The Application of ZFD Formula to Kinematic Control of Redundant Robot Manipulators with Guaranteed Motion Precision

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Abstract—Zhang et al recently design a special difference formula called the Zhang finite difference (ZFD) formula. This study applies the ZFD formula for the kinematic control of redundant robot manipulators, due to its good performance in approximating first-order derivatives. Specifically, the ZFD formula is employed to discretize the pseudoinverse-based (P-based) kinematic control scheme, a new P-based kinematic control scheme with guaranteed motion precision is developed for redundant robot manipulators. Simulation results based on a five-link planar robot manipulator which performs different end-effector paths validate the effectiveness of the new scheme and indicate the application prospects of the ZFD formula.

Index Terms—Zhang finite difference (ZFD) formula, kinematic control, pseudoinverse, redundant robot manipulators, motion precision

I. INTRODUCTION

Robotics have attracted considerable attention among researchers in the fields of science and engineering in recent years. The kinematic control of redundant robot manipulators, as a sub-topic of robotics, has been investigated by numerous researchers and practitioners [1]–[6]. Specifically, a corresponding joint trajectory \( \theta(t) \in \mathbb{R}^n \) should be determined in real time \( t \), given a desired end-effector path \( r_d(t) \in \mathbb{R}^m \). On this basis, many studies on the kinematic control of redundant robot manipulators have been reported [1]–[12].

A pseudoinverse-based (P-based) kinematic control approach is a conventional approach used for redundant robot manipulators. Such an approach is generally formulated as the summation of a minimum-norm particular solution and a homogeneous solution [1]–[4]. In particular, a P-based minimum velocity norm (MVN) scheme is the typical approach, which has been widely adopted for the kinematic control of redundant robot manipulators [2]:

\[
\dot{\theta} = J^\dagger(\theta) r_d,
\]

where \( \dot{\theta} \) is the joint velocity vector, \( J^\dagger \in \mathbb{R}^{n \times m} \) is the pseudoinverse of the Jacobian matrix \( J \in \mathbb{R}^{n \times n} \), and \( r_d \) is the time derivative of \( r_d \). However, scheme (1) may introduce a divergence phenomenon in the end-effector tracking error. A feedback can be added to prevent such an undesired phenomenon. On this basis, a P-based MVN scheme with feedback (a closed-loop scheme) is established as follows [2]:

\[
\dot{\theta} = J^\dagger(\theta) (r_d + \gamma (r_d - f(\theta))),
\]

where \( \gamma > 0 \in R \) is the feedback gain and \( f(\cdot) \) is a differentiable nonlinear mapping. The simulation and experiment results from the literature show the effectiveness of the P-based MVN scheme (2) in the kinematic control of redundant robot manipulators [1], [2], [13]–[15].

The P-based MVN scheme (2) is depicted as a continuous-time formulation. A corresponding discrete-time form should be designed when is applied on practical robot manipulators. Generally, the Euler difference formula is widely used to discretize a continuous-time scheme [16], [17]. However, the resultant discrete-time scheme may be ineffective in guaranteeing the motion precision for the kinematic control of redundant robot manipulators. Zhang et al recently designed a special difference formula called the Zhang finite difference (ZFD) formula [18]. The ZFD formula is theoretically proven to have an \( O(\tau^2) \) truncation error, where \( \tau \) denotes the sampling gap. Obviously, the ZFD formula exhibits better performance on the first-order derivative approximations than the Euler difference formula, that is, \( O(\tau) \) versus \( O(\tau^2) \). Thus, in the current study, the ZFD formula is employed to develop a kinematic control scheme with high motion precision for redundant robot manipulators.

Specifically, the ZFD formula is employed to discretize the P-based MVN scheme (2), which is investigated in this paper for redundant robot manipulators. On the basis of a five-link planar robot manipulator, simulation results are presented to validate the effectiveness of the new kinematic control scheme, and to indicate the application prospects of the ZFD formula. The main contributions of this paper are summarized below.

- A new P-based kinematic control scheme with high motion precision is developed for redundant robot manipulators. To the best of our knowledge, schemes that are similar or identical to the new scheme have not been reported in the literature.
- The ZFD formula is applied on the kinematic control of redundant robot manipulators. This process is a remarkable improvement in the robotics research, because

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it provides a valuable insight in the design of a high-motion-precision scheme for the kinematic control of redundant robot manipulators.

• The simulation results are illustrated to validate the effectiveness of the new kinematic control scheme. These results also indicate that the end-effector tracking error via the new scheme is in the order $O(\tau^3)$. Thus, the motion precision of the new scheme is guaranteed. As far as we know, this study is also the first attempt to provide a kinematic control scheme with an $O(\tau^3)$ tracking-error pattern for redundant robot manipulators.

The rest of this paper is organized as follows: Section II shows the detailed descriptions of the ZFD formula and the new kinematic control scheme; Section III presents the simulation results that are synthesized by the new scheme and Section IV concludes this study with the final remarks.

II. NEW KINEMATIC CONTROL SCHEME

In this section, the ZFD formula is presented and the new kinematic control scheme with guaranteed motion precision is developed for redundant robot manipulators.

A. ZFD formula

In a previous work [18], Zhang et al designed a ZFD formula for the first-order derivative approximation.

Lemma 1. The ZFD formula in [18] is formulated as follows:

$$\dot{\phi}(t_k) \approx \frac{26\phi(t_{k+1}) - 33\phi(t_k) + 18\phi(t_{k-1}) - 11\phi(t_{k-2})}{30\tau},$$

(3)

which has a truncation error of $O(\tau^2)$. In addition, $\phi(\cdot)$ is the objective function and $\tau = t_{k+1} - t_k = t_k - t_{k-1} = t_{k-1} - t_{k-2}$ with $k = 2, 3, 4, \ldots$.

Proof. By using a Taylor series expansion [16], we have

$$\phi(t_{k+1}) = \phi(t_k) + \tau \dot{\phi}(t_k) + \frac{\tau^2}{2} \ddot{\phi}(t_k) + O(\tau^3),$$

(4)

$$\phi(t_{k-1}) = \phi(t_k) - \tau \dot{\phi}(t_k) + \frac{\tau^2}{2} \ddot{\phi}(t_k) + O(\tau^3),$$

(5)

$$\phi(t_{k-2}) = \phi(t_k) - 2\tau \dot{\phi}(t_k) + 2\tau^2 \ddot{\phi}(t_k) + O(\tau^3),$$

(6)

where $\dot{\phi}(t_k)$ and $\ddot{\phi}(t_k)$ are the first-order and second-order time derivatives of $\phi(t)$ at time instant $t_k$, respectively.

By using an algebraic manipulation as “$13 \times (4) + 9 \times (5) - 5.5 \times (6)$”, the following result is obtained:

$$26\phi(t_{k+1}) - 33\phi(t_k) + 18\phi(t_{k-1}) - 11\phi(t_{k-2}) = 30\tau \dot{\phi}(t_k) + O(\tau^3),$$

which is rewritten as follows:

$$\dot{\phi}(t_k) = \frac{(26\phi(t_{k+1}) - 33\phi(t_k) + 18\phi(t_{k-1}) - 11\phi(t_{k-2}))}{30\tau} + O(\tau^2).$$

(7)

The elimination of $O(\tau^2)$ of (7) exactly yields the ZFD formula (3), indicating that such a formula has a truncation error of $O(\tau^2)$. Thus, the proof is completed.

B. Scheme formulation

On the basis of the ZFD formula (3) in Lemma 1, the following approximation of joint velocity $\dot{\theta}_k = \dot{\theta}(t_k = k\tau)$ at time instant $t_k$ is obtained:

$$\dot{\theta}_k \approx \frac{26\theta_{k+1} - 33\theta_k + 18\theta_{k-1} - 11\theta_{k-2}}{30\tau},$$

(8)

where $\theta_k = \theta(t_k = k\tau)$. By utilizing (8) to discretize the P-based MVN scheme (2), the new kinematic control scheme for redundant robot manipulators is thus developed as follows:

$$\theta_{k+1} = \frac{33}{26} \theta_k - \frac{9}{13} \theta_{k-1} + \frac{11}{26} \theta_{k-2} + \frac{15}{13} J^T(\theta_k) (\tau \hat{r}_{dk} + h(r_{dk} - f(\theta_k))),$$

(9)

where $\hat{r}_{dk} = \dot{r}_d(t_k = k\tau)$, $r_{dk} = r_d(t_k = k\tau)$, and $h = \tau \gamma > 0 \in R$ is the step-size. For kinematic control scheme (9), three states (i.e., $\theta_0, \theta_1$, and $\theta_2$) are required to start the iteration. In this case, the other two states for scheme (9) are determined based on a given initial joint state $\theta_0$, through the following iterations:

$$\begin{cases} 
\theta_1 = \theta_0 + J^T(\theta_0) (\tau \hat{r}_{d0} + h(r_{d0} - f(\theta_0))), \\
\theta_2 = \theta_0 + J^T(\theta_1) (\tau \hat{r}_{d1} + h(r_{d1} - f(\theta_1))).
\end{cases}$$

Lemma 2. Kinematic control scheme (9) processes a convergent characteristic and has a truncation error of $O(\tau^3)$.

Proof. Scheme (9) can be generalized from [18].

Remark 1. Considering the intrinsic nonlinearity in redundant robot manipulators and the transient error of the kinematic control scheme, the end-effector tracking error $\varepsilon(t) = f(\theta(t)) - r_a(t) \in R^n$ increases in the transient phase, especially when it reaches zero at several time instances (including the initial time instance) [15], [19]. The theoretical result stated in Lemma 2 guarantees that the magnitude of error $\varepsilon(t)$ synthesized by kinematic control scheme (9) can be maintained within the region of small values. In addition, error $\varepsilon(t)$ can be made sufficiently small by appropriately selecting the values of $\tau$ and $h$, (to be presented in the next section). Scheme (9) shows excellent performance on the kinematic control of redundant robot manipulators. Thus, the motion precision of kinematic control scheme (9) for redundant robot manipulators is thus guaranteed.

III. COMPARATIVE SIMULATIONS

In this section, the computer simulation results obtained with a five-link planar robot manipulator are presented to validate the effectiveness of kinematic control scheme (9). The ensuing simulations are realized via MATLAB R2008a on a digital computer with an Intel(R) Core(TM) i3-3110M @2.40 GHz CPU, 4 GB memory, and Windows 7 OS. For the robot manipulator in simulations, only the end-effector position is considered, and the initial joint state is set as $\theta_0 = [\pi/15; \pi/15; \pi/12; \pi/12; \pi/9]$ rad.
A. Tricuspid path tracking example

In this example, the kinematic control scheme (9) is simulated on the five-link planar robot manipulator with its end-effector tracking a tricuspid path. For comparison, the following kinematic control scheme is simulated, which is obtained by using the Euler difference formula [16] to discretize (2):

\[
\theta_{k+1} = \theta_k + J^\top(\theta_k)\left(\tau_{\text{r}} + h(\tau_{\text{d}} - f(\theta_k))\right).
\]

(10)

The related simulation results are presented in Figs. 1–3.

Fig. 1 illustrates the results provided by the kinematic control scheme (10) with \(\tau = 0.01\) and \(h = 0.5\). As shown in Fig. 1, the end-effector of the five-link planar robot manipulator effectively tracks the tricuspid path, and the maximal value of the tracking error is less than \(1.5 \times 10^{-4}\) m. In addition, Fig. 1(b) presents that the end-effector tracking error does not exhibit divergence, demonstrating the important role of the added feedback to scheme (10). Evidently, these simulation results indicate that scheme (10) is effective on the kinematic control of the five-link planar robot manipulator.

Fig. 2 presents the results provided by kinematic control scheme (9) with \(\tau = 0.01\) and \(h = 0.5\). As shown in Fig. 2, the robot’s end-effector effectively tracks the tricuspid path, in which the maximal value of the tracking error is less than \(2.7 \times 10^{-6}\) m. Fig. 2(b) shows that no divergence phenomenon exists in the end-effector tracking error. These results reflect the effectiveness of scheme (9) on the kinematic control of the five-link planar robot manipulator. The comparison of Figs. 1(b) and 2(b) reveals that the end-effector tracking error obtained via (9) is approximately 100 times smaller than that obtained via (10). Thus, the new kinematic control scheme (9) is superior to the kinematic control scheme (10).

Kinematic control scheme (9) is simulated by changing the value of \(\tau\) (i.e., from 0.01 to 0.001). The relevant results, shown in Fig. 3, validate the assumption that scheme (9) is effective on the kinematic control of the robot manipulator. The comparison of Figs. 2(b) and 3(b) shows that the end-effector tracking error becomes small with the decrease of the value of \(\tau\). Moreover, the decrease on \(\tau\) by 10 times...
leads to the error reduction by 1000 times (i.e., from $10^{-6}$ to $10^{-9}$). Thus, $\tau$ plays an important role in the kinematic control scheme (9), and should be sufficiently low to guarantee the motion precision of (9) in robotic applications.

Kinematic control scheme (9) is further simulated by changing the value of $h$, and the relevant results are presented in Fig. 4. These results indicate again the effectiveness of scheme (9) in term of small tracking errors (i.e., in the order $10^{-9}$). The close inspection of Figs. 3(b) and 4 denotes that the error becomes small with the increase of the value of $h$. Therefore, $h$ is also an important parameter in the kinematic control scheme.
Fig. 5. Simulation results via the new kinematic control scheme (9) with $\tau = 0.01$ and $h = 0.5$ for the robot manipulator tracking different paths. (9), and should be appropriately large to further improve the motion precision of (9).

In summary, these comparative simulation results indicate the effective and superior performance of kinematic control scheme (9) for redundant robot manipulators compared with scheme (10).

**B. Triangular and square path tracking examples**

In this subsection, the kinematic control scheme (9) is simulated for the five-link planar robot manipulator with its end-effector tracking the triangular and square paths.

The relevant results are presented in Fig. 5 by simulating the new kinematic control scheme (9) with $\tau = 0.01$ and $h = 0.5$, where the left subfigures show the simulated motion trajectories and the right subfigures show the end-effector tracking errors. As shown in Fig. 5, the end-effector of the robot manipulator effectively tracks the desired paths with small tracking errors (i.e., of order $10^{-8}$). The effectiveness of the kinematic control scheme (9) is thus validated. Scheme (9) is simulated by changing the value of $\tau$ to $0.001$. The corresponding results are illustrated in Fig. 6, which shows that the end-effector tracking error via scheme (9) can be maintained within the region of small values by using appropriate $\tau$ and $h$. Thus, the motion precision of the kinematic control scheme (9) for redundant robot manipulators is guaranteed by a small value of $\tau$ and a relatively large value of $h$.

In summary, the above comparative simulation results (i.e., Figs. 1–6) show the effectiveness and superiority of the new kinematic control scheme (9). Moreover, these results indicate the application prospects of the ZFD formula (3) for the effective kinematic control of redundant robot manipulators.

**IV. Conclusion**

This study applies the ZFD formula (3) on the kinematic control of redundant robot manipulators. Specifically, ZFD
formula is employed to discretize the P-based MVN scheme (2), and the new kinematic control scheme (9) is developed and investigated in this study. Theoretical results are given to guarantee the motion precision of scheme (9) for redundant robot manipulators. Simulation results based on the five-link planar robot manipulator which performs different end-effector paths validate the effectiveness of the scheme (9), and indicate the application prospects of the ZFD formula (3).

One future research direction can involve the investigation of the kinematic control scheme (9) with joint angle limits considered. Another future research direction is the design of more kinematic control schemes with high motion precision for redundant robot manipulators. As a follow-up to this study, (9) will be applied to a practical Epson robot manipulator.

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REFERENCES


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