the invariant zeros of the triple \( \{T \bar{A}_i, \bar{C}, TB_i\} \) will fall into the open left-hand complex plant, or
\[ \text{rank} \left( \begin{bmatrix} sI_9 - T \bar{A}_i & TB_i \\ \bar{C} & 0_{6 \times 3} \end{bmatrix} \right) = 12 \] (26)
holds, if and only if the rank condition (18) holds.

**Proof:** First, we define the following new matrices
\[ \Pi_1 = \begin{bmatrix} I_n & 0 & -sC^T \\ 0 & I_3 & -sD^T \\ 0 & 0 & I_6 \end{bmatrix}, \]
\[ \Pi_{2i} = \begin{bmatrix} sI_n + sC^T \bar{C} - (A + B_i K_i) & sC^T D \\ sD^T \bar{C} & sD^T D \end{bmatrix}. \]
Considering the matrix rank in left side of (26), in fact, we can derive that
The following condition holds

\[
\text{rank}\left(\begin{bmatrix}
sI_9 - T\bar{A}_i & TB_i \\
\bar{C} & 0_{3\times 3}
\end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix}
(sI_9 - (sE^T \bar{E})^{-1} \bar{C}) & 0_{9\times 6} & \bar{A}_i & 0_{3\times 6} & B_i
\end{bmatrix}\right)
\]

\[
= \text{rank}\left(\begin{bmatrix}
sI_9 - (sE^T \bar{E})^{-1} \bar{C} & 0_{9\times 6} & \bar{A}_i & 0_{3\times 6} & B_i
\end{bmatrix}\right)
\]

\[
= \text{rank}\left(\begin{bmatrix}
sI_9 - (sE^T \bar{E})^{-1} \bar{C} & 0_{9\times 6} & \bar{A}_i & 0_{3\times 6} & B_i
\end{bmatrix}\right)
\]

\[
= \text{rank}\left(\begin{bmatrix}
sI_9 - (sE^T \bar{E})^{-1} \bar{C} & 0_{9\times 6} & \bar{A}_i & 0_{3\times 6} & B_i
\end{bmatrix}\right)
\]

Assumption 4: The following condition holds

\[
\text{rank}(CTB_i) = \text{rank}(TB_i). 
\] (28)

Assumption 4 is important since from [30] we know that there exists a sliding mode observer for system (20) if and only if conditions (18) and (28) are guaranteed.

We provide the following lemma which is on the observer gains design for system (20).

**Lemma 2:** [30] Conditions (18) and (28) hold if and only if there exist symmetric positive definite matrices \( \bar{P}_i \in \mathbb{R}^{9\times 9} \), and matrices \( \bar{K}_i \in \mathbb{R}^{9\times 6} \) and \( H_i \in \mathbb{R}^{3\times 6} \) such that

\[
(T\bar{A}_i - \bar{K}_i\bar{C})^T \bar{P}_i + \bar{P}_i(T\bar{A}_i - \bar{K}_i\bar{C}) < 0, \tag{29}
\]

\[
(TB_i)^T \bar{P}_i = H_i\bar{C}. \tag{30}
\]

Notice that conditions (29) and (30) are difficult to be solved directly, here we use the method in [31] to solve this problem: conditions (29) and (30) is equivalent to the following minimization problem:

\[
\min \beta, \quad \text{subject to} \quad \begin{bmatrix}
\bar{A}_i^T \bar{A}_i + \bar{P}_i + \bar{P}_i(T\bar{A}_i - \bar{K}_i\bar{C}) - \bar{C}^TY_i^T - Y_i\bar{C} < 0,

\begin{bmatrix}
-\beta I_9 \\
-(TB_i)^T \bar{P}_i - H_i\bar{C}
\end{bmatrix}^T \bar{I}_3
\end{bmatrix} < 0, \tag{31}
\]

where \( \beta > 0 \) is the parameter to be minimized. It is shown that the linear matrices constraints (31)-(31) are linear and can be solved directly by Matlab toolbox.

Based on the above analysis, we now introduce the following extended sliding mode observer for system (20)

\[
\begin{align*}
\hat{\dot{x}}(t) &= \bar{F}_i\hat{x}(t) + \bar{L}_{oi}\hat{y}(t) + TB_iu_s(t), \\
\hat{\hat{x}}(t) &= z(t) + L_y(t),
\end{align*}
\]

where \( \hat{x}(t) = [\hat{x}^T(t) \hat{w}(t)]^T \) is the estimate of \( \bar{x}(t) \), \( z(t) \in \mathbb{R}^9 \) is the intermediate variable of the observer, \( \bar{F}_i \in \mathbb{R}^{9\times 9} \) and \( \bar{L}_{oi} \in \mathbb{R}^{1\times 9} \) refers to the observer gain to be synthesized. We define the following error vector

\[
e(t) = \hat{x}(t) - \bar{x}(t).
\]

Based on condition (21) it is shown that \( T\bar{E} + \bar{L}\bar{C} = I_9 \).

Therefore, we can derive the error system as follows

\[
\begin{align*}
\dot{\epsilon}(t) &= \hat{x}(t) - \bar{x}(t) \\
&= \dot{\bar{x}}(t) + \bar{L}\bar{C}\hat{x}(t) - \bar{x}(t) \\
&= \dot{\bar{x}}(t) - T\bar{E}\hat{x}(t) \\
&= \bar{F}_i(\hat{x}(t) - \bar{L}\bar{C}\bar{x}(t)) + \bar{L}_{oi}\bar{C}\bar{x}(t) + TB_iu_s(t) \\
&\quad - T\bar{A}_i\hat{x}(t) + B_i\hat{\psi}_i(x_i(t), d_i(t)) \\
&= \bar{F}_i\epsilon(t) + (\bar{F}_i - \bar{F}_i\bar{L}\bar{C} + \bar{L}_{oi}\bar{C} - T\bar{A}_i)\bar{x}(t) \\
&\quad + TB_i(u_s(t) - \hat{\psi}_i(x_i(t), f_i(t))) \tag{33}
\end{align*}
\]

We design the observer gain \( \bar{F}_i \) and \( \bar{L}_{oi} \) as follows

\[
\bar{F}_i \triangleq T\bar{A}_i - \bar{K}_i\bar{C}, \quad \bar{L}_{oi} \triangleq \bar{K}_i + \bar{F}_i\bar{L}, \tag{34}
\]

and it can be obtained that

\[
\bar{F}_i - \bar{F}_i\bar{L}\bar{C} + \bar{L}_{oi}\bar{C} - T\bar{A}_i \\
= T\bar{A}_i - \bar{K}_i\bar{C} - (T\bar{A}_i - \bar{K}_i\bar{C})\bar{L}\bar{C} + (\bar{K}_i + \bar{F}_i\bar{L})\bar{C} \\
- T\bar{A}_i \\
= 0.
\]

As a result, error system (33) becomes

\[
\begin{align*}
\dot{\epsilon}(t) &= \bar{F}_i\epsilon(t) + TB_i[u_s(t) - \hat{\psi}_i(x_i(t), f_i(t))] \tag{35}
\end{align*}
\]

We now investigate the design problem of the discontinuous input \( u_s(t) \) in error dynamic (35). We design the following sliding surface

\[
s_i(t) = (B_i)^T T^T \bar{P}_i\epsilon(t) = H_i\bar{C}\epsilon(t). \tag{36}
\]

where matrices \( T, \bar{P}_i \) and \( H_i \) have been defined in the aforementioned discussion.

Recalling that \( \hat{\psi}_i(x_i(t), f_i(t)) = (K_iD - m_i\bar{A}_i\bar{D})w(t) + d_i(t) \), to eliminate the effect of \( \hat{\psi}_i(x_i(t), f_i(t)) \), we design \( u_s(t) \) as the following form

\[
u_s(t) = -\left(\| (K_iD - m_i\bar{A}_i\bar{D})\| w_{\max} + f_{\max}

+ \alpha_i + \varepsilon_1\right) \times \text{sign}(s_i(t)), \tag{37}
\]

where \( \alpha_i > 0 \) is the gain parameter to be designed in Theorem 2 below, and \( \varepsilon_1 > 0 \) is a small scalar. The following theorem is on stability analysis of the error system (35).

**Theorem 1:** Considering the error system (33), it is asymptotically stable under the designed discontinuous term (37).

**Proof:** For the error system (35), define the following Lyapunov function \( V_1(t) = e^T(t)\bar{P}_i\epsilon(t) \), then we have

\[
\dot{V}(t) \leq e^T(t)(\bar{P}_i\dot{F}_i + \bar{F}_i\bar{P}_i)\epsilon(t) \\
+ 2e^T(t)\bar{P}_iTB_i(u_s(t) - \hat{\psi}_i(x_i(t), f_i(t))) \tag{38}
\]

From (29) it is known that \( \hat{P}_i \hat{F}_i + \hat{F}_i^T \hat{P}_i < 0 \) holds. On the other hand, the following derivation can be obtained

\[
2s_i^T(t)(u_s(t) - \hat{\psi}_i(x_i(t), f_i(t))) \\
\leq 2s_i^T(t)u_s(t) + 2|s_i(t)||\hat{\psi}_i(x_i(t), f_i(t))| \\
\leq -2s_i^T(t)((K_iD - m_ifA_{\varepsilon 2}D_1)||w(t)|| + |d_i(t)|| \\
+ \alpha_1 + \varepsilon_1)sgn(s_i(t)) \\
+ 2||s_i(t)||((K_iD - m_ifA_{\varepsilon 2}||w_{\text{max}} + f_{\text{max}}) \\
\leq -2\varepsilon_1||s_i(t)||.
\]

Hence, it can be proved that \( \dot{V}(t) \leq -2\varepsilon_1||s_i(t)|| < 0 \) for \( \forall s_i(t) \neq 0 \). As a result, the error system (35) is asymptotically stable.

**IV. EXTERNAL DISTURBANCE RECONSTRUCTION**

In the previous section, we have proposed an extended observer technique to obtain the estimation of state vector \( x(t) \) and output disturbance \( w(t) \). In this section, based on the proposed observer technique, we will reconstruct the external disturbance \( f_i(t) \) based on the equivalent output error injection approach.

Considering the state equation of error system (35)

\[
\dot{e}(t) = \hat{F}_i e(t) + TB_i[u_s(t) - \hat{\psi}_i(x_i(t), f_i(t))]
\]

(39)

We define the following new matrices

\[
S_i = \begin{bmatrix}
B_i \\
H \hat{C}_i
\end{bmatrix} \in \mathbb{R}^{9 \times 9},
\]

\[
\mathcal{A}_i = \begin{bmatrix}
\mathcal{A}_1 & \mathcal{A}_2 \\
\mathcal{A}_3 & \mathcal{A}_4
\end{bmatrix} = S_i(T \hat{A}_i - \hat{K} \hat{C}_i)S_i^{-1},
\]

(40)

where \( \mathcal{B}_i \in \mathbb{R}^{(9-m) \times 9} \) refers to the orthocomplement of \( TB_i \), which satisfies \( \mathcal{B}_i TB_i = 0 \). Then it can be derived that

\[
S_iTB_i = \begin{bmatrix}
B_i \\
H \hat{C}_i
\end{bmatrix}TB_i = \begin{bmatrix}
0 \\
B_i^T T^T \hat{P}_i TB_i
\end{bmatrix}.
\]

Pre-multiplying \( S_i \) on both sides of error system (39) we have

\[
S_i \dot{e}(t) = S_i(T \hat{A}_i - \hat{K} \hat{C}_i)S_i^{-1} S_i \dot{e}(t) + S_i TB_i[u_s(t) - \hat{\psi}_i(x_i(t), f_i(t))].
\]

(41)

Note that

\[
S_i \dot{e}(t) = \begin{bmatrix}
B_i \\
H \hat{C}_i
\end{bmatrix} \dot{e}(t) = \begin{bmatrix}
\mathcal{A}_1 \mathcal{A}_2 \\
\mathcal{A}_3 \mathcal{A}_4
\end{bmatrix} \begin{bmatrix}
\mathcal{B}_i \\
H \hat{C}_i
\end{bmatrix} \dot{e}(t) \\
+ \begin{bmatrix}
0 \\
B_i^T T^T \hat{P}_i TB_i
\end{bmatrix} \begin{bmatrix}
u_s(t) - \hat{\psi}_i(x_i(t), f_i(t))\end{bmatrix}.
\]

Hence, system (41) is can be rewritten as

\[
\begin{bmatrix}
\mathcal{B}_i \\
H \hat{C}_i
\end{bmatrix} \dot{e}(t) = \begin{bmatrix}
\mathcal{A}_1 \mathcal{A}_2 (\mathcal{A}_3 \mathcal{B}_i + \mathcal{A}_4 H \hat{C}_i) e(t) \\
\mathcal{A}_3 \mathcal{A}_4 (\mathcal{A}_3 \mathcal{B}_i + \mathcal{A}_4 H \hat{C}_i) e(t)
\end{bmatrix} + \begin{bmatrix}
0 \\
B_i^T T^T \hat{P}_i TB_i
\end{bmatrix} \begin{bmatrix}
u_s(t) - \hat{\psi}_i(x_i(t), f_i(t))\end{bmatrix}.
\]

Note that \( \dot{s}_i(t) = H \hat{C}_i \dot{e}(t) \), we thus can obtain the sliding surface equation as follows:

\[
\dot{s}_i(t) = (\mathcal{A}_3 \mathcal{B}_i + \mathcal{A}_4 H \hat{C}_i) e(t) + B_i^T T^T \hat{P}_i TB_i \begin{bmatrix}
u_s(t) - \hat{\psi}_i(x_i(t), f_i(t))\end{bmatrix}.
\]

(43)

In the following discussion, we should redesign the discontinuous input \( u_s(t) \) in error system (35) to forbid the trajectory of \( e(t) \) to arrive on sliding surface \( s(t) = 0 \) in finite time and then yields a sliding motion, which is in fact a design basis for the disturbance reconstruction.

**Theorem 2:** For the sliding surface equation (39), we design the parameter \( \alpha_1 \) in switching term (37) as

\[
\alpha_1 = \|B_i^T T^T \hat{P}_i TB_i\|^{-1} \times (\|\mathcal{A}_3 \mathcal{B}_i + \mathcal{A}_4 H \hat{C}_i\| \gamma_1),
\]

(44)

then the trajectory of error vector \( e(t) \) can be driven onto the sliding surface \( s_i(t) \) in finite time.

**Proof:** For the sliding surface equation (39), we define the Lyapunov function

\[
V_2(t) = 0.5 s_i^T(t) \begin{bmatrix}
B_i^T T^T \hat{P}_i TB_i
\end{bmatrix}^{-1} s_i(t),
\]

(45)

it is shown that

\[
\dot{V}_2(t) \leq s_i^T(t) \begin{bmatrix}
B_i^T T^T \hat{P}_i TB_i
\end{bmatrix}^{-1} \left[\left(\|\mathcal{A}_3 \mathcal{B}_i + \mathcal{A}_4 H \hat{C}_i\| e(t) \right) \\
+ B_i^T T^T \hat{P}_i TB_i \begin{bmatrix}
u_s(t) - \hat{\psi}_i(x_i(t), f_i(t))\end{bmatrix}\right]
\]

\[
\leq \|s_i(t)|| \begin{bmatrix}
B_i^T T^T \hat{P}_i TB_i
\end{bmatrix}^{-1} \| \times (\|\mathcal{A}_3 \mathcal{B}_i + \mathcal{A}_4 H \hat{C}_i\| \times \|e(t)|| + s_i^T(t) u_s(t) \\
+ \|s_i(t)|| \times \|((K_iD - m_{if} A_{\varepsilon 2})||w_{\text{max}} + f_{\text{max}})\|.
\]

(46)

On the other hand, in the previously discussion we have prove that under the discontinuous input \( \mathcal{B}_i \) the error system (35) is asymptotically stable. Therefore, there always exists a time instant \( t^* \) such that when \( t \geq t^* \) we have \( \|e(t)|| \leq \gamma_1 \) always holds, where \( \gamma_1 > 0 \) is any given constant. Due to this fact we have

\[
\dot{V}_2(t) \leq \gamma_1 \|s_i(t)|| \begin{bmatrix}
B_i^T T^T \hat{P}_i TB_i
\end{bmatrix}^{-1} \|\|\mathcal{A}_3 \mathcal{B}_i + \mathcal{A}_4 H \hat{C}_i\|| \\
+ \|s_i(t)|| \times \|((K_iD - m_{if} A_{\varepsilon 2})||w_{\text{max}} + f_{\text{max}})\| \\
+ s_i^T(t) u_s(t).
\]

(47)

by substituting the discontinuous term (37) into (47) it yields \( \dot{V}_2(t) \leq -\varepsilon_1 \|s_i(t)||^2 \) holds for \( \forall s_i(t) \neq 0 \). That means the trajectory of \( \dot{e}(t) \) will be globally driven onto sliding surface \( s(t) \) in finite time. We complete the proof.

Based on Theorem 2 we can conclude that a sliding motion (42) for the sliding surface \( s_i(t) \) must occur in finite time, and we
have that \( s_i(t) = 0 \) and \( s_i(t) = 0 \) hold in finite time. As a result, the sliding surface equation (43) becomes

\[
0 = B_t^T T T B_t \left( u_e(t) - \hat{w}_i(x_i, f_i(t)) \right),
\]

where \( u_e(t) \) is the so-called equivalent output error injection vector. As mentioned in [32], \( u_e(t) \) can be calculated as

\[
u_e(t) = - \left( (K_i D - m_i f_i A_i D_i) \right) \| w_{\text{max}} + f_{\text{max}} + \alpha_1 + \varepsilon_1 \times \frac{s(t)}{\| s(t) \| + \varepsilon_2},
\]

where \( \varepsilon_2 > 0 \) is a small parameter set here to avoid the chattering behavior. On the other hand, since \( B_t^T T T B_t \) is nonsingular, the external disturbance \( d_i(t) \) can be reconstructed as

\[
f_i(t) = - \left( (K_i D - m_i f_i A_i D_i) \right) \| w_{\text{max}} + f_{\text{max}} + \alpha_1 + \varepsilon_1 \times \frac{s(t)}{\| s(t) \| + \varepsilon_2},
\]

The external disturbance \( d(t) \) and output noise \( w(t) \) are given as

\[
d(t) = 0.2 \left[ 0.2 \cos(0.5t) \ 0.3 \cos(0.5t) \ 0.6 \sin(1.1t) \right]^T, \quad w(t) = 0.1 \left[ 0.5 \cos(t) \ -0.1 \cos(t) \ 0.2 \sin(5t) \right]^T.
\]

According to the design procedure given above, the estimation scheme is divided into the following several steps: (i) Check conditions (18) and (28), and they are satisfied. We select \( K_1 \) as

\[
K_1 = \begin{bmatrix}
-0.5722 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.5722 & 0 & 0 & 0 & -1.2843 \\
0 & 0 & 0 & 0 & 0 & -1.2843 \\
0 & 0 & 0 & 0 & 0 & -1.2843
\end{bmatrix},
\]

such that \( A + B_1 K_1 \) is Hurwitz.

According to equation (25), the parameter matrices \( T \) and \( \bar{L} \) are designed as

\[
T = \begin{bmatrix}
0.5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2
\end{bmatrix},
\]

\[
\bar{L} = \begin{bmatrix}
0.5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 10 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2
\end{bmatrix}.
\]

(ii) Solve the optimization problem (31)-(31) by applying Matlab toolbox and it is obtained \( \beta \approx 2 \times 10^{-5} \). This means that the constraint \( (TB_d)^T \bar{P}_i = H \bar{C} \) is satisfied. Besides, one can find feasible solutions as follows:

\[
\bar{K}_1 = \begin{bmatrix}
0.4522 & 0 & 0 & 0 & 0 & 0 \\
-1.1215 & 0 & 0 & 0 & 0 & 0 \\
0 & 2.6254 & 0 & 0 & 0 & 0 \\
0 & 0 & -2.2017 & 0 & 0 & 0 \\
1.4327 & 0 & 0 & 0 & 0 & -1.5764 \\
0 & 4.3551 & 0 & 0 & 0 & 0 \\
0 & 0 & 9.7575 & 0 & 0 & 0 \\
0 & 0 & 0 & 5.2816 & 0 & 0 \\
-4.6170 & 0 & 0 & 0 & 0 & -11.7880 \\
0 & -22.6798 & 0 & 0 & -19.6211 & 0 \\
0 & 0 & 22.0568 & 0 & 0 & -26.8590
\end{bmatrix}.
\]

According to (34), the observer gains \( \bar{F}_1 \) and \( \bar{L}_{\alpha 1} \) can be calculated and the values which are given below.

(iii) Design the discontinuous input \( u_e(t) \) according to (37), where the controller parameters are chosen as \( \varepsilon_1 = 10^{-4} \) and \( \varepsilon_2 = 5 \times 10^{-6} \).

(iv) The initial values are set as \( x(0) = [500 \ 750 \ 10 \ 200 \ 200 - 35]^T, \ z(0) = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T, \ \bar{x}(0) = [500 - 750 - 750 10 200 - 35.05 - 0.01 0.001]^T, \ e(0) = [5 - 15 - 10 -1 5 2 0.4 0.5 0.4]^T \). The simulation results are shown in Figures 1-10. The trajectory of the error system (35) is plotted in Figure 1; the trajectories of state vector \( x(t) \) and its estimation are depicted in Figures 2-7; the trajectories of external disturbance \( d(t) \) and its estimation are displayed in Figures 8-10. It can be seen that the tracking of system states \( x(t) \) and external disturbance \( d(t) \) has achieved an ideal performance.
\[
\bar{F}_1 = \begin{bmatrix}
-0.4522 & 1.1215 & 0 & 0 & 0.3395 & 0 & 0 & -0.0401 & 0 & 0 & 0.2302 & 0 & 0 & -0.7842 \\
0 & 0 & -2.2654 & 0 & 0 & 2.8017 & 0 & 0 & 2.0764 & 0 & 0 & 0 & 0 & -0.5282 \\
-1.4384 & 0 & 0 & -0.0574 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5.8940 \\
0 & -4.4008 & 0 & -9.7632 & 0 & -5.7044 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1139 \\
4.6391 & 0 & 0 & -0.4724 & 0 & 0 & 24.7496 & 0 & 0 & 0 & 0 & 0 & 0 & 1.9621 \\
0 & 0 & -22.0683 & 0 & 0 & 27.8333 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -14.4295 
\end{bmatrix},
\]

\[
\bar{L}_{o1} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1.1327 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2.1047 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 4.8759 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2.2971 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4.8759 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 11.0227 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 
\end{bmatrix},
\]

VI. CONCLUSION

This paper has investigated satellites autonomous navigation problem based on an extended state sliding mode observer approach. A descriptor system based sliding mode observer method has been developed to solve the addressed estimation problem. The designed observer method can reconstruct the state and disturbance vector simultaneously. Further work will be focused on extend the designed observer method to solve the fault tolerant control problem for spacecraft formation control system.

VII. ACKNOWLEDGMENTS

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Fig. 5: $x_4(t)$ and its estimation

Fig. 6: $x_5(t)$ and its estimation

Fig. 7: $x_6(t)$ and its estimation

Fig. 8: $d_1(t)$ and its estimation

Fig. 9: $d_2(t)$ and its estimation

Fig. 10: $d_3(t)$ and its estimation
REFERENCES


