Towards robust simultaneous actuator and sensor fault estimation for a class of nonlinear systems: Design and comparison

MARCIN PAZERA, MARCIN WITCZAK

Abstract: The paper aims at providing two competitive solutions to the problem of estimating state as well as sensor and actuator faults. The development starts with a common system description, which is transformed into a nonlinear descriptor system-like form. The appealing property of such an approach is that the sensor fault is eliminated from the output equation by suitably extending the state vector. Subsequently, a novel fault estimation scheme is proposed and its error dynamics are carefully described. Having such a unified framework, two competitive design procedures are proposed. One is based on the celebrated $\mathcal{H}_\infty$ while the other employs the quadratic boundedness strategy. With the estimators, the paper provides a unified framework for assessing the uncertainty of the resulting estimates in the form of uncertainty intervals. Thus, both solutions can be compared from the theoretical perspective. The final part of the paper contains a numerical simulation example concerning a twin-rotor system that clearly shows the comparison and performance of the above approaches. Subsequently, an experimental study concerning a multi-tank system using real data is reported.

Index Terms: fault diagnosis, fault location, state estimation, observers, decision making, stability, robustness, nonlinear systems

I. INTRODUCTION

In the permanently developing world, industrial companies are strongly oriented towards optimization relying on cost reduction along with concurrently increasing profits. They also increase their requirements pertaining reliability and safety. This causes a need for installing more sensitive and actuating devices. Indeed, with the rise of modern IoT (Internet of Things) smart sensors and actuators, such a growth becomes unquestionably evident. In spite of the apparent appeal of such an approach, it increases the likelihood of simultaneous actuator and sensor faults. This does not necessarily mean that, e.g., a given sensor fault induces a given actuator fault. In other words, such a simultaneous appearance may happen without any hardware correlation between them.

The research pertaining Fault Diagnosis (FD) has already received considerable scientific attention and a vast number of mature FD techniques are available [4], [12], [23]. Initially, FD was oriented towards Fault Detection and Isolation (FDI), but with the rising research attention towards Fault-Tolerant Control (FTC) [16], [29], [31], fault estimation became an emerging research area. Indeed, it can be perceived as a crucial component of the so-called active FTC [10], [18]. Thus, effective fault estimation is simply impossible without reliable fault estimation. Unfortunately, most of the fault estimation and FTC strategies present in the literature are dedicated to either actuator or sensor faults only. Thus, their application forces an unrealistic assumption stating that only one kind of faults can affect a system. The same situation can be observed in the research approaches concerning fault estimation [14], [33], [39]. The main trends towards developing fault estimation schemes are oriented towards various observer-based approaches [42]. They are especially attractive because they allow effective FDI and fault estimation.
Taking into account the above discussion, it is also natural that the problem of simultaneous actuator and sensor fault estimation receives growing research attention [9], [24], [27], [28], [40]. However, most of them are developed for linear systems while the number strategies capable of handling some classes of nonlinear systems is rather limited. Another unappealing feature of the existing approaches to simultaneous actuator and sensor fault estimation is that they do not provide sufficient information about the estimation quality. Indeed, a recent literature review clearly indicates the trends for settling simultaneous actuator and sensor fault estimation for nonlinear systems. In particular, [19] transformed a nonlinear Lipschitz system into a Linear Parameter Variation (LPV) one using the so-called reformulated Lipschitz property. The proposed design strategy provides optimal $H_\infty$ fault estimates in the finite-frequency domain. Another interesting approach for Lipschitz systems was proposed in [1]. The author decouples the system into two subsystems, each affected by either a sensor or an actuator fault. As a result, two separate Sliding Mode Observers (SMOs) are developed. Unfortunately, the obtained estimates are guaranteed to asymptotically converge to real faults without considering the robustness issue. A similar SMO was proposed in [20]. Contrarily to the preceding approach, the authors modelled uncertainties in the form of bounded disturbances. Another group of approaches treat nonlinear systems as Takagi–Sugeno fuzzy ones. A good example of such strategies is presented in [5]. The authors propose an adaptive fuzzy estimator and take into account various fault scenarios, i.e., bias, drift, and loss of accuracy along with loss of effectiveness. The approach is devoted to deterministic systems without any uncertain factors. Another interesting approach is developed in [17] and deals with Takagi–Sugeno systems with switching nonlinearities. Unfortunately, it exhibits the same drawback of the lack of robustness. Finally, in [34] a new Takagi–Sugeno multiple integral unknown input observer is proposed. Unlike in the previous approaches, it is capable of decoupling disturbances. The last group of representative approaches are the ones for polynomial LPV and LPV systems [6], [26]. The strategy boils down to converting the system into the so-called augmented-state one and then estimating it. The indicated approach copes with a process disturbance only.

In spite of the incontestable appeal of the presented approaches, they all inherit one common drawback pertaining their reliability. It can be measured by introducing the so-called uncertainty intervals. These describe a possible range of an unknown fault, which is consistent with the measured data. The above approaches can provide a point estimate of a fault but they cannot be applied to assess the uncertainty intervals. This means that, while comparing a set of fault estimators, the narrower uncertainty intervals are provided by the best of them.

The paper tackles the above issue by answering the following research problems:

1) How to describe the uncertainty present in the system?
2) How to design the fault estimator?
3) How to determine the uncertainty intervals of the obtained estimates?
4) How to develop the estimator providing the narrowest uncertainty intervals?

To settle the first issue, two competitive approaches are employed, i.e., one based on the celebrated $H_\infty$ approach and the other employing the Quadratic Boundedness (QB) strategy [3]. Subsequently, irrespective of the selected uncertainty description, a unified strategy for simultaneous sensor and actuator fault estimation is proposed along with its error dynamics. The third issue is addressed by a separate development of uncertainty intervals for $H_\infty$ and QB cases, respectively. The last question is answered by providing a step by step design procedure of the above estimators, which aims at minimizing the lengths of uncertainty intervals.

The paper is organized as follows: Section II outlines the preliminaries and the problem being considered. Section III shows how to transform the said system into a descriptor-like form along with a novel sensor and actuator fault estimator. Moreover, careful derivation of its error dynamics is provided. Subsequently, the design procedures for the $H_\infty$ and the QB cases are respectively shown. The development presented in this section allows formulating and solving the uncertainty interval determination task, which is detailed in Section IV. Subsequently, Section V portrays a numerical simulation example concerning a twin-rotor system, which clearly shows the performance and comparison of the above approaches. Finally, the experimental results concerning a multi-tank system are presented and compared with a selected approach.

II. PRELIMINARIES

Let us consider the following non-linear, discrete-time system, which involves both actuator and sensor faults:

$$x_{k+1} = Ax_k + Bu_k + Bf_{a,k} + g(x_k) + W_1 w_{1,k}, \quad (1)$$
$$y_k = Cx_k + Cf_{a,k} + W_2 w_{2,k}, \quad (2)$$

where $x_k \in \mathbb{R}^n$, $y_k \in \mathbb{R}^m$ and $u_k \in \mathbb{R}^r$ denote the state, output and input, respectively, $f_{a,k} \in F_a \subset \mathbb{R}^r$ and $f_{s,k} \in F_s \subset \mathbb{R}^{ns}$ stand for the actuator and sensor fault, while $C_f$ is the sensor fault distribution matrix. Moreover, $g(x_k) : \mathbb{R}^n \to \mathbb{R}$ represents a non-linear function with respect to the state. Furthermore, $w_{1,k}$ and $w_{2,k}$ are exogenous external disturbance vectors while $W_1$ and $W_2$ represent their distribution matrices. The model (1)–(2) is widely used in the literature for describing a class of nonlinear systems in which the input influences the state in a linear way while the evolution of the state is described by both linear $Ax_k$ and nonlinear components $g(x_k)$. Moreover, the output equation is linear with respect to the state. The system described by (1)–(2) is affected by faults pertaining either actuators or sensors. Note that (1) can be written with $B(u_k + f_{a,k})$.

Thus, if the actuator fault is equal to zero, then the actuator responds according to the control action, else its response is impaired by a fault [7], [13], [25], [37], [43].
As stated in the Introduction, the reliability of the proposed solutions is measured with the so-called uncertainty intervals, which are guaranteed to contain a real system fault. It should be noted that no statistical assumptions are made with respect to \(w_{1,k}\) and \(w_{2,k}\). Indeed, instead of deliberating about the distribution of these variables, the subsequent part of the paper examines two possible bounded-disturbance strategies, namely, \(\mathcal{H}_\infty\) and QB. Finally, the robustness of these approaches is exemplified by the so-called uncertainty intervals. Having two competitive approaches capable of handling the robustness problem, it is necessary to deal with the nonlinearity issue. Apart from the obvious linearization technique used to design the estimator. Subsequently, it splits into two cases: \(\mathcal{H}_\infty\) and QB. The main contribution of this paper compares the existing approaches and proposes a new solution, which overcomes their restrictions. Apart from its incontestable appeal, it is based on a relatively simple algebraic manipulation. Based on the above recommendations, it is employed in this paper.

For the purpose of further deliberations, let us recall the following lemma [41], which will be used to cope with nonlinearities:

**Lemma 1:** [41] For \(g(\cdot)\), the following statements are equivalent:

1. \(g(\cdot)\) is Lipschitz respect to their arguments with Lipschitz constant \(\gamma_h > 0\), i.e.,
   \[
   \| g(X) - g(Y) \| \leq \gamma_h \| X - Y \|, \quad \forall X, Y \in \mathbb{R}.
   \]

2. For all \(i, j = 1, \ldots, n\), there exist functions \(h_{i,j}: \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) and constants \(\hat{\gamma}_{h_{i,j}}\) such that, for each \(X, Y \in \mathbb{R}\)
   \[
   g(X) - g(Y) = \sum_{i=1}^{n} \sum_{j=1}^{n} h_{i,j}(X - Y),
   \]
   and
   \[
   0 \leq h_{i,j} \leq \hat{\gamma}_{h_{i,j}}, \quad h_{i,j}(X, Y) = c_i c_j^T g(X, Y), \quad H_{i,j} = c_i c_j^T,
   \]
   as well as a scalar function \(h_{i,j}\) given by
   \[
   h_{i,j}(X, Y) = \begin{cases} 0 & \text{if } x_j = y_j \\ g_j(X, Y) - g_j(x_j, y_j) & \text{if } x_j \neq y_j \end{cases}
   \]
   where \(c_i\) stands for the \(i\)-th column of the \(n\)-th order identity matrix while \(X^Y\), is defined by
   \[
   X^Y = \begin{bmatrix} y_1 \\ \vdots \\ y_i \\ x_{i+1} \\ \vdots \\ x_n \end{bmatrix} \quad \text{for } i = 1, \ldots, n
   \]

As a result of applying the above lemma, the following relation can be determined:

\[
\dot{\gamma}_h \leq \hat{\gamma}_h, \quad \forall h(i,j). \tag{5}
\]

Throughout the paper, the transformation of the system (1)–(2) into a nonlinear descriptor one is proposed. It combines the state and sensor fault into an extended state. As a result, a novel nonlinear descriptor-based estimator is proposed, although the actuator fault is estimated in an adaptive way.

Let us define an extended vector

\[
\tilde{\mathbf{x}}_k = \begin{bmatrix} x_k \\ f_s(k) \end{bmatrix} \quad \text{for } k = 1, \ldots, n
\]

The descriptor framework is utilized to construct the augmented system of (1)–(2):

\[
\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{k+1} \\ f_{s,k+1} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_k \\ f_s(k) \end{bmatrix} + \begin{bmatrix} B & 0 \end{bmatrix} u_k
+ \begin{bmatrix} B \\ 0 \end{bmatrix} f_{a,k} + \begin{bmatrix} 0 & I \end{bmatrix} \varepsilon_{s,k} + \begin{bmatrix} W_1 \\ 0 \end{bmatrix} w_{1,k}, \tag{8}
\]

\[
y_k = \begin{bmatrix} C_c \\ C_f \end{bmatrix} \begin{bmatrix} x_k \\ f_s(k) \end{bmatrix} + W_2 w_{2,k}, \quad \text{for } k = 1, \ldots, n
\]

with \(\varepsilon_{s,k} = f_{s,k+1} - f_{s,k}\). Finally, the above description can be rewritten in a simpler form:

\[
\begin{bmatrix} \bar{E} & \varepsilon_k \\ & \bar{B} \end{bmatrix} \tilde{\mathbf{x}}_k + \bar{D} u_k + \bar{F} f_{a,k} + \bar{G} g(x_k)
+ \bar{H} \varepsilon_{s,k} + \bar{W}_1 w_{1,k}, \tag{10}
\]

\[
y_k = \bar{C} \tilde{\mathbf{x}}_k + W_2 w_{2,k}, \quad \text{for } k = 1, \ldots, n \tag{11}
\]

**III. DESCRIPTOR-LIKE FAULT ESTIMATOR DESIGN**

The objective of the section is to propose a descriptor-like estimator which allows estimating state as well as actuator and sensor faults for the nonlinear system exhibited in Sec. II. The first part of this section concerns calculation of the estimation errors, and hence it is the same irrespective of the robustness technique used to design the estimator. Subsequently, it splits into two cases: \(\mathcal{H}_\infty\) and QB. The main contribution of this section is:
• a novel estimator structure for simultaneous state, actuator, and sensor fault estimation for a class of nonlinear systems (1)–(2);
• determination of the unified convergence analysis framework using either $H_\infty$ or QB approaches;
• determination of formal convergence proofs with either $H_\infty$ or QB approaches.

The proposed structure of the novel state and fault estimator is as follows:

$$z_{k+1} = N_2 z_k + M u_k + Ly_k + T_1 \tilde{B} f_{a,k} + T_1 \bar{G} g(\tilde{x}_k),$$

$$\hat{x}_k = z_k + T_2 y_k,$$

$$\hat{f}_{a,k+1} = \tilde{f}_{a,k} + F (y_k - \bar{C}\tilde{x}_k).$$

(12)\hspace{1cm} (13)\hspace{1cm} (14)

Let us start with the state estimation error

$$\tilde{e}_k = \bar{x}_k - \hat{x}_k = \bar{x}_k - z_k - T_2 \bar{C} \tilde{x}_k - T_2 W_2 w_{2,k}$$

$$= (I - T_2 C) \bar{x}_k - z_k - T_2 W_2 w_{2,k}$$

$$= T_1 \bar{E} \tilde{x}_k - z_k - T_2 W_2 w_{2,k}.$$ \hspace{1cm} (15)

From (15) it can be concluded that

$$z_k = T_1 \bar{E} \tilde{x}_k - \tilde{e}_k - T_2 W_2 w_{2,k}.$$ \hspace{1cm} (16)

Thus, taking into account (8)–(9) and (15)–(16), the dynamic of the estimation error obeys

$$\tilde{e}_{k+1} = T_1 \bar{E} \tilde{x}_{k+1} - z_{k+1} - T_2 W_2 w_{2,k+1}$$

$$= T_1 \bar{A} \bar{x}_k + T_1 \bar{B} u_k + T_1 \bar{B} f_{a,k} + T_1 \bar{G} g(x_k) + T_1 \bar{H} \tilde{e}_{s,k} + T_1 \bar{W} _1 w_{1,k}$$

$$- N_2 z_k - \bar{M} u_k - L \bar{C} \tilde{x}_k - LW_2 w_{2,k}$$

$$= (T_1 \bar{A} - N_2 T_1 \bar{E} - L \bar{C}) \tilde{x}_k + T_1 \bar{B} e_{a,k} + (T_1 \bar{B} - M) u_k + N_2 \tilde{e}_k + T_1 \bar{G} g(x_k)$$

$$- T_1 \bar{G} g(\tilde{x}_k) + T_1 \bar{H} \tilde{e}_{s,k} + T_1 \bar{W} _1 w_{1,k}$$

$$- T_2 W_2 w_{2,k+1} + (N_2 T_2 - L) W_2 w_{2,k}.$$ \hspace{1cm} (17)

Based on the (17), it is obvious that the following relationships should hold:

$$T_1 \bar{B} - M = 0,$$

$$T_1 \bar{A} - N_2 T_1 \bar{E} - L \bar{C} = 0,$$

or equivalently:

$$M = T_1 \bar{B},$$

$$T_1 \bar{A} = N_2 T_1 \bar{E} + L \bar{C},$$

and hence

$$T_1 \bar{A} = N_2 (I - T_2 \bar{C}) + L \bar{C},$$

$$T_1 \bar{A} = N_2 - N_2 T_2 \bar{C} + L \bar{C},$$

$$N_2 = T_1 \bar{A} + N_2 T_2 \bar{C} - L \bar{C},$$

$$N_2 = T_1 \bar{A} - (L - N_2 T_2) \bar{C},$$

$$N_2 = T_1 \bar{A} - K \bar{C}.$$ \hspace{1cm} (20)\hspace{1cm} (21)\hspace{1cm} (22)\hspace{1cm} (23)\hspace{1cm} (24)\hspace{1cm} (25)\hspace{1cm} (26)

where

$$K = L - N_2 T_2,$$ \hspace{1cm} (27)\hspace{1cm} (28)

As a consequence, the augmented state estimation error is given

$$\tilde{e}_{k+1} = (T_1 \bar{A} - K \bar{C}) \tilde{e}_k + T_1 \bar{B} e_{a,k} + T_1 \bar{G} g(x_k) + T_1 \bar{H} \tilde{e}_{s,k} + T_1 \bar{W} _1 w_{1,k}$$

$$+ T_1 \bar{H} \tilde{e}_{s,k} - T_2 W_2 w_{2,k+1} - K W_2 w_{2,k}.$$ \hspace{1cm} (29)

Subsequently, the dynamics of the actuator fault estimation error is governed by

$$e_{a,k+1} = f_{a,k+1} + \tilde{f}_{a,k+1} = f_{a,k+1} + f_{a,k} - f_{a,k}$$

$$- F \tilde{e}_k - F (y_k - \bar{C} \tilde{x}_k) = e_{a,k} + \tilde{e}_{a,k}$$

$$- F \bar{C} \tilde{e}_k - F W_2 w_{2,k},$$

with $\tilde{e}_{a,k} = f_{a,k+1} - f_{a,k}.$

Now, by defining:

$$\tilde{e}_k = \begin{bmatrix} \tilde{e}_k \\ e_{a,k} \end{bmatrix}, \quad \tilde{w}_k = \begin{bmatrix} w_{1,k} \\ w_{2,k} \\ w_{2,k+1} \\ e_{s,k} \\ e_{a,k} \end{bmatrix},$$

(30)

and applying Lemma 1 with (5)–(6), the estimation errors can be combined into a compact form

$$\tilde{e}_{k+1} = X(h) \tilde{e}_k + Z \tilde{w}_k,$$ \hspace{1cm} (31)

with $X(h) = \tilde{A}(h) - \bar{K} \tilde{C}$ and $Z = \tilde{W}_1 - \bar{K} \tilde{W}_2,$ where

$$\tilde{A}(h) = \begin{bmatrix} T_1 (\bar{A} + G A(h) G^T) & T_1 \bar{B} \\ 0 & I \end{bmatrix},$$

$$\tilde{K} = \begin{bmatrix} K \\ F \end{bmatrix}, \tilde{C} = [\tilde{C} \ 0],$$

$$\tilde{W}_1 = \begin{bmatrix} T_1 \bar{W}_1 & 0 & -T_2 W_2 & T_1 \bar{H} & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix},$$

$$\tilde{W}_2 = \begin{bmatrix} 0 & W_2 & 0 & 0 \end{bmatrix}.$$ \hspace{1cm} (32)

For the purpose of stability analysis, let us recall the Finsler’s Lemma [35]:

**Lemma 2:** The following expressions are equivalent:

1. $\bar{x}_k^T Q \bar{x}_k < 0, \forall \bar{x} \in \{\bar{x} \in \mathbb{R}^{n+z} | \bar{x} \neq 0, R \bar{x} = 0\},$

2. $\exists M \in \mathbb{R}^{n+z \times m}$ such that $Q + M R + R^T M^T < 0.$

Subsequently, let us denote the Lyapunov candidate function

$$V_k = \tilde{e}_k^T P \tilde{e}_k,$$ \hspace{1cm} (33)

with $P \succ 0.$ Moreover, let us define the following super-vector

$$\tilde{x}_k = \begin{bmatrix} \tilde{e}_k^T, \tilde{w}_k^T, \tilde{e}_{k+1}^T \end{bmatrix}^T.$$ \hspace{1cm} (34)

Based on (35) and (32), it can be shown that

$$R(h) = [X(h) Z - I],$$ \hspace{1cm} (36)
satisfying

\[ R(h) \bar{x}_k = 0. \]  

(37)

As it can be observed in the literature, a typical approach to robust observer design is to use the \( \mathcal{H}_\infty \) strategy. This reduces to satisfying [8]

\[ V_{k+1} - V_k + \bar{e}_k^T R_e \bar{e}_k - \mu^2 \bar{w}_k^T \bar{w}_k < 0, \]  

(38)

with \( R_e > 0 \) and \( \mu > 0 \) being a disturbance attenuation level. Note that the common choice is to set \( R_e = I \). A crucial assumption underlying this approach is that \( \bar{w}_k \in \mathbb{E}_2 \) whilst:

\[ l_2 = \{ w \in \mathbb{R}^n \mid \| w \|_{l_2} < +\infty \}, \]  

(39)

\[ \| w \|_{l_2} = \left( \sum_{k=0}^{\infty} \| w_k \|^2 \right)^{\frac{1}{2}}. \]  

(40)

This means that it can formally be used for \( \bar{w}_k \) with a finite energy. QB relaxes this assumption by requiring \( \bar{w}_k \) to be overbounded by an ellipsoid

\[ \mathbb{E}_w = \{ \bar{w}_k : \bar{w}_k^T Q_w \bar{w}_k \leq 1 \}, \]  

(41)

with \( Q_w > 0 \). The resulting stability condition is given as

\[ V_{k+1} - (1 - \alpha) V_k - \alpha \bar{w}_k^T Q_w \bar{w}_k < 0, \]  

(42)

where \( 0 < \alpha < 1 \). It is evident that the \( \mathcal{H}_\infty \) can be equivalent to the QB approach by setting \( V_k = \bar{e}_k^T P \bar{e}_k, \ P > 0, \ R_e = \alpha P, \ Q_w = \frac{\mu^2}{\alpha} I \). At this point, let us also recall the following definitions [3, 15]:

**Definition 1:** The system (32) is strictly quadratically bounded for all allowable \( \bar{w}_k \in \mathbb{E}_w, \ k \geq 0 \), if \( V_k > 1 \implies V_{k+1} - V_k < 0 \) for any \( \bar{w}_k \in \mathbb{E}_w \).

**Definition 2:** A set \( \mathbb{E} \) is a positively invariant set for (32) for all \( \bar{w}_k \in \mathbb{E}_w \), if \( \bar{e}_k \in \mathbb{E} \) implies \( \bar{e}_{k+1} \in \mathbb{E} \) for any \( \bar{w}_k \in \mathbb{E}_w \).

In both cases this boils down to the following theorems:

**Theorem 1:** For a given \( \bar{w}_k \) attenuation level \( \mu > 0 \), the \( \mathcal{H}_\infty \) estimator design task for the system (8)–(9) is feasible iff there exist \( N, U, P > 0 \) such that the following condition is satisfied:

\[
\begin{bmatrix}
I - P & 0 \\
0 & -\mu^2 I \\
U \bar{A}(h) - N \bar{C} & U \bar{W}_1 - N \bar{W}_2 & P - U - U^T
\end{bmatrix} < 0.
\]  

(43)

**Proof.** See Appendix.

**Theorem 2:** The QB estimator design task for the system (8)–(9) is feasible iff (32) is strictly quadratically bounded for all \( \bar{w}_k \in \mathbb{E}_w \). Thus, there exist matrices \( P > 0, U, N \) and a scalar \( \alpha \in (0, 1) \) such that the following inequality is satisfied:

\[
\begin{bmatrix}
-I + P & 0 \\
0 & -\alpha Q_w \\
U \bar{A}(h) - N \bar{C} & U \bar{W}_1 - N \bar{W}_2 & P - U - U^T
\end{bmatrix} < 0.
\]  

(44)

**Proof.** See Appendix.

IV. UNCERTAINTY INTERVALS

The objective of the previous section was to obtain the stability conditions of the proposed estimator. This section focuses on assessing uncertainty intervals of the estimates provided by the \( \mathcal{H}_\infty \) and QB estimators. It should be highlighted that the reliability of the proposed approach is measured with the length of the uncertainty interval of the fault estimates. Thus, it is natural to minimize their length/size, and hence make the obtained estimates as reliable as possible.

The first step towards this goal is to assess the lower and upper bounds of the state/fault estimation error. The preliminary objective is to show that its possible evolution is described with an ellipsoid of varying size. Using such ellipsoids, the lower and upper bounds of sensor and actuator uncertainty intervals are derived. Up to the authors’ knowledge, there is no approach present in the literature that provides such uncertainty intervals for the \( \mathcal{H}_\infty \) and quadratic boundedness case. Moreover, these approaches are described in a unified framework, which facilitates their comparison.

The deliberations are started with two corollaries, which provide tools for calculating upper bounds of the Lyapunov function for these two cases.

Let us start with the QB case:

**Corollary 1:** [3] If (32) satisfies (44), then there exists \( 0 < \alpha < 1 \) such that, for all allowable \( \bar{w}_k \in \mathbb{E}_w \)

\[ V_k \leq \xi_k(\alpha) \quad k = 0, 1, \ldots, \]  

(45)

with

\[ \xi_k(\alpha) = (1 - \alpha)^k \bar{e}_0 P \bar{e}_0 + 1 - (1 - \alpha)^k. \]  

(46)

To derive a similar bound for the \( \mathcal{H}_\infty \) case, it is necessary to assume that \( \bar{w}_k \in l_2 \) and \( \bar{w}_k \in \mathbb{E}_w \).

**Corollary 2:** If (32) satisfies (43), then there exists \( 0 < \gamma < 1 \) such that, for all allowable \( \bar{w}_k \in \mathbb{E}_w \)

\[ V_k \leq \beta_k(\gamma) \quad k = 0, 1, \ldots, \]  

(47)

with

\[ \beta_k(\gamma) = (1 - \gamma)^k \bar{e}_0 P \bar{e}_0 + \mu_p \sum_{i=0}^{k-1} (1 - \gamma)^i, \]  

(48)

where

\[ \mu_p = \mu^2 \sum_{i=1}^{n + m + r} q_{w,i,i}. \]  

(49)

Having the upper bounds for both cases, it is possible to formulate a general theorem providing uncertainty intervals.

**Theorem 3:** Let

\[ V_k \leq \eta_k, \]  

(50)
then uncertainty intervals for state, sensor and actuator faults are given by:

\[
\hat{x}_{i,k} - s_{i,k} \leq x_{i,k} \leq \hat{x}_{i,k} + s_{i,k}, \quad i = 1, \ldots, n,
\]

\[
\hat{f}_{j,s,k} - s_{i,k} \leq f_{j,s,k} \leq \hat{f}_{j,s,k} + s_{i,k},
\]

\[
\hat{f}_{l,a,k} - s_{i,k} \leq f_{l,a,k} \leq \hat{f}_{l,a,k} + s_{i,k},
\]

\[
\alpha
\]

\[
\eta_i = \left\{ \begin{array}{ll}
\beta_i(\gamma) & \text{for the } \mathcal{H}_\infty \text{ case}, \\
\xi_i(\alpha) & \text{for the QB case.}
\end{array} \right. \]

Finally, by analyzing (46), it is evident that \( \xi(\alpha) \) converges to 1. Similarly, as \( k \to \infty \), \( \beta(\gamma) \) converges to \( \frac{1}{\gamma} \).

The above remark clearly leads to the conclusion that the length of the uncertainty intervals depends solely on matrix \( P \), which shapes either the ellipsoid (45) or (47), respectively. This means that the minimum length of the uncertainty intervals is associated with the size of the above ellipsoid. This size can be expressed using various measures, but the most common approach is to assess it with the determinant of \( P \) [36]. Note that, it is inversely proportional to the volume of either (45) or (47), respectively.

This leads to the final procedure of the proposed estimator:

\( \mathcal{H}_\infty \) case:

1) Subs \( \mu^2 = \kappa \) to (43).
2) Set \( \kappa > 0 \)
3) Solve the optimization problem

\[
\max \det(P),
\]

under (43) and \( P > 0 \).
4) Calculate \( \mu = \sqrt{\kappa} \).

QB case:

1) Set \( \alpha \in (0,1) \)
2) Solve the optimization problem

\[
\max \det(P),
\]

under (44) and \( P > 0 \).

However, in order to satisfy (44) the matrix \( Q_w \) in (41) can be selected by shaping the size of the \( \bar{w}_k \) domain by assuming that

\[
\bar{w}_{i,k} \leq \tilde{w}_{i,k} \leq \bar{w}_{i,k} \quad i = 1, \ldots, s_w,
\]

with \( s_w = n + 2m + ns + r \), where \( \tilde{w}_{i,k} \) (for \( i = 1, \ldots, n + 2m \)) shape the possible maximum values of external exogenous disturbances \( w_{1,k}, w_{2,k} \) and \( w_{2,k+1} \). The subsequent \( \tilde{w}_{i,k} \) (for \( i = n + 2m + 1, \ldots, n + 2m + ns \)) correspond to the maximum rate of change of sensor faults whilst the last ones (for \( i = n + 2m + ns + 1, \ldots, s_w \)) stand for shaping the maximum rate of change of actuator faults. As a consequence

\[
Q_w = \text{diag}\left(\frac{1}{w_{1,k}}, \ldots, \frac{1}{w_{s_w,k}}\right).
\]

Finally, irrespective of the chosen robustness approach, the estimator gain matrix is calculated as follows

\[
\tilde{K} = U^{-1} N.
\]

The remaining matrices of the estimator can be obtained using:

\[
L = K + N_1 T_2,
\]

\[
M = T_1 \tilde{B},
\]

\[
N = T_1 \tilde{A} - K \tilde{C},
\]

where

\[
\begin{bmatrix}
T_1 & T_2
\end{bmatrix} = \begin{bmatrix}
\tilde{E}^\top \\
\tilde{C}
\end{bmatrix}.
\]

V. ILLUSTRATIVE EXAMPLES

A. SIMULATION STUDY

The proposed approach for simultaneous actuator and sensor fault estimation of nonlinear systems was verified with a laboratory Twin-Rotor MIMO System (TRMS), which is portrayed in Fig. 1. Such a system has been designed to practically verify identification, control and fault diagnosis methodologies for non-linear systems. It can be described by
a 6-th order highly nonlinear model with significantly cross-coupled axes using the differential equations as follows:

\[
\frac{d\omega_v}{dt} = \frac{k_a k_2}{J_{mr} R_a} u_v - \left( \frac{B_{mr}}{J_{mr}} + \frac{k_a^2}{J_{mr} R_a} \right) \omega_v - f_2(\omega_v),
\]

\[
\frac{d\Omega_v}{dt} = \frac{l_m f_v(\omega_v) + k_a \Omega_h f_5(\omega_v) \cos \theta_v - k_{ov} \Omega_v}{J_v} + \frac{g((K_A - \theta B) \cos \theta_v - K_C \sin \theta_v)}{J_v} - \frac{\Omega_v^2 K_H \sin \theta_v \cos \theta_v}{J_v} - \frac{k_l \left( \frac{k_r}{\pi_c} u_h - \left( B_{tr} + \frac{k_r^2}{\pi_c} \right) \omega_h - f_1(\omega_h) \right)}{J_v J_{tr}},
\]

\[
\frac{d\theta_v}{dt} = \Omega_v,
\]

\[
\frac{d\omega_h}{dt} = \frac{k_a k_1}{J_{tr} R_a} u_h - \left( \frac{B_{tr}}{J_{tr}} + \frac{k_r^2}{J_{tr} R_a} \right) \omega_h - f_1(\omega_h),
\]

\[
\frac{d\Omega_h}{dt} = \frac{K_D \cos^2 \theta_v + K_E \sin^2 \theta_v + K_F}{f_3(\theta_h) + f_6(\theta_v)} \Omega_h - \frac{k_m \omega_v \sin \theta_v \Omega_v (K_D \cos^2 \theta_v - K_E \sin^2 \theta_v)^2}{(K_D \cos^2 \theta_v + K_E \sin^2 \theta_v + K_F)^2} - \frac{K_D \cos^2 \theta_v + K_E \sin^2 \theta_v + K_F}{f_3(\omega_v)} - \frac{k_m \cos \theta_v \left( \frac{k_r}{\pi_c} u_v - \left( B_{mr} + \frac{k_r^2}{\pi_c} \right) \omega_v \right)}{J_{mr} (K_D \cos^2 \theta_v + K_E \sin^2 \theta_v + K_F)} - \frac{f_4(\omega_v)}{J_{mr} (K_D \cos^2 \theta_v + K_E \sin^2 \theta_v + K_F)},
\]

\[
\frac{d\theta_h}{dt} = \Omega_h.
\]

Finally, the discrete-time model is given as follows [38]:

\[
A = \begin{bmatrix}
0.8534 & 0 & 0 \\
0 & 0.9588 & -0.0352 \\
0 & 0.1000 & 1.0000
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
0.2782 & 0 \\
0 & 0.0026 \\
0 & 0.0845 \\
0 & 0 & 0
\end{bmatrix},
\]

along with a nonlinear function defined as

\[
g(x_k) = g(x_{eq}, x_k, u_k) = (A(x_k) - A)x_k,
\]

where \(x_{eq} = 0\) stands for the equilibrium point of the system, while \(A(x_k)\) is obtained by excluding it from (65)–(70). In particular, its numerical values are given as follows:

\[
A(x_k) = \begin{bmatrix}
T_s \cdot a_{11} + 1 & 0 & 0 \\
T_s \cdot a_{21} & T_s \cdot a_{22} + 1 & T_s \cdot a_{23} \\
0 & T_s & 1 \\
0 & 0 & 0 \\
T_s \cdot a_{51} & T_s \cdot a_{52} & T_s \cdot a_{53} \\
0 & 0 & 0 \\
T_s \cdot a_{44} + 10 & 0 & 0 \\
T_s \cdot a_{54} & T_s \cdot a_{55} + 1 & T_s \cdot a_{56} \\
0 & T_s & 1
\end{bmatrix}
\]
with $T_s$ standing for the sampling time, and where

$$a_{11} = - (0.0114 \cdot \omega_h)/0.0059,$$

$$a_{21} = (0.2820 \cdot \cos (\theta_v + \theta_{v,0}) \cdot 0.0566 \cdot \omega_h) / (0.0553 \cdot \cos (\theta_v + \theta_{v,0})^2 + 0.0058 \cdot \sin (\theta_v + \theta_{v,0})^2 + 0.0059),$$

$$a_{22} = -0.0185 / (0.0553 \cdot \cos (\theta_v + \theta_{v,0})^2 + 0.0059),$$

$$a_{23} = -0.0158 / (0.0553 \cdot \cos (\theta_v + \theta_{v,0})^2 + 0.0059),$$

$$a_{24} = -0.0017 \cdot \cos (\theta_v + \theta_{v,0}) \cdot 0.0195 \cdot \omega_v / (0.0254 \cdot (0.0553 \cdot \cos (\theta_v + \theta_{v,0})^2 + 0.0058 \cdot \sin (\theta_v + \theta_{v,0})^2 + 0.0059),$$

$$a_{25} = 0.0017 \cdot \omega_v \cdot \sin (\theta_v + \theta_{v,0}) \cdot (0.0553 \cdot \cos (\theta_v + \theta_{v,0})^2 - 0.0058 \cdot \sin (\theta_v + \theta_{v,0})^2 - 0.0059 \cdot 2 \cdot 0.0058 \cdot \cos (\theta_v + \theta_{v,0})^2 / (0.0553 \cdot \cos (\theta_v + \theta_{v,0})^2 + 0.0059),$$

$$a_{26} = 0.0623 \cdot \theta_v \cdot (0.0553 \cdot \cos (\theta_v + \theta_{v,0})^2 + 0.0059),$$

$$a_{44} = -(0.0195 \cdot \omega_v)/0.0254,$$

$$a_{51} = -8.651 \cdot 10^{-5} + 0.0027 \cdot \omega_v/(3.792 \cdot 10^{-4}),$$

$$a_{52} = (0.0764 \cdot \omega_v \cdot \cos (\theta_v + \theta_{v,0}) - \Omega_h \cdot 0.0591 \cdot \sin (\theta_v + \theta_{v,0}) \cdot \cos (\theta_v + \theta_{v,0}) / 0.0643,$$

$$a_{54} = 0.0939 \cdot \omega_v/0.0643,$$

$$a_{55} = -1.5963,$$

$$a_{56} = 9.81 \cdot ( -0.0157 \cdot \cos (\theta_v + \theta_{v,0}) - 0.0220 \cdot \sin (\theta_v + \theta_{v,0}) ) / 0.0321,$$

in which $\theta_{v,0}$ denotes an initial pitch angle. For more information the reader is referred to [38]. Finally, the output matrix is given as follows

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

which indicates that both angular velocities $\Omega_v$ and $\Omega_h$ were not measured. The disturbances $w_{1,k}$ and $w_{1,k}$ were generated according to the uniform distribution on the intervals $[-0.2, 0.2]$ and $[-0.3, 0.3]$, respectively.

In order to compare the results, both schemes are affected by the same fault scenario:

$$f_{a,1,k} = \begin{cases} -0.2 \cdot u_{1,k} & 80 \leq t[s] \leq 110 \\ 0 & \text{otherwise} \end{cases},$$

$$f_{a,2,k} = \begin{cases} -0.1 \cdot u_{2,k} & 40 \leq t[s] \leq 95 \\ 0 & \text{otherwise} \end{cases},$$

$$f_{s,1,k} = \begin{cases} y_k - 0.3 & 60 \leq t[s] \leq 90 \\ 0 & \text{otherwise} \end{cases},$$

$$f_{s,2,k} = \begin{cases} y_k - 0.05 & 50 \leq t[s] \leq 70 \\ 0 & \text{otherwise} \end{cases},$$

which means that the sensor faults impair the angle of rotation in both horizontal and vertical axes. Notice that the actuator faults appear somewhat at the same time. In other words, there were some moments of time when the system was performing incorrectly due to the presence of both sensor and actuator faults. Finally, the actuator faults might be interpreted as an intermittent degradation of the rotor efficiency. It should be also noted that the fault scenarios are designed in such a way as to have both periods of simultaneous and separate sensor and actuator faults, respectively. Such an approach allows exposing the ability of simultaneous fault estimation. It will also make it possible to prove that such an ability does not impair estimation of separate faults.

Moreover, the experiments were performed without the feedback and the constant control input obeying:

$$u_{1,k} = 0.3, \quad u_{1,k} \in (0, 1), \quad k = 1, \ldots,$$

$$u_{2,k} = 0.2, \quad u_{2,k} \in (0, 1), \quad k = 1, \ldots,$$

which means that the main rotor was operated as 30% of its maximum performance while the tail one as 20%. The reason behind such an open loop experiment is associated with the fact of providing the same operating conditions for the $H_{\infty}$ and QB cases, respectively.

Moreover, during the experiment the initial conditions for the state and the estimator were set to $x_0 = [0, 0, 0.001, 0, 0, 0.001]$ and $z_0 = [0, 0, 0.01, 0, 0, 0.01]$, respectively. Additionally, the actuator and sensor faults were initialized by $f_{a,0} = [0, 0]$ and $f_{s,0} = [0, 0]$, respectively.

Note that throughout the paper the real state or faults are depicted with a blue solid line while their estimates with a red dashed line. Furthermore, the uncertainty intervals overbounding the real and estimated values are indicated with black dash-dotted lines. In the figures presenting the response of the system, the measured output is displayed with a green dash-dotted line. Note that the uncertainty intervals are guaranteed to include the real and estimated values.

Figures 2–3 present the states of the TRMS under the sensor faults along with their estimates, measured values.
as well as uncertainty intervals. It can be observed that the states were correctly reproduced in spite of incorrect values measured by the faulty sensors. Moreover, the designed estimator allowed obtaining satisfactorily tight uncertainty intervals. In case of the state estimation problem, they can be perceived as some kind of sensor fault detection threshold with the following detection rule: *If the measurement exceeds the uncertainty interval, then the sensor fault is detected.*

Figure 4 shows the performance of the main rotor actuator fault estimation along with the uncertainty intervals obtained with the QB approach (Fig. 4a) and the $H_\infty$ one (Fig. 4b), whilst the tail rotor actuator fault is presented in Fig. 5 (QB-based: Fig. 5a, $H_\infty$-based: Fig. 5b). It can be observed that the QB approach results with faster convergence to the real fault, but this is realized as the cost of a minor impair in the disturbance attenuation. In order to compare the levels of fault estimation accuracy and noise attenuation, Tab. 1 details the statistical metrics, i.e., the mean values, median and Mean Squared Error (MSE) of the actuator fault estimation error for both the main and tail rotors. It presents the measured differences between the results obtained with the $H_\infty$ and QB approach. However, it cannot be clearly stated which one is better due to the fact that the values exhibited in Tab. 1 are very similar.

On the other hand, the superiority of either the $H_\infty$ or QB approach can be deduced while analyzing the associated size of uncertainty intervals. Moreover, such a superiority can be also measured with the convergence rate. Indeed, it is especially evident for the tail rotor. The estimator designed with the $H_\infty$ approach converges much slower than that obtained with the QB one.

Figures 6–7 show the sensor faults for the rotation angles $\theta_v$ and $\theta_h$, respectively, along with their estimates and uncertainty intervals. In both cases, i.e., QB and $H_\infty$, they were estimated very well. The fault estimate oscillations around the true values are caused by a considerable disturbance acting onto the system.

Taking into account the fact that the uncertainty intervals are symmetric, it is more profitable to analyze their upper bound. Figure 8 shows the convergence of sample upper bounds provided by the $H_\infty$ and QB for the actuator faults. It can be easily observed that those obtained with the $H_\infty$ (solid lines) for both the main and tail rotor faults are much wider than the ones acquired with the QB approach (dashed lines). Moreover, the $H_\infty$ ones converge much slower while the QB-based ones settle at the steady-state very quickly.

### TABLE 1: Actuator fault estimation error comparison for the TRMS.

<table>
<thead>
<tr>
<th></th>
<th>$H_\infty$</th>
<th>QB</th>
</tr>
</thead>
<tbody>
<tr>
<td>average $f_{a,1} - f_{a,1}$</td>
<td>9.5751 $\cdot 10^{-4}$</td>
<td>2.3890 $\cdot 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td>$f_{a,2} - f_{a,2}$</td>
<td>1.8105 $\cdot 10^{-4}$</td>
</tr>
<tr>
<td>median $f_{a,1} - f_{a,1}$</td>
<td>2.2225 $\cdot 10^{-4}$</td>
<td>1.0323 $\cdot 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td>$f_{a,2} - f_{a,2}$</td>
<td>3.1978 $\cdot 10^{-4}$</td>
</tr>
<tr>
<td>MSE</td>
<td>$f_{a,1} - f_{a,1}$</td>
<td>2.7983 $\cdot 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td>$f_{a,2} - f_{a,2}$</td>
<td>7.6759 $\cdot 10^{-4}$</td>
</tr>
</tbody>
</table>

**B. APPLICATION TO A REAL MULTI-TANK SYSTEM**

Motivated by the results presented in the preceding section, the QB-based approach was selected for validation with real system data. In particular, this section evaluates the effectiveness of the proposed solution using the Multi-Tank (MT) system presented in Fig. 9. It is intended to simulate a real industrial system within laboratory conditions [22]. It is composed of three separate tanks located one over another, and accompanied with drain valves and liquid level sensors, which are based on a hydraulic pressure measurement. All tanks have different shapes reflecting the system nonlinearities. A controlled-speed water pump is employed to fill the upper tank. The considered multi-tank system operates with an external, PC-based digital controller. The control computer is connected with a pump, sensor and valves with a dedicated I/O device. The I/O device is controlled by a real-time software operating under Matlab/Simulink. Finally,
it should be noted that the inlet flow is expressed in $m^3/s$. This means that, due to the relatively small size of multtank system, the actuator input and possible faults operate on relatively low values.

In order to design the estimator, the following discrete time model was used:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1.1428 \\ 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

along with the nonlinear function describing the behavior of the system denoted by

$$g(x_k) = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix},$$

where $H_1$, $H_1$, $H_3$ denote the water level in the upper, middle and lower tanks, respectively.

The proposed methodology was compared to the one presented in [30]. For that purpose, the following fault scenario

with

$$g_1 = -0.1149371429 \cdot 10^{-3} \cdot H_1^{1/2},$$

$$g_2 = \frac{0.10057 \cdot 10^{-5} \cdot H_2}{(0.03831 \cdot H_2)^{1/2}} \cdot H_1^{1/2},$$

$$g_3 = 0.00003418 \cdot (0.133225 - (0.35 - H_3)^2)^{-1/2} \cdot H_2^{1/2}$$

$$- 0.2800228571 \cdot 10^{-4} \cdot (0.133225 - (0.35 - H_3)^2)^{-1/2} \cdot H_3^{1/2} - (0.35 - H_3)^2,$$
concerning an intermittent sensor fault and an incipient actuator fault was employed:

\[
f_{a,k} = \begin{cases} 
-0.2 \left( k + 5000 \right) u_k & 50 \leq t[s] \leq 100 \\
0 & \text{otherwise}
\end{cases}, \quad (102)
\]

\[
f_{s,k} = \begin{cases} 
0.1 y_k & 70 \leq t[s] \leq 110 \\
0 & \text{otherwise}
\end{cases}, \quad (103)
\]

with the sensor fault distribution matrix

\[
C_f = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad (104)
\]

which means that the sensor fault impairs the measurements of the lower tank liquid level. In the above scenario, the sensor fault appears somewhat at the same time. It makes the measurements gathered from the sensor 10\text{cm} biased compared to the real water level. Moreover, the actuator
fault might be interpreted as a slow degradation of the pump efficiency from 0 up to 20%.

Figures 10–12 present the real water level of the appropriate tank along with its measurements and the estimates obtained with the methodology proposed in this paper. As can be observed, the real states were recovered with good accuracy despite the simultaneous actuator and sensor faults.

Figures 13–14 present the real actuator and sensor faults, along with their estimates. It can be observed that the actuator fault was recovered correctly. Only the initial transient phase exhibits bigger discrepancies. However, after that the estimator converged to the real value and then followed it properly even in the presence/absence of the incipient fault. It can be also seen that the sensor fault was estimated almost perfectly. Both actuator and sensor faults inaccuracies are caused by measurement noise as well as the model–reality mismatch.

Figure 15 presents a comparison of the actuator fault estimate achieved with the proposed approach (red dashed line) and the one obtained with the Recurrent Neural-Network Fault Estimator (RNNFE) proposed in [30] (black dash-dotted line). It can be observed that the RNNFE-based fault estimate is less accurate and converges slower to the real fault than the one obtained with the proposed approach. Note that the scheme proposed in [30] is able to estimate actuator faults only. This means that the comparison was performed with fault-free sensors. It is an obvious fact that, in the case of simultaneous actuator and sensor faults, the actuator fault
estimate provided by the RNNFE estimator [30] would be estimated incorrectly. Notice that the actuator fault estimate obtained with the proposed approach (Fig. 13) is not visibly impaired by the simultaneously occurring sensor fault. This means that very similar estimation quality is obtained with a fault-free sensor (Fig. 15).

TABLE 2: Actuator fault estimation error comparison for the real MT system.

<table>
<thead>
<tr>
<th></th>
<th>RNNFE</th>
<th>QB (proposed)</th>
</tr>
</thead>
<tbody>
<tr>
<td>average $f_{a,1} - \hat{f}_{a,1}$</td>
<td>$4.1393 \cdot 10^{-5}$</td>
<td>$1.0650 \cdot 10^{-6}$</td>
</tr>
<tr>
<td>median $f_{a,1} - \hat{f}_{a,1}$</td>
<td>$1.7849 \cdot 10^{-6}$</td>
<td>$3.6296 \cdot 10^{-7}$</td>
</tr>
<tr>
<td>MSE $f_{a,1} - \hat{f}_{a,1}$</td>
<td>$8.8597 \cdot 10^{-8}$</td>
<td>$1.4955 \cdot 10^{-11}$</td>
</tr>
</tbody>
</table>

For the purpose of further comparison, an estimator was obtained. It has the same structure as the proposed one but is nonoptimal in the QB sense. This means that it only guarantees the convergence of the estimator without taking care of the robustness issue. Figure 16 presents a comparison of the QB optimal and nonoptimal solutions. It can be easily observed that the nonoptimal one is impaired with lower noise attenuation and slower convergence than the optimal one.

Finally, Fig. 17 provides the histograms of a relative actuator fault estimation error obtained with the proposed approach (a) and the RNNFE [30] (b). These histograms, along with the statistics provided in Tab. 2, clearly exhibit the superiority of the proposed approach over the RNNFE one.

VI. CONCLUSIONS
The paper aimed at solving the problem of simultaneous estimation of state as well as sensor and actuator faults. The proposed methodology is based on transforming the original nonlinear state-space system into descriptor one. In particu-
lar, it combines the state and sensor fault into one extended vector. For such a system, the task boils down to estimating the extended vector and the actuator fault simultaneously. As a consequence, all of the desired parameters are estimated. Moreover, in order to ensure the robustness of the estimator the quadratic boundedness approach is that the rate of the convergence of the estimate and the disturbance attenuation level can be treated separately by adjusting the weight matrix. Contrarily, the $H_{\infty}$ approach typically assumes that it is equal to identity matrix, and as a consequence both are controlled through the one scalar parameter. Thus, the faster the convergence, the less noise damping, and vice-versa. However, as it was shown, the $H_{\infty}$ approach constitutes a special case of the quadratic boundedness one. This observation justifies the obtained results.

The final part of the paper presented an illustrative simulation example concerning a twin rotor system. The obtained results clearly confirm the efficiency of the proposed approach and indicate the quadratic boundedness as a superior strategy. Taking into account this observation, an experimental study concerning a multi-tank system was presented. The obtained results are based on real data gather from laboratory system. They clearly indicate the good performance of the proposed approach compared to an alternative one borrowed from the literature.

The future research will be oriented towards the development of a robust fault-tolerant controller utilizing the strategies designed and compared in this paper.

**APPENDIX**

**Proof 1**: Let the stability analysis of the $H_\infty$ estimator design begin with recalling that (32) should satisfy

$$
\Delta V_k + \epsilon_k^T \epsilon_k - \mu^2 \bar{w}_k^T \bar{w}_k < 0,
$$

(105)

with

$$
\Delta V_k = V_{k+1} - V_k.
$$

(106)

Let us set:

$$
Q = \begin{bmatrix} I - P & 0 & 0 \\ 0 & -\mu^2I & 0 \\ 0 & 0 & P \end{bmatrix},
$$

(107)

and

$$
M = \begin{bmatrix} 0 \\ 0 \\ U \end{bmatrix}.
$$

(108)

Applying Lemma 2 along with (105) leads to

$$
\begin{bmatrix} I - P & 0 & X(h)^T U^T \\ 0 & -\mu^2I & Z^T U^T \\ U X(h) & U Z & P - U - U^T \end{bmatrix} < 0.
$$

(109)

Substituting

$$
U X(h) = U \left( \hat{A}(h) - \hat{K} \hat{C} \right) = U \hat{A}(h) - N \hat{C},
$$

(110)

and

$$
U Z = U \left( \hat{W}_1 - \hat{K} \hat{W}_2 \right) = U \hat{W}_1 - N \hat{W}_2
$$

(111)

into (109) yields (43). Finally, the reverse can be proven by substituting $U = P$. In this case, applying Lemma 2 to (109) gives (105). This completes the proof.

**Proof 2**: Let us formulate the following QB-based Lyapunov function

$$
V_{k+1} - (1 - \alpha) V_k - \alpha \bar{w}_k^T Q_{w} \bar{w}_k < 0.
$$

(112)

Let us define

$$
Q = \begin{bmatrix} -P + \alpha P & 0 & 0 \\ 0 & -\alpha Q_w & 0 \\ 0 & 0 & P \end{bmatrix},
$$

(113)

and

$$
M = \begin{bmatrix} 0 \\ 0 \\ U \end{bmatrix}.
$$

(114)
Applying Lemma 2 along with (112) leads to
\[
\begin{bmatrix}
-P + \alpha P & 0 & X(h)^T U^T \\
0 & -\alpha Q_w & Z^T U^T \\
U X(h) & UZ & P - U - U^T
\end{bmatrix} < 0. 
\tag{115}
\]
Substituting
\[
U X(h) = U \left( \dot{A}(h) - \dot{K} \dot{C} \right) = U \dot{A}(h) - NU \dot{C}, 
\tag{116}
\]
\[
U Z = U \left( \dot{W}_1 - \dot{K} \dot{W}_2 \right) = U \dot{W}_1 - NU \dot{W}_2, 
\tag{117}
\]
into (115).

Finally, the reverse can be proven by substituting \( U = P \).

In this case, applying Lemma 2 to (115) gives (112). This completes the proof. \( \blacksquare \)

**Proof 3:** From (105) it is evident that
\[
V_{k+1} - V_k < \mu^2 \tilde{w}^T_k \tilde{w}_k - \tilde{e}^T_k \tilde{e}_k. 
\tag{118}
\]
Since \( \tilde{w}_k \in E_{w_2} \), i.e., \( \tilde{w}^T_k Q_w \tilde{w}_k \leq 1 \), it can be easily seen that
\[
V_{k+1} < \mu_p + V_k - \tilde{e}^T_k \tilde{e}_k, 
\tag{119}
\]
or
\[
V_{k+1} < \mu_p + \tilde{e}^T_k (P - I) \tilde{e}_k. 
\tag{120}
\]
Satisfying (43) causes that \( P - I > 0 \), which yields
\[
\tilde{e}^T_k P \tilde{e}_k > \tilde{e}^T_k \tilde{e}_k, 
\tag{121}
\]
and hence there exists \( 0 < \zeta < 1 \) such that
\[
\gamma \tilde{e}^T_k P \tilde{e}_k = \tilde{e}^T_k \tilde{e}_k. 
\tag{122}
\]
The above relation allows us to write
\[
V_{k+1} \leq \mu_p + (1 - \gamma) V_k. 
\tag{123}
\]
By applying induction, the above inequality is equivalent to (48), which completes the proof. \( \blacksquare \)

Note that the parameter \( \gamma \) in (122) can be obtained by applying the Rayleigh quotient, i.e.,
\[
\gamma = \frac{\tilde{e}^T_k \tilde{e}_k}{\tilde{e}^T_k P \tilde{e}_k} \leq \lambda_{\text{min}}(P)^{-1}, 
\tag{124}
\]
where \( \lambda_{\text{min}}(P) \) stands for the minimum eigenvalue of \( P \).

**Proof 4:** The overbounding values of \( \tilde{e}_{i,k} \) can be computed by maximizing/minimizing \( c^T \tilde{e}_k \). Applying the Lagrange framework, the following Lagrange functional can be formulated:
\[
h (\tilde{e}_k, \lambda) = c^T \tilde{e}_k + \lambda \left( c^T P \tilde{e}_k - \eta_k \right), 
\tag{125}
\]
where \( \lambda \) stands for the Lagrange multiplier. Differentiating (125) with respect to \( \tilde{e}_k \) and \( \lambda \) gives:
\[
\frac{\partial h (\tilde{e}_k, \lambda)}{\partial \tilde{e}_k} = c^T + 2 \lambda \tilde{e}^T_k P = 0, 
\tag{126}
\]
\[
\frac{\partial h (\tilde{e}_k, \lambda)}{\partial \lambda} = c^T_k P \tilde{e}_k - \eta_k. 
\tag{127}
\]
Therefore, from (126), it can be shown that
\[
\tilde{e}^T_k = -\frac{1}{2\lambda} \tilde{e}^T_k P^{-1}. 
\tag{128}
\]
Substituting (128) into (127) gives
\[
\lambda = \frac{1}{2} \left\{ \left( n_h^{1} c^T P^{-1} c \right)^{\frac{1}{2}} \right\}. 
\tag{129}
\]
Finally, injecting (129) into (128) yields
\[
-\delta_{i,k} \leq \tilde{e}_{i,k} \leq \delta_{i,k}, \quad i = 1, \ldots, n + n_s + r, 
\tag{130}
\]
where \( \delta_{i,k} \) is given by (54), which completes the proof. \( \blacksquare \)

**REFERENCES**


MARCIAN PAZERA was born in Poland in 1990. He received the M.Sc. degree in control engineering and robotics from the University of Zielona Góra (Poland) in 2015. He is currently a Ph.D. student at the Institute of Control and Computation Engineering, University of Zielona Góra. His current research interests include fault detection and isolation (FDI), fault-tolerant control (FTC), as well as experimental design and control theory.

MARCIAN WITCZAK was born in Poland in 1973, received the M.Sc. degree in electrical engineering from the University of Zielona Góra (Poland), the Ph.D. degree in automatic control and robotics from the Wroclaw University of Technology (Poland), and the D.Sc. degree in electrical engineering from the University of Zielona Góra, in 1998, 2002, and 2007, respectively. In 2015 he received a full professorial title. Since then, Marcin Witczak has been a professor of automatic control and robotics at the Institute of Control and Computation Engineering, University of Zielona Góra. His current research interests include computational intelligence, fault detection and isolation (FDI), fault-tolerant control (FTC), as well as experimental design and control theory. Marcin Witczak has published more than 200 papers in international journals and conference proceedings. He is an author of 4 monographs and 30 book chapters. Since 2015, he has been a member of the Committee on Automatic Control and Robotics Committee of the Polish Academy of Sciences. As of 2018, he is also an associated editor of ISA Transactions.

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