Generalized Riemann-Liouville $k$-fractional integrals associated with Ostrowski type inequalities and error bounds of Hadamard Inequalities

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ABSTRACT

Ostrowski inequality provides the estimation of a function to its integral mean. It is useful in error estimations of quadrature rules in numerical analysis. The objective of this paper is to define a more general form of Riemann-Liouville $k$-fractional integrals with respect to an increasing function which are used to obtain fractional integral inequalities of Ostrowski type. A simple and straightforward approach is followed to establish these inequalities. Applications of established results are also briefly discussed and succeeded to get bounds of some fractional Hadamard inequalities.

INDEX TERMS

Fractional inequalities, Hadamard inequality, Ostrowski inequality, Riemann-Liouville fractional integrals, Generalized fractional integrals

I. INTRODUCTION AND PRELIMINARY RESULTS

OSTrowski inequality is very important and useful in the subjects of mathematical analysis, numerical analysis and other fields of mathematics and engineering. It was introduced by Ostrowski in 1938 [18]. It is stated in the following theorem.

Theorem 1: Suppose that $g$ be a differentiable function on $J^0$, the interior of an interval $J$ in $\mathbb{R}$. For $a, b \in J^0$, $a < b$, let $|g'(t)| \leq M$ for all $t \in [a, b]$. Then for $x \in [a, b]$ the following inequality holds true

$$
|g(x) - \frac{1}{b-a} \int_a^b g(t) dt| \leq \left[ \frac{1}{4} + \frac{(x - a + b)^2}{(b-a)^2} \right] (b-a) M.
$$

As an application point of view it provides the estimations of the Hadamard inequality. It also establishes the error bounds, of relations in special means and of several numerical quadrature rules of integration like rectangular, trapezoidal, Simpson and other in very general form [1], [2]. Many authors have been working continuously on inequality (1) and have produced very interesting results (see, [2]–[5], [8], [10], [13] and references there in). We are motivated to study this inequality for the Riemann-Liouville $k$-fractional integrals in a general form with respect to an increasing function. As a result we get several fractional integral inequalities, such inequalities are useful in the theory of fractional differential equations. Also we apply these fractional inequalities to obtain bounds of the Hadamard inequalities for Riemann-Liouville fractional integrals given in [11], [12], [19], [20].

The Hadamard inequality is a fascinating interpretation of convex functions in the coordinate plane, and it is stated as follows:

Theorem 2: If $g$ is a convex function on an interval $J$ of real numbers, then the inequality

$$
g \left( \frac{a+b}{2} \right) \leq \frac{b}{2} \int_a^b g(t) dt \leq \frac{g(a) + g(b)}{2}
$$

for $a, b \in J$, $a < b$ holds.

The Ostrowski inequality is associated with the Hadamard inequality in the sense that this provides its estimations. The main goal of this paper is to develop fractional versions of the Ostrowski type inequalities and...
bounds of associated fractional Hadamard type inequalities. It is worthful to give a brief description of already developed work in this regard.

In [5] a different and straightforward proof of the Ostrowski inequality is given by utilizing its conditions in a sound way. By following this new and keen method in [4] some Riemann-Liouville fractional inequalities of Ostrowski type have been studied. These inequalities generalize the Ostrowski inequality in fractional calculus point of view. Also this method of studying Ostrowski inequality do not need to define kernels and establish identities for the sake of its proof. Fractional Ostrowski type inequalities are useful in studying bounds of the fractional Hadamard type inequalities. In [19] a fractional version of the Hadamard inequality using bounds of the fractional Hadamard type inequalities. In the next theorem we state this version to a number of its versions for convex and related functions in Riemann-Liouville fractional integrals is proved which leads in [12] a fractional version of the Hadamard inequality using bounds of the fractional Hadamard type inequalities. In this method of studying Ostrowski inequality do not need to
trowski inequality in fractional calculus point of view. Also This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2018.2878266, IEEE Access

Theorem 3: Suppose that \( g \) be a positive convex function defined on the interval \([a, b]\), \( 0 \leq a < b \), \( g \in L_1[a, b] \). Then the following inequalities hold for Riemann-Liouville fractional integrals
\[
\frac{g(a+b)}{2} \leq \frac{\Gamma(\alpha+1)}{2(b-a)\alpha} \left( I^\alpha_{a+} g(b) + I^{\alpha}_{b-} g(a) \right) \leq \frac{g(a) + g(b)}{2}
\]
with \( \alpha > 0 \).

In [12] it is generalized via Riemann-Liouville \( k \)-fractional integrals. In the next theorem we state \( k \)-fractional version of the Hadamard inequality for Riemann-Liouville \( k \)-fractional integrals defined in Definition 2.

Theorem 4: Suppose that \( g \) be a positive convex function defined on the interval \([a, b]\), \( 0 \leq a < b \), \( g \in L_1[a, b] \). Then the following inequalities hold for Riemann-Liouville \( k \)-fractional integrals
\[
\frac{g(a+b)}{2} \leq \frac{\Gamma_k(\alpha+k)}{2(b-a)\alpha} \left( I^{\alpha,k}_{a+} g(b) + I^{\alpha,k}_{b-} g(a) \right) \leq \frac{g(a) + g(b)}{2}
\]
with \( \alpha, k > 0 \).

Another fractional version of the Hadamard inequality is stated in next theorem [20].

Theorem 5: Suppose that \( g \) be a positive convex function defined on the interval \([a, b]\), \( 0 \leq a < b \), \( g \in L_1[a, b] \). Then the following inequalities hold for Riemann-Liouville fractional integrals
\[
\frac{g(a+b)}{2} \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{b-a} \left( I^{\alpha}_{a+} g(b) + I^{\alpha}_{b-} g(a) \right) \leq \frac{g(a) + g(b)}{2}
\]
with \( \alpha > 0 \).

In [12] the following \( k \)-fractional version of the above fractional Hadamard inequality is proved.

Theorem 6: Suppose that \( g \) be a positive convex function defined on the interval \([a, b]\), \( 0 \leq a < b \), \( g \in L_1[a, b] \). Then the following inequalities hold for Riemann-Liouville \( k \)-fractional integrals
\[
g \left( \frac{a+b}{2} \right) \leq \frac{\Gamma_k(\alpha+k)}{(b-a)\alpha} \left[ I^{\alpha,k}_{(a+b)} g(b) + I^{\alpha,k}_{(a+b)} g(a) \right] \leq \frac{g(a) + g(b)}{2}
\]
with \( \alpha, k > 0 \).

For more general results in fractional calculus we suggest the readers [6], [7], [9], [14], [15], [21] and their cited references.

The results of this paper are used for some estimations of these Hadamard type fractional integral inequalities stated in above theorems. The method we have adopted also provide the technique to prove such results independently.

In the following we give the definitions of Riemann-Liouville fractional integrals, Riemann-Liouville \( k \)-fractional integrals and general form of Riemann-Liouville fractional integrals with respect to an increasing function.

Definition 1: Let \( g \in L[a, b] \). Then the left-sided and right-sided Riemann-Liouville fractional integrals of order \( \alpha > 0 \) with \( a \geq 0 \) are defined as:
\[
I^\alpha_{a+} g(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (t-x)^{\alpha-1} g(t) dt, \quad x > a
\]
and
\[
I^\alpha_{b-} g(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} g(t) dt, \quad x < b
\]
where \( \Gamma(\cdot) \) is the Gamma function.

A generalization of Riemann-Liouville fractional integrals are the left-sided and right-side Riemann-Liouville \( k \)-fractional integrals defined as follows [17].

Definition 2: Let \( g \in L_1[a, b] \). Then the \( k \)-fractional integrals of order \( \alpha, k > 0 \) with \( a \geq 0 \) are defined as:
\[
I^{\alpha,k}_{a+} g(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (t-x)^{\alpha-1} g(t) dt, \quad x > a
\]
and
\[
I^{\alpha,k}_{b-} g(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\alpha-1} g(t) dt, \quad x < b
\]
where \( \Gamma_k(\cdot) \) is the \( k \)-Gamma function.

A generalization of the Riemann-Liouville fractional integrals with respect to an increasing function is given as follows [16].

Definition 3: Let \( f_1 : [a, b] \to \mathbb{R} \) be an integrable function. Also let \( f_2 \) be an increasing and positive function on \([a, b]\),
having a continuous derivative $f'_2$ on $(a, b)$. The left-sided and right-sided fractional integrals of a function $f_1$ with respect to another function $f_2$ on $[a, b]$ of order $\alpha > 0$ are defined as:

$$T^\alpha_{f_2,a+}f_1(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (f_2(x) - f_2(t))^{\alpha-1} f'_2(t) f_1(t) dt, \quad x > a$$

and

$$T^\alpha_{f_2,b-}f_1(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (f_2(t) - f_2(x))^{\alpha-1} f'_2(t) f_1(t) dt, \quad x < b.$$

We organize the paper as follows:

In Section II we define a more general form of Riemann-Liouville $k$-fractional integrals with respect to an increasing function and use them to obtain Ostrowski-type inequalities. Utilizing a simple inequality via an increasing function and assumptions of Ostrowski inequality several fractional integral inequalities are obtained. These results provide the Ostrowski type inequalities for Riemann-Liouville fractional integrals which are published in [4]. In Section III fractional versions of inequalities of Section II are presented. As an application point of view in Section IV some of the results are applied to find estimations of the Hadamard type fractional inequalities for Riemann-Liouville fractional and $k$-fractional integrals.

II. $k$-FRACTIONAL INTEGRAL INEQUALITIES IN A GENERAL FORM

We define a more general form of Riemann-Liouville $k$-fractional integrals with respect to an increasing function as follows:

**Definition 4:** Let $f_1 : [a, b] \rightarrow \mathbb{R}$ be an integrable function. Also let $f_2$ be an increasing and positive function on $(a, b)$, having a continuous derivative $f'_2$ on $(a, b)$. The left-sided and right-sided fractional integrals of a function $f_1$ with respect to another function $f_2$ on $[a, b]$ of order $\alpha > 0$ are defined as

$$T^{\alpha,k}_{f_2,a+}f_1(x) = \frac{1}{k \Gamma(\alpha)} \int_a^x (f_2(x) - f_2(t))^{\frac{\alpha}{k}} f'_2(t) f_1(t) dt, \quad x > a$$

and

$$T^{\alpha,k}_{f_2,b-}f_1(x) = \frac{1}{k \Gamma(\alpha)} \int_x^b (f_2(t) - f_2(x))^{\frac{\alpha}{k}} f'_2(t) f_1(t) dt, \quad x < b,$$

where $\Gamma_k(.)$ is the $k$-Gamma function.

**Remark 1:** In the above Definition 4.

(i) If we take $k = 1$, then we get the Definition 3 of Riemann-Liouville fractional integrals with respect to an increasing function.

(ii) If we take $f_2(x) = x$, then we get the Definition 2 of Riemann-Liouville $k$-fractional integrals.

(iii) If we take $f_2(x) = x$ and $k = 1$, then we get the Definition 1 of Riemann-Liouville fractional integrals.

Next we give Ostrowski-type inequality due to Riemann-Liouville $k$-fractional integrals with respect to an increasing function.

**Theorem 7:** Let $f_1 : J \rightarrow \mathbb{R}$ where $J$ is an interval in $\mathbb{R}$, be a function differentiable in $J^0$, the interior of $J$ and $a, b \in J^0$, $a < b$. Also let $f_2 : [a, b] \rightarrow \mathbb{R}$ be a differentiable and strictly increasing function with $f'_2 \in L[a, b]$ and $|f'_2(t)| \leq M$ for all $t \in [a, b]$. Then for $\alpha, \beta \geq 0$ and $k > 0$, the following inequality for general form of Riemann-Liouville $k$-fractional integrals holds

$$\left| f_1(x) \left( f_2(b) - f_2(x) \right)^\frac{\alpha}{k} + (f_2(x) - f_2(a)) \frac{\beta}{k} \right| \leq \left| - (\Gamma_k(\beta + k) T^{\beta,k}_{f_2,a+}f_1(x) + L_k(\alpha + k) T^{\alpha,k}_{f_2,a+}f_1(x)) \right| \leq M \left( x \left( f_2(x) - f_2(a) \right)^\frac{\beta}{k} - (f_2(b) - f_2(x)) \frac{\beta}{k} \right) + \Gamma_k(\beta + k) T^{\beta,k}_{f_2,a+}I(x) - \Gamma_k(\alpha + k) T^{\alpha,k}_{f_2,a+}I(x) \right|,$$

where $I$ is the identity function.

**Proof 1:** Let $x \in [a, b]$ and $t \in [a, x]$. Since the function $f_2$ is strictly increasing therefore for $\alpha \geq 0$ and $k > 0$, the following inequality holds true

$$\left( f_2(x) - f_2(t) \right)^\frac{\alpha}{k} \leq \left( f_2(x) - f_2(a) \right)^\frac{\alpha}{k}.$$

From (5) and the boundedness condition on $f'_1$, the following inequalities are the simple consequences

$$\int_a^x (M - f'_1(t))(f_2(x) - f_2(t))^{\frac{\alpha}{k}} dt \leq \int_a^x (f_2(x) - f_2(a))^{\frac{\alpha}{k}} \int_a^x (M - f'_1(t)) dt.$$

From (6) and (7) after integrating and simple calculation by using Definition 4, we get the following resulting inequalities

$$\left( f_2(x) - f_2(a) \right)^\frac{\alpha}{k} f_1(x) - \Gamma_k(\alpha + k) T^{\alpha,k}_{f_2,a+}f_1(x) \leq M \left( x \left( f_2(x) - f_2(a) \right)^\frac{\beta}{k} - \Gamma_k(\alpha + k) T^{\alpha,k}_{f_2,a+}I(x) \right)$$

and

$$\Gamma_k(\alpha + k) T^{\alpha,k}_{f_2,a+}f_1(x) - \left( f_2(x) - f_2(a) \right)^\frac{\beta}{k} f_1(x) \leq M \left( x \left( f_2(x) - f_2(a) \right)^\frac{\beta}{k} - \Gamma_k(\alpha + k) T^{\alpha,k}_{f_2,a+}I(x) \right).$$

Therefore from (8) and (9) we have the following modulus inequality

$$\left( f_2(x) - f_2(a) \right)^\frac{\alpha}{k} f_1(x) - \Gamma_k(\alpha + k) T^{\alpha,k}_{f_2,a+}f_1(x) \leq M \left( x \left( f_2(x) - f_2(a) \right)^\frac{\beta}{k} - \Gamma_k(\alpha + k) T^{\alpha,k}_{f_2,a+}I(x) \right).$$

Now on the other hand let $x \in [a, b]$, $t \in [x, b]$ and $\beta \geq 0$ and $k > 0$. Then the following inequality holds true

$$\left( f_2(t) - f_2(x) \right)^\frac{\alpha}{k} \leq (f_2(b) - f_2(x))^{\frac{\alpha}{k}}.$$

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From (11) and the boundedness of \( f'_1 \), the following inequalities are their simple consequences
\[
\int_x^b (M - f'_1(t))(f_2(t) - f_2(x)) \frac{dt}{\tau} \leq (f_2(b) - f_2(x))^\frac{1}{\tau} \int_x^b (M - f'_1(t))dt \tag{12}
\]
\[
\int_x^b (M + f'_1(t))(f_2(t) - f_2(x)) \frac{dt}{\tau} \leq (f_2(b) - f_2(x))^\frac{1}{\tau} \int_x^b (M + f'_1(t))dt. \tag{13}
\]
Following the same way as we have adopted for (5), (6) and (7) one can get from (11), (12) and (13) the following modulus inequality
\[
\left| (f_2(b) - f_2(x))^\frac{1}{\tau} f_1(x) - \Gamma_k(\beta + k)I_{f_2,b}^{\beta,k}f_1(x) \right| \leq M \left( \Gamma_k(\beta + k)I_{f_2,b}^{\beta,k}I(x) - x(f_2(b) - f_2(x))^\frac{1}{\tau} \right). \tag{14}
\]
Modulus inequalities (10) and (14) constitute the inequality (4).

**Corollary 1**: Under the same assumptions of Theorem 7 for \( \alpha = \beta \) in (4) the following fractional integral inequality holds true
\[
\left| f_1(x)\left( \left( f_2(b) - f_2(x) \right)^{\frac{1}{\tau}} + \left( f_2(x) - f_2(a) \right)^{\frac{1}{\tau}} \right) - \left( f_2(b) - f_2(x) \right)^{\frac{1}{\tau}} \right| - \Gamma_k(\alpha + k)I_{f_2,b}^{\alpha,k}f_1(x) \leq M \left( f_2(b) - f_2(x) \right)^{\frac{1}{\tau}} - \left( f_2(b) - f_2(x) \right)^{\frac{1}{\tau}} + \Gamma_k(\alpha + k)I_{f_2,b}^{\alpha,k}I(x). \tag{15}
\]

**Corollary 2**: If we take \( f_2(x) = x \) in (4), then we get the following fractional integral inequality for Riemann-Liouville \( k \)-fractional integrals
\[
\left| f_1(x) \left( (b - x)^{\frac{1}{\tau}} + (a - x)^{\frac{1}{\tau}} \right) - \left( f_2(b) - f_2(x) \right)^{\frac{1}{\tau}} \right| - \Gamma_k(\beta + k)I_{b}^{\beta,k}f_1(x) + \Gamma_k(\alpha + k)I_{a}^{\alpha,k}f_1(x) \leq M \left( \frac{\beta}{\beta + k} (b - x)^{\frac{1}{\tau} + 1} + \frac{\alpha}{\alpha + k} (x - a)^{\frac{1}{\tau} + 1} \right). \tag{16}
\]

**Remark 2**: (i) If we take \( f_2(x) = x \) and \( k = 1 \) in (4), then we get fractional integral inequality for Riemann-Liouville fractional integrals [4, Theorem 1.2].

(ii) If we take \( \alpha = \beta = k = 1 \) and \( f_2(x) = x \) in (4), then we get Ostrowski inequality (1).

In the following we give a more general form of fractional Ostrowski type inequality due to Riemann-Liouville \( k \)-fractional integrals with respect to an increasing function.

**Theorem 8**: Let \( f_1 : J \rightarrow \mathbb{R} \) where \( J \) is an interval in \( \mathbb{R} \), be a mapping differentiable in \( J^o \), the interior of \( J \) and \( a, b \in J^o \), \( a < b \). Also let \( f_2 : [a,b] \rightarrow \mathbb{R} \) be differentiable and strictly increasing function with \( f'_2 \in L[a,b] \). If \( m < f'_1(t) \leq M \) for all \( t \in [a,b] \), then for \( \alpha, \beta \geq 0 \) and \( k > 0 \), the following inequalities for general form of Riemann-Liouville \( k \)-fractional integrals hold
\[
\left((f_2(x) + f_2(a))^\beta - (f_2(b) - f_2(x))^\beta \right) f_1(x) \leq (f_2(b) - f_2(x))^\beta \int_a^x (M - f'_1(t))(f_2(t) - f_2(x))^\beta dt \tag{17}
\]
\[
\leq M \left( x(f_2(x) - f_2(a))^\beta - \Gamma_k(\beta + k)I_{f_2,a}^{\alpha,k}I(x) \right) - m \left( \Gamma_k(\beta + k)I_{f_2,a}^{\beta,k}I(x) - x(f_2(b) - f_2(x))^\beta \right) \tag{18}
\]
and
\[
\left((f_2(b) - f_2(x))^\beta - (f_2(b) - f_2(a))^\beta \right) f_1(x) \leq (f_2(b) - f_2(x))^\beta \int_a^x (M - f'_1(t))(f_2(t) - f_2(x))^\beta dt \tag{19}
\]
\[
\leq M \left( x(f_2(x) - f_2(a))^\beta - \Gamma_k(\beta + k)I_{f_2,a}^{\alpha,k}I(x) \right) - m \left( \Gamma_k(\beta + k)I_{f_2,a}^{\beta,k}I(x) - x(f_2(b) - f_2(x))^\beta \right) \tag{20}
\]
\[
\leq (f_2(x) - f_2(a))^\beta \int_a^x (M - f'_1(t))(f_2(t) - f_2(x))^\beta dt \tag{21}
\]
From (20) and (21) after integrating and simple calculation and by using Definition 4, we get
\[
\Gamma_k(\alpha + k)\mathcal{I}^{\alpha,k}_{a,x}f_1(a) - (f_2(x) - f_2(a))^{\frac{\alpha}{k}} f_1(a) 
\leq M \left( \Gamma_k(\alpha + k)\mathcal{I}^{\alpha,k}_{a,x} - a(f_2(x) - f_2(a))^{\frac{\alpha}{k}} \right) 
\]
(22)

and
\[
(f_2(x) - f_2(a))^{\frac{\alpha}{k}} f_1(a) - \Gamma_k(\alpha + k)\mathcal{I}^{\alpha,k}_{a,x} f_1(a) 
\leq M \left( \Gamma_k(\alpha + k)\mathcal{I}^{\alpha,k}_{a,x} - a(f_2(x) - f_2(a))^{\frac{\alpha}{k}} \right) 
\]
(23)

Now on the other hand let \( x \in [a, b] \), \( t \in [x, b] \) and \( \beta \geq 0 \) and \( k > 0 \), the following inequality holds true
\[
(f_2(b) - f_2(t))^{\frac{\alpha}{k}} \leq (f_2(b) - f_2(x))^{\frac{\alpha}{k}}. 
\]
(25)

From (25) and given condition on \( f'_1 \) following inequalities can be obtained
\[
\int_x^b (M - f'_1(t))(f_2(b) - f_2(t))^{\frac{\alpha}{k}} dt 
\leq (f_2(b) - f_2(x))^{\frac{\alpha}{k}} \int_a^x (M - f'_1(t)) dt 
\]
and
\[
\int_x^b (M + f'_1(t))(f_2(b) - f_2(t))^{\frac{\alpha}{k}} dt 
\leq (f_2(b) - f_2(x))^{\frac{\alpha}{k}} \int_a^x (M + f'_1(t)) dt 
\]
(26)

Following the same way as we have adopted for (19), (20) and (21) one can get from (25), (26) and (27) the following inequality
\[
((f_2(b) - f_2(x))^{\frac{\alpha}{k}} f_1(b) - \Gamma_k(\beta + k)\mathcal{I}^{\beta,k}_{a,x} f_1(b) 
\leq M \left( b(f_2(b) - f_2(x))^{\frac{\alpha}{k}} - \Gamma_k(\beta + k)\mathcal{I}^{\beta,k}_{a,x} + b \right) 
\]
(28)

Modulus inequalities (24) and (28) constitute the inequality (18).

**Corollary 3:** Under the same assumptions of Theorem 9 for \( \alpha = \beta = 1 \) in (18) we get the following fractional integral inequality
\[
(((f_2(b) - f_2(x))^{\frac{\alpha}{k}} f_1(b) + (f_2(x) - f_2(a))^{\frac{\alpha}{k}} f_1(a) 
- \Gamma_k(\alpha + k)\mathcal{I}^{\alpha,k}_{a,x} f_1(b) + \Gamma_k(\alpha + k)\mathcal{I}^{\alpha,k}_{a,x} f_1(a) 
\leq M \left( b(f_2(b) - f_2(x))^{\frac{\alpha}{k}} - \alpha(f_2(x) - f_2(a))^{\frac{\alpha}{k}} 
+ \Gamma_k(\alpha + k)(\mathcal{I}^{\alpha,k}_{a,x} - \mathcal{I}^{\alpha,k}_{a,x}) \right) 
\]
Corollary 4: If we take \( f_2(x) = x \) in (18), then we get the following fractional integral inequality for Riemann-Liouville \( k \)-fractional integrals
\[
\left| \left( (b - x)\frac{\alpha}{k} f_1(b) + (x - a)\frac{\alpha}{k} f_1(a) \right) 
- \left( \Gamma_k(\beta + k)\mathcal{I}^{\beta,k}_{a,x} f_1(b) + \Gamma_k(\alpha + k)(\mathcal{I}^{\alpha,k}_{a,x} f_1(a) \right) \right| 
\leq M \left( \frac{\beta}{\beta + k} (b - x)\frac{\alpha}{k} f_1(b) + \frac{\alpha}{\alpha + k} (x - a)\frac{\alpha}{k} f_1(a) \right). 
\]
Remark 4: (i) If we take \( f_2(x) = x \) and \( k = 1 \), then we get fractional integral inequality for Riemann-Liouville fractional integral [4, Theorem 1.4].
(ii) A more general form of Theorem 9 like Theorem 8 holds which we leave for the reader.

**III. FRACTIONAL INTEGRAL INEQUALITIES IN GENERAL FORM**

In this section we present the particular results of previous section. These results can also be proved independently following the method used in previous section. Theorem 7 takes the particular form as follows:

**Theorem 10:** Suppose that the assumptions of the Theorem 7 hold true. Then we have
\[
\left| \left( (f_2(b) - f_2(x))^{\beta} + (f_2(x) - f_2(a))^{\alpha} \right) f_1(x) 
- \left( \Gamma(\beta + 1)\mathcal{I}^{\beta}_{a,b} f_1(x) + \Gamma(\alpha + 1)\mathcal{I}^{\alpha}_{a,b} f_1(x) \right) \right| 
\leq M \left( (f_2(x) - f_2(a))^{\alpha} + (f_2(b) - f_2(x))^{\beta} \right) 
+ \Gamma(\beta + 1)\mathcal{I}^{\beta}_{a,b} \cdot \beta_{[a,b]}(x) - \Gamma(\alpha + 1)\mathcal{I}^{\alpha}_{a,b} \cdot \alpha_{[a,b]}(x) \right) ; 
\]
\( x \in [a, b] \),
where \( \beta_{[a,b]} \) denotes identity function on \([a, b] \).
**Theorem 12:** Suppose that the assumptions of Theorem 9 hold true. Then we have
\[
\left|(f_2(b) - f_2(x))^\beta + (f_2(x) - f_2(a))^\alpha \right| f_1(x) \quad (30)
\]
\[
\leq M \left( (f_2(b) - f_2(x))^\beta - (f_2(x) - f_2(a))^\alpha \right) a
\]
\[
+ \Gamma(\alpha + 1) I_{a^+}^\alpha f_1(x) ; x \in [a, b],
\]

All these results of this section also provide the fractional integral inequalities for Riemann-Liouville fractional integrals when the function \( f_2 \) behaves as an identity function.

**IV. APPLICATIONS**

In this section we give applications of the results proved in Section II. First we apply Theorem 7 and get the following result.

**Theorem 13:** Under the assumptions of Theorem 7, we have
\[
\left|(f_1(a) f_2(b) - f_2(a))^\beta + f_1(b) (f_2(b) - f_2(a))^\alpha \right| \quad (31)
\]
\[
- \left( \Gamma(\alpha + k) I_{a^+}^\alpha f_1(x) + \Gamma(\alpha + k) I_{a^+}^\alpha f_1(b) \right)
\]
\[
\leq M \left( (b - a) \left(f_2(b) - f_2(a))^\beta + (f_2(b) - f_2(a))^{\alpha} \right) \right)
\]
\[
+ \Gamma(\alpha + k) I_{a^+}^\alpha f_1(x) - \Gamma(\alpha + k) I_{a^+}^\alpha f_1(b).
\]

**Proof 4:** By taking \( x = a \) and \( x = b \) in (16), then adding resulting inequalities we get (31).

**Corollary 5:** Under the assumptions of Theorem 7 for \( \alpha = \beta \) in (31), we have
\[
\left|(f_1(a) + f_1(b)) (f_2(b) - f_2(a))^\alpha \right| \quad (32)
\]
\[
- \Gamma(\alpha + k) \left(I_{b^+}^\alpha f_1(x) + I_{b^+}^\alpha f_1(b) \right)
\]
\[
\leq M \left( (b - a) \left(f_2(b) - f_2(a))^\alpha \right) \right)
\]
\[
+ \Gamma(\alpha + k) \left(I_{b^+}^\alpha f_1(x) - I_{b^+}^\alpha f_1(b) \right).
\]

In the following an estimation of the Hadamard inequality for Riemann-Liouville \( k \)-fractional integrals [11, Theorem 2.1] is established.

**Corollary 6:** Under the assumptions of Theorem 7 for \( f_2(x) = x \) in (32) an estimation of the Hadamard inequality for Riemann-Liouville \( k \)-fractional integrals is obtained as follows
\[
\left| f_1(a) + f_1(b) \right| \quad (33)
\]
\[
- \frac{\Gamma(\alpha + k)}{2(b - a)^\alpha} \left(I_{b^+}^\alpha f_1(x) + I_{b^+}^\alpha f_1(b) \right)
\]
\[
\leq \frac{M \alpha(b - a)}{\alpha + k}.
\]

In the following an estimation of the Hadamard inequality for Riemann-Liouville fractional integral [19, Theorem 2] is established.

**Corollary 7:** Under the assumptions of Theorem 7 for \( f_2(x) = x \) and \( k = 1 \) in (32) an estimation of the Hadamard inequality for Riemann-Liouville fractional integrals is obtained as follows
\[
\left| f_1(a) + f_1(b) \right| \quad (34)
\]
\[
- \frac{2^{\alpha - 1/2} \Gamma(\alpha + k)}{(b - a)^{\alpha - 1/2}} \left(I_{b^+}^\alpha f_1(x) + I_{b^+}^\alpha f_1(b) \right)
\]
\[
\leq \frac{M \alpha(b - a)}{2(\alpha + k)}.
\]

Next we apply Theorem 9 to study some more \( k \)-fractional and fractional integral inequalities and the estimations of fractional Hadamard integral inequalities.

**Theorem 14:** Under the assumptions of Theorem 9, we have
\[
\left| f_1(b) \left(f_2(b) - g \left( \frac{a + b}{2} \right) \right)^\alpha + f_1(a) \left( g \left( \frac{a + b}{2} \right) - f_2(a) \right) \right| \quad (35)
\]
\[
- \left( \Gamma(\alpha + k) I_{b^+}^\alpha f_1(x) + \Gamma(\alpha + k) I_{b^+}^\alpha f_1(b) \right)
\]
\[
\leq M \left( b \left(f_2(b) - g \left( \frac{a + b}{2} \right) \right)^\alpha - a \left(g \left( \frac{a + b}{2} \right) - f_2(a) \right)^\alpha \right)
\]
\[
+ \Gamma(\alpha + k) I_{b^+}^\alpha f_1(x) - \Gamma(\alpha + k) I_{b^+}^\alpha f_1(b).
\]

**Proof 5:** By taking \( x = \frac{a + b}{2} \) in (30) resulting inequality (33) can be obtained.

**Corollary 7:** Under the assumptions of Theorem 9 for \( \alpha = \beta \) in (33), we have
\[
\left| f_1(b) \left(f_2(b) - g \left( \frac{a + b}{2} \right) \right)^\alpha + f_1(a) \left( g \left( \frac{a + b}{2} \right) - f_2(a) \right) \right| \quad (34)
\]
\[
- \left( \Gamma(\alpha + k) I_{b^+}^\alpha f_1(x) + \Gamma(\alpha + k) I_{b^+}^\alpha f_1(b) \right)
\]
\[
\leq M \left( b \left(f_2(b) - g \left( \frac{a + b}{2} \right) \right)^\alpha - a \left(g \left( \frac{a + b}{2} \right) - f_2(a) \right)^\alpha \right)
\]
\[
+ \Gamma(\alpha + k) I_{b^+}^\alpha f_1(x) - \Gamma(\alpha + k) I_{b^+}^\alpha f_1(b).
\]

In the following an estimation of the Hadamard inequality for Riemann-Liouville \( k \)-fractional integrals [12, Theorem 2.1] is established.

**Corollary 8:** Under the assumptions of Theorem 9 for \( f_2(x) = x \) in (34) an estimation of the Hadamard inequality for Riemann-Liouville \( k \)-fractional integrals is obtained as follows
\[
\left| f_1(a) + f_1(b) \right| \quad (35)
\]
\[
- \frac{2^{\alpha - 1/2} \Gamma(\alpha + k)}{(b - a)^{\alpha - 1/2}} \left(I_{b^+}^\alpha f_1(x) + I_{b^+}^\alpha f_1(b) \right)
\]
\[
\leq \frac{M \alpha(b - a)}{2(\alpha + k)}.
\]
Note that all the applications of this section also hold for Section III. Remarks of this section are also applicable for the Section III.

CONCLUDING REMARKS

This paper provides a new and elegant technique to prove fractional integral inequalities of Ostrowski type. Here Riemann-Liouville $k$-fractional integrals in a general form are utilized. Applying Theorem 7 and Theorem 9 some interesting results have been obtained which are connected to already published work. Several similar results can be obtained as an application of Theorem 7, Theorem 8 and Theorem 9. Some of these results are actually very useful to establish the error bounds of the Hadamard inequalities in fractional calculus. Adopting the method developed in this paper several fractional integrals can be used to establish new fractional integral inequalities.

REFERENCES

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