High Torque Density Macro-Scale Electrostatic Rotating Machines: Electrical Design, Generalized d-q Framework & Demonstration

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Abstract—The use of advanced dielectric liquids and manufacturing techniques for enhanced surface area per unit volume, along with other multiplicative gains, is enabling macro-scale electrostatic machines with competitive torque density. This emergence prompts the need for circuit modeling that guides machine design and drive controls, rooted in the canon of well-established electromagnetic machinery practices. Prior dielectric and manufacturing innovations are combined here with a newly developed unified electrostatic machine dq-axis framework to form a machine that demonstrates industrial utility. Design, equivalent circuits and performance are validated by an experimental prototype intended for low speed direct drive servo actuators. A prototype electrostatic machine was constructed entirely of aluminum and printed circuit boards and possesses active material torque densities ≥ 1.4 N·m/kg and ≥ 2.65 N·m/L without the need for forced air or liquid cooling. Additional features include near zero loss at stall and low torque ripple.

Keywords— Electrostatic machines, d-q equivalent circuits, electric machines, capacitance, medium voltage

 NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbols</th>
<th>Meaning</th>
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<tbody>
<tr>
<td>j, a</td>
<td>( e^{j2\pi/3} ), constant complex vector</td>
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<tr>
<td>v, i</td>
<td>transient voltage [V] and current [A]</td>
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<tr>
<td>( v_r, i_r )</td>
<td>steady state voltage [V] and current [A]</td>
</tr>
<tr>
<td>( d_s, d_r )</td>
<td>duty ratio of stator and rotor traces</td>
</tr>
<tr>
<td>N</td>
<td>number of rotor plates</td>
</tr>
<tr>
<td>C, G</td>
<td>capacitance [F], conductance [S]</td>
</tr>
<tr>
<td>r</td>
<td>resistance [( \Omega )] or radius [m]</td>
</tr>
<tr>
<td>( P, \lambda )</td>
<td>pole number, angular period [rad]</td>
</tr>
<tr>
<td>( r_i, r_o, g )</td>
<td>active inner radius, outer radius, gap [m]</td>
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<tr>
<td>( \varepsilon_0, \varepsilon_x )</td>
<td>vacuum and material permittivities [F/m]</td>
</tr>
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<td>( \gamma )</td>
<td>torque angle [rad]</td>
</tr>
<tr>
<td>( \omega_r, \omega_e )</td>
<td>rotor position [rad] &amp; angular speed [rad/s]</td>
</tr>
<tr>
<td>( T_e, P_e )</td>
<td>shaft torque [N·m] and power [W]</td>
</tr>
<tr>
<td>( i_{lk}, Q )</td>
<td>leakage current [A], charge [C]</td>
</tr>
</tbody>
</table>

Table 1: Multiplicative Gains to Close the Electric – Magnetic Macro – scale Torque Gap

<table>
<thead>
<tr>
<th>Multiplier</th>
<th>Dielectric liquids for higher electric shear stress [2] ~ 300</th>
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<tbody>
<tr>
<td></td>
<td>High surface area via cascaded gaps [3]–[5] ~ 10</td>
</tr>
<tr>
<td></td>
<td>Electrode optimization [6] ~ 3</td>
</tr>
<tr>
<td></td>
<td>Materials &amp; manufacturing [6], [7] ~ 3</td>
</tr>
<tr>
<td></td>
<td>Field torque utilization [8] ~ 5</td>
</tr>
<tr>
<td></td>
<td>Net Multiplicative Gain ~ 1.35 \times 10^3</td>
</tr>
</tbody>
</table>

This leap in scale and performance for electrostatic machines now prompts development of macro scale models using the electromagnetic d-q canon as a guide. Prior work has focused on three specific machines 1) switched and synchronous elstactance (variable capacitance), 2) induction machines, and more recently 3) synchronous machines, each with their own model. Equivalent circuit modeling has been the approach of choice for synchronous elstactance machines [9], while induction machines have relied more on field models of traveling potential waves [10], [11]. The traveling wave induction machine model was further developed into a two phase \((\alpha - \beta)\) model with slip [12]. Separately excited synchronous machine dq-axis models appear in power systems as equivalencies of power electronic inverters or synchronous condensers [13], but are not developed from the physical
capacitances of machines to provide torque production insight. The primary contributions of this paper are divided among the following sections:

**Section II – Electrostatic Machine Form and Design.**
This section presents a form factor for an industrial electrostatic machine and develops the basic design equations that govern its torque capability.

**Section III – Electrostatic dq-Axis Model.**
This section derives the general dq-axis model for a salient-pole-separately-excited synchronous machine, the corresponding torque equation containing field and elastance (saliency) components and solves for maximum torque per volt. The primary equivalent lumped elements concerning power conversion are identified. This section also establishes duality between electrostatic and magnetic dq-axis models to form a basis for future drive development and charge oriented control.

**Section IV – Experimental Results & Demonstration.**
This section presents an experimental prototype machine and validates the design/modeling equations. Active material torque densities ≥ 1.4 N-m/kg and ≥ 2.65 N-m/L are demonstrated. A peak torque of 7.3 N-m only requires 5 W input power at stall and can be sustained indefinitely with no need for forced air or liquid cooling. This level of performance is an order of magnitude higher than prior macro-scale work spanning years 1933-2016 in reference [2] and documented with numeric metrics in Table 1 of reference [6].

II. ELECTROSTATIC MACHINE FORM & DESIGN

A. Proposed Machine Topology

An axial-flux style three-phase synchronous electrostatic machine using printed circuit boards (PCB) as the active torque producing components is proposed and shown in Fig. 1(a). The machine consists of a cascaded stack of 7 stator boards and 6 rotor boards, all of which are double sided and two layers except for the stators at the ends of the stack. In practice any number could be stacked to achieve a given output but the proposed number constitutes 12 cascaded axial gaps and demonstrates the modular design well. The arrangement of poles (traces) on the PCBs is evident in Fig. 1(b), where a two phase rotor is excited with bipolar DC potential and a 3-phase stator is excited with AC potential. This is the electrostatic dual of the magnetic wound field synchronous machine. Balanced three phase stator voltages are fed from the rear of the machine to the stator board while rotor voltage is supplied through slip rings and a hollow shaft. Any number of inductive or capacitive approaches could be used to excite rotor brushlessly if desired. The stator-rotor gap is filled with a dielectric liquid possessing high permittivity, high breakdown strength, low conductivity and low viscosity [2] and an aluminum shell with shaft seals retains the liquid.

B. Design Approach

Traditionally, the design of either electromagnetic machines or electrostatic machines has been done by calculating the torque producing inductance [14] or capacitance. However, the solution of an electrostatic field governed by Laplace’s equation is usually complicated and often analytically unavailable for multi-conductor discontinuous surfaces. A generic and easy to implement design approach is proposed here to circumvent such difficulty. It is derived from the ideal traveling wave model assuming the stator and rotor surface are excited with pure sinusoidal waves \( V_s \) and \( V_r \) (peak) as in Fig. 1c and is then applied to Fig. 1(b) [10]. For two waves enclosing a gap of thickness of \( g \) and permittivity of \( \varepsilon_g \), J. R. Melcher showed that the torque per unit area at radius \( r \) is simply (1).
\[
\tau_z = -\frac{\varepsilon_g P^2}{2r \sinh \left( \frac{Pg}{r} \right)} V_r \cos(\gamma) \tag{1}
\]

The angle \(\gamma\), traditionally known as the torque angle, will be explored later in Section III. The maximum achievable torque is calculated by setting \(\gamma\) to \(\pi\) and dropping the cosine term. Now referring to Fig. 1b, the total torque generated in \(\gamma\) cascaded rotor plates (2\(N\) fluid gaps) may be obtained by integrating over the active area,

\[
T_{e,\max} = 2N \int_{r_1}^{r_2} \int_0^{2\pi} \tau_z |_{\gamma = \pi} \cdot r \, d\theta \cdot dr \tag{2}
\]

where \(r_1\) and \(r_2\) are the inner and outer radii of the overlapping active facing area of the stator and rotor. In the ideal case, the above integration can be evaluated directly since \(V_s\) and \(V_r\) are constant. However, in reality, \(V_s\) and \(V_r\) are facilitated by symmetrically arranged discrete traces (poles) whose dimensions are a function of the radius for two reasons:

1. Because the potential drop across two adjacent traces is independent of the radius \(r\), a constant clearance (distance, not angle), as the magnified area shows in Fig.1b, in the radial direction is necessary for optimal usage of the space;
2. The potential over the circumferential span of the traces/poles is constant as shown in Fig.2c. This is analogous to the concentrated winding in traditional magnetic machines.

Naturally, it leads to evaluating the fundamental component of the stator and rotor potential distribution. To avoid finding the exact solution of the potential distribution between the traces, it is approximated as a linear distribution in Fig.2 (dashed line). It will be shown later that this approximation is accurate enough for design purposes.

\[\begin{align*}
V_{s}(t) & = v_{cs}(t) + v_{cs}(t) \theta + v_{cs}(t) - v_{cs}(t) \\
V_{r}(t) & = v_{bs}(t) + v_{bs}(t) \theta + v_{bs}(t) - v_{bs}(t) \\
\end{align*}\]

and via \(d_s\) for the rotor field:

\[\begin{align*}
v_{sf} - v_{-f} & = v_{sf} - v_{-f} \\
v_{sf} & = \frac{1}{2} + \frac{1}{2} d_s \frac{d_s}{d_r} \theta \\
v_{-f} & = \frac{1}{2} - \frac{1}{2} d_s \frac{d_s}{d_r} \theta \\
\end{align*}\]

By applying Fourier decomposition on these waveforms, the fundamental components of the stator and rotor excitation are obtained as (4) and (5).

\[\begin{align*}
V_s & = \frac{9\sqrt{3}}{2\pi^2} \cos \left( \frac{\pi d_s}{6} + \frac{\pi}{6} \right) V_{sp} \tag{4} \\
V_r & = \frac{8}{\pi^2} \cos \left( \frac{\pi d_r}{2} \right) V_{rp} \tag{5} \\
\end{align*}\]

The maximum torque of a given design is calculated by substituting (1) and (3) - (5) back into (2), provided that parameter inputs \(V_{sp}, V_{rp}, E_{\max}, r_1, r_2, P, g\) and \(\varepsilon_g\) are known or swept over a feasible space.

**Remark 1:** It seems reasonable to substitute (3) into (4) and (5) to obtain even simpler relations:

\[\begin{align*}
V_s & = \frac{3r}{\pi} \cos \left( \frac{\pi d_s}{6} + \frac{\pi}{6} \right) E_{\max} \tag{4'} \\\nV_r & = \frac{4r}{\pi} \cos \left( \frac{\pi d_r}{2} \right) E_{\max} \tag{5'} \\
\end{align*}\]

both of which imply to minimize the duty ratios or trace width to maximize the fundamental components. However, the peak
voltage $V_{sp}$ and $V_{rp}$ in this case may be far beyond the capability of the drive circuitry for a fixed electric field. Usually, this works in the opposite way, i.e. first determine $V_{sp}$ and $V_{rp}$ based on the power electronics, especially for medium voltage, and then find the optimal duty ratios according to (4) and (5). Figure 3 plots $V_{s}/V_{sp}$ and $V_{r}/V_{rp}$ as functions of $d_s$ and $d_r$ respectively. It shows the fundamental components actually increase as the traces become wider with fixed peak voltages.

Remark 2: Observing that the potential falls off the trace edge dramatically and tends to stay in the lowest energy state, the linear distribution approximation is believed to result in an over estimation of the fundamental components. Two extreme cases, under and over estimations, are also included in Figure 3. The first one fills the gap region with zero potential and results in a lower bound of the torque production. The second one keeps up the trace potential to the gap region and gives an upper bound. The linear approximation proposed here is tighter to the upper bound.

Remark 3: Lastly, the torque derived in this section only corresponds to the field torque in Section III. Unlike the large permeability contrasts in magnetic machines between back iron, air or permanent magnets, the proposed machine does not possess a permittivity contrast with spatial dependency. A saliency ratio close to one is the result and this machine is therefore a dual of the round rotor synchronous machine.

III. ELECTROSTATIC MACHINE DQ-AXIS MODEL

A dq-axis model can be derived step by step for the three phase electrostatic machines following R. H. Park’s approach [15], [16] and D. W. Novotny and T. A. Lipo’s conventions defined in [17]. This model will create an intuitive bridge from the field design just established and the operation of the electrostatic machine within the context of a drive. A general equivalent circuit that encompasses field, saliency and induction torque will be derived along with the corresponding torque equation. This general model can be reduced to any specific machine incarnation, e.g. a synchronous machine.

A. Terminal Equations

Similar to a traditional electromagnetic machine, a three phase AC electrostatic synchronous machine equipped with a DC field excitation and dampers can be abstracted as a set of three phase stator electrodes, two out of phase rotor electrodes and dielectric damping insulations, the latter two are shown separately in Fig. 4(a) and 4(b). Figure 4(a) is an embodiment of the separately excited salient pole synchronous machine (similar to Fig. 1(b)) while Fig. 4(b) is an induction machine utilizing a coating of some conductivity and relative permittivity [10]. Alternatively, the dampers could take the form of discrete electrodes connected together via discrete resistors. In practice, a damper coating/surface could be placed on the poles of a synchronous machine, but they are drawn separately for two reasons: i) modern drive technologies, i.e. field oriented control and direct torque and flux control, do not require a synchronous machine physically equipped with dampers [17]; and ii) the dampers are included for a more general derivation, the result of which can also be reduced to the dq-axis model of the electrostatic induction machine. In duality to an electromagnetic machine, an electrostatic machine is voltage driven, the current is the quantity used to regulate the relevant voltages and the charge residing in the electrodes varies continuously. The general machine has nine terminal equations that are contained in (6) and (7) (three stator terminals, two field terminals, four damping terminals). Here subscripts + and − stand for positive and negative direct current electrodes, $lk$ for the leakage current due to the nonzero conductivity of the dielectric medium and $p$ is the derivative operator. Notice that the damping terminals are not physically presented in Fig. 1(b). Nevertheless, the potential traveling wave generated by the polarized (induced) charges can be equivalently represented by the two phase damping terminals. For the sake of brevity, the following derivation will be based on (6), initially neglecting the dampers. However, the symmetry of problem allows the dampers to be reintroduced into the final results by inspection.

The derivation process mirrors the circuit derivation of a general electromagnetic machine. However, rather than inductances, the derivation here begins from capacitances. Fig. 5 shows the capacitance coupling between the terminals without the dampers (it would be combinatorial $\binom{9}{2} + 9 = 45$ coupling capacitors with the dampers included!). The relationship between charge or leakage current at/through each terminal and the terminal voltages can be expressed as the matrix (8) using the coupling capacitances or conductance defined on the right hand side of Fig. 5. This matrix yields the resistive and capacitance characteristics of the machine given the current field and electric field are both governed by Laplace’s equation.
\[\begin{align*}
\mathbf{\Phi} &= \begin{bmatrix}
\varphi_{1s} \\
\varphi_{2s} \\
\varphi_{3s} \\
\varphi_{4s} \\
\varphi_{5s} \\
\varphi_{6s} \\
\varphi_{7s} \\
\varphi_{8s}
\end{bmatrix} = \begin{bmatrix}
\Phi_{1s} \\
\Phi_{2s} \\
\Phi_{3s} \\
\Phi_{4s} \\
\Phi_{5s} \\
\Phi_{6s} \\
\Phi_{7s} \\
\Phi_{8s}
\end{bmatrix} = \begin{bmatrix}
\varphi_{1s} + \varphi_{2s} + \varphi_{3s} + \varphi_{4s} + \varphi_{5s} + \varphi_{6s} + \varphi_{7s} + \varphi_{8s}
\end{bmatrix}
\end{align*}\]

Figure 4. (a) The generalized model of three phase salient pole electrostatic synchronous machine with field excitation; (b) the generalized model of three phase electrostatic round rotor induction machine.

\[
\begin{align*}
\mathbf{i}_{as} &= \mathbf{i}_{rk,as} + p Q_{as} \\
\mathbf{i}_{bs} &= \mathbf{i}_{rk,b} + p Q_{bs}, \text{ field:} \\
\mathbf{i}_{cs} &= \mathbf{i}_{rk,cs} + p Q_{cs} \\
\mathbf{i}_{+f} &= \mathbf{i}_{rk,+f} + p Q_{+f} \\
\mathbf{i}_{-f} &= \mathbf{i}_{rk,-f} + p Q_{-f}
\end{align*}
\] (6)

\[
\begin{align*}
\mathbf{d}-\text{axis damper:} & \quad \mathbf{i}_{+d} = \mathbf{i}_{rk,+d} + p Q_{+d} \\
\mathbf{i}_{-d} &= \mathbf{i}_{rk,-d} + p Q_{-d} \\
\mathbf{i}_{+q} &= \mathbf{i}_{rk,+q} + p Q_{+q} \\
\mathbf{i}_{-q} &= \mathbf{i}_{rk,-q} + p Q_{-q}
\end{align*}
\] (7)

Figure 5. Capacitance/conductance coupling in the three phase separately excited electrostatic synchronous machine without damping terminals.

\[
\begin{align*}
\mathbf{K}_{as} &= \begin{bmatrix}
\mathbf{A}_{as} & \mathbf{A}_{asbs} & \mathbf{A}_{asc} & \mathbf{A}_{as+f} & \mathbf{A}_{as-f} \\
\mathbf{A}_{bsas} & \mathbf{A}_{bs} & \mathbf{A}_{bscs} & \mathbf{A}_{bs+f} & \mathbf{A}_{bs-f} \\
\mathbf{A}_{csas} & \mathbf{A}_{csbs} & \mathbf{A}_{cs} & \mathbf{A}_{cs+f} & \mathbf{A}_{cs-f} \\
\mathbf{A}_{f+as} & \mathbf{A}_{f+bs} & \mathbf{A}_{f+cs} & \mathbf{A}_{f+} & \mathbf{A}_{f-f} \\
\mathbf{A}_{f-as} & \mathbf{A}_{f-bs} & \mathbf{A}_{f-cs} & \mathbf{A}_{f-} & \mathbf{A}_{f-f}
\end{bmatrix} \\
\mathbf{K}_{bs} &= \begin{bmatrix}
\mathbf{v}_{as} \\
\mathbf{v}_{bs} \\
\mathbf{v}_{cs} \\
\mathbf{v}_{+f} \\
\mathbf{v}_{-f}
\end{bmatrix}
\end{align*}
\] (8)

Note: \(\kappa\) stands for the charge \(Q\) or the leakage current \(i_{rk}\) and correspondingly, \(A\) stands for the capacitance \(C\) or the conductance \(G\).

B. Definition of Capacitance/Conductance Profile

The analytical calculation of capacitance/conductance in multi-conductor systems is difficult even for modestly complex arrangements, thus capacitance/conductance profiles are determined according to waveforms obtained through FEA on the axial flux structures in Fig. 1 and Fig. 2 or radial flux configurations in Fig. 4 and Fig. 5. The symmetrical structure of the terminal capacitances and conductances in (8) is defined in Table 2 with a handful parameters (ignoring higher order harmonics not contributing any average torque).

Table 2. Matrix Element Definitions for Machine Model

<table>
<thead>
<tr>
<th></th>
<th>stator self:</th>
<th>rotor field self:</th>
<th>stator mutual:</th>
<th>stator-rotor mutual:</th>
<th>rotor field mutual:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(A_{as} = A_{s0} - A_{s2}\cos(2\theta_r)),</td>
<td>(A_{as} = \frac{+\pi}{3}A_{bs2} + \frac{2\pi}{3}A_{cs})</td>
<td>(A_{bs} = -(A_{sm0} + A_{sm2}\cos(2\theta_r))),</td>
<td>(A_{as} = -(A_{sfm0} + A_{sfm1}\sin(\theta_r))),</td>
<td>(A_{fs} = -A_{fm0})</td>
</tr>
<tr>
<td></td>
<td>(\frac{-2\pi}{3}A_{bs}, \frac{+2\pi}{3}A_{cs})</td>
<td></td>
<td>(\frac{+2\pi}{3}A_{csas}, \frac{+2\pi}{3}A_{asbs})</td>
<td>(\frac{-2\pi}{3}A_{csas}, \frac{+2\pi}{3}A_{asbs})</td>
<td></td>
</tr>
</tbody>
</table>
The following should be noted about the elements defined in Table 2 and the governing equivalent circuit equations:

- Given the current field and electric field are both governed by Laplace’s equation, the capacitance and the conductance profile shape are identical.

- $q \Rightarrow A_{xy}$ means the mutual capacitance or conductance $A_{xy}$ between conductors $x$ and $y$ is obtained by shifting the leftmost capacitance waveform by phase angle $q$.

- $A_{xy} = A_{yx}$ terms are combined to be concise since these capacitances and conductances are symmetrical.

- $0, 1, 2$ in the subscript represents the spatial harmonic order of the given capacitance or conductance.

- Subscripts are self-evident that “s” and “f” are used to distinguish self- and mutual-capacitances.

### C. Complex Vector Form of the Stator Terminal Equations

Assuming the three phase stator leakage resistances are constant (in practice they can vary in a salient machine), the stator terminal equations in (1) can be written in a vector form:

$$i_{abc} = i_{ik,abc} + pQ_{abc}$$  \hspace{1cm} (9)

where again $p$ is the derivative operator, $f_{abc} = [f_{a}; f_{b}; f_{c}]$ and $f$ stands for $i, i_k, v$ and $Q$. The complex vector operator is defined as $a = e^{j2\pi/3}$ yielding (10).

$$f_{abc} = \frac{2}{3}(f_{a} + af_{b} + a^2f_{c})$$  \hspace{1cm} (10)

The complex form (10) has a particular meaning in the machine analysis. In an electromagnetic machine, $i_{abc}$ supports the MMF traveling wave that develops torque. Correspondingly in an electrostatic machine, $v_{abc}$ supports the EMF traveling wave and assumes the torque role.

$$i_{abc} = i_{ik,abc} + pQ_{abc}$$  \hspace{1cm} (11)

Equation (11) results from applying (10) on (9) and implies that $i_{abc}$ and $Q_{abc}$ are also traveling waves since the above differential equation is linear and the source $v_{abc}$ is trigonometric. Furthermore, when the source $v_{abc}$ in the stationary reference frame is traveling with a constant speed in time and a constant amplitude in space, $v_{abc}$, $i_{abc}$ and $Q_{abc}$ are all constant observing from the rotor reference frame. Being a synchronous machine, the analysis naturally falls into the rotor or excitation reference frame. Multiplying both sides of (10) by a rotating vector $e^{j\omega t}$ defines the complex form (12).

$$f'_{abc} = f'_{a} - jf'_{b} = f_{abc} e^{-j\omega t}$$  \hspace{1cm} (12)

Finally the stator terminal (11) turns into (13) where $\omega_r = p\theta_r$.

$$i'_{abc} = i'_{ik,abc} + pQ'_{abc}$$  \hspace{1cm} (13)

Substituting the capacitance/conductance defined in the last subsection in Table 2 into (8) and turning $Q_{abc}$ in (9) into $Q'_{abc}$, and then $Q'_{Q'_{abc}}$, where $v_r = v_s - v_f$, yields (14).

$$k'_{qds} = (A_{qds} + A_{smd})v'_{qds} - \left(\frac{A_{ss2}}{2} + A_{sm2}\right)(E'_{qds})' + jA_{sfm1}v_r$$  \hspace{1cm} (14)

Equations (13) and (14) constitute the intermediate terminal equations in the rotor reference frame for the stator side.

### D. Complex Vector Form of the Rotor Terminal Equation

Since the positive and negative rotor electrodes are just $\pi$ radians out of phase, the rotor terminal equations in (6) can be reduced to (15).

$$i_{fr} = i_{fr} + p(Q_{s+} - Q_{s-}) = i_{fr} + pQ_{fr}$$  \hspace{1cm} (15)

Following a similar charge derivation process as in the last subsection yields (16).

$$k_{fr} = (A_{fr0} + A_{frs})v_{fr} - 3A_{sfm1}v_{fr}^2$$  \hspace{1cm} (16)

Equations (15) and (16) constitute the intermediate terminal equations in the rotor reference frame for the rotor side.

### E. Compact Dynamic Equations

Even though the intermediate dynamic equations look quite close to the ones of the electromagnetic synchronous machine, the capacitance coefficients are cumbersome and may be more elegantly stated. To further simplify the dynamic equations, the condensation of capacitances and fractions together into the following definitions provides a more compact and inviting form factor.

$$\begin{align*}
Q'_{fr} &= \frac{1}{3}Q_{fr} \\
A'_{ds} &= A_{ds} + A_{dt0} + A_{dt0} + A_{dtm} \\
A_{fr} &= \frac{1}{3}(A_{fr0} + A_{frs}) \\
A_{mf} &= A_{mf1} \\
A_{fr} &= A_{fr} - A_{mf}
\end{align*}$$  \hspace{1cm} (17)

Here, prime superscripts denote stator referred quantities. Although there is no direct turns ratio in an electrostatic machine, the convention of using primes remains as a general notation for condensed quantities. Finally, substituting the parameters and variables of (17) into the intermediate results (15) and (16), the terminal equations simplify to (18) and (19).

stator referred dynamic equations in the rotor reference frame

$$\begin{align*}
\frac{1}{3}v'_{qs} &= i_{ik,qs} + pQ'_{qs} + \omega_r v'_r \\
\frac{1}{3}v'_{ds} &= i_{ik,ds} + pQ'_{ds} - \omega_r v'_qs \\
\frac{1}{3}v'_{fr} &= i_{fr} + pQ'_{fr} \\
K'_{qs} &= A_{qs} v'_{qs} + A_{ds} v'_{ds} - A_{mf} v_{fr} \\
K'_{fr} &= A_{fr} v'_{fr} - A_{mf} v_{fr}
\end{align*}$$  \hspace{1cm} (18)

F. Equivalent Circuit

According to the dynamic equation (18) and charge equation (19), the dq-axis equivalent circuit can be drawn as Fig. 6. When it is equipped with dielectric dampers, the corresponding equivalent circuit is expanded to Fig. 7.
Comparing these to the equivalent circuit of the electromagnetic synchronous machine, beautiful dualities are established. Namely, the interchange of parallel and series connections of circuit elements as well as voltage and current sources. Referring to Fig. 8 (where dampers are included), even the $d$-axis “Y” circuit in the electromagnetic synchronous machine (Fig. 8(b)) becomes a $d$-axis “Δ” circuit in the electrostatic synchronous machine (Fig. 8(a)). This is not unexpected since the electrostatic field is a potential referenced field. In the electromagnetic synchronous machine, $L_m$ is the shunt where the stator and rotor current meet and merge, meaning the total sum of the stator and rotor current contribute to the magnetizing flux.
\[ T_e = \frac{3P}{2} \left\{ Q_{qds}^r Q_{dqs} - \frac{3}{2} [Q_{qds} v_{qf} - Q_{dqs} v_{df}] \right\} \]  

where \( P \) stands for the number of poles. Notice that (21) differs from the electromagnetic torque equation by a factor of 2. This is due to the existence of monopoles in electric fields and their absence in magnetic fields. Another way to think about this is by observing Fig. 1. Each phase is a single terminal surface, which couples to other phases/surfaces via displacement current. This is contrary to the phase windings in a magnetic machine, which are two terminals. With the charge defined in (19), the torque equation (21) can be expanded as (22).

\[
T_e = \frac{3P}{2} \left( \frac{C_{ds} - C_{qS}}{2} v_{dS} v_{dS} - \frac{C_{mfs} v_{qS} v_{fr}}{2} \right) - \frac{C_{mds} v_{qS} v_{rd} + C_{mqS} v_{rd} v_{r}}{2}
\]

Equation (22) consists of elastance torque (due to saliency), electrostatic field torque (due to field excitation) and electrostatic induction torque (due to the inclusion of the dampers). Comparing to the reluctance torque, electromagnetic field torque and electromagnetic induction torque, this constitutes another duality. Substituting the capacitance definition (17) back into (22), the three torque components in terms of the terminal capacitances are obtained as (23).

\[
\begin{align*}
T_{es} &= \frac{3P}{2} (C_{ss2} + 2C_{sm2}) v_{qS} v_{dS} \\
T_{fd} &= -\frac{3P}{2} C_{sfr} v_{qS} v_{fr} \\
T_{ind} &= \frac{3P}{2} (C_{sfr} v_{r} v_{r} - C_{adm1} v_{qS} v_{rd})
\end{align*}
\]

These equations provide great insight to the machine designer since they reveal which capacitances are important for a particular type of machine. Specifically, for the salient pole separately excited synchronous machine (without damping terminals) proposed here, \( C_{sfr} \), the fundamental component of the mutual capacitance between the stator and rotor electrodes, \( C_{ss2} \), the second order harmonic of the stator self capacitance, and \( C_{adm2} \), the second order harmonic of the mutual capacitance between stator electrodes all determine the total torque production. The rest of the coupling capacitances in Fig. 5 are irrelevant to the average torque production, but they contribute to the leakage capacitance and affect the power factor looking through the machine terminals, which is out of the scope of this paper. Due to the symmetrical arrangement of the electrodes, only three independent capacitance components (for example, \( C_{as}, C_{ast} \) and \( C_{as+f} \)) need to be calculated while sweeping the machine design space to maximize the torque production.

\[ H. Maximum Torque Per Volt \]

The maximum continuous torque is determined by converting the dynamic equations into their steady state counterparts (24).
The superscript r’s and primes are omitted, and lower case dynamic variables are changed to upper case steady state variables for elucidation. The corresponding phasor diagram is drawn in Fig. 9. The rotor induced back-MMF (a back-current) vector is orthogonal to the rotor charge vector. The rotor charge (without damping terminals), \(T_f\) is solved as (26).

\[
\begin{align*}
\mathbf{I}_q &= \frac{1}{r_2} \mathbf{V}_q + \frac{\mathbf{V}_{ds}}{X_{ds}} - \frac{\mathbf{V}_{fr}}{X_{mf}} \\
\mathbf{I}_d &= \frac{1}{r_2} \mathbf{V}_d - \frac{\mathbf{V}_q}{X_q} \\
\mathbf{I}_{fr} &= \frac{1}{r_{fr}} \mathbf{V}_{fr}
\end{align*}
\] (24)

\(I_{qs}\) is the angle between \(\mathbf{V}_q\) and \(\mathbf{I}_{fr}\). The rotor induced back-MMF (a back-current) vector is orthogonal to the rotor charge vector. The rotor charge is aligned with the \(d\)-axis.

To keep the meaning of the torque angle \(\gamma\) consistent, it is defined as the angle between the \(q\)-axis and the phasor \(\mathbf{V}_{2ds}\) (it is the angle between \(q\)-axis and the phasor \(\mathbf{I}_{qd}\) in the electromagnetic synchronous machines). The torque equation at the steady state can be further written as (25).

\[
T_e = -\frac{3P}{2} \left[ (C_{ss2} + 2C_{sm2})V_s^2 \sin \gamma \cos \gamma + C_{sfm1}V_{fr} \cos \gamma \right]
\] (25)

By differentiating (25) with respect to \(\gamma\) and setting the differential to zero, the maximum torque producing angle is solved as (26).

\[
\gamma_{\text{max}} = -\arcsin \left[ \frac{C_{sfm1}V_{fr} \pm \sqrt{C_{sfm1}^2V_{fr}^2 + 8(C_{ss2} + 2C_{sm2})^2V_s^2}}{4(C_{ss2} + 2C_{sm2})V_s} \right]
\] (26)

The maximum torque is then calculated by substituting (26) into (25). The two resulting angles correspond to the generating and motoring torques respectively. General field and elistance torque components are plotted as functions of the torque angle \(\gamma\) in Fig. 10. Yet another duality emerging in Fig. 10 is the generating mode occurs in the first and fourth quadrants and the motoring mode occurs in the second and third quadrants, while they are switched for the electromagnetic synchronous machine. In the case of a non-salient synchronous machine (without damping terminals), \(C_{ss2}\) and \(C_{sm2}\) vanish to zero and \(T_{frd}\) is the only torque component. The maximum torque occurs when \(\gamma = 0\) or \(\pi\) and the corresponding torque is (27), which agrees with (23).

\[
T_e = \pm \frac{3P}{2} C_{sfm1}V_sV_{fr}
\] (27)

Note that (27) is in exact agreement with equations (1) and (2), where the torque is a function of electrostatic shear over an area. When (1) is integrated over a surface in (2), the integration of the geometric terms in (1) result in the mutual capacitance multiplied by the product of stator and rotor voltage, i.e. the same form as (27).

### IV. EXPERIMENTAL RESULTS

#### A. Synchronous Electrostatic Machine Prototype

A prototype machine was constructed to validate the multiplicative gain principles in Table 1 and the electrical theory presented in the prior sections. This was the goal of the prototype and no specific target specifications or optimization for an application were intended aside from general characteristics that are attractive for servos. The primary servo attributes are the ability to hold full torque during stall indefinitely without cooling, low torque ripple and possess competitive specific torque density relative to typical magnetic machines of similar ratings. Servo applications reflective of these properties fall in the class of “position and hold,” which include actuators, robots, flight surfaces, etc. These applications are inherently low average speed and spend much of their time at stall, thereby lessening the impact of liquid drag. Low speed direct drive applications in the hundreds of rpm and less, e.g. large diameter fans and pumps, may also be a good match.

The design of the machine is heavily influenced by the available materials. To realize the vision in Fig. 1(a), the active materials of the machine are PCB substrate (FR4) and dielectric liquids. (Tradeoffs between the physical characteristics of the dielectric fluids and their impact on machine performance is discussed in [2].) FR4 has a relative permittivity of 4.4 while the liquid is 3.8. The poles/traces are 1 oz copper and are very thin compared to the gap. Given the similarity in permittivity among materials and the desire for a high pole count, the opportunity for saliency torque with these materials is lessened and an approach that entirely focuses on field torque was
pursued. Within the context of a servo, cogging torque is to be avoided, thus this design decision is aligned with that application. As liquid dielectric materials emerge with higher permittivity, future machine designs can exploit them in combination with innovative flux barriers. The analytical design procedure outlined in Section II was used to size a machine. The input constraints on the design include the target torque, operating voltage, diameter and gap. Table 3 summarizes the input constraints on electrical quantities and dimensions of the machine that was prototyped. The torque and diameter were selected based on comparable permanent magnet motors of similar sizing and the gap length was determined by tolerance capabilities. The dielectric fluid HT101 was made available from C-Motive Technologies Inc.

Table 3. Input Design Constraints for Prototype

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diameter</td>
<td>~250 mm</td>
</tr>
<tr>
<td>Gap distance</td>
<td>0.76 mm</td>
</tr>
<tr>
<td>Max Torque</td>
<td>&gt;6 Nm</td>
</tr>
<tr>
<td>DC Rotor Voltage</td>
<td>±3.5 kV (7 kV)</td>
</tr>
<tr>
<td>AC Stator Voltage</td>
<td>3.5 kVpk</td>
</tr>
<tr>
<td>Dielectric Fluid</td>
<td>HT101, $\varepsilon_r = 3.8$, $E_{\text{max}} &gt; 20$ kV/mm, $\sigma &lt; 10^{-10}$ S/m</td>
</tr>
</tbody>
</table>

The complete prototype parameters that resulted from the electrical design process in section II are listed in Table 4 and the resulting prototype is pictured in Fig. 11(a). While the design of the active torque producing elements is presented, the finer details of the surrounding mechanical enclosure is beyond the scope of this paper. However, a CAD profile view is provided Fig. 11(b) to display the interior layout of the machine. Fig. 11(b) and 11(c) can serve as a pictorial representation of the five multiplicative gains (at least those mechanical related aspects) listed in Table 1.

1. **Liquid:** The fluid resides between the stators and rotors to amplify the electric shear stress. The case serves as a fluid retention system and shaft seals prevent leaks around the shaft.
2. **Surface Area:** 6 rotor PCBs are nested within 7 stator PCBs for enhanced surface area per unit volume. No “end turns,” or “back-iron” to take up space.
3. **Electrode Fidelity:** Photos of the stator, Fig. 11(c) and rotor Fig. 11(d) PCB traces illustrate the small feature size afforded by PCB manufacturing to realize machine poles.
4. **Materials & Manufacturing:** Aside from the shaft and bearings, all machine components are plastic, aluminum or composite for light weight.
5. **Separate Excitation:** The rotor is excited with high DC potential. A hallow shaft facilitates the connections of the rotors to slip rings at the rear of the machine.

Other features noticeable from Fig. 11 are an encoder (red) to read the shaft position and high voltage BNC connectors that serve as the power terminals.
Table 4. Final Design Parameters for Prototype

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>radius $r_i, r_o$</td>
<td>47.5, 107.5</td>
<td>mm</td>
</tr>
<tr>
<td>gap thickness $g$</td>
<td>0.76</td>
<td>mm</td>
</tr>
<tr>
<td>stator trace clearance</td>
<td>0.69</td>
<td>mm</td>
</tr>
<tr>
<td>rotor trace clearance</td>
<td>0.80</td>
<td>mm</td>
</tr>
<tr>
<td>pole number $P$</td>
<td>96</td>
<td></td>
</tr>
<tr>
<td>FR4 substrate thickness</td>
<td>2.28</td>
<td>mm</td>
</tr>
<tr>
<td>Active stack length</td>
<td>40</td>
<td>mm</td>
</tr>
<tr>
<td>Case axial length</td>
<td>165</td>
<td>mm</td>
</tr>
<tr>
<td>Case outer diameter</td>
<td>256</td>
<td>mm</td>
</tr>
<tr>
<td>number of stator plates</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>number of rotor plates</td>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>

B. Back MMF, Characteristic Voltage, Equivalent Circuit

In keeping with traditional machine testing, open and short circuit tests were performed during rotation on a dynamometer to determine the equivalent circuit. The first test was to measure the back MMF (back current) of the machine during short circuit, i.e. the dual of the open circuit back EMF of a magnetic machine. The measured back MMF is plotted in Fig. 12. The back MMF is observed to be linear with speed and excitation, as expected. Additionally, the total harmonic distortion (THD) of the back MMF was measured out to the 51st harmonic, and it is less than $< 1\%$. The low THD supports the ability for high performance servo applications with minimal torque ripple.

An open circuit test reveals the characteristic voltage of the machine, which is the dual of a characteristic current of a magnetic machine in short circuit. The measured open circuit peak voltage versus excitation is plotted in Fig. 13, revealing a linear relationship. The slight imbalance among the phases is caused by discrepancies in the parasitic leakage paths for each phase given the construction of the machine.

Figure 13. Measured open circuit peak voltage vs. excitation at 100 rpm.

Table 5: Measured Equivalent Circuit Parameters

<table>
<thead>
<tr>
<th>$C_m$</th>
<th>$C_s$</th>
<th>$r_s$</th>
<th>$r_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0 nF</td>
<td>13.8 nF</td>
<td>1.7 MΩ</td>
<td>50 MΩ</td>
</tr>
</tbody>
</table>

The data in Fig. 12 and Fig. 13 can be used to determine the per phase equivalent circuit in Fig. 14. Over a variety of speeds the short circuit test reveals $C_m$ while the open circuit test reveals $C_s$ and $r_s$. Static tests with DC voltage were used to measure the leakage resistance of the rotor. The measured equivalent circuit diagram of the machine is pictured in Fig. 14, with measured parameters listed in Table 5. The circuit is in perfect topological agreement with the theory developed in Section III, specifically the non-salient version of the synchronous machine resulting from a simplified Fig. 6.

C. Torque Measurements, Stall Loss, Viscous Drag, modeled efficiency while motoring

Torque was measured on a dynamometer to verify the field design equations in Section II and the circuit level torque equations of Section III. The first test examined the torque during fixed excitation and varying torque angle. The test was repeated multiple excitation levels with results plotted in Fig. 15. The measured data correlates with the field design approach, circuit theory and reflects the field torque versus angle characteristic plotted in Fig. 10. The max torque of the machine versus the stator-rotor voltage product was measured while holding the torque angle at $\gamma = -180$ electrical degrees to...
verify the max torque line. The measured torque is shown in Fig. 16 and it correlates with the trajectory predicted by (27). The maximum developed torque measured was 7.3 N-m (6.8 N-m net torque accounting for 0.5 N-m of static friction). The active components of the machine include the rotors, stators, dielectric liquid, shaft and bearings, i.e. everything but the aluminum housing. The active mass is 5.1 kg and the active volume is 2.76 L, which yields torque densities $\geq 1.4$ N-m/kg and $\geq 2.65$ N-m/L.

Figure 16. Measured and calculated torque vs. voltage product at maximum torque angle.

Power loss at stall versus torque was measured to evaluate the machine’s “position and hold” capability in a servo application and is plotted in Fig. 17. Under stall, a developed torque of 7.3 N-m (6.8 N-m net torque accounting for 0.5 N-m of static friction) only requires 5W input power, negating the need for any form of active cooling.

Figure 17. Measured power loss vs. torque under stall conditions at maximum torque angle.

The machine is filled with a liquid, which naturally leads to viscous drag losses during rotation. Additionally the shaft seals that retain the liquid within the machine add static and kinematic friction. The drag torque was measured on the dynamometer with no electrical excitation applied to the electrostatic machine. The drag torque curve is plotted in Fig. 18, which clearly indicates a squared relationship with speed. The static friction torque at stall is 0.5 N-m, mostly caused by the seals. Subtracting the drag torque from the electrical torque yields the net torque the machine can deliver to the load for a given speed.

V. CONCLUSIONS

A shear stress analytical design approach for separately excited synchronous electrostatic machines was developed that accommodates real world printed circuit board manufacturing practices. A generalized $dq$-axis model for all electrostatic machines was also presented, creating a framework for future machine and drive development that establishes a duality between electrostatic and magnetic $dq$-axis models. The model describes field, elastance (saliency) and induction (damping) torque components. From the machine design perspective, three coupling capacitances are identified as crucial for optimizing the torque production in synchronous machines, i.e. the fundamental component of the stator-rotor field mutual capacitance $C_{sfm1}$, the second order harmonic of the stator self-capacitance $C_{sn2}$ and the second order harmonic of the stator...
mutual capacitance $C_{ni}$. The design and modeling efforts were validated with a prototype electrostatic machine constructed entirely of aluminum and printed circuit boards that possesses active component torque densities ≥ 1.4 N-m/kg and ≥ 2.65 N-m/L without the need for forced air or liquid cooling. A developed torque of 7.3N-m at stall only requires 5 W input electrical power and this loss is distributed uniformly within the machine. While outside the scope of this work, a cost comparison between magnetic and electrostatic machine designs may be made on the material cost of designs with equal specific torque density (e.g. $$/kg of copper or magnets versus aluminum, plastic or composite). The prototype machine is the first demonstration of a sum of multiplicative gains in electrostatic machine design that closes the 4 – 5 orders of magnitude performance gap. These advancements can extend the high performance drive characteristics of electromagnetic machinery to electrostatic machines.

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The author D.C. Ludois has an ownership stake in C-Motive Technologies Inc., a company that licenses this technology.

REFERENCES

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