Interval-Valued Pythagorean Fuzzy Maclaurin Symmetric Mean Operators in Multiple Attribute Decision Making

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Abstract—Interval-valued Pythagorean fuzzy (IVPF) set is one the successful extension of the existing theories for handling the uncertainties during the decision-making process. Under that environment, various aggregation operators have been developed by the authors to aggregate the different preferences of the decision makers under the different attributes. But these studies have conducted under the assumption that their corresponding pairs are independent and don’t consider the interaction between the pairs of the membership degrees. In the present manuscript, these conditions have been relaxed by considering the interrelationship between the different inputs by using Maclaurin symmetric mean (MSM) operator. Further, based on the input and MSM operator, we proposed two aggregation operators namely, IVPF Maclaurin symmetric mean (IVPFMSM) and IVPF weighted Maclaurin symmetric mean (IVPFWMSM) operators and studied their desirable properties. A decision-making method based on these operators has been discussed for solving the decision-making problems under IVPF set environment. Finally, an illustrative example and a comparative analysis have been presented to demonstrate the proposed approach.

Index Terms—multiple attribute decision making (MADM); MSM operator; IVPFS; Aggregation operators; potential evaluation; emerging technology commercialization

1. INTRODUCTION

Atanassov[1] presented the concept of the intuitionistic fuzzy set (IFS) by considering the pairs of the degree of the membership, nonmembership, by generalizing the concept of the fuzzy set [2] such that their sum is not greater than one. After their existence, researchers have applied these theories in different disciplines [3-17] and found that they are more profitable to handle the uncertainties during the analysis. Since the above-defined theories have been successfully defined but in some case, it is unable to handle the situation by IFS. For instance, if a decision maker (DM) may take the membership degrees of any element is 0.8 and 0.5 then clearly their sum is not less than one. Hence, under such types of cases, IFS have some sort of deficiencies. In order to resolve it, Pythagorean fuzzy set (PFS) [18, 19], an extension of IFSs, has emerged as an effective tool for depicting the uncertainty in the data. In this set, the condition of the sum of the degrees is replaced with their sum of squares is less than one and hence the PFS is more general than the IFS. Further, it is clearly that $0.8^2 + 0.5^2 \leq 1$ and hence PFS stand for such cases. After their existence, Zhang and Xu [20] presented the mathematical expression for PFS and developed the Pythagorean fuzzy TOPSIS (Technique for Order Preference by Simplicity to Ideal Solution) technique for solving the decision-making problems. Zhang [21] presented Pythagorean fuzzy weighted and ordered weighted aggregation operators and a similarity measure based decision-making approach for solving multi criteria decision-making problems under the Pythagorean fuzzy environment. Peng and Yang [22] developed some fundamental properties of the PFNs. Reformat and Yager [23] applied the PFNs in handling the collaborative-based recommender system. Garg [25, 26] proposed the new generalization of Pythagorean fuzzy information aggregation by using Einstein Operations. Zhang [27] extended the PFS to the interval-valued PFSs (IVPFSs). Garg [28] presented the averaging and geometric aggregation operators under the interval-valued PF (IVPF) environment. Also, a novel accuracy function has been presented to rank the IVPF numbers. However, in terms of the information measure theory, a novel accuracy function [28], correlation coefficient [29], improved accuracy function [30], improved score function [31] have been defined under the PFS and IVPFS and applied them to solve the decision-making problems. Recently, an aggregation operation for the PFSs by incorporating the confidence level of the decision makers during the decision-making process named as confidence based Pythagorean fuzzy weighted average and geometric operators have been proposed by Garg [32].

Apart from that, some others researchers are working in the field of the PFS or IPFS and proposed various types of decision making approaches which are summarized by [24, 27, 30, 31, 33-36].

All the above-mentioned operators and measures are based on the assumption that input argument which we want to aggregate are independent and hence, in sometimes, they may be unable to justify the decision maker goals. On the other hand, in our real-life situation, it may be possible that there are interactions among the different attributes in
a decision making process. To address such type of issues, Maclaurin symmetric mean (MSM) was originally introduced by Maclaurin [37] and then developed by Detemple and Robertson [38], has prominent characteristics to capture the interrelationship among the multi-input arguments. Further, MSM operator satisfies the certain property that, with respect to the parameter, it is monotonically decreasing and hence can reflect the risk attitude during the process of the decision makers. In the past few years, the MSM has received more and more attentions, many important results both in theory and application are developed [39-41].

Therefore, by considering the advantages of the IVPFS and the MSM operator during the information fusion process, the present study enhanced the work in that direction. In it, IVPFS has been used to handle the uncertainties in the data in the form of IVPFNs while MSM operator is used to considering the interrelationships between the different attributes. As far as authors are aware, there is no research conducted under this direction and hence it is meaningful to pay any attention to it. So, to consider the advantages of interrelationships among any number of the attributes in IVPFS environment, in this manuscript we proposed two aggregation operators namely, IVPF Maclaurin symmetric mean (IVPFMMSM) and IVPF weighted Maclaurin symmetric mean (IVPFWMMSM). Further, some of their desirable properties have also been addressed. Finally, based on these operators, a decision-making approach has been presented under IVPFS environment and illustrate with a numerical example to validate the approach through some comparative study with the existing approaches.

To do so, the rest of the paper is organized as follows. Some basic concepts on PFS and IVPFSs have been summarized in the next section. Section 3, presented the MSM operators under IVPFS environment namely, IVPFWMSM and IVPFMMSM along with their certain properties. In Section 4, a group decision making approach has been presented based on these proposed operators. Section 5 deal with a practical example to illustrate the proposed approach and deliver a comparative analysis to validate it. Section 6 gives the concluding remarks.

2. PRELIMINARIES
In this section, some basic concepts of the PFS and IVPFS have been reviewed over a fix set $X$.

2.1. PYTHAGOREAN FUZZY SET
Definition 1 [18, 20]. A PFS $P$ is defined as

\[
P = \left\{ (x, (\mu_p(x), \nu_p(x))) \mid x \in X \right\}
\]

where the functions $\mu_p : X \rightarrow [0,1]$ and $\nu_p : X \rightarrow [0,1]$ defines the degrees of membership and non-membership of the element $x \in X$ to $P$, such that for each $x$, the condition $\left( \mu_p(x) \right)^2 + \left( \nu_p(x) \right)^2 \leq 1$, holds.

Definition 2 [20]. The $p = (\mu, \nu)$ be called as Pythagorean fuzzy number (PFN) and defined the score and accuracy functions as $S(p) = \mu^2 - \nu^2$ and $H(p) = \mu^2 + \nu^2$. In order to compare two or more PFNs $p_1$ and $p_2$, a comparison law is defined as

1. if $S(p_1) < S(p_2)$, then $p_1 < p_2$;
2. if $S(p_1) = S(p_2)$, then
   1. if $H(p_1) = H(p_2)$, then $p_1 = p_2$;
   2. if $H(p_1) < H(p_2)$, then $p_1 < p_2$.

2.2. INTERVAL-VALUED PYTHAGOREAN FUZZY SET
Zhang [27] extended the PFS to the IVPFSs which is defined as followed over the fix set $X$.

Definition 4 [27]. An IVPFS $\tilde{P}$ is defined as

\[
\tilde{P} = \left\{ (x, (\tilde{\mu}_p(x), \tilde{\nu}_p(x))) \mid x \in X \right\}
\]

where

\[
\tilde{\mu}_p(x) = \left[ \mu_p^L(x), \mu_p^R(x) \right], \quad \tilde{\nu}_p(x) = \left[ \nu_p^L(x), \nu_p^R(x) \right]
\]

are the interval numbers of $[0, 1]$ with the condition $0 \leq \left( \mu_p^L(x) \right)^2 + \left( \nu_p^L(x) \right)^2 \leq 1$, $\forall x \in X$. The pair $\tilde{p} = \left( [u_p^L, u_p^R], [v_p^L, v_p^R] \right)$ is called as an IVPF number (IVPFN), where $u_p, v_p \subseteq [0,1]$ and $(u_p^R)^2 + (v_p^R)^2 \leq 1$.

Definition 5 [28]. For three IVPFNs $\tilde{P}_1 = \left( [u_{1L}, u_{1R}], [v_{1L}, v_{1R}] \right)$, $\tilde{P}_2 = \left( [u_{2L}, u_{2R}], [v_{2L}, v_{2R}] \right)$, and $\tilde{P} = \left( [u_p^L, u_p^R], [v_p^L, v_p^R] \right)$, the basic operational laws are defined as follows:
Based on the Definition 5, Garg [28] derived the following properties easily.

Theorem 1. Let \( \tilde{p}_1 = \left[ u_{p_1}^L, u_{p_1}^R, v_{p_1}^L, v_{p_1}^R \right] \) and 
\( \tilde{p}_2 = \left[ u_{p_2}^L, u_{p_2}^R, v_{p_2}^L, v_{p_2}^R \right] \) be two IVPFNs, 
\( \lambda, \lambda_1, \lambda_2 > 0 \), be three real numbers, then

1. \( \tilde{p}_1 \oplus \tilde{p}_2 = \tilde{p}_2 \oplus \tilde{p}_1 \);
2. \( \tilde{p}_1 \otimes \tilde{p}_2 = \tilde{p}_2 \otimes \tilde{p}_1 \);
3. \( \lambda (\tilde{p}_1 \oplus \tilde{p}_2) = \lambda \tilde{p}_1 + \lambda \tilde{p}_2 \);
4. \( (\tilde{p}_1 \otimes \tilde{p}_2)^\lambda = (\tilde{p}_1)^\lambda \otimes (\tilde{p}_2)^\lambda \);
5. \( \lambda_1 \tilde{p}_1 \oplus \lambda_2 \tilde{p}_1 = (\lambda_1 + \lambda_2) \tilde{p}_1 \);
6. \( (\tilde{p}_1 \otimes \tilde{p}_1)^\lambda = (\tilde{p}_1)^{(\lambda_1 + \lambda_2)} \);
7. \( ((\tilde{p}_1 \otimes \tilde{p}_1)^\lambda)^\lambda = (\tilde{p}_1)^{(\lambda_1)^\lambda} \).

Definition 6. For an IVPFN \( \tilde{p} = \left[ u_p^L, u_p^R, v_p^L, v_p^R \right] \),
the score and accuracy functions of it are defined as
\[
S(\tilde{p}) = \frac{\lambda}{4} \left[ (u_p^L)^2 + (u_p^R)^2 + (v_p^L)^2 + (v_p^R)^2 \right] 
\]
and \( H(\tilde{p}) = \frac{(u_p^L)^2 + (u_p^R)^2 + (v_p^L)^2 + (v_p^R)^2}{2} \) respectively.

Further, in order to compare two different IVPFNs \( \tilde{p}_1 \) and \( \tilde{p}_2 \), an order relation is defined as

1. if \( S(\tilde{p}_1) < S(\tilde{p}_2) \), then \( \tilde{p}_1 < \tilde{p}_2 \);
2. if \( S(\tilde{p}_1) = S(\tilde{p}_2) \), then
   (i) if \( H(\tilde{p}_1) = H(\tilde{p}_2) \), then \( \tilde{p}_1 = \tilde{p}_2 \);
   (ii) if \( H(\tilde{p}_1) < H(\tilde{p}_2) \), then \( \tilde{p}_1 < \tilde{p}_2 \).

2.3. Maclaurin symmetric mean (MSM) operator

Definition 7 [37]. For a collection of non-negative numbers \( a_j (j = 1, 2, \ldots, n) \) and \( k = 1, 2, \ldots, n \). If

\[
\text{MSM}^k(a_1, a_2, \ldots, a_n) = \left( \frac{\sum_{i=1}^{k} \prod_{j=1}^{i} a_j}{C_n^k} \right)^{\frac{1}{k}}
\]
then \( \text{MSM}^k \) is called the Maclaurin symmetric mean (MSM) operator, where \( (i_1, i_2, \ldots, i_k) \) traversal all the \( k \)-tuple combination of \( (1, 2, \ldots, n) \), \( C_n^k \) is the binomial coefficient.

Obviously, the \( \text{MSM}^k \) have the following properties:

(i) \( \text{MSM}^k(0, 0, \ldots, 0) = 0 \);
(ii) \( \text{MSM}^k(a, a, \ldots, a) = a \);
(iii) \( \text{MSM}^k(a_1, a_2, \ldots, a_n) \leq \text{MSM}^k(b_1, b_2, \ldots, b_n) \) if \( a_i \leq b_i \) for all \( i \);
(iv) \( \min_i a_i \leq \text{MSM}^k(a_1, a_2, \ldots, a_n) \leq \max_i a_i \).

3. INTERVAL-VALUED PYTHAGOREAN FUZZY MACLAURIN SYMMETRIC MEAN OPERATORS

In this section, we have presented some aggregation operators for IVPFNs by using MSM operator as follows:

3.1. IVPFSMSM\(^{(k)}\) OPERATOR

Definition 8. For a collection of IVPFNs \( \tilde{p}_j, j = 1, 2, \ldots, n \), a mapping \( \text{IVPFSMSM}\(^{(k)}\) : \Omega^n \rightarrow \Omega \) is called an IVPF Maclaurin symmetric mean operator and given by
where \( \Omega \) be a collection of IVPFNs.

Now, based on the operations law of IVPFNs, the
\[
IVPFMSM^{(k)}(\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_n) = \left( \bigoplus_{1 \leq i \leq k} \left( \bigotimes_{j=1}^{k} \bar{p}_{ij} \right) \right)^{1/k}
\]

following results have been derived as

**Theorem 2.** The aggregated value for a collection of IVPFNs
\[
\bar{p}_j = \left( [\mu_{ij}^L, \mu_{ij}^R], [\nu_{ij}^L, \nu_{ij}^R] \right) (j = 1, 2, \ldots, n)
\]

by using Definition 8 is also an IVPFN, and given by

\[
IVPFMSM^{(k)}(\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_n)
\]

\[
= \left[ \left[ \prod_{1 \leq i \leq k} \mu_{ij}^L, \prod_{1 \leq i \leq k} \mu_{ij}^R \right] \right]^{1/k},
\]

**Proof:** By the operational laws of the IVPFNs, we have

\[
\bigotimes_{j=1}^{k} \bar{p}_{ij} = \left[ \prod_{j=1}^{k} \mu_{ij}^L, \prod_{j=1}^{k} \mu_{ij}^R \right],
\]

and

\[
\bigoplus_{1 \leq i \leq k} \left( \bigotimes_{j=1}^{k} \bar{p}_{ij} \right) = \left[ \prod_{1 \leq i \leq k} \mu_{ij}^L, \prod_{1 \leq i \leq k} \mu_{ij}^R \right],
\]

then we obtain

\[
\frac{1}{C_k} \bigoplus_{1 \leq i \leq k} \left( \bigotimes_{j=1}^{k} \bar{p}_{ij} \right) = \left[ \prod_{1 \leq i \leq k} \mu_{ij}^L, \prod_{1 \leq i \leq k} \mu_{ij}^R \right],
\]

Therefore,
IVPFMSM\(^{(k)}\) \((\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n)\)

\[
\begin{bmatrix}
\sqrt{1 - \prod_{I \leq k \leq n} (1 - (\mu^L_i)^2)} \\
\sqrt{1 - \prod_{I \leq k \leq n} (1 - (\mu^R_j)^2)} \\
\end{bmatrix}
\]

It can be easily proved that the IVPFMSM operator has the following properties.

**Property 1.** (Idempotency) If all \(\tilde{p}_j\) \((j = 1, 2, \cdots, n)\) are equal, i.e. \(\tilde{p}_j = \tilde{p}\) for all \(j\), then

IVPFMSM\(^{(k)}\) \((\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n) = \tilde{p}\) \((9)\)

**Proof.** Since \(\tilde{p} = ([\mu^L_i, \mu^R_j], [\nu^L_i, \nu^R_j])\), based on Theorem 2, we have

IVPFMSM\(^{(k)}\) \((\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n) = \tilde{p}\) \((9)\)
which complete the proof of Property 1.

**Property 2.** (Commutativity) If \( \tilde{p}_j \) is any permutation of IVPFN \( \tilde{p}_j (j = 1,2,\ldots,n) \), then

\[
\text{IVPFMSM}^{(k)} (\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n) = \text{IVPFMSM}^{(k)} (\tilde{p}_1', \tilde{p}_2', \cdots, \tilde{p}_n')
\]

(10)

**Proof.** Since \( \tilde{p}_j' \) is permutation of \( \tilde{p}_j (j = 1,2,\ldots,n) \), based on the definition of IVPFM in Eq.(5), we have

\[
\text{IVPFMSM}^{(k)} (\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n) = \left( \bigoplus_{1 \leq i_1, \ldots, i_k \leq n} \left( \bigotimes_{j=1}^{k} \tilde{p}_{i_j} \right) \right)^{1/k}
\]

\[
= \left( \bigoplus_{1 \leq i_1, \ldots, i_k \leq n} \left( \bigotimes_{j=1}^{k} \tilde{p}_{i_j}' \right) \right)^{1/k}
\]

\[
= \text{IVPFMSM}^{(k)} (\tilde{p}_1', \tilde{p}_2', \cdots, \tilde{p}_n')
\]

Thus the proof is complete.

**Property 3.** (Monotonicity) Let \( \tilde{p}_j', \tilde{p}_j'' (j = 1,2,\ldots,n) \) be two collections of IVPFNs, if \( \mu_j^L \geq \mu_j^L'', \mu_j^R \geq \mu_j^R'', \) \( v_j^L \geq v_j^L'', v_j^R \geq v_j^R'', \) for all \( j \), then

\[
\text{IVPFMSM}^{(k)} (\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n) \geq \text{IVPFMSM}^{(k)} (\tilde{p}_1', \tilde{p}_2', \cdots, \tilde{p}_n')
\]

(11)

**Proof.** Since \( k \geq 1 \), \( \mu_j^L \geq \mu_j^L'', \mu_j^R \geq \mu_j^R'', \) \( v_j^L \leq v_j^L, v_j^R \leq v_j^R, \) then we have

\[
1 \geq \mu_j^L \geq \mu_j^L'' \geq 0.1 \geq \mu_j^R \geq \mu_j^R'' \geq 0
\]

\[
0 \leq v_j^L \leq v_j^L', \quad 0 \leq v_j^R \leq v_j^R',
\]

Based on the assumption condition, for all \( i, j (i = 1,2,\ldots,n; \ j = 1,2,\ldots,k) \), we obtain
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IVPFMSM\(^{(k)}\)\((\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n)\)
\[\geq IVPFMSM\(^{(k)}\)\((\tilde{p}^-, \tilde{p}^-, \cdots, \tilde{p}^-) = \tilde{p}^-\)
IVPFMSM\(^{(k)}\)\((\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n)\)
\[\leq IVPFMSM\(^{(k)}\)\((\tilde{p}^+, \tilde{p}^+, \cdots, \tilde{p}^+) = \tilde{p}^+\)

Thus the proof is complete.

In the below, we shall check the monotonicity of the proposed operator with \(k\).

**Lemma 1.** (Maclaurin inequality) \([42]\). Let \(a_i\) \((i = 1, 2, \cdots, n)\) be a set of non-negative real numbers, and for \(k = 1, 2, \cdots, n\), then

\[
MSM^{(1)}(a_1, a_2, \cdots, a_n) \\
\geq MSM^{(2)}(a_1, a_2, \cdots, a_n) \\
\cdots \geq MSM^{(n)}(a_1, a_2, \cdots, a_n)
\]

with equality if and only if \(a_1 = a_2 = \cdots = a_n\).

**Lemma 2** \([42]\). Let \(a_i > 0\), \(b_i > 0\) \((i = 1, 2, \cdots, n)\), and

\[
\sum_{i=1}^{n} b_i = 1, \text{ then}
\]

\[
\prod_{i=1}^{n} (a_i)^{b_i} \leq \sum_{i=1}^{n} a_i b_i
\]

with equality if and only if \(a_1 = a_2 = \cdots = a_n\).

**Theorem 3.** For a given IVPFNs \(\tilde{p}_j\) \((j = 1, 2, \cdots, n)\) and positive finite integer \(k\), the proposed IVPFMSM is monotonically decreasing with \(k\).

**Proof.** Based on the Theorem 1, we have

\[
\tilde{p}^- \leq IVPFMSM^{(k)}(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n) \leq \tilde{p}^+ \quad (12)
\]

**Proof.** Based on the Property 1 and 3, we have

\[
IVPFMSM^{(k)}(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n) = \left[\begin{array}{c}
\left(1 - \prod_{i=1}^{k} \left(1 - \left(\min \mu^L_i\right)^2\right)\right) \frac{1}{\sqrt[k]{k}} \\
\left(1 - \prod_{i=1}^{k} \left(1 - \left(\max \nu^R_i\right)^2\right)\right) \frac{1}{\sqrt[k]{k}}
\end{array}\right]\]

(14)
Let 
\[
    f(k) = \left( 1 - \prod_{\in \subset \cup \alpha} \left( 1 - \left( \prod_{j=1}^{k} \mu_{i_j} \right)^2 \right) \right)^{1/2}
\]
(15)
and
\[
    g(k) = \left( 1 - \prod_{\in \subset \cup \alpha} \left( 1 - \left( \prod_{j=1}^{k} \left( 1 - \nu_{i_j} \right)^2 \right) \right) \right)^{1/2}
\]
(16)
Now, based on the Lemmas 1 and 2, we have
\[
    f(k) = \left( 1 - \sum_{\in \subset \cup \alpha} \frac{1 - \left( \prod_{j=1}^{k} \mu_{i_j} \right)^2}{C_n^k} \right)^{1/2} \geq \left( 1 - \sum_{\in \subset \cup \alpha} \frac{1 - \left( \prod_{j=1}^{k} \mu_{i_j} \right)^2}{C_n^k} \right)^{1/2} = \left( \sum_{\in \subset \cup \alpha} \frac{\left( \prod_{j=1}^{k} \mu_{i_j} \right)^2}{C_n^k} \right)^{1/2}
\]
Assume that \( f(k) \) is monotonically increasing with \( k \), i.e.,
\[
    f(n) > f(n-1) > \cdots > f(1)
\]
(17)
Also since
\[
    f(1) \geq \sum_{\in \subset \cup \alpha} \frac{\left( \prod_{j=1}^{1} \mu_{i_j} \right)^2}{C_n^1} = \frac{\sum_{i=1}^{n} \mu_i^2}{n}
\]
(18)
which follows that
IVPFMSM\(^{(i)}\left(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n\right)\)

\[
\begin{align*}
= & \left[\sqrt{1 - \left(\prod_{i=1}^{n} \left(1 - \left(\prod_{j=1}^{i} \mu^L_{ij}\right)^2\right)^{\frac{1}{2\mu^L_{ij}}^2}\right)}\right]^{\frac{1}{i}} \cdot \left[\sqrt{1 - \left(\prod_{j=1}^{n} \left(1 - \left(\prod_{i=1}^{j} \mu^R_{ij}\right)^2\right)^{\frac{1}{2\mu^R_{ij}}^2}\right)}\right]^{\frac{1}{j}} \\
= & \left[\sqrt{1 - \left(\prod_{i=1}^{n} \left(1 - \left(\prod_{j=1}^{i} \left(1 - (v^L_{ij})^2\right)^2\right)^{\frac{1}{2\mu^L_{ij}}^2}\right)}\right]^{\frac{1}{i}} \cdot \left[\sqrt{1 - \left(\prod_{j=1}^{n} \left(1 - \left(\prod_{i=1}^{j} (1 - (v^R_{ij})^2))\right)^2\right)^{\frac{1}{2\mu^R_{ij}}^2}\right)}\right]^{\frac{1}{j}} \\
= & \left[\sqrt{1 - \left(\prod_{i=1}^{n} \left(1 - (\mu^L_i)^2\right)^{\frac{1}{n}}\right)}\right] \cdot \left[\sqrt{1 - \left(\prod_{i=1}^{n} \left(1 - (\mu^R_i)^2\right)^{\frac{1}{n}}\right)}\right] \cdot \left[\prod_{i=1}^{n} \left(\frac{1}{\mu^L_i} \cdot \sqrt{n} \cdot \sqrt{\prod_{i=1}^{n} \left(1 - (\mu^R_i)^2\right)^{\frac{1}{n}}:\right)\right] (let \ i = i) \\
= & \left[\sqrt{1 - \left(\prod_{i=1}^{n} \left(1 - (\mu^L_i)^2\right)^{\frac{1}{n}}\right)}\right] \cdot \left[\sqrt{1 - \left(\prod_{i=1}^{n} \left(1 - (\mu^R_i)^2\right)^{\frac{1}{n}}\right)}\right] \cdot \left[\prod_{i=1}^{n} \left(\mu^L_i + \mu^R_i\right)^{\frac{1}{n}}\right]
\end{align*}
\]

(2) When \(k = 2\), IVPFMSM\(^{(k)}\) operator becomes IVPF Bonferroni mean (IVPFBM) operator as follows:
IVPFMSM$^{(2)}$ ($\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n$)

\[
\begin{aligned}
&= \left( \sqrt{1 - \prod_{i=1}^{n} \left( 1 - \left( \frac{2}{C_i} \mu^L_{i,j} \right)^2 \right)^{C_i^2} \left( 1 - \left( \frac{2}{C_i} \mu^L_{i,j} \right)^2 \right)^{C_i^2} \left( 1 - \left( \frac{2}{C_i} \mu^L_{i,j} \right)^2 \right)^{C_i^2}} \right) \left( \sqrt{1 - \prod_{i=1}^{n} \left( 1 - \left( \frac{2}{C_i} \mu^R_{i,j} \right)^2 \right)^{C_i^2} \left( 1 - \left( \frac{2}{C_i} \mu^R_{i,j} \right)^2 \right)^{C_i^2} \left( 1 - \left( \frac{2}{C_i} \mu^R_{i,j} \right)^2 \right)^{C_i^2}} \right)
\end{aligned}
\]
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(3) When $k = n$, IVPFMSM$^{(k)}$ operator reduces to the IVPF geometric mean (IVPFGM) operator as

$$\text{IVPFMSM}^{(n)}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n)$$

follows:

$$= \text{IVPFBM}^{11}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n)$$

Example 1.
Let
\[ \mathbf{p}_1 = ([0.3, 0.4], [0.6, 0.7]), \quad \mathbf{p}_2 = ([0.4, 0.5], [0.3, 0.4]), \quad \mathbf{p}_3 = ([0.7, 0.8], [0.2, 0.3]), \quad \mathbf{p}_4 = ([0.5, 0.6], [0.2, 0.3]) \]
be an arbitrary four IVPFNs.

Without loss of generality, we take \( k = 2 \), and applied IVPFWSM\(^{(k)}\) operator to aggregate these numbers. Then we have

\[ \mathbf{p}_1 \otimes \mathbf{p}_2 = \left( [0.3 \times 0.4, 0.4 \times 0.5], \frac{\sqrt{0.6^2 + 0.3^2 - 0.6 \times 0.3^2}, \sqrt{0.7^2 + 0.4^2 - 0.7^2 \times 0.4^2}}{0.1200, 0.2000}, [0.6462, 0.7560] \right) \]
\[ \mathbf{p}_1 \otimes \mathbf{p}_3 = \left( [0.3 \times 0.7, 0.4 \times 0.8], \frac{\sqrt{0.6^2 + 0.2^2 - 0.6 \times 0.2^2}, \sqrt{0.7^2 + 0.3^2 - 0.7^2 \times 0.3^2}}{0.2100, 0.3200}, [0.6210, 0.7321] \right) \]
\[ \mathbf{p}_1 \otimes \mathbf{p}_4 = \left( [0.3 \times 0.5, 0.4 \times 0.6], \frac{\sqrt{0.6^2 + 0.2^2 - 0.6 \times 0.2^2}, \sqrt{0.7^2 + 0.3^2 - 0.7^2 \times 0.3^2}}{0.1500, 0.2400}, [0.6210, 0.7321] \right) \]
\[ \mathbf{p}_2 \otimes \mathbf{p}_3 = \left( [0.4 \times 0.7, 0.5 \times 0.8], \frac{\sqrt{0.3^2 + 0.2^2 - 0.3 \times 0.2^2}, \sqrt{0.4^2 + 0.3^2 - 0.4^2 \times 0.3^2}}{0.2800, 0.4000}, [0.3555, 0.4854] \right) \]
\[ \mathbf{p}_2 \otimes \mathbf{p}_4 = \left( [0.4 \times 0.5, 0.5 \times 0.6], \frac{\sqrt{0.3^2 + 0.2^2 - 0.3 \times 0.2^2}, \sqrt{0.4^2 + 0.3^2 - 0.4^2 \times 0.3^2}}{0.2000, 0.3000}, [0.3555, 0.4854] \right) \]
\[ \mathbf{p}_3 \otimes \mathbf{p}_4 = \left( [0.7 \times 0.5, 0.8 \times 0.6], \frac{\sqrt{0.2^2 + 0.2^2 - 0.2 \times 0.2^2}, \sqrt{0.5^2 + 0.3^2 - 0.5^2 \times 0.3^2}}{0.3500, 0.4800}, [0.2800, 0.4146] \right) \]

Therefore, by Eq. (5), we get

\[ \text{IVPFWSM}^{(2)}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = \left( [0.4829, 0.5834], [0.5113, 0.5957] \right) \]

### 3.2. IVPFWSM\(^{(k)}\) OPERATOR

The above proposed aggregation operator has been extended to the weighted aggregation operator by taking into the account the priority of the elements and named as IVPF weighted MSM IVPFWSM\(^{(k)}\) operator.

**Definition 9.** Let \( \mathbf{p}_j, j = 1, 2, \ldots, n \) be a collection of IVPFNs, \( \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T \) be the weight vector of its such that \( \omega_j > 0 \), \( \sum_{j=1}^{n} \omega_j = 1 \). A mapping

\[ \text{IVPFWSM}^{(k)}_{\omega} : \Omega^n \rightarrow \Omega \]

is given by

\[ \text{IVPFWSM}^{(k)}_{\omega}(\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n) = \left( \bigoplus_{1 \leq i \leq n} \bigoplus_{1 \leq j \leq n} \left( \mathbf{p}_i \otimes \mathbf{p}_j \right)^{\alpha_{ij}} \right) \]

**Theorem 4.** For a collection of IVPFNs \( \mathbf{p}_j = \left( [\mu_j^L, \mu_j^R], [\nu_j^L, \nu_j^R] \right) \) \( (j = 1, 2, \ldots, n) \) and finite positive integer \( k \), the aggregated value by using IVPFWSM\(^{(k)}\) operator is also IVPFN and given by
IVPFWMSE\(^{(k)}\) \((\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_n)\)

\[
\mathcal{C}_n^k \left( \bigotimes_{j=1}^{k} (\tilde{P}_{ij})^{a_{ij}} \right) = \left[ \prod_{i=1}^{k} \sqrt{1 - \left( \prod_{j=1}^{k} \left( \mu_{ij}^{L \theta_j} \right)^{a_{ij}} \right)^2} \right] \left[ \prod_{i=1}^{k} \sqrt{1 - \left( \prod_{j=1}^{k} \left( \mu_{ij}^{R \theta_j} \right)^{a_{ij}} \right)^2} \right]
\]

**Proof:** By the operational laws of the IVPFNs, we have

\[
\left[ \prod_{j=1}^{k} \left( \mu_{ij}^{L \theta_j} \right)^{a_{ij}} \right] \left[ \prod_{j=1}^{k} \left( \mu_{ij}^{R \theta_j} \right)^{a_{ij}} \right] = \left[ \sqrt{1 - \prod_{j=1}^{k} \left( 1 - \left( \mu_{ij}^{L \theta_j} \right)^{a_{ij}} \right)^2} \right] \left[ \sqrt{1 - \prod_{j=1}^{k} \left( 1 - \left( \mu_{ij}^{R \theta_j} \right)^{a_{ij}} \right)^2} \right]
\]

and

\[
\mathcal{C}_n^k \left( \bigotimes_{j=1}^{k} (\tilde{P}_{ij})^{a_{ij}} \right) = \left[ \prod_{i=1}^{k} \sqrt{1 - \left( \prod_{j=1}^{k} \left( \mu_{ij}^{L \theta_j} \right)^{a_{ij}} \right)^2} \right] \left[ \prod_{i=1}^{k} \sqrt{1 - \left( \prod_{j=1}^{k} \left( \mu_{ij}^{R \theta_j} \right)^{a_{ij}} \right)^2} \right]
\]

then we obtain

\[
\frac{1}{\mathcal{C}_n^k} \left( \bigotimes_{j=1}^{k} (\tilde{P}_{ij})^{a_{ij}} \right)
\]

\[
\left[ \prod_{i=1}^{k} \sqrt{1 - \left( \prod_{j=1}^{k} \left( \mu_{ij}^{L \theta_j} \right)^{a_{ij}} \right)^2} \right] \left[ \prod_{i=1}^{k} \sqrt{1 - \left( \prod_{j=1}^{k} \left( \mu_{ij}^{R \theta_j} \right)^{a_{ij}} \right)^2} \right]
\]

Therefore,
Further, it can be easily deduced that this operator also satisfies the property of idempotency, boundedness, monotonicity etc., for IVPFNs. Apart from these, we have deduced some of the existing operators from our proposed one as follows for different value of $k$:

(1) When $k = 1$, $\text{IVPFWMSM}_{\omega}^{(k)}$ operator reduces to the following form:
IVPFWMSM\(\alpha_i^1\)(\(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n\))

\[
\sqrt[\frac{1}{m}]{\prod_{i \in S_k, j \in S_k} \left(1 - \left(\frac{1}{\mu_{\mu_i} - \mu_{\mu_j}}\right)^2\right)} \cdot \sqrt[\frac{1}{m}]{\prod_{i \in S_k, j \in S_k} \left(1 - \left(\frac{1}{\mu_{\mu_i} - \mu_{\mu_j}}\right)^2\right)}
\]

\[
\left(\frac{1}{m} \prod_{i \in S_k, j \in S_k} \left(1 - \left(\frac{1}{\mu_{\mu_i} - \mu_{\mu_j}}\right)^2\right)\right)^{\frac{1}{m}}
\]

\[
\left(\prod_{i \in S_k, j \in S_k} \left(1 - \left(\frac{1}{\mu_{\mu_i} - \mu_{\mu_j}}\right)^2\right)\right)^{\frac{1}{m}}
\]

When \(k = 2\), IVPFWMSM\(\alpha_i^1\) operator reduces to the IVPF Bonferoni mean (IVPFBM) operator as follows:
IVPFWMSM\(^{(2)}\) \((\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n)\)

\[= \left[ \prod_{1 \leq i < j \leq n} \left( \left( \prod_{j=1}^{n} (\mu_j^L)^{a_j} \right)^2 \right) \prod_{1 \leq i < j \leq n} \left( \left( \prod_{j=1}^{n} (\nu_j^L)^{a_j} \right)^2 \right) \right]^{1/2}, \]

\[\left[ \prod_{1 \leq i < j \leq n} \left( \left( \prod_{j=1}^{n} (\mu_j^R)^{a_j} \right)^2 \right) \prod_{1 \leq i < j \leq n} \left( \left( \prod_{j=1}^{n} (\nu_j^R)^{a_j} \right)^2 \right) \right]^{1/2} \]

IVPFWBM\(^{(1,1)}\) \((\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n)\)

(3) When \(k = n\), IVPFWMSM\(^{(k)}\) operator becomes

IVPF weighted geometric mean (IVPFWGM) operator as follows:

IVPFWMSM\(^{(n)}\) \((\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n)\)

\[= \left[ \prod_{1 \leq i < j \leq n} \left( \left( \prod_{j=1}^{n} (\mu_j^L)^{a_j} \right)^2 \right) \prod_{1 \leq i < j \leq n} \left( \left( \prod_{j=1}^{n} (\nu_j^L)^{a_j} \right)^2 \right) \right]^{1/2}, \]

\[\left[ \prod_{1 \leq i < j \leq n} \left( \left( \prod_{j=1}^{n} (\mu_j^R)^{a_j} \right)^2 \right) \prod_{1 \leq i < j \leq n} \left( \left( \prod_{j=1}^{n} (\nu_j^R)^{a_j} \right)^2 \right) \right]^{1/2} \]

The above defined operator has been illustrated with a numerical example as follows:

**Example 2.** Consider a four IVPFNs \(\tilde{p}_1 = ([0.3, 0.4], [0.6, 0.7])\) and \(\tilde{p}_2 = ([0.4, 0.5], [0.3, 0.4])\), and \(\omega = (0.2, 0.1, 0.3, 0.4)\) is the weight of its. Then, without loss of generality, we assume \(k = 2\) and by IVPFWWSM\(^{(k)}\) operator, we get
be the set of attributes with \( \omega \), \( \sum_{j=1}^{n} \omega_j = 1 \). Suppose that a decision maker has evaluated these alternatives under the given attributes and present their rating values in terms of IVPFNs \( \tilde{r}_{ij} \) where \( [\mu^L_{ij}, \mu^R_{ij}] \) and \( [v^L_{ij}, v^R_{ij}] \) represents the degree that alternative \( A_i \) satisfies and doesn’t satisfy to the attribute \( G_j \) such that \( [\mu^L_{ij}, \mu^R_{ij}] \subseteq [0,1] \) and \( [v^L_{ij}, v^R_{ij}] \subseteq [0,1] \), \( \mu^L_{ij} + \mu^R_{ij} \leq 1 \), \( i = 1, 2, \ldots, m \), \( j = 1, 2, \ldots, n \). The collective information of all are summarized in the decision matrix \( R = (r_{ij})_{mn} \). Then, the various steps of the proposed approach has been summarized as follows, for solving the decision making problems for potential evaluation of emerging technology commercialization under IVPF environment.

4. MODEL FOR MADM WITH IVPF INFORMATION

In this section, we shall utilize the proposed operator to MADM under IVPF environment. For it, the following assumptions or notations are used to represent the MADM problems for potential evaluation of emerging technology commercialization.

Let \( A = \{A_1, A_2, \ldots, A_n\} \) be a discrete set of alternatives, and \( G = \{G_1, G_2, \ldots, G_n\} \) be the set of attributes with \( \omega = (\omega_1, \omega_2, \ldots, \omega_n) \) be its weight vector, where \( \omega_j \in [0,1], j = 1, 2, \ldots, n \), \( \sum_{j=1}^{n} \omega_j = 1 \). Suppose that a decision maker has evaluated these alternatives under the given attributes and present their rating values in terms of IVPFNs \( \tilde{r}_{ij} \) where \( [\mu^L_{ij}, \mu^R_{ij}] \) and \( [v^L_{ij}, v^R_{ij}] \) represents the degree that alternative \( A_i \) satisfies and doesn’t satisfy to the attribute \( G_j \) such that \( [\mu^L_{ij}, \mu^R_{ij}] \subseteq [0,1] \) and \( [v^L_{ij}, v^R_{ij}] \subseteq [0,1] \), \( \mu^L_{ij} + \mu^R_{ij} \leq 1 \), \( i = 1, 2, \ldots, m \), \( j = 1, 2, \ldots, n \). The collective information of all are summarized in the decision matrix \( R = (r_{ij})_{mn} \). Then, the various steps of the proposed approach has been summarized as follows, for solving the decision making problems for potential evaluation of emerging technology commercialization under IVPF environment.

Thus, we get

\[
IVPWFSM(2) \left( \tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4 \right) = \bigoplus \left( \tilde{a}_1 \circ \tilde{a}_2 \circ \tilde{a}_3 \circ \tilde{a}_4 \right)
\]

where \( \omega_j \in [0,1], j = 1, 2, \ldots, n \), \( \sum_{j=1}^{n} \omega_j = 1 \). Suppose that a decision maker has evaluated these alternatives under the given attributes and present their rating values in terms of IVPFNs \( \tilde{r}_{ij} \) where \( [\mu^L_{ij}, \mu^R_{ij}] \) and \( [v^L_{ij}, v^R_{ij}] \) represents the degree that alternative \( A_i \) satisfies and doesn’t satisfy to the attribute \( G_j \) such that \( [\mu^L_{ij}, \mu^R_{ij}] \subseteq [0,1] \) and \( [v^L_{ij}, v^R_{ij}] \subseteq [0,1] \), \( \mu^L_{ij} + \mu^R_{ij} \leq 1 \), \( i = 1, 2, \ldots, m \), \( j = 1, 2, \ldots, n \). The collective information of all are summarized in the decision matrix \( R = (r_{ij})_{mn} \). Then, the various steps of the proposed approach has been summarized as follows, for solving the decision making problems for potential evaluation of emerging technology commercialization under IVPF environment.
Step 1. Aggregate the different preferences \( r_{ij} (j=1,2,\ldots,n) \) of the alternative \( A_i \) under attribute \( G_j \) to overall value \( r_i (i=1,2,\ldots,m) \) by using

\[
\tilde{r}_i = \left( \left[ \mu^L_i, \mu^R_i \right], \left[ v^L_i, v^R_i \right] \right)
\]

\[
= \text{IVPFWMSM}^{(k)}_{w} \left( \tilde{r}_i, \tilde{r}_2, \ldots, \tilde{r}_m \right)
\]

\[
= \sum_{i=1}^{n} \left( \frac{1}{C_n^k} \prod_{j=1}^{k} \left( \frac{1}{\mu_j^L} + \frac{1}{\mu_j^R} \right) \right \uplus \prod_{j=1}^{k} \left( \frac{1}{v_j^L} + \frac{1}{v_j^R} \right) \right)
\]

Step 2. Compute the score value of the aggregated IVPFN \( r_i \) by using

\[
S(r_i) = \frac{1}{4} \left[ 1 + (\mu_i^L)^2 - (v_i^L)^2 + 1 + (\mu_i^R)^2 - (v_i^R)^2 \right].
\]

Step 3. Rank the alternatives \( A_i (i=1,2,\ldots,m) \) based on the comparison laws defined in Definition 6 and hence select the best one(s).

Step 4. End.

5. NUMERICAL EXAMPLE AND COMPARATIVE ANALYSIS

The above mentioned approach has been illustrated with a numerical example as follows.

5.1. NUMERICAL EXAMPLE

\[
\begin{align*}
\tilde{r}_1 &= \left( \left[ 0.3, 0.4 \right], \left[ 0.2, 0.3 \right] \right),
\tilde{r}_2 &= \left( \left[ 0.5, 0.6 \right], \left[ 0.2, 0.3 \right] \right),
\tilde{r}_3 &= \left( \left[ 0.1, 0.2 \right], \left[ 0.3, 0.4 \right] \right),
\tilde{r}_4 &= \left( \left[ 0.5, 0.6 \right], \left[ 0.2, 0.3 \right] \right),
\tilde{r}_5 &= \left( \left[ 0.3, 0.4 \right], \left[ 0.2, 0.3 \right] \right).
\end{align*}
\]

\[
\tilde{R} = \left( \begin{array}{cccc}
0.3, 0.4 & 0.2, 0.3 & 0.1, 0.2 & 0.3, 0.4 \\
0.5, 0.6 & 0.2, 0.3 & 0.3, 0.4 & 0.5, 0.6 \\
0.1, 0.2 & 0.3, 0.4 & 0.1, 0.2 & 0.3, 0.4 \\
0.5, 0.6 & 0.2, 0.3 & 0.3, 0.4 & 0.3, 0.4 \\
0.3, 0.4 & 0.2, 0.3 & 0.3, 0.4 & 0.2, 0.3 \\
\end{array} \right)
\]

Consider a MADM problem in which the emerging technology enterprises (ETE) \( A_i (i=1,2,3,4,5) \) are considered to be the five possible alternatives which are going to be evaluated under the set of four different attributes namely, 1) \( G_1 \) is “the technical advancement”; 2) \( G_2 \) is “the potential market and market risk”; 3) \( G_3 \) is “the industrialization infrastructure, human resources and financial conditions”; 4) \( G_4 \) is “the employment creation and the development of science and technology”. In order to find the best possible ETE, an expert has evaluated these five enterprises and summarized their rating values in terms of IVPF decision matrix \( \tilde{R} \) as follows.

Step 1. Without loss of generality, take \( k=2 \) and the weight vector of the attribute is \( \omega = (0.2, 0.1, 0.3, 0.4)^T \).

Utilize IVPFWMSM\(^{(k)}_{w}\) operator to compute the overall values \( \tilde{r}_i, i=1,2,\ldots,5 \) of the ETE \( A_i (i=1,2,3,4,5) \) as:
The score values of \( \hat{r}_i, i = 1, 2, \ldots, 5 \) is computed as

\[
S(\hat{r}_1) = 0.7613, \quad S(\hat{r}_2) = 0.8304, \quad S(\hat{r}_3) = 0.8034, \quad S(\hat{r}_4) = 0.7725, \quad S(\hat{r}_5) = 0.8943.
\]

**Step 2.** The scores values of \( \hat{r}_i, i = 1, 2, \ldots, 5 \) are computed as

\[
S(\hat{r}_1) = 0.7613, \quad S(\hat{r}_2) = 0.8304, \quad S(\hat{r}_3) = 0.8034, \quad S(\hat{r}_4) = 0.7725, \quad S(\hat{r}_5) = 0.8943.
\]

**5.2. COMPARATIVE ANALYSIS**

In order to compare the proposed approach results with one of the existing approaches result to validate the stability of the proposed work. For it, we have considered the weighted averaging and geometric aggregation operators under IVPF environment named as IVPFWA and IVPFWG as proposed by Garg [28] and are defined as follows.

**Definition 7** [28]. Let \( \hat{p}_j = \left( \mu_j^l, \mu_j^u, v_j^l, v_j^u \right) \) \( (j = 1, 2, \ldots, n) \) be a collection of IVPFNs, and \( \omega = \left( \omega_1, \omega_2, \ldots, \omega_n \right)^T \) be the normalized weight vector. Then IVPF weighted average (IVPFWA) and IVPF weighted geometric (IVPFWG) are defined as

\[
\text{IVPFWA}_\omega (\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n) = \left[ \prod_{j=1}^{n} \left( \frac{1}{1-(\mu_j^l)^2} \right)^{\omega_j} \right] \left[ \prod_{j=1}^{n} \left( \frac{1}{1-(\mu_j^u)^2} \right)^{\omega_j} \right] \left[ \prod_{j=1}^{n} \left( \frac{1}{1-(v_j^l)^2} \right)^{\omega_j} \right] \left[ \prod_{j=1}^{n} \left( \frac{1}{1-(v_j^u)^2} \right)^{\omega_j} \right]
\]

and

\[
\text{IVPFWG}_\omega (\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n) = \left[ \prod_{j=1}^{n} \left( \frac{1}{1-(\mu_j^l)^2} \right)^{\omega_j} \right] \left[ \prod_{j=1}^{n} \left( \frac{1}{1-(\mu_j^u)^2} \right)^{\omega_j} \right] \left[ \prod_{j=1}^{n} \left( \frac{1}{1-(v_j^l)^2} \right)^{\omega_j} \right] \left[ \prod_{j=1}^{n} \left( \frac{1}{1-(v_j^u)^2} \right)^{\omega_j} \right]
\]

If we aggregate the considered data \( \hat{r}_j : j = 1, 2, 3, 4 \) as given in the matrix \( \hat{R} \) by using these operators then their corresponding aggregated IVPF values are summarized in Table 2.

**Table 2.** The aggregating results of the ETE by the IVPFWA & IVPFWG operators

<table>
<thead>
<tr>
<th></th>
<th>IVPFWA [28]</th>
<th>IVPFWG [28]</th>
</tr>
</thead>
<tbody>
<tr>
<td>A₁</td>
<td>(0.2484, 0.3444), [0.1625, 0.2685])</td>
<td>(0.2072, 0.3157), [0.2025, 0.2969])</td>
</tr>
<tr>
<td>A₂</td>
<td>(0.3751, 0.4742), [0.3492, 0.4542])</td>
<td>(0.3478, 0.4507), [0.3943, 0.4930])</td>
</tr>
<tr>
<td>A₃</td>
<td>(0.3796, 0.4778), [0.2352, 0.3474])</td>
<td>(0.3259, 0.4315), [0.2981, 0.3936])</td>
</tr>
<tr>
<td>A₄</td>
<td>(0.3145, 0.4109), [0.1813, 0.2891])</td>
<td>(0.2595, 0.3699), [0.2638, 0.3568])</td>
</tr>
<tr>
<td>A₅</td>
<td>(0.6222, 0.7246), [0.2352, 0.3366])</td>
<td>(0.6042, 0.7050), [0.2456, 0.3445])</td>
</tr>
</tbody>
</table>

The score values of these aggregated numbers are computed by the Definition 2.7 in Ref. [28] and their results are summarized in Table 3 along with their ranking order. Thus, from this study, it has been observed that the best alternative remains the same i.e., A₅ while other alternatives have changed it slightly depending on the aggregation operator used.

**Table 3: Overall score value and rating values of ETE**

<table>
<thead>
<tr>
<th></th>
<th>Overall rating values of ETE</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>Existing</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A₁</td>
<td>(0.7009, 0.7659)</td>
<td></td>
</tr>
<tr>
<td>A₂</td>
<td>(0.7830, 0.8307)</td>
<td></td>
</tr>
<tr>
<td>A₃</td>
<td>(0.7685, 0.8196)</td>
<td></td>
</tr>
<tr>
<td>A₄</td>
<td>(0.7214, 0.7847)</td>
<td></td>
</tr>
<tr>
<td>A₅</td>
<td>(0.8833, 0.9173)</td>
<td></td>
</tr>
</tbody>
</table>

Step 3. Based on the Definition 6, we rank the given ETE \( A_i \ (i = 1, 2, 3, 4, 5) \) and get \( A_5 > A_2 > A_3 > A_4 > A_1 \), and thus the most desirable ETE is \( A_5 \).

However, in order to reflect the influence of the proposed approach with different values of \( k \), we have implemented the proposed approach for different \( k \)'s and their overall rating values along with their ranking are summarized in Table 1. From this table, it has been concluded that the final ranking order of the alternatives changes slightly with the change of the decision maker preferences.
The following observations have been noted from this comparative study which is summarized as follows:

1) The existing operators proposed by Garg [28], is independent of the preference of the decision maker (DM) attitudinal character while proposed one does have. Furthermore, the proposed operators are monotonically decreasing with this parameter. Thus, based on it, if DM wants to take a decision either on the basis of risk preference or aversion, then they can choose the value of as large as or as small as possible respectively. Therefore, this feature will give the DM flexibility to take a decision more precisely according to their desired goals.

2) The major advantage of the proposed approach is to consider the interrelationship between the different pairs of input arguments, while the existing method [28] doesn’t. Also, it has been observed that several existing operators such as averaging, geometric, Bonferroni mean etc., are deduced from the proposed operators. Therefore, it is more generalized and flexible in the phase of the information aggregation process.

6. CONCLUSION

The present paper deals to handle the uncertainties in the data during the decision-making process. For it, the IVPF set environment have been considered, which is generalized than the other existing theories, for rating the different preferences of the object. Further, by incorporating the idea of the interrelationship between the input arguments, the present paper developed some new aggregation operators namely and to solve the MADM problems by using IVPF operation laws and MSM operator. Various properties and some of the special cases of these operators have been investigated in details and found that existing operators are the special cases of it. Finally, based on these proposed operators, we have presented a MADM approach under the IVPF set environment and validate with an illustrative example. A comparative study with some of the existing approaches supports the efficiency of the proposed approach. In the future, we shall extend the proposed operators to the different fields as well as to propose some new aggregation operators under the uncertain environment [43-55].

ACKNOWLEDGMENT

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