Mathematical Modeling of Semicircular Linear Motor Based on Vector Potential with Landen’s Transformation

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Abstract—A semicircular linear motor that realizes a motion along circumference of the circle has been developed. To settle the optimal design of this motor by mathematical analysis, the theoretical equation of motor performance has to be clarified. In this paper, the mathematical modeling of semicircular linear motor is carried out from the viewpoint of Lorentz force and magnetic flux density derived by vector potential. By comparing calculated, analyzed, and measured magnetic flux density and thrust force, the validity of derived mathematical model is confirmed. In addition, the cause of difference among three results is also discussed.

Index Terms—AC motor, Analytical models, Magnetic field analysis, Magnetic flux density, Mathematical modeling, Thrust characteristics.

I. INTRODUCTION

Various actuators are developed to realize a multiple-degree-of-freedom application, for example helical motor [1], two-degree-of-freedom motor [2], [3], unique magnet array motor [4], [5]. A semicircular linear motor (called circular shaft motor (CSM) in this paper) is one of the developed actuators [6]. The CSM is direct-drive motor that moves along circumference of the circle, and it is also expected to realize a multiple-degree-of-freedom application. To implement the CSM in various applications, the characteristics of the CSM should be analyzed. In addition, further improvement of the performance of the CSM by optimizing structure is also required. Because the structure is designed by trial and error based on result of electro magnetic analysis, precise mathematical modeling of motor is required to improve the performance of motor theoretically [7]-[9].

Various researches realized the mathematical modeling and performance improvement of motors [10]. The magnetic induction field by cylindrical magnets and coils are analyzed mathematically [11]. The vector potential of permanent magnet is generally utilized for mathematical analysis [12]. In rotary machine with Halbach array magnets, the optimal ratio of width of radially and parallel magnets is derived by vector potential [13]. The permeance model of motor with original structure derived the expected performance [14], [15]. Maxwell’s equations by magnetic vector potential of divided field domain clear the magnetic field distribution and cogging force by slot width and end effects [16]. The magnetic equivalent circuit is also utilized for analytical model. Hybrid model of nonlinear magnetic circuit by 2-dimensional finite element method model and experimental characterization indicated correct characteristics [17]. Motor performance is also expected by differential of magnetic field energy with respect to mechanical position variance [18]. The optimal design for maximized torque of rotary motor from the perspective of Lorentz force is also reported [19]. Derived mathematical model helps design of structure by finite element analysis [20].

This paper focuses on the mathematical modeling of the CSM toward optimization of structure based on analysis by solving constrained extremal problem. Constrained extremal problem by Lagrange multipliers method reveals the optimized design [21]. The CSM was already analyzed theoretically in previous research by using thrust equation based on inductance matrix [22]. To solve the thrust equation, the vector potential is introduced to model the coil and magnets. However, derived equation is not adapted for Lagrange multipliers method because of complexity of equations. Then, in this paper, the simplified thrust equation of the CSM is derived by Landen’s transformation [23]. To evaluate the validity of calculated magnetic flux density and thrust force, comparison with finite element analysis by electro magnetic analysis and measured result by actual equipment is carried out. The cause of difference among calculation, analysis, measurement result is also discussed.

This paper is organized as follows. In section II, the structure of the CSM is represented. In section III, the derivation flow of theoretical magnetic flux density by cylinder magnets is explained. In section IV, the derivation flow of theoretical thrust force is clarified. The difference between calculation, analysis, measurement result is also mentioned. Section V concludes this paper.

II. STRUCTURE OF CIRCULAR SHAFT MOTOR

The actual equipment of the CSM system is shown in Fig. 1. The mover part of the CSM moves along with
Fig. 1. Overall structure of the CSM system. The mover moves along with circumferential stator pipe.

Fig. 2. Cross-section diagram of the stator and mover parts in the CSM system. Three-phase coil moves along with guide rail under the mover case.

Fig. 3. Model of the trapezoidal cylindrical magnet, where the magnet region for each pole pitch angle is split into 20 pieces.

TABLE I

<table>
<thead>
<tr>
<th>Specifications of Stator and Mover Parts in the CSM System</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Stainless bent pipe</strong></td>
</tr>
<tr>
<td>Material</td>
</tr>
<tr>
<td>Outside diameter [mm]</td>
</tr>
<tr>
<td>Pipe wall thickness [mm]</td>
</tr>
<tr>
<td>Bend radius [mm]</td>
</tr>
<tr>
<td><strong>Permanent magnet</strong></td>
</tr>
<tr>
<td>Material</td>
</tr>
<tr>
<td>Pole pitch angle [deg]</td>
</tr>
<tr>
<td>Radius [mm]</td>
</tr>
<tr>
<td>Top side [mm]</td>
</tr>
<tr>
<td>Bottom side [mm]</td>
</tr>
<tr>
<td><strong>Mover bobbin</strong></td>
</tr>
<tr>
<td>Number of mover slots</td>
</tr>
<tr>
<td><strong>Coil</strong></td>
</tr>
<tr>
<td>Connection</td>
</tr>
<tr>
<td>Coil diameter [mm]</td>
</tr>
<tr>
<td><strong>Coil turns</strong></td>
</tr>
<tr>
<td>Inside diameter [mm]</td>
</tr>
<tr>
<td>Outside diameter [mm]</td>
</tr>
<tr>
<td>Coil turns for each slot</td>
</tr>
<tr>
<td>Coil turns for each phase</td>
</tr>
<tr>
<td><strong>Optical linear encoder</strong></td>
</tr>
<tr>
<td>Manufacturer</td>
</tr>
<tr>
<td>Model number</td>
</tr>
<tr>
<td>Resolution [μm]</td>
</tr>
</tbody>
</table>

This paper, the theoretical magnetic flux density and thrust force are derived based on the specifications of actual equipment listed in TABLE I. The outside diameter of coil part is measured from the actual system.

III. DERIVATION FLOW OF THEORETICAL MAGNETIC FLUX DENSITY BY CYLINDRICAL TRAPEZOIDAL MAGNETS

This section explains the derivation flow of theoretical magnetic flux density by trapezoidal cylindrical magnets. Firstly, magnet region for each pole pitch angle is approximated by dividing into 20 pieces along circumferential direction like Fig. 3. Each divided piece has 0.375 degrees because pole pitch angle 7.50 degrees is divided by 20. Secondly, the magnetic flux density by one piece of magnet is derived in following first subsection. The derived equation of magnetic flux density includes the first and second complete elliptic integral. These elliptic integral are...
Fig. 4. Magnetic flux distribution by a trapezoidal cylindrical magnet. The top view is trapezoidal and side view is circular because this magnet is made by cutting both edge of a cylindrical magnet.

Fig. 5. Axis definitions for calculation of vector potential by equivalent coil. Magnet piece is assumed as circular coil with equivalent magnetization current \( I_m \).

where the parameters are listed in TABLE II. Fig. 5 also indicates the axis definitions for derivation of vector potential by coil with \( I_m \). This axis definitions are based on cylindrical coordinate that the origin is set as the center of equivalent coil. The components in cylindrical coordinate (z′r plane) are converted to tangential component and normal component in circular arc coordinate (zr plane) that the origin is set as the center of circular arc stator after the next subsection.

From these definitions, the vector potential along with \( \phi \) direction \( A_\phi \) at point P is calculated by

\[
A_\phi = \frac{\mu_0 I_m}{2\pi} \int_0^\pi \frac{-a \cos \phi}{\sqrt{r^2 + z'^2 + a^2 - 2ar' \cos \phi}} \, d\phi.
\]  

Finally, using the first complete elliptic integral \( Z1(k_0) \) and the second one \( Z2(k_0) \) that have the parameter \( k_0 \), (2) becomes

\[
A_\phi = \frac{\mu_0 I_m a}{\pi \alpha} \left\{ \left( \frac{2}{k_0^2} - 1 \right) Z1(k_0) - \frac{2}{k_0^2} Z2(k_0) \right\},
\]  

A. Magnetic flux density derived by vector potential

The magnetic flux density by one piece magnet shown in Fig. 3 is calculated mathematically based on vector potential by following theory. Fig. 4 explains the magnetic flux distribution in one magnet. The circle in Fig. 5 defines the cross-section of magnet piece, that is same as the side view of Fig. 4. The magnet piece is equivalently considered as circular coil with equivalent magnetization current \( I_m \) that depends on remnant magnetic flux density \( B_{rem} \) and magnet width \( \theta_m \). By this assumption, the equivalent coil generates the same magnetic flux density with magnet piece, and \( I_m \) is derived by

\[
I_m = \frac{B_{rem} 2\pi (a + r_{CSM}) \theta_m}{\mu_0 \mu_s}. \tag{1}
\]  

TABLE II

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Magnetic permeability of vacuum</td>
<td>( \mu_0 )</td>
<td>1.26</td>
</tr>
<tr>
<td>Relative permeability of neodymium</td>
<td>( \mu_s )</td>
<td>1.05</td>
</tr>
<tr>
<td>Radius of magnet [mm]</td>
<td>( a )</td>
<td>18.5</td>
</tr>
<tr>
<td>Remanent magnetic flux density</td>
<td>( B_{rem} )</td>
<td>1.33</td>
</tr>
<tr>
<td>Width of one piece magnet [deg]</td>
<td>( \theta_m )</td>
<td>11.755</td>
</tr>
<tr>
<td>Radius of circular arc [mm]</td>
<td>( r_{CSM} )</td>
<td>400</td>
</tr>
</tbody>
</table>

The rotation of vector potential becomes the magnetic flux density. In the cylindrical coordinate system, the rotation of vector potential is represented as

\[
\text{rot} A = \begin{bmatrix} \text{rot} A_{r'} \, \text{rot} A_{\phi} \, \text{rot} A_z \end{bmatrix}^T
\]  

The symbol \( A_{r'} \), \( A_{\phi} \), and \( A_z \) refer the vector potential along \( r' \), \( \phi \), and \( z' \) direction at point P. Some vector potential becomes 0 because in the cylindrical coordinate system, the magnetic field is axial symmetry to the \( z' \) axis as follows:

\[
A_{r'} = A_z = \frac{\partial A_\phi}{\partial \phi} = 0.
\]
Then, by vector potential $A_{z'}$, the magnetic flux density in $z'$ direction $B_{z'}$ and that in $r'$ direction $B_{r'}$ are derived as

$$\begin{align*}
B_{z'} &= \text{rot}A_{z'} = \frac{\partial A_{\phi}}{\partial r'} + \frac{A_{\phi}}{r'} - \frac{\partial A_{\phi}}{r'} = \frac{\mu_0 I_{n} a}{\pi \alpha^3} \{ \frac{2 r'}{k_0^2} Z_1(k_0) + \frac{(a + r') k_0^2 - 2 r'}{k_0^2(1 - k_0^2)} Z_2(k_0) \}, \\
B_{r'} &= \text{rot}A_{r'} = -\frac{\partial A_{\phi}}{\partial z'} - \frac{\partial A_{\phi}}{\partial z'} = \frac{\mu_0 I_{n} a z'}{\pi \alpha^3} \{ - \frac{2}{k_0^2} Z_1(k_0) + \frac{-k_0^2 + 2}{k_0^2(1 - k_0^2)} Z_2(k_0) \}.
\end{align*}$$

(10)

These equations are unfit for mathematical analysis and design methodology because (10) and (11) include complete elliptic integral. For further simplification of these equations, following subsection explains the Landen's transformation.

B. Landen's transformation

The elliptic integral has complex calculation because of parameter $k_0$. Then, Landen's transformation, the method based on the arithmetic geometric mean [23], is introduced to simplify the elliptic integral. The recurrence formula of first and second elliptic integral based on Landen's transformation are defined as

$$\begin{align*}
Z_1(k_n) &= \left( 1 + \frac{1 - \sqrt{1 - k_n^2}}{1 + \sqrt{1 - k_n^2}} \right) Z_1(k_{n+1}), \\
Z_2(k_n) &= \left( 1 + \frac{1 - \sqrt{1 - k_n^2}}{1 + \sqrt{1 - k_n^2}} \right) Z_2(k_{n+1}) - \sqrt{1 - k_n^2(1 + k_{n+1})} Z_1(k_{n+1}),
\end{align*}$$

(12), (13)

where the subscript $n$ indicates the integer number for the definition of recurrence formula about $k_n$. The recurrence formula of $k_n$ is defined as

$$k_{n+1} = \frac{1 - \sqrt{1 - k_n^2}}{1 + \sqrt{1 - k_n^2}},$$

(14)

where the beginning value of $n$ is 0. When the $n$ is 0, the $k_n$ indicates $k_0$ that appears in (4) and (5). Therefore, $k_0$ is the first term for a sequence of numbers by (14), that is derived by (6) and (7).

In this time, $k_1$, $k_2$, $k_3$, and $k_4$ are derived by (14) because $k_4$ is smaller than $10^{-15}$, almost 0. It should be noted that $k_5$, $k_6$, $k_7$, ... become smaller than $k_4$. When the $k_n$ is almost 0, the first and second complete elliptic integral are approximated as $\frac{\pi}{2}$ because as follows

$$\begin{align*}
Z_1(k_n) &= Z_1(0) = \int_{0}^{\pi} \frac{1}{\sqrt{1 - 0^2 \sin^2 \phi}} d\phi = \frac{\pi}{2}, \\
Z_2(k_n) &= Z_2(0) = \int_{0}^{\pi} \sqrt{1 - 0^2 \sin^2 \phi} d\phi = \frac{\pi}{2}.
\end{align*}$$

(15), (16)

Therefore, $Z_1(k_4)$ and $Z_2(k_4)$ are approximated by $\frac{\pi}{2}$.

As a result, from (12), (14), and (15), $Z_1(k_0)$ is derived as

$$\begin{align*}
Z_1(k_0) &= (1 + k_1)(1 + k_2) \cdots (1 + k_n) Z_1(k_n) \\
&= (1 + k_1)(1 + k_2)(1 + k_3)(1 + k_4) Z_1(k_4) \\
&= (1 + k_1)(1 + k_2)(1 + k_3)(1 + k_4) \frac{\pi}{2}.
\end{align*}$$

(17)

On the other hand, $Z_2(k_0)$ is derived in reverse order by (13). Firstly, $Z_2(k_3)$ is derived as

$$\begin{align*}
Z_2(k_3) &= (1 + \sqrt{1 - k_4^2}) Z_2(k_4) - \sqrt{1 - k_3^2(1 + k_4)} Z_1(k_4) \\
&= (1 + \sqrt{1 - k_4^2}) Z_2(k_4) - \sqrt{1 - k_3^2(1 + k_4)} \frac{\pi}{2}.
\end{align*}$$

(18)

Also, $Z_2(k_2)$ is calculated as

$$\begin{align*}
Z_2(k_2) &= (1 + \sqrt{1 - k_3^2}) Z_2(k_3) - \sqrt{1 - k_2^2(1 + k_3)} Z_1(k_3) \\
&= (1 + \sqrt{1 - k_3^2}) Z_2(k_3) - \sqrt{1 - k_2^2(1 + k_3)} \frac{\pi}{2}.
\end{align*}$$

(19)

Then, $Z_2(k_1)$ becomes as

$$\begin{align*}
Z_2(k_1) &= (1 + \sqrt{1 - k_2^2}) Z_2(k_2) - \sqrt{1 - k_1^2(1 + k_2)} Z_1(k_2) \\
&= (1 + \sqrt{1 - k_2^2}) Z_2(k_2) - \sqrt{1 - k_1^2(1 + k_2)} \frac{\pi}{2}.
\end{align*}$$

(20)

Finally, $Z_2(k_0)$ can be derived by

$$\begin{align*}
Z_2(k_0) &= (1 + \sqrt{1 - k_1^2}) Z_2(k_1) - \sqrt{1 - k_0^2(1 + k_1)} Z_1(k_1) \\
&= (1 + \sqrt{1 - k_1^2}) Z_2(k_1) - \sqrt{1 - k_0^2(1 + k_1)} \frac{\pi}{2}.
\end{align*}$$

(21)

By substituting (17) and (21) into (10) and (11), the magnetic flux density along $z'$ and $r'$ direction is calculated.
In the following subsection, the coordinate conversion is discussed to consider the difference of motor coordinate and magnet coordinate because of circular arc structure.

C. Conversion from magnet coordinate to CSM coordinate

In this subsection, the magnetic flux density along z' and r' direction, \(B_{z'}\) and \(B_{r'}\), in cylindrical coordinate(\(z'\)r' plane) are converted to tangential and normal component, \(B_z\) and \(B_r\), in circular arc coordinate(\(z\)r plane). Fig. 6 indicates the two axes definitions that belong circular arc coordinate(xy plane) and magnet coordinate(\(z'\)r' plane). From already derived equations, the magnetic flux density along with \(z'\) axis and \(r'\) axis is able to be calculated. Then, the coordinates of point P at magnet coordinate, \(z'\) and \(r'\), have to be transformed to the coordinates at magnet coordinate \(P(z',r')\). The \(P(x,y)\) is represented by radius \(r_p\) and angle \(\theta_p\) as

\[
x = r_p \cos \theta_p, \quad y = r_p \sin \theta_p.
\]

By origin shift shown in (23) and (24) and coordinate rotation, following relational equations are derived as,

\[
x' = r_p \cos \theta_p - r_{CSM} \cos \theta_c, \quad (23)
\]

\[
y' = r_p \sin \theta_p - r_{CSM} \sin \theta_c, \quad (24)
\]

\[
z' = x \cos (\theta_c - \frac{\pi}{2}) + y \sin (\theta_c - \frac{\pi}{2}), \quad (25)
\]

\[
r' = -x \sin (\theta_c - \frac{\pi}{2}) + y \cos (\theta_c - \frac{\pi}{2}), \quad (26)
\]

where \(\theta_c\) is the angle of magnet piece in polar coordinate based on circular arc coordinate. From these equation, when the coordinate of point P are \(r_p\) and \(\theta_p\) in polar coordinate, the coordinate of point P at magnet coordinate, \(z'\) and \(r'\) are derived as follows:

\[
z' = r_p \sin (\theta_c - \theta_p), \quad (27)
\]

\[
r' = r_p \cos (\theta_c - \theta_p) - r_{CSM}. \quad (28)
\]

Then, the magnetic flux density along with \(z'\) and \(r'\) direction at point P are calculated by substituting \(z'\) and \(r'\) coordinate derived from (27) and (28) into (10) and (11). As a result, (10) and (11) are rewritten as (29) and (30) with (31) and (32). Finally, magnetic flux density along radial direction at P, \(B_r\), have to be clarified because the Lorentz force is generated by \(B_r\). Magnetic flux density \(B_{z'}\) and \(B_{r'}\) are the component of overall magnetic flux density at P, and from Fig. 7, the angle between \(B_{z'}\), \(B_{r'}\), and \(B_r\) are clarified. Then, the magnetic flux density along radial direction \(B_r\) is derived by

\[
B_r = B_{z'} \cos (\theta_p - \theta_c + \frac{\pi}{2}) + B_{r'} \cos (\theta_c - \theta_p). \quad (33)
\]

To obtain further validity of derived equations, the magnetic flux density along circumferential direction \(B_z\) is also
Fig. 8. Comparison of calculated, analyzed, and measured magnetic flux density along radial direction $B_r$.

Fig. 9. Comparison of calculated, analyzed, and measured magnetic flux density along circumferential direction $B_z$.

calculated by

$$B_z = B_{z'} \sin(\theta_p - \theta_c + \frac{\pi}{2}) - B_{r'} \sin(\theta_c - \theta_p).$$

(34)

By (33) and (34), the magnetic flux density at point P by one magnet piece is derived. To calculate overall magnetic flux density at point P by all magnet pieces, (33) and (34) with $\theta_c$ in increments of 0.375 degrees from 0 to 179.625 (it means 180-0.375) degrees are totalized. It takes 480 times calculation and its summation becomes overall magnetic flux density.

D. Calculated, analyzed, measured magnetic flux density

This subsection represents the comparison of calculated, analyzed, and measured magnetic flux density. Observation point $P$ is from 82.5 degree to 97.5 degree that is twice the length of pole–pitch angle. Fig. 8 shows the calculation result by (33), analysis result by electromagnetic analysis, and measurement result from actual equipment shown in Fig. 1. The gap length between magnet and observation point P is set as 6.200 mm that is on the surface of mover bobbin.

From this result, the measured peak value of wave of magnetic flux density is 14.5% larger than calculated data. One of the cause of this error is misalignment of two retainers at edge of stator. Because the actual system is large, the variance of magnetic gap by misalignment of retainers causes large errors. For example, the calculated magnetic flux density when the magnetic gap is 5.20 mm that is 1 mm smaller than Fig. 8, the maximum value becomes 0.471[T].

In addition, the analyzed peak value is 4.34% larger than calculated peak. One of the cause of this difference is disregard of relative permeability of neodymium. Some of the area between magnet and point $P$ include the area inside of magnet that is lower magnetic resistance thanks to relative permeability of neodymium $\mu_s$. Then, the disregard of relative permeability leads to higher magnetic resistance and lower theoretical magnetic flux density. The average of absolute value of error between calculated value and analyzed value $|Error_Ave|_{B_r}$ and the standard deviation of absolute value of error $Error_{SD_{B_r}}$ are calculated as

$$|Error_Ave|_{B_r} = 0.0104,$$

(35)

$$Error_{SD_{B_r}} = 0.00572.$$  

(36)

In addition, Fig. 9 represents the calculation result by (34), analysis result by electromagnetic analysis, and measurement result from actual equipment. The gap length between magnet and observation point P is set as 8.200 mm that is 2.000 mm longer than $B_r$ condition due to the structure of the Gauss meter. The analyzed peak value is 8.28% larger and measured peak value is 10.7% larger than calculated peak by same reason in the case of $B_r$. The average of absolute value of error between calculated value and analyzed value $|Error_Ave|_{B_z}$ and the standard deviation of absolute value of error $Error_{SD_{B_z}}$ are calculated as

$$|Error_Ave|_{B_z} = 0.00915,$$

(37)

$$Error_{SD_{B_z}} = 0.00452.$$  

(38)

On the other hand, the distribution of three results matches each other. Especially, the analysis result is almost same with calculation result from the perspective of wave form of magnetic flux density.

IV. DERIVATION FLOW OF THEORETICAL THRUST FORCE

A. Calculation of theoretical thrust force

This subsection explains the derivation flow of theoretical thrust force of CSM by already derived magnetic flux density. Fig. 10 indicates separated magnetic flux density $B_r$ to 3–phase. Magnetic flux density $B_r$ shown in Fig. 8 repeats every 15 degrees from 0 to 180 degrees by all 24
magnets pairs. The theoretical thrust force is calculated by following theory. The coil region is approximated by dividing 8 parts with different diameter as shown in Fig. 11. This coil approximation is similar way to magnet approximation in [25]. The \( r' \) axis is same with that is in Fig. 5. The overall thrust force is calculated by the summation of thrust force in 8 layers. The thrust force in each layer is derived by Lorentz force that is calculated by current, length of conductor, and magnetic flux density.

Firstly, in the case of U–phase, the current \( I_U(\theta_U) \) is represented as

\[
I_U(\theta_U) = \sqrt{2} \sin \left( \frac{2 \pi \theta_U}{15} + \frac{\pi}{2} \right),
\]

where 15 in denominator indicates cycle of magnetic flux density, \( \theta_U \) is defined as center angle of U–phase coil like Fig. 10. V–phase and W–phase current are also defined in the similar way to U–phase.

Secondly, the length of conductor in layer \( i \), \( L_i \) is defined as circumferential length of coil like

\[
L_i = 2\pi \times r_{\text{coil} - i},
\]

\[
r_{\text{coil} - i} = 24.2 + 0.8(i - 1),
\]

\[
i = 1, 2, \ldots, 8,
\]

where \( i \) indicates the number of layer, and \( r_{\text{coil} - i} \) is the radius of \( i \) layer coil as shown in Fig. 11. \( i \) is the integer from 1 to 8, as (42).

Thirdly, each 1 layer has 50 turns because 400 turns coil is divided into 8 layers. Then, coil turns in layer \( i \), \( N_i \) is

\[
N_i = \frac{N}{8} = 50,
\]

where \( N \) is coil turns in one slot, as listed in Table I. In addition, overall thrust force for 1 layer in U–phase, \( F_{U_i}(\theta_U) \) is calculated by the summation of minute Lorentz force \( \Delta F_{U_i}(\theta_U) \) like

\[
F_{U_i}(\theta_U) = \sum_{j=0}^{30} \Delta F_{U_i}(\theta_U) = \sum_{j=0}^{30} \{ I_U(\theta_U)L_i B_{r-U}(\theta_U - 2.5 + \frac{j}{6})N_i \},
\]

\[
= I_U(\theta_U)L_i N_i \sum_{j=0}^{30} \{ B_{r-U}(\theta_U - 2.5 + \frac{j}{6}) \},
\]

(44)

where \( B_{r-U}(\theta_U - 2.5 + j/6) \) defines the radial magnetic flux density in U–phase when the \( \theta_U \) is \( 2.5 + j/6 \) degrees. The angle definition of U–phase coil is from \( \theta_U - 2.5 \) to \( \theta_U + 2.5 \) degrees. In this time, the beginning value of \( j \) is 0 and end value is 30 because there are 31 plotted data in one phase as shown in Fig. 10. When \( j \) is 0, \( \theta_U - 2.5 + j/6 \) becomes \( \theta_U - 2.5 \), and when \( j \) is 30, \( \theta_U - 2.5 + j/6 \) becomes \( \theta_U + 2.5 \). Then, the summation in (44) is considered that the plotted magnetic flux density from \( \theta_U - 2.5 \) to \( \theta_U + 2.5 \) degrees are totaled. That is because U–phase current \( I_U(\theta_U) \), the length of conductor \( L_i \), and coil turns per one layer \( N_i \) is constant value between \( \theta_U - 2.5 \) and \( \theta_U + 2.5 \) degrees.

Finally, the thrust force in U–phase by all layers, \( F_U(\theta_U) \) are derived by the summation of \( F_{U_i}(\theta_U) \) in all layers like

\[
F_U(\theta_U) = \sum_{i=1}^{8} F_{U_i}(\theta_U) = \sum_{i=1}^{8} \{ I_U(\theta_U)L_i N_i \sum_{j=0}^{30} \{ B_{r-U}(\theta_U - 2.5 + \frac{j}{6}) \} \},
\]

(45)

In a similar way, the thrust force by V–phase, \( F_V(\theta_V) \) is calculated by

\[
F_V(\theta_V) = \sum_{i=1}^{8} \{ I_V(\theta_V)L_i N_i \sum_{j=0}^{30} \{ B_{r-V}(\theta_V - 2.5 + \frac{j}{6}) \} \},
\]

(46)

where

\[
I_V(\theta_V) = \sqrt{2} \sin \left( \frac{2 \pi \theta_V}{15} + \frac{\pi}{2} \right),
\]

\[
\theta_V = \theta_U + 10,
\]

(47)

(48)
and $B_{r-V}(\theta_V - 2.5 + j/6)$ defines the radial magnetic flux density in V-phase. Also the thrust force by W-phase, $F_W(\theta_W)$ is

$$F_W(\theta_W) = \sum_{i=1}^{8} \left\{ I_W(\theta_W)L_iN_i \sum_{j=0}^{30} B_{r-W}(\theta_W - 2.5 + j/6) \right\},$$

where

$$I_W(\theta_W) = \sqrt{2} \sin(\frac{2\pi\theta_W}{15} + \frac{\pi}{2}),$$

and $B_{r-W}(\theta_W - 2.5 + j/6)$ defines the radial magnetic flux density in W-phase.

Therefore, overall thrust force by 3-phase coils, $F_{cal}(\theta_U; \theta_V; \theta_W)$ is

$$F_{cal}(\theta_U; \theta_V; \theta_W) = 2\{F_U(\theta_U) + F_V(\theta_V) + F_W(\theta_W)\},$$

where 2 indicates 2 pairs of 3-phase coils, as shown in Fig. 2. By calculating (52) for 15 degrees that is twice of pole-pitch angle, thrust ripple is also clarified. The initial angle of $\theta_U$ is set as 85 degrees. Then, $\theta_U$ varies from 85 to 100, $\theta_V$ is from 95 to 110, and $\theta_W$ changes from 90 to 105 degrees in this calculation.

B. Analysis conditions

This subsection is for electromagnetic analysis of CSM. The electromagnetic analysis with the conditions listed in Table I, II, and III is carried out.

C. Comparison of calculated and analyzed thrust force

From previous two subsections, theoretical thrust force is calculated and analyzed. The calculation and analysis results shown in Fig. 12 represent that calculated thrust force agree well with analyzed thrust force, especially about thrust ripple. Thrust ripple is calculated from maximum $F_{max}$, minimum $F_{min}$, and average $F_{ave}$ of thrust force by

$$\text{Ripple}[\%] = \frac{F_{max} - F_{min}}{2F_{ave}} \times 100. \quad (53)$$

The cause of 2.56% error about thrust average is the difference of magnetic flux density mentioned in Fig. 8.

D. Measurement of thrust force in actual system

Fig. 13 shows the experimental setup for thrust characteristics test. Thrust characteristics test is carried out with a motor driver, load cell, and output meter. Specifications of motor driver, load cell, and output meter are listed in Table IV. The motor driver supplies 3-phase AC current to the CSM system. The generated thrust force is measured by load cell and displayed by output meter. Thrust force is measured one minute after feeding current to observe steady-state thrust force. In addition, thrust force is measured by supplying current in increments of 0.05 A from 0.10 A to 0.7 A. To prevent decline of magnetic flux density by heat from coil, maximum feeding current is set as 0.7 A. The slope of linear approximation line about measured data is derived as thrust average to consider the influence of static friction.

Fig. 14 and Fig. 15 shows the measured thrust force along with positive(clockwise) and negative(anticlockwise) direction. Thrust force derived by linear approximation in positive direction is 71.7 N/A and that in negative direction is 75.9 N/A. Hand-rolled coil turns cause the error of these thrust force difference. The average of two thrust forces is derived as 73.8 N/A. Table V groups derived magnetic flux density and thrust force together. Measured thrust force is 18.1% larger than calculated thrust force in Fig. 12. Main cause of this error is the difference of derived magnetic flux density that is 14.5% shown in Fig. 8. In other words, the correct magnetic flux density derives same thrust force.
TABLE IV
SPECIFICATIONS OF EXPERIMENTAL SETUP

<table>
<thead>
<tr>
<th></th>
<th>Model</th>
<th>Load cell</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manufacturer</td>
<td>Sveland corporation</td>
<td>Kyowa Electronic instruments</td>
</tr>
<tr>
<td>Model</td>
<td>SVFM1</td>
<td>LMA-A-50N</td>
</tr>
<tr>
<td>Rated current [rms]</td>
<td>0.800</td>
<td>50.0</td>
</tr>
<tr>
<td>Maximum current [rms]</td>
<td>2.00</td>
<td></td>
</tr>
</tbody>
</table>

TABLE V
DIFFERENCES BETWEEN CALCULATED, ANALYZED, AND MEASURED MAGNETIC FLUX DENSITY AND THRUST FORCE.

<table>
<thead>
<tr>
<th></th>
<th>Calculation</th>
<th>Analysis</th>
<th>Measurement</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_r$ peak [T]</td>
<td>0.414</td>
<td>0.432</td>
<td>0.474</td>
</tr>
<tr>
<td>Error from calculation [%]</td>
<td>-4.34</td>
<td>14.5</td>
<td></td>
</tr>
<tr>
<td>Thrust average [N/A]</td>
<td>62.5</td>
<td>64.1</td>
<td>73.8</td>
</tr>
<tr>
<td>Error from calculation [%]</td>
<td>-2.56</td>
<td>18.1</td>
<td></td>
</tr>
</tbody>
</table>

This paper explained mathematical modeling flow of semicircular linear motor. Firstly, the structure of the CSM was explained. Neighboring opposite magnetized magnets array produce magnetic flux density that is utilized to generate thrust force.

Secondly, the magnetic flux density by cylindrical magnet was derived by the integration of vector potential. The derived magnetic flux density included the first and second complete elliptic integral. By utilizing Landen’s transformation to elliptic integral, theoretical equation of magnetic flux density was simplified. The magnetic flux density along with radial direction was focused on, and calculated with the consideration of coordinate transformation from magnet coordinate to motor coordinate. The cause of difference of peak value about calculation, analysis, and measurement result was also discussed.

Thirdly, theoretical thrust force was derived by already identified magnetic flux density. Calculated thrust force agreed well with analyzed thrust force, especially about thrust ripple. Measured thrust force was higher than calculated thrust force due to the difference of derived magnetic flux density.

From the comparison and discussions of these results, the validity of derived mathematical model was confirmed. Derived model is validate for optimal design based on constrained extreme problem.

REFERENCES


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