Adaptive Fuzzy Output Tracking Control of a Class of Uncertain Fractional Order Systems Subject to Unknown Disturbance

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Abstract—In this paper, a novel adaptive fuzzy backstepping control method is designed for a class of incommensurate fractional order nonlinear systems with unknown nonlinearities and external disturbance, in which the frequency distributed model is employed so that the indirect Lyapunov method can be used to design the controller. In each step of the backstepping, the complicated unknown nonlinear function coming from a fractional differential function is approximated using a fuzzy logic system, and the virtual control law and parameters update law are designed, in which the order of the parameters update law cannot be fixed to the system order and more degree of freedom can be obtained. In the last step, an adaptive fuzzy controller is given, which can ensure the convergence of the tracking error. Compared with the previous methods, the proposed backstepping method is first adopted to solve the tracking problem of incommensurate order systems with unknown nonlinearities and external disturbance, in which the stabilization and tracking can be achieved. Two careful simulation studies are provided to illustrate the effectiveness of this novel scheme.

Index Terms—Adaptive backstepping control, Fractional order systems, Fuzzy logic system, Frequency distributed model, Indirect Lyapunov method.

I. INTRODUCTION

The fractional order calculus is originated from a letter written by Leibniz in 1695, in which the non-integer order derivative is mentioned for the first time [1]. The fractional order calculus is as an extension of the conventional integer order calculus, which has interesting properties and potential applications [2–5]. Recently, the number of the theoretical research results (such as fractional order stability analysis and fractional order control) and engineering applications of fractional order control systems is growing continuously [6–11]. The foremost reason is that the fractional order nonlinear systems can precisely describe the phenomenon of viscoelastic structures and heat conduction [12]. In addition, the control performance of fractional order controllers is better than the classical controllers due to their usual forms, hereditary and non-locality [13]. In fact, with the existence of parameter uncertainties and noises, a precise physical model of the engineering plant is difficult to obtain. Therefore, the uncertain fractional order system control is an attractive and challenging research field, and several research results have been presented in recent years, such as the PI^D^ control [14, 15], the sliding mode control [16, 17], the adaptive control [18–21], the neural network control [22, 23] and the fuzzy control [24, 25].

The adaptive backstepping control method is an important technique to control uncertain integer order nonlinear systems with triangular structure [26, 27], which establishes a systematic framework by using intermediate variables recursively and constructs a Lyapunov function to ensure the stability of the closed loop system. Considering the advantages, such as superior tracking, transient performance and global stability, the adaptive backstepping technique has been extended to control fractional order nonlinear systems by some scholars.

Since the adaptive backstepping technique is first extended to the fractional order systems in [28], there are mainly two classes of Lyapunov functions used to analyze the stability of the closed loop system: the direct Lyapunov method [4, 29, 30] and the indirect Lyapunov method [20, 21, 31–33]. For the direct Lyapunov method, an inequality lemma [34] involving the fractional order extension of the Lyapunov direct method is used to find Lyapunov candidate functions to demonstrate the stability of the fractional order systems. For the indirect Lyapunov method, the frequency distributed model of the fractional order system [35] is introduced so that the indirect Lyapunov method can be used to design the controller.

Employing fuzzy system to approximate the system uncertainty is an effective control method for the uncertain fractional order system. The fuzzy systems considered as a universal approximator can estimate any functions defined on some compact set. Therefore, multifarious adaptive fuzzy backstepping controllers have been presented to design the controller of the integer order uncertain nonlinear systems [36–40]. So far, there are many research results on the adaptive fuzzy control of uncertain fractional order systems with nonlinearity. In [41], an adaptive fuzzy controller is designed for uncertain fractional order nonlinear systems through the direct Lyapunov method.

For a class of uncertain fractional order nonlinear systems with disturbances, an adaptive fuzzy backstepping control method is developed in [42], in which the direct Lyapunov method is adopted to analyze the stability. However, the above method
can only be applied to the commensurate fractional order systems, and the orders of the parameter update laws are fixed to the system order. Such limitation in application makes significant conservatism by using these methods.

According to the above discussions, an adaptive fuzzy backstepping control method is presented for a class of incommensurate fractional order systems with unknown nonlinearities and external disturbance in this paper. In each step, the fuzzy logic system is employed to approximate the fractional order nonlinear function produced by differentiating a compound function. The contributions of this paper are summarized as:

1) With the fractional order nonlinear functions approximated by the fuzzy logic systems, an adaptive fuzzy backstepping method is presented for incommensurate fractional order nonlinear systems with unknown nonlinearities and external disturbance. 2) The stability of the proposed adaptive fuzzy backstepping method is analyzed by using the indirect Lyapunov method with frequency distributed model. 3) The orders of fractional order adaptation laws are not fixed to the order of the fractional order system, in which more degree of freedom and better control performance can be obtained. 4) The proposed method can be applied to both commensurate fractional order systems and the incommensurate fractional order nonlinear systems, which has more broadly applicable compared to [42].

The remainder of this article is organized as follows. In Section 2, the basic definitions on the fractional integrals and derivatives, the preliminary results on fractional order systems and the description of fuzzy logic systems are presented. The detailed process of the controller design and the stability analysis are given in Section 3. In Section 4, the validity of the proposed approach is illustrated by the provided simulation results. Section 5 gives the Conclusions.

II. PRELIMINARY RESULTS

A. Fractional order integration and differentiation

The fractional order integro-differential operator is an extended concept of the integer order integro-differential operator [43]. The fractional order integral of continuous function \( f(t) \) with respect to \( t \) and the lower terminal \( t_0 \) is defined as follow:

\[
s_0D^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_{t_0}^{t} (t - \tau)^{n-1-\alpha} f(\tau) d\tau
given \quad n - 1 < \alpha < n, n \in \mathbb{Z}^+
\]

where \( \Gamma(n - \alpha) = \int_0^\infty e^{-x}x^{n-1-\alpha} dx \) is the well-known Euler's Gamma function. The \( \alpha-th \) Caputo fractional derivative is defined by:

\[
s_0D^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_{t_0}^{t} (t - \tau)^{n-1-\alpha} f^{(n)}(\tau) d\tau
given \quad n - 1 < \alpha < n, n \in \mathbb{Z}^+
\]

where \( n - 1 < \alpha < n, n \in \mathbb{Z}^+ \) and \( t_0D^\alpha f(t) \) is the classical \( \alpha-th \) order derivative operator. When \( t_0 = 0, s_0D^\alpha f(t) \) can be abbreviated as \( D^\alpha f(t) \).

A fractional order system can be represented by an infinite-dimensional system generally called diffusive representations.

Lemma 1 [35]: The fractional order nonlinear differential equation

\[
D^\alpha x(t) = f(x(t)), \quad \alpha \in (0, 1), x(t) \in \mathbb{R}^n
\]

is essentially a continuous frequency distributed model described by

\[
\begin{cases}
\frac{d(\omega t)}{dt} = -\omega \zeta(\omega t) + f(x(t)) \\
x(t) = \int_0^\infty \mu_\omega(\omega) \zeta(\omega t) d\omega
\end{cases}
\]

where \( \zeta(\omega t) \) is the infinite dimension distributed state variable and \( \mu_\omega(\omega) = \left(\frac{\sin(\alpha \pi)}{\alpha \pi}\right)^\omega \) is the weighting function.

Remark 1. Lemma 1 implying the nature of fractional order systems is that the dimension of fractional order state is infinite. Although Lemma 1 is deduced with zero initial condition, the frequency distributed model with specific initial value can be found from the fractional order system defined by Riemann-Liouville or Caputo [44]. With the help of equivalent frequency distributed model, the stability analysis of fractional order systems can be done through the Lyapunov technique expediently [35].

B. Description of Fuzzy Logic Systems

In the mathematical point of view, fuzzy logic systems can be used as practical function approximators to approximate a continuous function \( f(x) \) defined on some compact set [38].

There are four parts in a fuzzy logic system: the knowledge base, the fuzzifier, the fuzzy inference engine and the defuzzifier. The knowledge base is composed of a collection of fuzzy IF-THEN rules as follows:

\[
R^l_i: \quad \text{if } x_1 \text{ is } F_{1}^l \text{ and } x_2 \text{ is } F_{2}^l \cdots \text{ and } x_n \text{ is } F_{n}^l, \text{ then } y \text{ is } G^l, \quad l = 1, 2, ..., N
\]

where \( x = [x_1, ..., x_n]^T \) and \( y \) are fuzzy logic system input and output, respectively. \( \mu_{F_i}(x_1) \) and \( \mu_{G_i}(y) \) are the membership functions of fuzzy sets \( F_i \) and \( G_i \), respectively. \( N \) is the number of inference rules. Through singleton fuzzifier, center average defuzzification and product inference, the fuzzy logic system can be expressed as

\[
y(x) = \frac{\sum_{l=1}^{N} \bar{y}_l \left(\prod_{i=1}^{n} \mu_{F_i}^l(x_i)\right)}{\sum_{l=1}^{N} \left(\prod_{i=1}^{n} \mu_{F_i}^l(x_i)\right)} \tag{5}
\]

where \( \bar{y}_l = \max_{y \in R} \mu_{G_i}(y) \).

Define the fuzzy basis functions as

\[
\varphi_i(x) = \frac{\prod_{i=1}^{n} \mu_{F_i}^l(x_i)}{\sum_{l=1}^{N} \left(\prod_{i=1}^{n} \mu_{F_i}^l(x_i)\right)}, l = 1, 2, ..., N \tag{6}
\]

Then, fuzzy logic system (5) can be rewritten as

\[
y(x) = \bar{\theta}^T \varphi(x) \tag{7}
\]

where \( \theta^T = [\bar{y}_1, \bar{y}_2, ..., \bar{y}_N] = [\theta_1, \theta_2, ..., \theta_N] \) and \( \varphi(x) = [\varphi_1(x), \varphi_2(x), ..., \varphi_N(x)]^T \).

The following lemma is demonstrating that the fuzzy logic system is a universal approximator for any smooth functions on a compact space.

Lemma 2 [45]: For any continuous function \( f(x) \) defined over a compact set \( \Omega \) and any given positive constant \( \varepsilon \), there exists a fuzzy logic system (7) and an ideal parameter vector \( \theta^* \) such that

\[
\sup_{x \in \Omega} \left| f(x) - \theta^{T*} \varphi(x) \right| \leq \varepsilon \tag{8}
\]
III. ADAPTIVE FUZZY BACKSTEPPING CONTROLLER

Consider a class of incommensurate fractional order nonlinear systems which can be described as follows

\[
\begin{aligned}
D^\alpha x_i &= x_{i+1} + f_i(x_1, \ldots, x_i) + g_i(x_1, \ldots, x_i), \\
&D^\alpha x_0 = bu + f_0(x) + g_0(x) + d(t) \\
&y = x_1
\end{aligned}
\]  

where \(0 < \alpha_i < 1\) is the system incommensurate fractional order, \(x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n\) is the measurable state vector, \(y \in \mathbb{R}\) is the output variable, \(f_i(\cdot) \in \mathbb{R}\) is an unknown continuous nonlinear function, \(g_i(\cdot) \in \mathbb{R}\) is a known nonlinear function, \(u \in \mathbb{R}\) is the control input, and \(d(t) \in \mathbb{R}\) is an unknown external disturbance.

Let \(y_d\) be a known reference signal. Our target is to build a proper controller \(u\) such that the tracking error \(e_1 = y - y_d\) converges to zero. Then, a recursive backstepping algorithm is given, which can be separated as the following steps.

Step 1: Based on Lemma 2, the unknown function \(f_i(x_1)\) from (9) can be approximated by a fuzzy logic system as follow:

\[
f_i(x_1, \theta_1) = \theta_i^T \vartheta_1(x_1)
\]

where parameter estimation \(\hat{\theta}_1 \in \mathbb{R}^{m_1}\). The ideal parameter \(\theta_1^*\) is given by

\[
\theta_1^* = \arg \min_{\theta_1} \left[ \sup_{x_1} \left| f_i(x_1) - f_i(x_1, \theta_1) \right| \right]
\]

In this paper, \(\theta_1^*\) is only for analysis purpose, which is not needed in the control design. Let

\[
\vartheta_1 = \theta_1^* - \theta_1 \\
e_1(x_1) = f_i(x_1, \theta_1^*) - f_i(x_1)
\]

be the parameter estimation error and the optimal approximation error, respectively. According to [46], the optimal approximation error is bounded. Then, one can obtain

\[
|e_1(x_1)| \leq \tilde{e}_1
\]

where \(\tilde{e}_1\) is an unknown positive constant. Therefore, one can get

\[
\begin{aligned}
f_i(x_1, \theta_1) &= f_i(x_1, \theta_1^*) + f_i(x_1, \theta_1^*) - f_i(x_1) \\
&= \theta_i^T \vartheta_1(x_1) + \vartheta_i^T \vartheta_1(x_1) + e_1(x_1) \\
&= -\vartheta_i^T \theta_1(x_1) + e_1(x_1)
\end{aligned}
\]

Due to the unknown \(\theta_1^*\), the control method must perform the function of both the controller and estimator. According to the estimated error \(\hat{\vartheta}_1 = \theta_1^* - \theta_1\) from (12), the following equation can be obtained on the basis of Caputo’s definition

\[
D^\beta \hat{\vartheta}_1 = D^\beta \theta_1^* - D^\beta \theta_1 = -D^\beta \theta_1
\]

where \(0 < \beta_1 < 1\). Based on Lemma 1, (15) can be transformed into the frequency distributed model as follows

\[
\begin{aligned}
\frac{d\vartheta_i(\omega)}{dt} &= -\omega \vartheta_i(\omega, t) + D^{\beta_i} \theta_1 \\
\vartheta_i(0) &= \mu_i(\omega) \vartheta_i(\omega, t) d\omega
\end{aligned}
\]

with \(\vartheta_i(\omega, t) \in \mathbb{R}^{m_1}\) and \(\mu_i(\omega) = \frac{\sin(\alpha_i \omega)}{\alpha_i^2 \pi}\). On the other hand, it follows from (9) and (14) that

\[
\begin{aligned}
D^\alpha e_1 &= D^\alpha x_1 - D^\alpha y_d \\
&= x_2 + f_1(x_1) + g_1(x_1) - D^\alpha y_d \\
&= x_2 + \theta_1^T \vartheta_1(x_1) - e_1(x_1) + \theta_1^T \vartheta_1(x_1) + g_1(x_1) - D^\alpha y_d
\end{aligned}
\]

Let a virtual control input \(u_1(e, x_1, y_d)\) be

\[
u_1(e, x_1, y_d) = -\theta_1^T \vartheta_1(x_1) - k_{11} e_1 - k_{21} \text{sign}(e_1) - g_1(x_1) + D^\alpha y_d
\]

where \(k_{11}\) and \(k_{12}\) are design parameters. Let

\[
e_2 = x_2 - u_1
\]

Substituting (18) and (19) into (17) gives

\[
D^\alpha e_1 = e_2 - k_{11} e_1 - k_{21} \text{sign}(e_1) + \theta_1^T \vartheta_1(x_1) - e_1(x_1)
\]

After transformation into the frequency distributed model according to Lemma 1, the equation (20) will be

\[
\begin{aligned}
\frac{d\vartheta_1(\omega)}{dt} &= -\omega \vartheta_1(\omega, t) + e_2 - k_{11} e_1 \\
&= -\omega \vartheta_1(\omega, t) + \vartheta_1^T \vartheta_1(x_1) - e_1(x_1)
\end{aligned}
\]

with \(\mu_1(\omega) = \frac{\sin(\alpha_1 \omega)}{\alpha_1^2 \pi}\). Selecting the Lyapunov function \(V_1\) as

\[
V_1 = \frac{1}{2\sigma_1} \int_{-\infty}^{\infty} \mu_1(\omega) \vartheta_1^T(\omega, t) \Lambda_1^{-1} \vartheta_1(\omega, t) d\omega
\]

where \(\sigma_1 > 0\), then the derivative of \(V_1\) based on frequency distributed model (16) and (21) is expressed as

\[
\begin{aligned}
\dot{V}_1 &= -\frac{1}{\sigma_1^2} \int_{-\infty}^{\infty} \mu_1(\omega) \vartheta_1^T(\omega, t) \Lambda_1^{-1} \vartheta_1(\omega, t) d\omega \\
&= -\int_{-\infty}^{\infty} \omega \vartheta_1 \vartheta_1^T(\omega, t) \Lambda_1^{-1} \vartheta_1(\omega, t) d\omega \\
&= -\int_{-\infty}^{\infty} \omega \vartheta_1 \vartheta_1^T(\omega, t) \Lambda_1^{-1} \vartheta_1(\omega, t) d\omega
\end{aligned}
\]

According to inequality (23) and LaSalle invariance principle [47], if \(e_2 = 0, k_{11} > 0, k_{21} > \tilde{e}_1\) (\(\tilde{e}_1\) is a known positive constant satisfying \(|\vartheta_1(x_1)| \leq \tilde{e}_1\)) and a fractional order adaptation law is designed as

\[
D^\beta \theta_1 = \sigma_1 \Lambda_1 \theta_1(x_1) e_1
\]
One gets $\dot{V}_1 < 0$. According to Lyapunov stability theory, we know that the equilibrium $z_0(\omega, t) = 0$ and $z_1(\omega, t) = 0$ are stable, and the parameter estimation $\dot{\theta}_i$ is bounded.

Step 2: From (9) and (19), one has

$$\dot{V}_2 = \dot{V}_1 - \frac{1}{\sigma_2} \int_0^\infty \omega \dot{\theta}_i(\omega, t) \Lambda_2^{-1} \dot{z}_0(\omega, t) \, d\omega$$

$$+ \int_0^\infty \omega \mu_\beta(\omega) \Lambda_2^{-1} \dot{z}_0(\omega, t) \, d\omega$$

where $F_2(x_1, x_2) = f_2(x_1, x_2) - D^\mu v_1$ is an unknown function. Just like the procedures in step 1, $F_2(x_1, x_2)$ is approximated by a fuzzy logic system as follow

$$\dot{F}_2(x_1, x_2, \theta_2) = \theta_2^T \theta_2(x_1, x_2)$$

where parameter estimation $\dot{\theta}_2 \in \mathbb{R}^m$.

Define the estimated error $\dot{\theta}_2 = \dot{\theta}_2^* - \dot{\theta}_2$, and the following equation can be obtained according to Caputo’s definition

$$\dot{V}_2 = \dot{V}_1 - \frac{1}{\sigma_2} \int_0^\infty \omega \dot{\theta}_i(\omega, t) \Lambda_2^{-1} \dot{z}_0(\omega, t) \, d\omega$$

$$- \int_0^\infty \omega \dot{\theta}_i(\omega, t) \Lambda_2^{-1} \dot{z}_0(\omega, t) \, d\omega$$

$$+ \int_0^\infty \omega \mu_\beta(\omega) \Lambda_2^{-1} \dot{z}_0(\omega, t) \, d\omega$$

Due to inequality (35) and LaSalle invariance principle, if $e_1 = 0, k_{12} > 0, k_{22} > 0$ (32) is a known positive constant satisfying $||e_2(x_1, x_2)|| \leq \tilde{e}_2$ and a fractional order adaptation law is designed as

$$\dot{\theta}_2 = \sigma_2 \Lambda_2 \dot{v}_2(x_1, x_2)$$

One can obtain $\dot{V}_2 < 0$.

Step i, 3 ≤ i ≤ n - 1: Let

$$e_i = x_i - u_{i-1}$$

Just like the procedures in step 1 and 2, one has

$$\dot{V}_2 = \dot{V}_1 - \frac{1}{\sigma_2} \int_0^\infty \omega \dot{\theta}_i(\omega, t) \Lambda_2^{-1} \dot{z}_0(\omega, t) \, d\omega$$

$$- \int_0^\infty \omega \dot{\theta}_i(\omega, t) \Lambda_2^{-1} \dot{z}_0(\omega, t) \, d\omega$$

$$+ \int_0^\infty \omega \mu_\beta(\omega) \Lambda_2^{-1} \dot{z}_0(\omega, t) \, d\omega$$

where $\dot{F}_i(x_1, x_2, \theta_i) = \theta_i^T \theta_i(x_1, x_2)$

$$\dot{v}_2 = \dot{v}_1 - \frac{1}{\sigma_2} \int_0^\infty \omega \dot{\theta}_i(\omega, t) \Lambda_2^{-1} \dot{z}_0(\omega, t) \, d\omega$$

$$- \int_0^\infty \omega \dot{\theta}_i(\omega, t) \Lambda_2^{-1} \dot{z}_0(\omega, t) \, d\omega$$

$$+ \int_0^\infty \omega \mu_\beta(\omega) \Lambda_2^{-1} \dot{z}_0(\omega, t) \, d\omega$$

where $\sigma_2 > 0$, then its derivative based on frequency distributed model (29) and (33) is
Let a virtual control input be
\[ u_1 = -\theta_1^T \theta_1 (x_1, \ldots, x_i) - k_1 e_i \]
where $k_1$ and $k_2$ are design parameters. Substituting (37) and (43) into (42) gives
\[
\dot{\theta}_1 = -\omega z_i (\omega, t) + e_{i+1} - k_1 e_i - k_2 \text{sign} (e_i) + \frac{1}{\omega} \int_0^\infty \mu_{\theta_1} (\omega) z_i (\omega, t) d\omega
\]
Its frequency distributed model corresponds to
\[
\frac{\partial \tilde{z}_i (\omega, t)}{\partial \omega} = \tilde{z}_i (\omega, t) - \bar{\theta}_1 (x_1, \ldots, x_i) e_i - e_{i-1}
\]
where $\sigma_i > 0$, then its derivative based on frequency distributed model (41) and (45) is expressed as
\[
\dot{V}_i = V_{i-1} - \int_0^\infty \left( \omega \mu_{\theta_1} (\omega) z_i^T (\omega, t) \Lambda_i^{-1} \tilde{z}_i (\omega, t) d\omega \right) + \frac{1}{2} \int_0^\infty \left( \omega 
\mu_{\theta_1} (\omega) z_i^2 (\omega, t) d\omega \right)
\]
where $\sigma_i > 0$. Let us build the controller $u$ as follow
\[
u_i = -\theta_1^T \theta_1 (x_1, \ldots, x_i) - k_1 e_i - k_2 \text{sign} (e_i)
\]
where $u_{n-1}$ is a virtual control input, and $F_n (x) = f_n (x) - D^\alpha u_{n-1}$ is an unknown function. Let
\[
\bar{F}_n (x, \theta_1) = \theta_1^T \theta_1 (x, \theta_1) = \theta_1^T \theta_1 (x, \theta_1)
\]
with parameter estimation $\theta_1 \in \mathbb{R}^m$. Define the estimated error $\bar{\theta}_1 = \theta_1^* - \theta_1$, and the following equation is obtained in view of Caputo’s definition
\[
D^\beta \bar{\theta}_1 = D^\beta \theta_1 - D^\beta \bar{\theta}_1 = -D^\beta \bar{\theta}_1
\]
where $0 < \beta_i < 1$. Based on Lemma 1, (52) will be
\[
\frac{\partial z_i (\omega, t)}{\partial \omega} = -\omega z_i (\omega, t) + \bar{\theta}_1 (x_1, \ldots, x_i) d\omega
\]
with $z_i (\omega, t) \in \mathbb{R}^m$ and $\bar{\theta}_1 (\omega) = \sigma_i \frac{\partial \theta_1 (\omega, t)}{\partial \omega}$. From (51), (50) can be rewritten as
\[
D^\nu e_{n-1} = bu + g_n (x) + d (t) + \bar{\theta}_1 (x_1, \ldots, x_i) - e_n (x) + \bar{\theta}_1 (x_1, \ldots, x_i)
\]
where $e_{n-1}$ is the disturbance $\dot{d} (t)$ is bounded, i.e., $|d (t)| \leq \tilde{d}$, $\forall t \geq 0$ where $\tilde{d} > 0$ is an unknown constant. Let us build the controller $u$ as follow
\[
u_i = -\theta_1^T \theta_1 (x, \theta_1) - k_1 e_i - k_2 \text{sign} (e_i) - g_n (x) - e_{n-1}
\]
where $k_{1n}$ and $k_{2n}$ are designed parameters, and $\tilde{d}$ is the estimation of the unknown constant $d$. Let $\tilde{d} = \tilde{d} - d$, then the following equation can be obtained
\[
D^\nu \tilde{d} = D^\nu \tilde{d} - D^\nu d = -D^\nu \tilde{d}
\]
with $0 < \nu < 1$. Based on Lemma 1, (56) will be
\[
D^\nu \tilde{d} = \frac{\partial z_i (\omega, t)}{\partial \omega} = -\omega z_i (\omega, t) - D^\nu \tilde{d}
\]
with $\mu_i (\omega) = \frac{\partial \theta_1 (\omega, t)}{\partial \omega}$. Substituting (49) and (55) into (54) gives
\[
D^\nu e_{n-1} = -k_1 e_i - k_2 \text{sign} (e_i) - e_{n-1} + d (t) + \bar{\theta}_1 (x_1, \ldots, x_i) - e_n (x)
\]
and the following equation is obtained for $\mu_i (\omega) = \sigma_i \frac{\partial \theta_1 (\omega, t)}{\partial \omega}$. Selecting the Lyapunov function $V_n$ as
\[
V_n = V_{n-1} + \frac{1}{2} \int_0^\infty \mu_{\theta_1} (\omega) z_i^2 (\omega, t) d\omega + \frac{1}{2} \int_0^\infty \mu_{\theta_1} (\omega) \bar{\theta}_1^2 (\omega, t) d\omega
\]
with $\bar{\theta}_1 (\omega, t) \in \mathbb{R}^m$ and $\mu_{\theta_1} (\omega) = \sigma_i \frac{\partial \theta_1 (\omega, t)}{\partial \omega}$.
where $\sigma_n, \rho > 0$.

Just like the procedures in step $i$, $3 \leq i \leq n-1$, the derivative of $V_n$ based on frequency distributed model (53), (57) and (59) is

$$V_n = V_{n-1} - \frac{1}{\sigma_n} \int_{t_{n-1}}^{\infty} \omega \mu_{n,i}(\omega) \bigg( \frac{\Lambda_{i-1}^n z_{\theta_n}(\omega, t) d\omega}{\sigma_n^2} \bigg)$$

and under a proper choice of design parameters $k_i > 0$ and $k_{\theta_n} > \tilde{E}_n$, a positive constant satisfying $e_n(x) \leq \tilde{E}_n$. One can obtain $\tilde{E}_n < 0$.

LaSalle invariant principle, the system state $z_i(\omega, t)$ and the estimated error $z_\theta(\omega, t), z_d(\omega, t)$ can be close to the set of all points where $V_n = 0$. One can deduce $z_i(\omega, t) = 0, z_\theta(\omega, t) = 0$ and $z_d(\omega, t) = 0$ from $V_n = 0$, which is the only equilibrium point. Therefore, $z_i(\omega, t) = 0, z_\theta(\omega, t) = 0$ and $z_d(\omega, t) = 0$ are convergent to the zero asymptotically, that is the error variables $e_i$, the estimated error $\bar{e}_i$ and $d$ convergent to zero asymptotically. Then, all the signals in the closed-loop adaptive system are uniformly bounded, and the tracking error $e_1 = y - y_d$ tend to zero asymptotically.

Remark 1. Theorem 1 proposes an adaptive fuzzy backstepping control algorithm for a kind of fractional order systems with unknown nonlinearities and external disturbance. If $\alpha_i = \alpha$, $i = 1, 2, \ldots, n$, this Theorem reduce to the commensurate case. Meanwhile, the order of the parameter estimation law (35), $i = 1, 2, \ldots, n$ from the new method is not fixed to the system order ($\alpha_i, i = 1, 2, \ldots, n$), in which more degree of freedom and better control performance can be obtained.

Remark 3. $D^{\alpha_i}u_{i-1}(i = 2, \ldots, n)$ contains the parameter estimation $\theta_{i-1}$ for the ideal parameter $\theta_{i-1}$ to approximate unknown function $f_i(x_1, \ldots, x_i)$, which also has the error form the estimated error $\hat{\theta}_{i-1} = \theta_{i-1} - \theta_{i-1}$. Therefore, $D^{\alpha_i}u_{i-1}(i = 2, \ldots, n)$ is considered as the unknown function in this paper.

IV. SIMULATION

To show the effectiveness of the presented method, two simulation examples are provided in this section.

A. Example 1

Consider an incommensurate fractional order nonlinear system as follows

$$D^{\alpha_1}x_1 = x_2 - 0.08x_1^2 + 0.15x_1$$

$$D^{\alpha_2}x_2 = x_3 + \frac{5 - 0.3x_3}{1 + x_3^2} - 0.05 \cos(x_1)$$

$$D^{\alpha_3}x_3 = bx - e^{-x_1/\gamma}x_1 \sin(5x_3) + 0.3 \sin(x_2) + d$$

where the control input coefficient $b = 3$, and the initial state $x(0) = \begin{bmatrix} 0.5 \ 1 \ 2 \end{bmatrix}^T$. $g_1(x_1) = 0.15x_1$, $g_2(x_1) = -0.05 \cos(x_1)$ and $g_3(x_2) = 0.3 \sin(x_2)$ are the known functions. $f_1(x_1) = -0.08x_1^2$, $f_2(x_1, x_2) = \frac{5 - 0.3x_3}{1 + x_3^2}$ and $f_3(x_1, x_2, x_3) = e^{-x_1/\gamma}x_1 \sin(5x_3)$ are the unknown nonlinear functions. As shown in Fig. 1, when $u = d = 0$, the uncontrolled system (64) is unstable.

The reference signal $y_d$ is chosen as $\sin(x_1 + 0.3)$ and will be tracked by the system output, the disturbance signal $d(t)$ is chosen as 0.1. In order to avoid the chattering.
Tracking error

Fig. 1: The response of system states when \( u = d = 0 \).

phenomenon, \( \text{sign}(\cdot) \) is replaced by \( 2\arctan(10\cdot)/\pi \) in the presented controller \( 2\arctan(10\cdot)/\pi \) equals to \( 2\arctan(10\zeta)/\pi \) with input \( \zeta \).

There are three fuzzy systems used in the presented controller. The first fuzzy system uses \( x_1 \) as its input and defines the Gaussian membership functions as

\[
\exp\left(-\frac{1}{2}\left(\frac{x_1 - x_{k_1}}{0.166}\right)^2\right), x_{k_1} \in \{0.5k_1 - 2|k_1 = 1, 2, \ldots, 6\} \tag{65}
\]

where the initial condition is \( \theta_1(0) = 0_{6\times1} \). The second fuzzy system employs \( x_1 \) and \( x_2 \) as its inputs. For the input \( x_1 \), the membership functions are chosen as (65). For the input \( x_2 \), the membership functions are chosen as

\[
\exp\left(-\frac{1}{2}\left(\frac{x_2 - x_{k_2}}{0.333}\right)^2\right), x_{k_2} \in \{k_2 - 2|k_2 = 1, 2, 3\} \tag{66}
\]

where the initial condition is \( \theta_2(0) = 0_{3\times1} \). The third fuzzy system uses \( x_1, x_2 \) and \( x_3 \) as its inputs. For the input \( x_1 \) and \( x_2 \), the membership functions are chosen as the same as the second one. For the input \( x_3 \), the membership functions are chosen as

\[
\exp\left(-\frac{1}{2}\left(\frac{x_3 - x_{k_3}}{0.166}\right)^2\right), x_{k_3} \in \{0.5k_3 - 1.5|k_3 = 1, 2, \ldots, 5\} \tag{67}
\]

where the initial condition is \( \theta_3(0) = 0_{5\times1} \).

The design parameters are chosen as \( k_{11} = k_{12} = k_{13} = 13, k_{21} = k_{22} = k_{23} = 0.01, \sigma_1 = \sigma_2 = \sigma_3 = 3, \Lambda_1 = I_6, \Lambda_2 = I_{18}, \Lambda_3 = I_{90} \) and \( \rho = 2 \).

Simulation results are presented in Fig. 2, in which the tracking error \( e_1 = y - y_d \) has a rapid convergence, and the controller works well in a disturbance environment with a partially unknown system model. However, \( e_1 \) cannot converge to zero but has tiny fluctuations near zero. To check the tracking performances fully, we zoom in to Fig. 2 and Fig. 3 is obtained. It can be found that the tracking error \( e_1 \) does not stop at zero, instead it still has a tiny fluctuations near zero.

There are three reasons for this result: 1) fuzzy logic systems have approximation error itself for unknown nonlinear functions; 2) \( \text{sign}(\cdot) \) is replaced by \( 2\arctan(10\cdot)/\pi \), then the asymptotical convergence of the tracking error cannot be achieved; 3) the varying reference and varying control input causes that the tracking error can not converge to zero. If the control accuracy is higher than the sensor accuracy in practice, the tracking performances can be satisfactory.

In order to evaluate the boundedness of the signals in the closed-loop adaptive system, the control input is presented in Fig. 4. Accordingly, the norm of parameters estimation of the fuzzy systems is presented in shown in Fig. 5, and the estimation of the upper bound of the external disturbance \( d(t) \) is presented in shown in Fig. 6. It is demonstrating that the signals in the closed-loop adaptive system are uniformly bounded. All of the above-mentioned results verify that the asymptotic tracking problem of such incommensurate fractional order systems with unknown nonlinearities and external disturbance can be solved by the proposed adaptive fuzzy backstepping controller effectively.

### B. Example 2

Consider an incommensurate fractional order Chua’s system [48]
where $\alpha_1 = 0.98$, $\alpha_2 = 0.96$, $\alpha_3 = 0.94$, and the initial state $\mathbf{x}(0) = \begin{bmatrix} 0.6 & 0.1 & -0.6 \end{bmatrix}^T$. $f_1(x_1) = -x_1 - x_1^2$, $f_2(x_1, x_2) = x_1 - x_2$ and $f_3(x_1, x_2, x_3) = -11x_2 - 0.1x_3$ are the unknown nonlinear functions.

The reference signal $y_d$ is chosen as $\sin(t+0.1)$ and will be tracked by the output of the system, the disturbance signal $d(t)$ is chosen as $0.3(\sin(t) + \cos(t))$. There are three fuzzy systems used in the presented controller. The first fuzzy system employs $x_1$ as its input and defines the Gaussian membership functions as

$$
\exp\left(-\frac{1}{2} \frac{(x_1 - x_{k_1})^2}{0.498}\right), x_{k_1} \in \{1.5k_1 - 6 | k_1 = 1, 2, \ldots, 6\} (69)
$$

where the initial condition is $\theta_1(0) = 0_{6\times1}$. The second fuzzy system uses $x_1$ and $x_2$ as its inputs. For the input $x_1$, the membership functions are chosen as the same as the first one. For the input $x_2$, the membership functions are chosen as

$$
\exp\left(-\frac{1}{2} \frac{(x_2 - x_{k_2})^2}{0.999}\right), x_{k_2} \in \{3k_2 - 6 | k_2 = 1, 2, 3\} (70)
$$

where the initial condition is $\theta_2(0) = 0_{18\times1}$. The third fuzzy system uses $x_1$, $x_2$ and $x_3$ as its inputs. For the input $x_1$, the membership functions are chosen as the same as the second one. For the input $x_3$, the membership functions are chosen as

$$
\exp\left(-\frac{1}{2} \frac{(x_3 - x_{k_3})^2}{0.498}\right), x_{k_3} \in \{1.5k_3 - 4.5 | k_3 = 1, 2, \ldots, 5\} (71)
$$

where the initial condition is $\theta_3(0) = 0_{90\times1}$. The design parameters are chosen as $k_{11} = k_{12} = k_{13} = 35$, $k_{21} = k_{22} = k_{23} = 0.03$, $\sigma_1 = \sigma_2 = \sigma_3 = 0.5$, $\Lambda_1 = I_{6}$, $\Lambda_2 = I_{18}$, $\Lambda_3 = I_{90}$ and $\rho = 0.3$.

Simulation results are presented in Fig. 7, in which the tracking error $e_1$ has a rapid convergence zero with tiny fluctuations. The control input is presented in Fig. 8. Accordingly, the norm of parameters estimation of the fuzzy systems is presented in shown in Fig. 9, and the estimation of the upper bound of the external disturbance $d(t)$ is presented in shown in Fig. 10. It is demonstrating that the signals in the closed-loop adaptive system are uniformly bounded.

V. CONCLUSION

In this paper, a new adaptive fuzzy backstepping control algorithm for a class of incommensurate fractional order nonlinear systems with unknown nonlinearities and external disturbance is presented. With regard to the frequency distribute model and the fuzzy logic system employed to approximate an unknown nonlinear function, an adaptive fuzzy controller guaranteeing the convergence of the tracking error is established based on the indirect Lyapunov method. The results show that the backstepping control technique can be extended to incommensurate fractional order nonlinear systems by using the frequency distribute model and Lyapunov stability criterion.
Fig. 7: Simulation results of $y$, $y_d$ and tracking error.

Fig. 8: Evolution of the control input.

Fig. 9: Evolution of the fuzzy parameters norm.

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