Synthesis of sliding mode control for a class of uncertain singular fractional-order systems based restricted equivalent

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ABSTRACT This paper has researched the sliding mode control (SMC) problem for a class of restricted singular fractional-order system (SFOS) with matched uncertainties. The condition of system decomposition based on controller separation is given at first. Then fractional order (FO) sliding mode (SM) function and controller for the restricted equivalent system are designed, and the stabilization of the system is studied by FO Lyapunov method. Finally, the feasibility of the proposed method is verified by a numerical example.

INDEX TERMS Sliding mode control, restricted equivalent decomposition, singular fractional-order systems, uncertainties

I. INTRODUCTION

SMC is a special variable structure control (VSC), originated from the former Soviet union in the 1960s, and made great progress through the vigorous promotion of Itkis, Utkin, Gao, Edwards and Spurgeon [1]–[3]. As one of the robust control technologies, SMC is possible the most successful method to deal with bounded internal uncertainties or external disturbances and parasitic dynamics in modern control theory. It has admirable properties including insensitivity to matched disturbances or uncertainties and robustness to parasitic dynamics. The idea of SMC is to steer system state trajectories into SM surface properly chosen and then maintain SM motion on the surface thereafter by means of control [4]. As researchers pay attention to SM, also senior SMC techniques are spawned, such as, the second-order SMC, high-order SMC, super-twisting SMC, adaptive SMC, etc [5]–[9]. Besides being widely used in linear systems, SMC is also widely used in singular systems (SS) and fractional order systems (FOS).

SS has extensive applications in electrical circuit, economical systems and many other fields, and has attracted much attention in the literatures [10]–[13]. It is also called descriptor systems (DS). Many scholars pay attention to the research of SS. For example, refs [14], [15] have studied the feedback (or output-feedback) control problem for linear SS (or DS), refs [16]–[19] have focused on the research of SMC technology for SS.

FOS have been studied by many researchers in mathematics, engineering, physics and other fields from the perspective of theory and application [20]. In FOS, the state of systems is pseudo, which can be estimated by a continuous filter and approximated by some functions of Matlab Toolbox [21]–[24]. SMC is also often used for FOS, such as FOSM controller and observer have been designed for FOS (or complex FOS) using kinds of senior methods [25]–[27].

Currently, several scholars have paid attention to SFOs, descriptor FOS or FO differential algebraic systems. Ref [28] has studied existence of solution of SFOS based on Caputo and related fractional derivatives. Meanwhile, SFOS has been applied to practical systems. For example, mathematical model of some electrical circuit, pendulum, Newtonian fluid have been expressed as SFOS [29], [30]. For $\alpha \in (0, 1)$, some researchers have discussed the admissibility, stability, robust stabilization and output controller for SFOS [31]–[34]. But for $\alpha \in (0, 2)$, other researchers has studied the stabilization and normalization for SFOS based on uncertainties [35]–
[37]. Ref [38] has researched estimation problem for discrete-time stochastic SFOS by Kalman filter.

At present, paper about SMC of SFOS is not found as we know. In this paper, we discuss the synthesis problem of SMC for SFOS. This manuscript will be arranged as hereunder mentioned. In section II, one gives system expressions and some preparatory works. In section III, firstly, one builds the restricted equivalent sufficient and necessary condition for SFOS. Then, SMC function and SMC controller are designed for SFOS. In section IV, one utilizes a numerical example to verify the method effectiveness. Some conclusions are talked over in section V.

Notations: In the sequel, R, C denotes respectively real and complex set. Iₙ ∈ Rⁿ×ₙ is a identity matrix. rank(E) is rank of matrix E. D⁰, D⁻α are respective abbreviations of 0Dᵗ α (Caputo FO derivative operator) and 0Dᵗ⁻α (Riemann-Liouville FO integral operator) without confusion. range(B) is range of matrix B. spec(A, α) represents spectrum of det(sI − A) = 0. spec(E, A, α) represents spectrum of det(sE − A) = 0. ∥ · ∥ represents matrix norm.

II. SYSTEM DESCRIPTION AND PRELIMINARIES
Considering the nonlinear SFOS,

\[ ED^\alpha x(t) = Ax(t) + Bu(t) + f(x(t), t), \]  

(1)

where 0 < α ≤ 1, x ∈ Rⁿ is pseudo semi-state vector, u(t) ∈ Rᵐ is input vector, E ∈ Rⁿ×ⁿ is singular matrix, A ∈ Rⁿ×ⁿ and B ∈ Rᵐ×ⁿ are system matrices. In this paper, one assumes that rank(E) = r < n and nonlinear term f(x(t), t) ∈ Rⁿ is bounded. x(0) = x₀ is system initial condition. If f = 0 and u = 0, one denotes (E, A, α) as system (1). If E (= I) is nonsingular and f = 0, system (1) becomes FOS,

\[ D^\alpha x(t) = Ax(t) + Bu(t). \]  

(2)

If α = 1 and f = 0, system (1) is SS,

\[ E\dot{x}(t) = Ax(t) + Bu(t). \]  

(3)

Next, some definitions, lemmas, assumptions and properties are given firstly.

Definition 1. [20] The Riemann-Liouville FO integral is defined as

\[ 0D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau)(t - \tau)^{\alpha - 1} d\tau, \]

the Caputo’s FO derivative is defined as

\[ 0D_t^\alpha f(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t f'(\tau)(t - \tau)^{-\alpha} d\tau, \]

where 0 < α < 1, Γ(·) is Euler’s Gamma function.

Definition 2. [35] For system (1), one gives regular, impulse free, stable definitions.

- If there exists at least one complex number s ∈ C such that the pseudo-polynomial det(sⁿE − A) is not identically zero, then the system (E, A, α) is regular.

- If deg(det(sE − A)) = rank(E) is satisfied, then the system (E, A, α) is impulse free.

- If all the roots of det(sE − A) = 0 satisfy the condition

\[ |\arg(spec(E, A, \alpha))| > \alpha \frac{\pi}{2}, \]

then the system (E, A, α) is stable.

The regularity ensures system (1) solutions are existential and unique under f = 0 and u = 0.

Lemma 1. [31] If the system (E, A, α) is regular, then one can find two nonsingular matrices M₁, N₁ satisfying

\[ M_1 EN_1 = \begin{bmatrix} I_r & 0 \\ 0 & J_{n-r} \end{bmatrix}, \]

\[ M_1 AN_1 = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}, \]

\[ M_1 B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \]  

(4)

where Jₙ₋ᵣ is a nilpotent matrix.

Let N⁻¹₁x(t) = [x₁ᵀ(t), x₂ᵀ(t)]ᵀ. Under lemma 1, the system (1) with f = 0 can be rewritten as

\[ \begin{cases} D^\alpha x_1(t) = \tilde{A}_1 x_1(t) + B_1 u(t) \\ J_{n-r} D^\alpha x_2(t) = x_2(t) + B_2 u(t) \end{cases} \]  

(5)

One calls the above decomposition as Weierstrass decomposition or slow-fast decomposition. The first part is called slow system, another is fast system.

Assumption 1. Suppose rank(sⁿE − A, B) = n, for any s ∈ C.

Assumption 2. Suppose rank(E, B) = n and B is column full rank.

Lemma 2. [39] If assumption 1 and 2 is satisfied, then system (1) with f = 0 is completely controllable.

Assumption 3. Suppose f ∈ range(B) and ∥f∥ ≤ ρ(∥E∥x₀), where ρ is continuous and ρ(0) = 0.

Assumption 4. Suppose K₁ satisfying

\[ |\arg(spec(A_{11} + A_{12}K_1))| > \alpha \frac{\pi}{2}. \]

Assumption 5. Suppose K₂ satisfying

\[ |\arg(spec(E_2, K_2, \alpha))| > \alpha \frac{\pi}{2}. \]

III. MAIN RESULTS
A. RESTRICTED SYSTEM EQUIVALENT DECOMPOSITION
Systems decomposition in terms of state variables are presented in the above results, but often bring inconvenience to the controller u design. Aiming at this defect, Ref [40] proposed a new restricted system equivalent decomposition of linear SS, namely the original n dimension linear SS is
decomposed into a $n - m$ dimension without controller and a $m$ dimension control related SS, which realize the completely separation of control variables. The design of system is divided into the control of the isolated two steps, thus the problem of high-dimensional turned into low-dimension of the same problem. one will extend these results to SFOS (1) with $f = 0$.

**Theorem 1.** For system (1) with $f = 0$, if assumption 2 is satisfied, then one can find two nonsingular matrices $M_2, N_2$ satisfying

$$
M_2E\overline{N}_2 = \begin{bmatrix}
I_{n-m} & 0 \\
0 & E_2
\end{bmatrix},
$$

$$
M_2AN_2 = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix},
$$

$$
M_2B = \begin{bmatrix}
0 \\
B_2
\end{bmatrix},
$$

where $E_2 \in \mathbb{R}^{m \times m}$, $B_2 \in \mathbb{R}^{(n-m) \times m}$ is nonsingular. Let $N_2^{-1}x(t) = \left[x_1^T(t), x_2^T(t)\right]^T, x_1(t) \in \mathbb{R}^{n-m}, x_2(t) \in \mathbb{R}^m$, then system (1) with $f = 0$ can be restricted equivalent with

$$
\begin{cases}
D^\alpha x_1(t) = A_{11}x_1(t) + A_{12}x_2(t), \\
E_2D^\alpha x_2(t) = A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t)
\end{cases}
$$

(7)

**Proof.** If $B$ is column full rank, then let

$$
B = \begin{bmatrix}
B_1 \\
B_2
\end{bmatrix},
$$

where $B_2 \in \mathbb{R}^{m \times m}$ is nonsingular matrix, $B_1 \in \mathbb{R}^{(n-m) \times m}$.

Choosing nonsingular matrix $\bar{\Phi} = \begin{bmatrix}
I_{n-m} & -B_1B_2^{-1} \\
0 & I_m
\end{bmatrix}$, then one supposed

$$
\bar{\Phi} = \begin{bmatrix}
\bar{E}_{11} & \bar{E}_{12} \\
\bar{E}_{21} & \bar{E}_{22}
\end{bmatrix},
$$

where $\bar{E}_{11} \in \mathbb{R}^{(n-m) \times (n-m)}, \bar{E}_{12} \in \mathbb{R}^{(n-m) \times m}, \bar{E}_{21} \in \mathbb{R}^{m \times (n-m)}, \bar{E}_{22} \in \mathbb{R}^{m \times m}$. If $\text{rank}(E, B) = n$, then

$$
\text{rank}M_2(E, B) = \text{rank}\begin{bmatrix}
\bar{E}_{11} & \bar{E}_{12} \\
\bar{E}_{21} & \bar{E}_{22} & B_2
\end{bmatrix} = \text{rank}(\bar{E}_{11}, \bar{E}_{12}) + m.
$$

From the above equation, one can get $\text{rank}(E_{11}, E_{12}) = n - m$. Therefore there exist a matrix $\bar{N}_2$ satisfying

$$
\bar{M}_2E\bar{N}_2 = \begin{bmatrix}
\bar{E}_{11} & \bar{E}_{12} \\
\bar{E}_{21} & \bar{E}_{22}
\end{bmatrix},
$$

where $\bar{E}_{11} \in \mathbb{R}^{(n-m) \times (n-m)}$ is nonsingular, $\bar{E}_{12} \in \mathbb{R}^{(n-m) \times m}, \bar{E}_{21} \in \mathbb{R}^{m \times (n-m)}, \bar{E}_{22} \in \mathbb{R}^{m \times m}$. Choosing nonsingular matrices

$$
\bar{M}_2 = \begin{bmatrix}
I_{n-m} & 0 \\
\bar{E}_{21}\bar{E}_{11}^{-1} & I_m
\end{bmatrix}, \bar{N}_2 = \begin{bmatrix}
I_{n-m} & 0 \\
0 & \bar{E}_{21}\bar{E}_{11}^{-1}
\end{bmatrix}.
$$

Therefore, one can get

$$
\dot{\mathbf{M}}_2\mathbf{E}\omega_2\bar{N}_2 = \begin{bmatrix}
I_{n-m} & 0 \\
\bar{E}_{21}\bar{E}_{11}^{-1}E_1 & I_m
\end{bmatrix}, \dot{\mathbf{E}}_{11}, \dot{\mathbf{E}}_{12} \begin{bmatrix}
\dot{E}_{11} & \dot{E}_{12} \\
\dot{E}_{21} & \dot{E}_{22}
\end{bmatrix},
$$

where $E_2 = \dot{E}_{22} - \dot{E}_{21}\dot{E}_{11}^{-1}\dot{E}_{12}$.

Finally, one can choose $M_2 = M_2\bar{M}_2, N_2 = \bar{N}_2\bar{N}_2$, such that equation (6) is obtained.

**Remark 1.** On Theorem 1, $E_2$ is singular matrix, so the first equation of (7) is linear FOS without $u$, the second equation of (7) is SFOS with $u$.

If there exists external disturbance of system (1), i.e., $f \neq 0$, but satisfied assumption 3. For $f \in \text{range}(B)$, then one can find a vector $\bar{f}$ satisfying $f = \bar{B}\bar{f}$. Based on Theorem 1, one can get

$$
M_2f = M_2B\bar{f} = \begin{bmatrix}
0 \\
B_2\bar{f}
\end{bmatrix}.
$$

Denoting $f_2 = B_2\bar{f}$, one gets the following corollary.

**Corollary 1.** If assumption 2 and 3 are satisfied, then system (1) can be restricted equivalent with

$$
\begin{cases}
D^\alpha x_1(t) = A_{11}x_1(t) + A_{12}x_2(t), \\
E_2D^\alpha x_2(t) = A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t) + f_2(t)
\end{cases}
$$

(8)

**B. SLIDING MODE FUNCTION**

According to above discussion, if the restricted condition $\text{rank}(E, B) = n$ is satisfied, the system (1) can be rewritten as (8). In the section, one try to find appropriate SM function for system (1).

Let $C = [K, I_m]$, then $CM_2Ex(t) = Kx_1(t) + E_2x_2(t)$.

The SM function was designed as

$$
\begin{cases}
s(t) = Kx_1(t) + E_2x_2(t) + \xi(t), \\
D^\alpha \xi(t) = - (K_{11} + E_2K_{11} - K_{21}K_{11})x_1(t) \\
- (K_2 + E_2K_{12} + K_{12})x_2(t)
\end{cases}
$$

(9)

where gain matrices $K_1, K_2$ respectively satisfied assumption 4 and 5. The design of gain matrix $K$ has a lot of freedom, and various expectations of SM performance can be realized. Meanwhile, additional optimization requirements of the system can also be implemented, which will be considered in the future.

Taking $\alpha$-order derivative of $s(t)$ along with system (8),

$$
D^\alpha s(t) = KD^\alpha x_1(t) + E_2D^\alpha x_2(t) = D^\alpha \xi(t)
$$

$$
= K(A_{11}x_1(t) + A_{12}x_2(t)) + A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t) + f_2(t) \\
- (K_{11} + E_2K_{11} - K_{21}K_{11})x_1(t) \\
- (K_2 + E_2K_{12} + K_{12})x_2(t)
$$

(10)

$$
= \Phi(t) + B_2u(t) + f_2(t),
$$
where \( \Phi(t) = (A_{21} - E_2 K_1 A_{11} + K_2 K_1)x_1(t) + (A_{22} - K_2 - E_2 K_1 A_{22})x_2(t). \)

If suppose \( D^\alpha s(t) = 0 \), then one can get the equivalent controller

\[
u_{eq}(t) = -B_2^{-1}(\Phi(t) + f_2(t)).\tag{11}\]

Taking the \( u_{eq} \) into the system (8), one can get the idea SM dynamical equation as

\[
\begin{align*}
D^\alpha x_1(t) &= A_{11}x_1(t) + A_{12}x_2(t), \\
E_2 D^\alpha x_2(t) &= (E_2 K_1 A_{11} - K_2 K_1)x_1(t) + (K_2 + E_2 K_1 A_{12})x_2(t).
\end{align*}
\tag{12}\]

Let \( y(t) = x_2(t) - K_1x_1(t) \), then system (12) becomes

\[
\begin{align*}
D^\alpha x_1(t) &= (A_{11} + A_{12}K_1)x_1(t) + A_{12}y(t), \\
E_2 D^\alpha y(t) &= K_2y(t).
\end{align*}
\tag{13}\]

One can choose appropriate \( K_2 \) satisfying assumption 5, the second equation of (13) is stable. Then, one also chooses appropriate \( K_1 \) satisfied assumption 4, the first equation of (13) is also stable. Finally, ideal SM equation is stable. Next, we can get the asymptotic stability of the SM equation based on some definitions and lemmas.

**Definition 3.** [41] If one can find the smallest non-negative integer \( q \) satisfying \( \text{rank} \bar{E}_2^q = \text{rank} E_2^{q+1} \), then one call \( q \) as the index of matrix \( \bar{E}_2 \in \mathbb{R}^{m \times m} \).

**Definition 4.** [41] If matrix \( \bar{E}_2 \) satisfies

\[
\begin{align*}
\bar{E}_2 E_2 &= \bar{E}_2^2 E_2, \\
E_2^2 \bar{E}_2 E_2^2 &= E_2^2, \\
\bar{E}_2 E_2^{q+1} &= \bar{E}_2^q,
\end{align*}
\]

then one calls \( \bar{E}_2^q \) as the Drazin inverse of \( \bar{E}_2 \in \mathbb{R}^{m \times m} \), where \( q \) is defined by definition 3.

**Lemma 3.** [41] If the pair \((E_2, K_2)\) is regular, then the solution of \( E_2 D^\alpha y(t) = K_2y(t) \) can be derived by

\[
y(t) = \Phi_0(t) \bar{E}_2 E_2^D y_0,\tag{14}\]

where \( \det(cE_2 - K_2) \neq 0, c \in \mathbb{C}, \bar{E}_2 = (cE_2 - K_2)^{-1}E_2, K_2 = (cE_2 - K_2)^{-1}K_2, \Phi_0(t) = \sum_{i=0}^{\infty} \frac{(\bar{E}_2^q K_2)^{\tau^\alpha}}{\Gamma(\alpha+1)}, y_0 \) is the initial condition.

**Lemma 4.** [20] One parameter Mittag-Leffler function

\[
E_\alpha(t) = \sum_{i=0}^{\infty} \frac{t^\alpha}{\Gamma(\alpha+1)}
\]

can be estimated if one can find two non-negative constants \( \frac{\alpha}{2} < \mu < \min \{ \pi, \alpha \pi \} \) and \( \bar{\epsilon} \) satisfying

\[
|E_\alpha(t)| \leq \frac{\bar{\epsilon}}{1 + |t|},
\tag{15}\]

|t| \geq 0, \mu \leq |\arg(t)| \leq \pi.

One can choose \( K_2 \) satisfying \( \bar{E}_2^q K_2 \leq 0 \). Based on lemma 4, \( \|\Phi_0(t)\| \leq \frac{\bar{\epsilon}}{1 + |t|} \) can be obtained. Then \( \lim_{t \to \infty} y(t) = 0 \).

**Lemma 5.** [20] The solution of \( D^\alpha x_1(t) = (A_{11} + A_{12}K_1)x_1(t) + A_{12}y(t) \) can be derived by

\[
x_1(t) = t^{\alpha-1}E_{\alpha}[(A_{11} + A_{12}K_1)t^\alpha] x_{10} + \int_{0}^{t}(t - \tau)^{\alpha-1}E_{\alpha}[(t - \tau)^\alpha(A_{11} + A_{12}K_1)A_{12}y(\tau)] d\tau,
\tag{16}\]

where \( x_{10} = x_1(0) \) is the initial condition.

One can choose \( K_1 \) to satisfy the assumption 4. Based on lemma 4-5, one can get \( \|x_1(t)\| \leq \frac{\bar{\epsilon}}{1 + |t|} (\|x_{10}\| + \|A_{12}\||y_0\|) \). Then, \( \lim_{t \to \infty} x_1(t) = 0 \). Finally, \( \lim_{t \to \infty} x_2(t) = \lim_{t \to \infty} K_1x_1(t) + \lim_{t \to \infty} y(t) = 0 \).

So, one get the theorem as

**Theorem 2.** If assumption 4-5 is satisfied, then one can find two matrices \( K_1, K_2 \) satisfying system (13) is asymptotically stable.

**Corollary 2.** If \( E_2 = 0 \) is satisfied, one can select SM function as

\[
\begin{align*}
s(t) &= K_1 x_1(t) + \xi(t), \\
D^\alpha \xi(t) &= -(KA_{11} + K_1)x_1(t),
\end{align*}
\tag{17}\]

On the SM hyper-surface \( s = 0 \), closed system is stable under the equivalent controller (11). The ideal SM equation is

\[
\begin{align*}
D^\alpha x_1(t) &= A_{11}x_1(t) + A_{12}x_2(t), \\
0 &= x_2(t) - K_1x_1(t)
\end{align*}
\tag{18}\]

**C. SLIDING MODE CONTROLLER**

For system (8), combing equivalent control with switching law, one can give the SM controller

\[
u(t) = -B_2^{-1}(\Phi(t) + \varepsilon s + \|M_2\|\rho(\|Ex(t)\|)\text{sign} s),
\tag{19}\]

where \( \varepsilon > 0, \text{sign}(\cdot) \) is a signal function which is defined as

\[
\text{sign} = \left\{\begin{array}{ll}
1 & s > 0, \\
-1 & s < 0,
\end{array}\right.
\text{sign} = \left\{\begin{array}{ll}
1 & s > 0, \\
-1 & s < 0.
\end{array}\right.
\]

Now, one can give main contributions of the paper.

**Theorem 3.** For system (8), if the assumption 3-5 is satisfied and SM controller (19) is given, then closed loop system is asymptotically stable, where \( s \) is SM function chosen by equation (9).

**Proof.** One proves the conclusion in two steps.
Firstly, the accessibility of the SM in finite-time is guaranteed. Let \( V(t) = \frac{1}{2} T s^T s \), taking the \( \alpha \)-order derivative of \( V(t) \) along with system (8) under controller \( u(t) \) of (19)

\[
D^\alpha V \leq s^T D^\alpha s \\
= s^T (\Phi(t) + B_2 u(t) + f_2(t)) \\
= s^T (\Phi(t) + B_2 (-B_2^{-1} \Phi(t) + \varepsilon s + ||M_2|| \rho(\|Ex(t)\|) \text{signs}) + f_2(t)) \\
= s^T (-\varepsilon s - ||M_2|| \rho(\|Ex(t)\|) \text{signs}) + s^T f_2(t) \\
\leq -2 \varepsilon V - ||M_2|| \rho(\|Ex(t)\|) ||s^T|| + ||s^T|| ||f_2|| \\
\leq -2 \varepsilon V - ||M_2|| ||s^T|| \rho(\|Ex(t)\|) - ||f|| \\
< -2 \varepsilon V,
\]

where \( ||f_2|| = ||M_2|| ||f|| \). According to Theorem 1 in Ref [42], then \( s \) is asymptotically stable, i.e., \( \lim_{t \to \infty} s = 0 \). This ensures the finite-time accessibility of SM.

Secondly, one would prove the closed systems’ asymptotic stability. Taking SM controller (19) into system (8), one can obtain closed system as

\[
D^\alpha x_1(t) = A_1 x_1(t) + A_2 x_2(t), \\
E_2 D^\alpha x_2(t) = (E_2 K_1 A_1 - K_2 K_1) x_1(t) + (K_2 + E_2 K_1 A_2) x_2(t) - \varepsilon s + f_2(t) \\
- ||M_2|| \rho(\|Ex(t)\|) \text{signs} \\
(20)
\]

According to SM theory, the stability for dynamic part of closed system can be guaranteed. If one supposes

\[
X = \begin{bmatrix} I_{n-m} & 0 \\ 0 & E_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ E_2 x_2(t) \end{bmatrix}
\]

then \( \lim_{t \to \infty} X = 0 \).

Let \( y(t) = x_2(t) - K_1 x_1(t) \), one can obtain closed system as

\[
D^\alpha x_1(t) = (A_1 + A_2 K_1) x_1(t) + A_2 y(t) \\
E_2 D^\alpha y(t) = K_2 y(t) - \varepsilon s + f_2(t) \\
- ||M_2|| \rho(\|Ex(t)\|) \text{signs} \\
(22)
\]

For system (22), according to Weierstrass decomposition, one can find two invertible matrices \( M_3, N_3 \) such that system (22) becomes

\[
D^\alpha z_1(t) = (A_{11} + A_{12} K_1) z_1(t) - \varepsilon s z_1 + g_1(t) \\
- M_{12} ||M_2|| \rho(\|Ex(t)\|) \text{signs}, \\
0 = z_2(t) - \varepsilon s z_2 + g_2(t) \\
- M_{22} ||M_2|| \rho(\|Ex(t)\|) \text{signs}, \\
(23)
\]

where

\[
M_3 \begin{bmatrix} I_{n-m} & 0 \\ 0 & E_2 \end{bmatrix} N_3 = \begin{bmatrix} I_r & 0 \\ 0 & I_m \end{bmatrix},
\]

\[
M_3 \begin{bmatrix} A_{11} + A_{12} K_1 & A_{12} \\ 0 & K_2 \end{bmatrix} N_3 = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix},
\]

\[
M_3 \begin{bmatrix} 0 \\ f_2 \end{bmatrix} = \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}, g_1(t) \in \mathbb{R}^r, g_2(t) \in \mathbb{R}^{n-r},
\]

\[
M_3 \begin{bmatrix} 0 \\ s \end{bmatrix} = \begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix}, s_1(t) \in \mathbb{R}^r, s_2(t) \in \mathbb{R}^{n-r},
\]

\[
M_3 = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \begin{bmatrix} x_1(t) \\ y(t) \end{bmatrix} = N_3 \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix},
\]

\[
z_1(t) \in \mathbb{R}^r, z_2(t) \in \mathbb{R}^{n-r}, M_{12}(t) \in \mathbb{R}^{(n-m) \times m}.
\]

Therefore,

\[
X = \begin{bmatrix} I_{n-m} & 0 \\ 0 & E_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\
= \begin{bmatrix} I_{n-m} & 0 \\ 0 & E_2 \end{bmatrix} \begin{bmatrix} I_{n-m} & 0 \\ K_1 & I_m \end{bmatrix} \begin{bmatrix} x_1(t) \\ y(t) \end{bmatrix} \\
= \begin{bmatrix} I_{n-m} & 0 \\ E_2 K_1 & E_2 \end{bmatrix} \begin{bmatrix} I_{n-m} & 0 \\ I_m & E_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ y(t) \end{bmatrix} \\
= \begin{bmatrix} I_{n-m} & 0 \\ E_2 K_1 & I_m \end{bmatrix} M_3^{-1} M_3 \begin{bmatrix} I_{n-m} & 0 \\ E_2 \end{bmatrix} \\
N_3 N_3^{-1} \begin{bmatrix} x_1(t) \\ y(t) \end{bmatrix} \\
= \begin{bmatrix} I_{n-m} & 0 \\ E_2 K_1 & I_m \end{bmatrix} M_3^{-1} \begin{bmatrix} I_r & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}
\]

On account of \( \lim_{t \to \infty} X = 0 \), then \( \lim_{t \to \infty} z_1(t) = 0 \). From equation (23), one can get

\[
||z_2(t)|| \leq \varepsilon ||s_2(t)|| + ||g_2(t)|| \\
+ ||M_2|| \rho(\|Ex(t)\|) \\
\leq a_1 ||s(t)|| + 2b_1 \rho(\|Ex(t)\|),
\]

where \( a_1 = ||M_2||, b_1 = ||M_2|| M_2 ||.\)

According to assumption 3, \( \lim_{t \to \infty} EX = 0, \rho(0) = 0, \lim_{t \to \infty} s = 0 \) and \( \rho(\|Ex(t)\|) \) is continuous, then \( \lim_{t \to \infty} ||z_2(t)|| = 0 \), i.e., \( \lim_{t \to \infty} z_2(t) = 0 \). And that

\[
x(t) = N_2 \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\
= N_2 \begin{bmatrix} I_{n-m} & 0 \\ K_1 & I_m \end{bmatrix} \begin{bmatrix} x_1(t) \\ y(t) \end{bmatrix} \\
= N_2 \begin{bmatrix} I_{n-m} & 0 \\ K_1 & I_m \end{bmatrix} N_3 \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}.
\]

Finally, \( \lim_{t \to \infty} x(t) = 0 \).

\[\Box\]

**Corollary 3.** For system (8), if \( E_2 = 0 \) and assumption 3-5 are satisfied, then one can find a SM controller

\[
u(t) = -B_2^{-1} [(I_m + A_{22}) x_2(t) - (K_1 - A_{21}) x_1(t) + \varepsilon s + M_2 \rho(Ex(t)) \text{signs}],
\]

satisfying closed loop system is asymptotic stable, where \( s \) is SM function chosen by equation (17).
IV. NUMERICAL SIMULATIONS

Considering uncertain SFOS in (1) satisfying assumption 3 with parameters

\[
ED^\alpha x(t) = Ax(t) + B(u(t) + f(x(t), t)),
\]

\[
E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},
A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},
\]

\[
f \leq 0.2x_3e^{-t}.
\]

Where \( \alpha = 0.8 \), \( \text{rank}(E) = 2 \), \( \text{det}(sE - A) = s - 1 = 0 \), satisfying \( \text{rank}(E) = \text{deg}(\text{det}(sE - A)) = 1 \), system (1) with \( f = 0 \) is regular, impulse-free and unstable, and \( \text{rank}(E, B) = 3 \) can be obtained. According to Theorem 1, one can choose two matrices

\[
M_2 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
N_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},
\]

system (8) can be decomposed by control separation.

\[
\begin{align*}
D^{0.8}\bar{x}_1(t) &= \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \bar{x}_1(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \bar{x}_2(t), \\
0 &= \begin{bmatrix} 0 & 1 \end{bmatrix} \bar{x}_1(t) + u(t) + f,
\end{align*}
\]

where \( \bar{x}_1(t) = \begin{bmatrix} x_1(t) \\ x_3(t) \end{bmatrix}, \bar{x}_2(t) = x_2(t) \). Then one defines the SM function (17) as

\[
s = x_1(t) + x_3(t) + \xi,
\]

\[
D^{0.8}\xi = 3x_1(t) - 2x_2(t) - 2x_3(t).
\]

SM controller (24) can be chosen as

\[
u(t) = 6x_1(t) - x_2(t) - 5x_3(t) - 0.5s - 0.2x_3^2(t)\text{sign}x.
\]

At last, the state trajectories \( x_i, i = 1, 2, 3 \), SM s, SM controller u are alternatively shown in Fig.1-3. In Fig.1, it’s just as easy to see the system state responses converge to zero, then the closed system is stable. In Fig.2, SM motion is also stable in finite time.

V. CONCLUSION

The SMC problem for a class of restricted SFOS with external matching uncertainties is studied in this paper. FO is selected in \( \alpha \in (0, 1) \). The restricted conditions have been given to satisfy system equivalent decomposition. SM surface can be given with FO switching function has been designed. SMC for SFOS has been investigated later. In the future, we will study SFOS with switching terms [43], [44]. Meanwhile, these results would be extended to SMC of SFOS (0 < \( \alpha < 2 \)).

REFERENCES


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