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Stabilizing model predictive control scheme for piecewise affine systems with maximal positively invariant terminal set

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Abstract: An efficient algorithm is proposed for computing the solution to the constrained finite time optimal control (CFTOC) problem for discrete-time piecewise affine (PWA) systems with a quadratic performance index. The maximal positively invariant terminal set, which is feasible and invariant with respect to a feedback control law, is computed as terminal target set and an associated Lyapunov function is chosen as terminal cost. The combination of these two components guarantees constraint satisfaction and closed-loop stability for all time. The proposed algorithm combines a dynamic programming strategy with a multi-parametric quadratic programming solver and basic polyhedral manipulation. A numerical example shows that a larger stabilizable set of states can be obtained by the proposed algorithm than precious work.

Keywords: constrained optimal predictive control, multi-parametric quadratic programming, dynamic programming, receding horizon control, positively invariant set.

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1. Introduction

In the last few years, several different techniques were developed for the analysis and controller synthesis for hybrid systems [1–5].

Piecewise affine systems have attracted much interest in the research community since they can approximate nonlinear systems and because of their equivalence to many classes of hybrid systems. They are obtained by partitioning the state space into polyhedral regions and associating to each region a different affine state update equation [2].

A large part of the literature has focused on end-point constraints to guarantee stability of the closed-loop system [3,6,7]. This type of constraint generally requires the use of large prediction horizons for the controller to cover the maximal attractive set, such that the complexity quickly becomes prohibitive. Other methods computational [8] only provide stability guarantees if the origin is contained in the interior of one of the dynamics.

For any dynamical system, stability is guaranteed if an invariant set is imposed as a terminal state constraint and terminal cost corresponds to a Lyapunov function for that set. Moreover, the decay rate of the "terminal Lypunov function" must be greater than the stage cost.

In this paper, we show how to compute a maximal positively invariant set with the associated Lyapunov function such that stability and constraint satisfaction of receding horizon control (RHC) is guaranteed. The scheme is based on the results in [9–11].

Based on the results above, we propose an efficient algorithm for computing the solution to the constrained finite time optimal control (CFTOC) problem for Piecewise affine (PWA) systems. The algorithm uses a dynamic programming together with multi-parametric programming solver.

2. Constrained finite time optimal control of piecewise affine systems

Consider discrete-time constrained PWA systems of the following form

$$x(t+1) = f_{\mathrm{PWA}}(x(t), u(t)) = A_i x(t) + B_i u(t) + f_i$$

$$\text{if } \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in D_i := \left\{ \begin{bmatrix} x \\ u \end{bmatrix} | H_i x + J_i u \leqslant K_i \right\},\ i \in I$$

where $t \geqslant 0$, $x \in X \subset R^n$ is the state, $u \in U \subset R^m$ is the control input, and $\{D_i\}_{i=1}^s$ is the polyhedral partition of the sets of the extended state+input space R^{n+m}. Furthermore, we assume that X and U are compact and polytopic which contain the origin in their interiors and let the union of the polyhedral partitions be $D := \bigcup\limits_{i=1}^{s} D_i$. The set is defined as $I \triangleq \{1, 2, \ldots, D\}$ where D denotes the number of different dynamics. We will henceforth assume that

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the sets D_i are non-intersecting. Note that linear state and input constrains of the general form $Kx(t) + Lu(t) \leqslant M$ are incorporated in the description of D_i.

Additionally, define the following cost function

$$J(U_0^{T-1}, x(0)) := \|Q_{x_N} x(T)\|_p + \sum_{k=0}^{T-1} \|Qx(k)\|_p + \|Ru(k)\|_p \quad (1)$$

and consider the finite constrained optimal control problem

$$J^*(x(0)) := \min_{U_0^{T-1}} J(U_0^{T-1}, x(0))$$

$$\text{s.t.} \begin{cases} x(t+1) = f_{\mathrm{PWA}}(x(t), u(t)) \\ x(T) \in \chi^f \end{cases} \quad (2)$$

where the column vector $U_0^{T-1} := [u(0)', \ldots, u(T-1)']' \in R^{mT}$ is the optimization vector, χ^f is the terminal target region and p in (1) denotes the corresponding standard vector norm, in particular, 2-norm. Additionally, we assume that R, Q, and Q_{x_N} are of full column rank.

We summarize the main result of the solution to the CFTOC problems (1)–(2) which is proved in [5].

Theorem 1 (Solution to the CFTOC) The solution to the optimal control problem (1)–(2) with $p = 2$ is a piecewise affine state feedback control law of the form

$$u^*(t) = f_t(x(t)) = F_i^t x(t) + G_i^t \quad \text{if } x(t) \in \Re_i^t \quad (3)$$

where $\Re_i^t$ $(i = 1, \ldots, N^t)$ is a polyhedral partition of the set χ^t of feasible states $x(t)$ at time $t = 0, \ldots, T-1$.

When we apply the receding horizon control policy in which only the first element of the computed optimal control sequence is applied to the systems, (3) can be furthermore denoted as

$$u^*(0) = f_0(x(0)) = F_i^0 x(t) + G_i^0 if x(0) \in \Re_i^0 \quad (4)$$

3. Computation of maximal positively invariant set O_∞^{PWA}

In this section, we will show how terminal set χ^f and corresponding weight matrix P can be computed such that stabilizing RHC controllers can be constructed for generic PWA systems.

We assume that the origin is an equilibrium state of the PWA system. Specifically, it would be necessary to require that the feedback law K_i associated with each dynamic satisfy the input constraints over the entire state space.

In a first step, we select all dynamics $i \in I_0$ which contain the origin, i.e. $I_0 \triangleq \{i \in I | 0 \in D_i\}$.

The search of stabilizing piecewise linear feedback controllers K_i and an associated common quadratic Lyapunov function $V(x) = x'Px$ can now be posed as

$$x^{\mathrm{T}} P x \geqslant 0, \ \forall x \in X$$

$$x^{\mathrm{T}}(A_i + B_i K_i)^{\mathrm{T}} P (A_i + B_i K_i) x - x^{\mathrm{T}} P x \leqslant -x^{\mathrm{T}} Q x - x^{\mathrm{T}} K_i^{\mathrm{T}} R K_i x, \forall x \in D_i, \forall i \in I_0 \quad (5)$$

If we relax this condition by setting $D_i = R^n, \forall i \in I_0$, the problem can be rewritten as an semi-definite programming (SDP) by using Schur complements and introducing the new variables $Y_i = K_i Z$ and $Z = \frac{1}{\gamma} P^{-1}$ (see [9] for details)

$$\min \gamma, \text{s.t.} \quad (6a)$$

$$Z \succ 0 \quad (6b)$$

$$\begin{bmatrix} Z & A_i Z + B_i Y_i & (Q^{0.5} Z)^{\mathrm{T}} & (R^{0.5} Y_i)^{\mathrm{T}} \\ (A_i Z + B_i Y_i)^{\mathrm{T}} & Z & 0 & 0 \\ Q^{0.5} Z & 0 & \gamma I & 0 \\ R^{0.5} Y_i & 0 & 0 & \gamma I \end{bmatrix} \succ 0, \quad \forall i \in I_0 \quad (6c)$$

where the scalar γ is introduced to optimize for the worst case performance. Note that it may not be possible for the worst case switching sequence considered in (6) to occur in practice, since not all dynamics are defined over the entire state space.

In a second step, we will compute the maximal positively invariant set O_∞^{PWA} of the PWA system subject to the feedback controllers K_i with the method in [10], which is guaranteed to terminate in finite time for the problem at hand, since the closed-loop system is asymptotically stable.

Here, we will summary the method in [10].

Let $k(x) = K_i x$ if $x \in D_i^*, \ \forall i \in I_0$, where $D_i^* \triangleq \{x | K_i x \in D_i\}, i \in I_0$ and we define

$$X_0 \triangleq \bigcup_{i \in I_0} D_i^* \quad (7)$$

So we consider the following autonomous system

$$x(t+1) = f_a(x) \triangleq (A_i + B_i K_i) x, x \in D_i^*, \ i \in I_0 \quad (8)$$

For the system above , we use $\phi(k; x_0)$ to denote the solution of $x(t+1) = f_a(x)$ at time k if the initial state is x_0 and $Pr\ e_a(S)$ to denote the set of the states that evolves to $S \subseteq X_0$ in one step, i.e.

$$Pr\ e_a(S) \triangleq \{x \in X_0 | f_a(x) \in S\} \quad (9)$$

moreover, if $S \subseteq X_0$ is a P-collection, then the set $Pr\ e_a(S)$, defined in (9), is also a P-collection.

Definition 1 (The maximal positively invariant set O_∞^{PWA}) The maximal positively invariant set, O_∞^{PWA}, for the discrete time system $x(t+1) = f_a(x)$, where $f_a(\cdot)$ is defined in (8) subject to the constrains in (7), is defined by

$$O_\infty^{\mathrm{PWA}} \triangleq \{x \in X_0 | \phi(k; x) \in X_0, \forall k \in N\}$$

The following algorithm provides a procedure for computing the maximal positively invariant subset of X_0 [12–14].

Algorithm 1 (Computation of O_∞^{PWA})

(a) $\Omega_0 = X_0$.

(b) $\Omega_{k+1} = \Pr e_a(\Omega_k)$.

(c) If $\Omega_{k+1} = \Omega_k$, return; set $k = k + 1$and go to (b).

The algorithm generates the set sequence $\{\Omega_k\}$ satisfying $\Omega_{k+1} \subseteq \Omega_k, \forall k \in N$ and it terminates if $\Omega_{k+1} = \Omega_k$ so that Ω_k is the maximal positively invariant set O_∞^{PWA}.

The proposed computation scheme above is summarized in the following algorithm:

Algorithm 2

(a) Identify all dynamics which contain the origin, i.e. $i \in I_0$.

(b) Solve (6) for all $i \in I_0$ to obtain K_i and P. If (6) is infeasible, abort the algorithm.

(c) Compute the maximal positively invariant set O_∞^{PWA} of the PWA system subject to feedback controllers K_i with the Algorithm 1.

(d) Return the target set O_∞^{PWA}, the feedback laws K_i and the associated matrix P.

4. Computation of the CFTOC solution via an efficient dynamic program

Here we show that how the considered finite time optimal control problems (1) and (2) can be solved with an efficient dynamic programming approach.

The equivalent dynamic program is of the following form

$$J_j^*(x(j)) := \min_{u(j)} \|Qx(j)\|_2 + \|Ru(j)\|_2 + J_{j+1}^*(f_{\mathrm{PWA}}(x(j), u(j)))$$
$$\text{s.t. } f_{\mathrm{PWA}}(x(j), u(j)) \in \chi^{j+1} \tag{10}$$

for $j = T-1, \dots, 0$, with boundary conditions

$$\chi^{\mathrm{T}} = \chi^f$$
$$J_T^*(x(T)) = \|Q_{x_N} x(T)\|_2 \tag{11}$$

where $\chi^j = \left\{x \in R^n | \exists u, f_{\mathrm{PWA}}(x, u) \in \chi^{j+1}\right\}$ is the set of all initial states for which the problem (10) is feasible.

4.1 The dynamic programming strategy

The dynamic programming problems (10) and (11) can be solved by using a multi-parametric quadratic program solver going backwards in time starting from the target region χ^f.

Consider the first step of the dynamic program (10), (11)

$$J_{T-1}^*(x(T-1)) := \min_{u(T-1)} \|Qx(T-1)\|_2 + \|Ru(T-1)\|_2 + J_T^*(f_{\mathrm{PWA}}(x(T-1), u(T-1)))$$
$$\text{s.t. } f_{\mathrm{PWA}}(x(T-1), u(T-1)) \in \chi^f \tag{12}$$

The cost-to-go function $J_T^*(x)$ in (12) is piecewise affine, the terminal region χ^f is a polyhedron and the constraints are piecewise affine, thus, problem (12) is a many s to one problem that can be solved with s mp-QPs.

At the second step, $j = T-2$, the cost-to-go function $J_{T-1}^*(x)$ is polyhedral piecewise affine and the terminal set χ^{T-1} is a union of N_{T-1}^r polyhedra, where N_{T-1}^r is the number of polyhedra of J_{T-1}^*. Note that the constrains are still piecewise but χ^{T-1} is not necessarily a convex set. Problems (10) and (11) become a many s to many N_{T-1}^r problem and can be solved by solving sN_{T-1}^r mp-QPs. From the third step $j = T-3$ to the last one $j = 0$, the cost-to-go function $J_j^*(x)$ is polyhedral piecewise affine with a certain multiplicity d_j, the terminal set χ^j is again the union of N_j^r polyhedra and the constrains are piecewise affine. Therefore, problems (10) and (11) are a many s to many N_j^r problem with multiplicity d_j that can be solved by solving $sN_j^r d_j$ mp-QPs. The resulting optimal solution will have the piecewise affine form (3).

5. Numerical example

In order to illustrate the proposed procedure, we consider a 2-dimensional PWA system

$$x(k+1) = \begin{cases} A_1 x(k) + B_1 u(k) + f_1, & x_1(k) \leqslant 1 \\ A_2 x(k) + B_2 u(k) + f_2, & x_1(k) > 1 \end{cases}$$

where

$$A_1 = \begin{bmatrix} 0.6 & -0.2 \\ -0.1 & 0.8 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, f_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0.7 & -0.3 \\ 0.8 & 0.6 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, f_2 = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}$$

subject to constrains

$$-x_1(k) + x_2(k) \leqslant 15$$
$$-3x_1(k) - x_2(k) \leqslant 25, \ -6 \leqslant x_1(k) \leqslant 8$$
$$0.2x_1(k) + x_2(k) \leqslant 9, \ -1 \leqslant u(k) \leqslant 1$$

Weight matrices in cost function were chosen as $Q = I_2$ and $R = 1$. The prediction and control horizon are considered to be $N = 3$. By solving the SDP in (6), the following stabilizing feedback controller and associated common quadratic Lyapunov function are obtained

$$K = [0.156\,0 \quad -0.565\,9]$$

$$V(x) = x^{\mathrm{T}} P x = x^{\mathrm{T}} \begin{bmatrix} 1.620\,4 & -0.445\,9 \\ -0.445\,9 & 2.016\,4 \end{bmatrix} x$$

The terminal set $\chi^f = O_\infty^{\mathrm{PWA}}$ was obtained by computing the maximal positively invariant set with Algorithm 1 (see Fig. 1) and corresponding weight matrix is chosen as P.

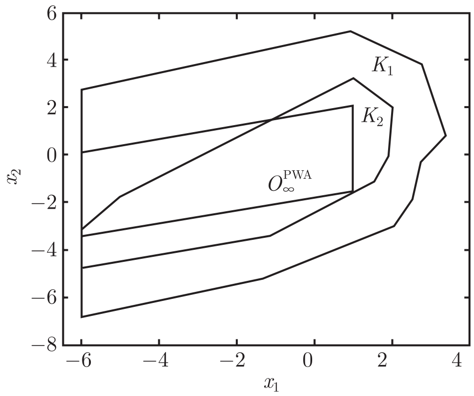

Fig. 1 Maximal positively invariant set and stabilizable sets

Then, by solving the CFTOC problem (1) with procedure aforementioned, the resulting stabilizable set denoted as K_1 is also depicted in Fig. 1.

Fig. 2 shows the states evolution for some initial states for the CFTOC problem above. As it can be seen, the states evolve asymptotically to the origin.

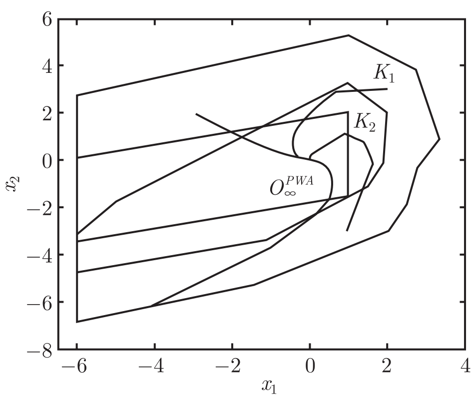

Fig. 2 State trajectories of the system of example with O_∞^{PWA}

In comparison with the maximal positively invariant set, a smaller invariant set around the origin is chosen as $\chi^f \triangleq \{x \in R^2, \|x\|_\infty \leqslant 0.5\}$ while the other parameters of CFTOC problem are the same. The resulting stabilizable set K_2 is depicted in Fig. 1.

It is obvious to observe that the solution to the CFTOC problem with maximal positively invariant set has larger stabilizable set of states than that with smaller positively invariant set.

6. Conclusion

In this paper, an efficient algorithm for computing the solution to the CFTOC problem with maximal positively invariant terminal set for discrete-time PWA systems is proposed. The algorithm combines dynamic programming strategy with multi-parameters quadratic programming solvers and basic polyhedral manipulation.

A maximal positively invariant chosen as terminal target set with respect to the local stabilizing controller is computed efficiently and the terminal cost is an associated Lyapunov function so that the stability of closed-loop system is guaranteed. Through a numerical example, the proposed algorithm is shown to have larger stabilizable region.

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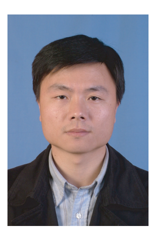
Biographies



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