Proper Processing of β-Quaternion Wide-Sense Markov Signals From Randomly Lost Observations

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***Abstract*—This letter addresses the problem of estimating wide-sense Markov signals within the β-quaternion domain, under conditions of first-order properness. This property offers a significant advantage that cannot be achieved in the real fields: a reduction in the dimensionality of the state-space model, resulting in lower computational load when running estimation algorithms. Once the equation of the wide-sense Markov signal model has been obtained, and assuming that the observations used to estimate the signal are affected by packet dropouts, a filtering algorithm based on first-order β-quaternion processing is devised for the optimal estimator. The effectiveness of the estimators proposed is illustrated through numerical comparison with those obtained by using traditional quaternion processing.**

***Index Terms*—Estimation, packet dropouts, properness conditions, β-quaternion processing, wide-sense Markov signals.**

I. Introduction

The linear least mean-squared error (LLMSE) estimation problem has traditionally focused on wide-sense Markov (WSM) processes, when Gaussianity is not satisfied. The interest in assuming this Markov property lies in its ability to facilitate a state-space representation for the signal. Numerous papers have been published on WSM signals, mainly in the real and complex fields (see e.g. [1], [2], [3]).

Over the last few years, the scientific community has shown great interest in studying 4D hypercomplex algebras since some real-world problems can be better modeled by using these kinds of signals than with real or complex ones (see e.g. [4], [5], [6], [7], [8]). The most widely used hypercomplex algebra is the quaternion algebra [9], [10], [11], [12], [13], [14], [15]. However, recently, other hypercomplex algebras have emerged [16], [17], [18], [19]. These new structures have provided better results than those obtained with the quaternion algebra. In particular, the β-quaternion algebra is a generalization of the quaternions (case $\beta = -1$), that makes possible the modeling of a higher variety of practical problems. For instance, it has proven to be a useful alternative framework for studying adaptive filtering problems [19].

A wide range of literature on the estimation problem concerning quaternion signals exists, where different assumptions are made on the observations used for the estimation [20], [21], [22], [23], [24], [25]. One of these assumptions involves observations affected by packet dropouts, which model many real-life situations where there is a probability that some observations are lost [26], [27], [28], [29], [30].

The study of properness cases is of great interest when the problem of estimating hypercomplex signals is being faced, as these cases can lead to a significant reduction in the computational burden of the estimation algorithms devised, that cannot be achieved in the real case [14], [31], [32], [33]. Thus, the estimation problem under properness conditions in the β-quaternion domain has been addressed in [19]. In general, widely linear ($\mathtt{WL}$) processing, based on the augmented processes, is considered the optimal approach. However, under first-order properness, the optimal approach shifts to $\mathtt{Q}_1$ processing, based solely on process information. In fact, $\mathtt{Q}_1$ and $\mathtt{WL}$ processing are equivalent under first-order properness.

This paper aims to address the LLMSE estimation problem of WSM signals from observations affected by packet dropouts, under first-order properness conditions. The main contributions obtained from this research are summarized in the following points: 1) Proper WSM β-quaternion signals of order $n \geq 1$ are introduced; 2) A state-space model is provided for these signals; 3) Computationally efficient reduced dimension filtering algorithms are proposed to solve the problem; 4) Simulations confirm that the estimators proposed perform better than those obtained in the quaternion domain.

The rest of the paper is organized as follows. After a brief review of the β-quaternion processing (Section II), a representation of a WSM signal of order $n \geq 1$ in the state-equation form is obtained under first-order properness (Section III). Subsequently, in Section IV, a filtering algorithm is proposed, whose computational load is significantly reduced under properness conditions. The efficiency of the estimators is illustrated in Section V, in comparison with those obtained by quaternion processing in two different examples: one with simulated data and another with a real-life application model.

II. β-Quaternion Signal Processing: A Review

Throughout the paper, lightface lowercase, boldface lowercase, and boldface uppercase letters are used to represent scalars, vectors, and matrices, respectively. So, a $p \times 1$ real (respectively, β-quaternion) vector $\mathbf{r}$, will be written as $\mathbf{r} \in \mathbb{R}^p$ (respectively, $\mathbf{r} \in \mathbb{Q}_\beta^p$). Moreover, $\mathcal{R}\{\cdot\}$ denotes the real part, $E[\cdot]$ the expectation operator, $\otimes$ the Kronecker product, superscript "T" represents the transpose, δ_{nl} the Kronecker delta function, $\mathbf{0}_{p\times q}$ is the $p \times q$ zero matrix, $\mathbf{I}_p$ the $p \times p$ identity matrix, and $\mathbf{1}_p$ the $p \times 1$ vector of ones. In addition, all the random variables are assumed to have zero-mean.

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Definition 1: A random β-quaternion signal vector $\mathbf{x}(t) \in \mathbb{Q}_\beta^p$, $t \in \mathbb{N}$, is a hypercomplex stochastic process stated as:

$$\mathbf{x}(t) = \mathbf{x}_{\mathrm{r}}(t) + \mathrm{i}\mathbf{x}_{\mathrm{i}}(t) + \mathrm{j}\mathbf{x}_{\mathrm{j}}(t) + \mathrm{k}\mathbf{x}_{\mathrm{k}}(t),$$

where $\mathbf{x}_\mu(t)$, for $\mu = \mathrm{r}, \mathrm{i}, \mathrm{j}, \mathrm{k}$ are p-dimensional real stochastic processes and the imaginary units satisfy the following rules:

$$\mathrm{i}^2 = -1,\ \mathrm{j}^2 = \mathrm{k}^2 = \beta,\ \mathrm{ij} = \mathrm{k},\ \mathrm{jk} = -\beta\mathrm{i},\ \mathrm{ki} = \mathrm{j},\ \mathrm{ijk} = \beta,$$

with $\beta \in \mathbb{R} - \{0\}$.

For any $\mathbf{x}(t) \in \mathbb{Q}_\beta^p$, the next auxiliary processes are defined:

$$\mathbf{x}^{\mathrm{i}}(t) = \mathbf{x}_{\mathrm{r}}(t) + \mathrm{i}\mathbf{x}_{\mathrm{i}}(t) - \mathrm{j}\mathbf{x}_{\mathrm{j}}(t) - \mathrm{k}\mathbf{x}_{\mathrm{k}}(t),$$

$$\mathbf{x}^{\mathrm{j}}(t) = \mathbf{x}_{\mathrm{r}}(t) - \mathrm{i}\mathbf{x}_{\mathrm{i}}(t) + \mathrm{j}\mathbf{x}_{\mathrm{j}}(t) - \mathrm{k}\mathbf{x}_{\mathrm{k}}(t),$$

$$\mathbf{x}^{\mathrm{k}}(t) = \mathbf{x}_{\mathrm{r}}(t) - \mathrm{i}\mathbf{x}_{\mathrm{i}}(t) - \mathrm{j}\mathbf{x}_{\mathrm{j}}(t) + \mathrm{k}\mathbf{x}_{\mathrm{k}}(t),$$

$$\mathbf{x}^{(\alpha)}(t) = \mathbf{x}_{\mathrm{r}}(t) - \mathrm{i}\mathbf{x}_{\mathrm{i}}(t) + \frac{\mathrm{j}}{\alpha}\mathbf{x}_{\mathrm{j}}(t) + \frac{\mathrm{k}}{\alpha}\mathbf{x}_{\mathrm{k}}(t),\ \alpha \in \mathbb{R} - \{0\}.$$

Moreover, the associated real and augmented vectors are respectively defined by $\mathbf{x}^r(t) = [\mathbf{x}_{\mathrm{r}}^{\mathrm{T}}(t), \mathbf{x}_{\mathrm{i}}^{\mathrm{T}}(t), \mathbf{x}_{\mathrm{j}}^{\mathrm{T}}(t), \mathbf{x}_{\mathrm{k}}^{\mathrm{T}}(t)]^{\mathrm{T}}$ and $\bar{\mathbf{x}}(t) = [\mathbf{x}^{\mathrm{T}}(t), \mathbf{x}^{\mathrm{k}^{\mathrm{T}}}(t), \mathbf{x}^{\mathrm{i}^{\mathrm{T}}}(t), \mathbf{x}^{\mathrm{j}^{\mathrm{T}}}(t)]^{\mathrm{T}}$.

In addition, the pseudo cross-correlation matrices of $\mathbf{x}(t), \mathbf{y}(t) \in \mathbb{Q}_\beta^p$ are defined as follows:

$$\mathbf{\Upsilon}_{\mathbf{xy}}^{1\beta}(t,s) = E[\mathbf{x}(t)\mathbf{y}^{\mathrm{H}_1}(s)], \qquad \forall\beta > 0,$$

$$\mathbf{\Upsilon}_{\mathbf{xy}}^{2\beta}(t,s) = E[\mathbf{x}(t)\mathbf{y}^{\mathrm{H}_{-1}}(s)], \qquad \forall\beta < 0,$$

where $\mathbf{y}^{\mathrm{H}_\alpha}(s) \triangleq \mathbf{y}^{(\alpha)^{\mathrm{T}}}(s)$. The pseudo autocorrelation function of $\mathbf{x}(t) \in \mathbb{Q}_\beta^p$, $\mathbf{\Upsilon}_{\mathbf{xx}}^{\nu\beta}(t,s)$, will be denoted by $\mathbf{\Upsilon}_{\mathbf{x}}^{\nu\beta}(t,s)$, for $\nu = 1, 2$, and $\mathbf{\Upsilon}_{\mathbf{x}}^{\nu\beta}(t,t) \triangleq \mathbf{\Upsilon}_{\mathbf{x}}^{\nu\beta}(t)$.

Next, the first-order properness concept, introduced in [19], is recalled.

Definition 2: Two random signals $\mathbf{x}(t) \in \mathbb{Q}_\beta^{p_1}$ and $\mathbf{y}(t) \in \mathbb{Q}_\beta^{p_2}$ are said to be:

- Cross $P_{\beta_0}^1$-proper (respectively, cross $N_{\beta_0}^1$-proper), if there exists a value $\beta_0 > 0$ (respectively, $\beta_0 < 0$) such that $\mathbf{\Upsilon}_{\mathbf{xy}^\mu}^{1\beta_0}(t,s)$ (respectively, $\mathbf{\Upsilon}_{\mathbf{xy}^\mu}^{2\beta_0}(t,s)$), for $\mu = \mathrm{i}, \mathrm{j}, \mathrm{k}$, vanish, $\forall t, s$. By setting $\mathbf{y} = \mathbf{x}$, the definition of $\mathbf{x}(t)$ as $P_{\beta_0}^1$-proper (respectively, $N_{\beta_0}^1$-proper) is established.
- Jointly $P_{\beta_0}^1$-proper (respectively, jointly $N_{\beta_0}^1$-proper), if and only if, they are $P_{\beta_0}^1$-proper and cross $P_{\beta_0}^1$-proper (respectively, $N_{\beta_0}^1$-proper and cross $N_{\beta_0}^1$-proper).

The first-order properness assumption results in a reduction in the dimensionality of the problem, which has no place in the real field, thereby leading to computational advantages in computing the LLMSE estimator (for further details, see [19]). Specifically, under $P_{\beta_0}^1$-properness conditions, the optimal processing is $P_{\beta_0}^1$ processing, which reduces the computational burden by a quarter. Similar considerations can be made for $N_{\beta_0}^1$-processing under $N_{\beta_0}^1$-properness conditions.

III. Proper WSM β-Quaternion Signals

Consider a random signal $\mathbf{x}(t) \in \mathbb{Q}_\beta^p$, and the vector:

$$\boldsymbol{\eta}(t) = \left[\mathbf{x}^{\mathrm{T}}(t), \mathbf{x}^{\mathrm{T}}(t-1), \ldots, \mathbf{x}^{\mathrm{T}}(t-n+1)\right]^{\mathrm{T}}, \qquad (1)$$

such that there exists a value $\beta_0 > 0$ if $\nu = 1$ (respectively, $\beta_0 < 0$ if $\nu = 2$) satisfying that $\det(\mathbf{\Upsilon}_{\boldsymbol{\eta}}^{\nu\beta_0}(t)) \neq 0$.

Definition 3: A random signal $\mathbf{x}(t) \in \mathbb{Q}_\beta^p$ is said to be $P_{\beta_0}^1$WSM of order $n \geq 1$, denoted as $P_{\beta_0}^1$WSM(n), if it is $P_{\beta_0}^1$-proper and satisfies that

$$\hat{\mathbf{x}}^{P_{\beta_0}^1}(t|\tau \leq s) = \hat{\mathbf{x}}^{P_{\beta_0}^1}(t|s, s-1, \ldots, s-n+1),\ \forall s \leq t,$$

where the left-term denotes the estimator of $\mathbf{x}(t)$, obtained under $P_{\beta_0}^1$-processing, from the information supplied by $\{\mathbf{x}(\tau), \tau \leq s\}$, and the right-term is the estimator, also derived from $P_{\beta_0}^1$-processing, based on $\{\mathbf{x}(s), \ldots, \mathbf{x}(s-n+1)\}$.

Analogously, a $N_{\beta_0}^1$WSM(n) signal is defined.

The advantage of using a Markov signal is that it verifies a state-space model, as shown in the following result.

Proposition 1: A random signal $\mathbf{x}(t) \in \mathbb{Q}_\beta^p$ is $P_{\beta_0}^1$WSM(n) if, and only if, $\boldsymbol{\eta}(t)$ in (1), can be expressed with the equation

$$\boldsymbol{\eta}(t+1) = \mathbf{K}(t)\boldsymbol{\eta}(t) + \mathbf{w}(t), \quad t \geq n-1, \qquad (2)$$

where $\mathbf{K}(t) \triangleq \mathbf{K}(t+1,t)$, $\mathbf{K}(t,s) = \mathbf{\Upsilon}_{\boldsymbol{\eta}}^{1\beta_0}(t,s)\mathbf{\Upsilon}_{\boldsymbol{\eta}}^{1\beta_0^{-1}}(s)$, $\mathbf{w}(t)$ is a white noise uncorrelated with $\boldsymbol{\eta}(n-1)$, for $t \geq n-1$, and $\mathbf{w}(t)$ and $\boldsymbol{\eta}(n-1)$ are jointly $P_{\beta_0}^1$-proper. Consequently,

$$\mathbf{x}(t+1) = \mathbf{K}^{(p)}(t)\boldsymbol{\eta}(t) + \mathbf{w}^{(p)}(t), \quad t \geq n-1, \qquad (3)$$

where $\mathbf{K}^{(p)}(t)$ is the matrix formed by the first p-rows of $\mathbf{K}(t)$ in (2) and $\mathbf{w}^{(p)}(t)$ given by the first p-elements of $\mathbf{w}(t)$ in (2).

A similar result can be established for a $N_{\beta_0}^1$WSM(n) random signal, by substituting $P_{\beta_0}^1$ with $N_{\beta_0}^1$.

Proof: Let $\mathbf{x}(t) \in \mathbb{Q}_\beta^p$ be a $P_{\beta_0}^1$WSM(n) signal. Then, $\boldsymbol{\eta}(t)$ fulfills (2), where $\mathbf{K}(t,s)$ satisfies the condition $\mathbf{K}(t,s) = \mathbf{K}(t,\zeta)\mathbf{K}(\zeta,s)$, for $t \geq \zeta \geq s$. From (2), Property 1 in [19], and properness conditions, it is deduced that $\mathbf{w}(t)$ and $\boldsymbol{\eta}(n-1)$ are jointly $P_{\beta_0}^1$-proper. In a similar way, using the principle of recursion in (2), sufficient condition holds.

The proof for $N_{\beta_0}^1$WSM(n) signals is similar. ■

IV. Proper WSM β-Quaternion Signal Processing

In this section, a classical signal estimation problem from observations affected by packet dropouts is approached (see e.g. [26], [27], [28], [29], [30], among others).

A. Problem Statement

Let $\mathbf{x}(t) \in \mathbb{Q}_\beta^p$ be a $P_{\beta_0}^1$WSM(n) signal and consider the observation equation:

$$\begin{aligned}\mathbf{y}(t) &= \boldsymbol{\gamma}(t) \star \mathbf{z}(t) + (\mathbf{1}_p - \boldsymbol{\gamma}(t)) \star \mathbf{y}(t-1), \quad t \geq n;\\ \mathbf{y}(n-1) &= \mathbf{z}(n-1),\\ \mathbf{z}(t) &= \mathbf{x}(t) + \mathbf{v}(t), \quad t \geq n. \end{aligned} \qquad (4)$$

Product $\star$ between two β-quaternion vectors, for example, $\boldsymbol{\gamma}(t) \star \mathbf{z}(t)$, is another vector whose components, $\gamma_l(t) \star z_l(t)$, for $l = 1, \ldots, p$, are computed as $\gamma_l(t) \star z_l(t) = \gamma_{l\mathrm{r}}(t)z_{l\mathrm{r}}(t) + \mathrm{i}\gamma_{l\mathrm{i}}(t)z_{l\mathrm{i}}(t) + \mathrm{j}\gamma_{l\mathrm{j}}(t)z_{l\mathrm{j}}(t) + \mathrm{k}\gamma_{l\mathrm{k}}(t)z_{l\mathrm{k}}(t)$. Moreover, $\gamma_{l\mu}(t)$, for $\mu = \mathrm{r}, \mathrm{i}, \mathrm{j}, \mathrm{k}$, are independent Bernoulli random variables with parameters $p_{l\mu}(t)$, with value 1 if the observation is updated, and 0 if it is lost and substituted by last available one.

In addition, $\boldsymbol{\gamma}(t)$, $\mathbf{x}(t)$ and $\mathbf{v}(t)$ are assumed to be mutually independent. Furthermore, $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are considered to be jointly $P_{\beta_0}^1$ WSM (n), meaning that $\mathbf{v}(t)$ must be $P_{\beta_0}^1$ WSM(n), and $p_{l\mu}(t) = p_l(t)$, $\forall t, l, \mu$. These assumptions reduce the dimension of the augmented observation equation by a quarter.

When considering $N_{\beta_0}^1$WSM(n) signals, the same problem can be easily formulated.

B. Filtering Problem

This section approaches the LLMSE filtering problem of the signal, $\mathbf{x}(t)$, from the observations $\{\mathbf{y}(0), \ldots, \mathbf{y}(t)\}$. If the $P^1_{\beta_0}$ processing is used, the filter obtained is denoted by $\hat{\mathbf{x}}^{P^1_{\beta_0}}(t|t)$ (similarly, $\hat{\mathbf{x}}^{N^1_{\beta_0}}(t|t)$ denotes the filter calculated from $N^1_{\beta_0}$ processing). Next result shows a recursive algorithm to calculate these estimators. The only difference in how it is applied is that $\nu = 1$ is employed when $P^1_{\beta_0}$ processing is used, and $\nu = 2$ is employed when $N^1_{\beta_0}$ processing is used.

Theorem 1: The filter $\hat{\mathbf{x}}^{\mathtt{Q}_1}(t|t)$, for $\mathtt{Q}_1 = P^1_{\beta_0}, N^1_{\beta_0}$ is formed by the first p components of $\hat{\boldsymbol{\eta}}(t|t)$, recursively calculated as:

$$\hat{\boldsymbol{\eta}}(t|t) = \hat{\boldsymbol{\eta}}(t|t-1) + \boldsymbol{\theta}(t)\boldsymbol{\Omega}^{-1}(t)\boldsymbol{\varepsilon}(t),\ t \geq n, \quad (5)$$

$$\hat{\boldsymbol{\eta}}(t+1|t) = \mathbf{K}(t)\hat{\boldsymbol{\eta}}(t|t),\ t \geq n-1, \quad (6)$$

with $\hat{\boldsymbol{\eta}}(n-1|n-1) = \boldsymbol{\Upsilon}^{\nu\beta_0}_{\boldsymbol{\eta}}(n-1)\boldsymbol{\Upsilon}^{\nu\beta_0^{-1}}_{\mathbf{y}}(n-1)\mathbf{y}(n-1)$ as initial condition, and $\mathbf{K}(t)$ given in (2). Moreover, for $t \geq n$:

$$\boldsymbol{\varepsilon}(t) = \mathbf{y}(t) - \boldsymbol{\Pi}(t)\hat{\boldsymbol{\eta}}(t|t-1) - (\boldsymbol{\Delta}_p - \boldsymbol{\Pi}(t))\,\bar{\mathbf{y}}(t-1), \quad (7)$$

$$\boldsymbol{\theta}(t) = \mathbf{P}(t|t-1)\boldsymbol{\Pi}^{\mathtt{T}}(t), \quad (8)$$

$$\boldsymbol{\Omega}(t) = \boldsymbol{\Sigma}(t) + \boldsymbol{\Pi}(t)\mathbf{P}(t|t-1)\boldsymbol{\Pi}^{\mathtt{T}}(t), \quad (9)$$

with initial conditions $\boldsymbol{\varepsilon}(n-1) = \mathbf{y}(n-1)$, $\boldsymbol{\theta}(n-1) = \boldsymbol{\Upsilon}^{\nu\beta_0}_{\boldsymbol{\eta}}(n-1)$, $\boldsymbol{\Omega}(n-1) = \boldsymbol{\Upsilon}^{\nu\beta_0}_{\mathbf{y}}(n-1)$, denoting by $\boldsymbol{\Delta}_p = [\mathbf{I}_p|\mathbf{0}_{p\times p(n-1)}]$. Also, $\boldsymbol{\Pi}(t) = [\mathcal{P}(t)|\mathbf{0}_{p\times p(n-1)}]$, $\mathcal{P}(t) = \mathrm{diag}(p_1(t), \ldots, p_p(t))$, and $\boldsymbol{\Sigma}(t) = \mathrm{diag}(\xi_1(t) \ldots, \xi_p(t))$, whose elements $\xi_l(t)$ are defined as follows:

$$\xi_l(t) = 4p_l(t)\left\{(1-p_l(t))\left(\mathrm{E}\left[x_l^2(t)\right] - 2\mathrm{E}\left[x_l(t)y_l(t-1)\right] + \mathrm{E}\left[{y_l}^2(t-1)\right]\right) + \mathrm{E}\left[v_l^2(t)\right]\right\}, \quad l = 1, \ldots, p,$$

where $E[x_l^2(t)]$, $E[x_l(t)y_l(t-1)]$ and $E[{y_l}^2(t-1)]$ are obtained, respectively, from the element $l \times l$ of the matrices $\boldsymbol{\Upsilon}^{\nu\beta_0}_{\boldsymbol{\eta}}(t)$, $\boldsymbol{\Upsilon}^{\nu\beta_0}_{\boldsymbol{\eta}\mathbf{y}}(t, t-1)$, and $\boldsymbol{\Upsilon}^{\nu\beta_0}_{\mathbf{y}}(t-1)$. These matrices satisfy the following recursive formulas, for $t \geq n$:

$$\boldsymbol{\Upsilon}^{\nu\beta_0}_{\boldsymbol{\eta}}(t) = \mathbf{K}(t-1)\boldsymbol{\Upsilon}^{\nu\beta_0}_{\boldsymbol{\eta}}(t-1)\mathbf{K}^{\mathtt{H}_{3-2\nu}}(t-1) + \boldsymbol{\Upsilon}^{\nu\beta_0}_{\mathbf{w}}(t-1), \quad (10)$$

$$\boldsymbol{\Upsilon}^{\nu\beta_0}_{\boldsymbol{\eta}\mathbf{y}}(t+1, t) = \mathbf{K}(t)\left\{\boldsymbol{\Upsilon}^{\nu\beta_0}_{\boldsymbol{\eta}}(t)\boldsymbol{\Delta}_p^{\mathtt{T}}\mathcal{P}(t) + \boldsymbol{\Upsilon}^{\nu\beta_0}_{\boldsymbol{\eta}\mathbf{y}}(t, t-1)\left(\mathbf{I}_p - \mathcal{P}(t)\right)\right\}, \quad (11)$$

$$\boldsymbol{\Upsilon}^{\nu\beta_0}_{\mathbf{y}}(t) = \mathcal{P}(t)\left(\boldsymbol{\Delta}_p\boldsymbol{\Upsilon}^{\nu\beta_0}_{\boldsymbol{\eta}}(t)\boldsymbol{\Delta}_p^{\mathtt{T}} + \boldsymbol{\Upsilon}^{\nu\beta_0}_{\mathbf{v}}(t)\right) + \left(\mathbf{I}_p - \mathcal{P}(t)\right)\boldsymbol{\Upsilon}^{\nu\beta_0}_{\mathbf{y}}(t-1), \quad (12)$$

with initial conditions $\boldsymbol{\Upsilon}^{\nu\beta_0}_{\boldsymbol{\eta}}(n-1)$, $\boldsymbol{\Upsilon}^{\nu\beta_0}_{\boldsymbol{\eta}\mathbf{y}}(n, n-1) = \mathbf{K}(n-1)\boldsymbol{\Upsilon}^{\nu\beta_0}_{\boldsymbol{\eta}}(n-1)\boldsymbol{\Delta}_p^{\mathtt{T}}$ and $\boldsymbol{\Upsilon}^{\nu\beta_0}_{\mathbf{y}}(n-1) = \boldsymbol{\Delta}_p\boldsymbol{\Upsilon}^{\nu\beta_0}_{\boldsymbol{\eta}}(n-1)\boldsymbol{\Delta}_p^{\mathtt{T}} + \boldsymbol{\Upsilon}^{\nu\beta_0}_{\mathbf{v}}(n-1)$. Moreover, $\mathbf{P}(t|t)$ is recursively computed from:

$$\mathbf{P}(t|t) = \mathbf{P}(t|t-1) - \boldsymbol{\theta}(t)\boldsymbol{\Omega}^{-1}(t)\boldsymbol{\theta}^{\mathtt{H}_{3-2\nu}}(t),\ t \geq n, \quad (13)$$

$$\mathbf{P}(t+1|t) = \mathbf{K}(t)\mathbf{P}(t|t)\mathbf{K}^{\mathtt{H}_{3-2\nu}}(t) + \boldsymbol{\Upsilon}^{\nu\beta_0}_{\mathbf{w}}(t),\ t \geq n-1, \quad (14)$$

with initial condition:

$$\mathbf{P}(n-1|n-1) = \boldsymbol{\Upsilon}^{\nu\beta_0}_{\boldsymbol{\eta}}(n-1) - \boldsymbol{\Upsilon}^{\nu\beta_0}_{\boldsymbol{\eta}\mathbf{y}}(n-1) \times \boldsymbol{\Upsilon}^{\nu\beta_0^{-1}}_{\mathbf{y}}(n-1)\boldsymbol{\Upsilon}^{\nu\beta_0^{\mathtt{H}_{3-2\nu}}}_{\boldsymbol{\eta}}(n-1).$$

Finally, the error associated to each element of $\mathbf{x}(t)$ is:

$$\epsilon^{\mathtt{Q}_1}(x_j(t)) = \frac{|\beta_0| + 1}{2|\beta_0|}\mathcal{R}\left\{\left[\mathbf{P}(t|t)\right]_{jj}\right\}, \quad j = 1, \ldots, p. \quad (15)$$

Proof: As known, the LLMSE linear filter is expressed as:

$$\hat{\bar{\boldsymbol{\eta}}}(t|t) = \sum_{i=1}^{t}\bar{\boldsymbol{\theta}}(i)\boldsymbol{\Omega}^{-1}(i)\boldsymbol{\varepsilon}(i), \quad (16)$$

with $\bar{\boldsymbol{\theta}}(i) = E[\bar{\boldsymbol{\eta}}(t)\boldsymbol{\varepsilon}^{\mathtt{H}_{3-2\nu}}(i)]$, and $\boldsymbol{\Omega}(i) = E[\boldsymbol{\varepsilon}(i)\boldsymbol{\varepsilon}^{\mathtt{H}_{3-2\nu}}(i)]$. In [19], the existence and uniqueness of the estimator are guaranteed (this result is used in this proof to derive all the estimators). Then, from (16), this expression is obtained:

$$\hat{\bar{\boldsymbol{\eta}}}(t|t) = \hat{\bar{\boldsymbol{\eta}}}(t|t-1) + \bar{\boldsymbol{\theta}}(t)\boldsymbol{\Omega}^{-1}(t)\boldsymbol{\varepsilon}(t), \quad (17)$$

and then, (5) is deduced from (17) by applying properness. Now from (2), (6) is easily derived under properness.

Moreover, the augmented form of (4), taking the augmented processes, under properness, can be reduced as

$$\mathbf{y}(t) = \boldsymbol{\mathcal{D}}^{\gamma}(t)\boldsymbol{\Gamma}_p\bar{\boldsymbol{\eta}}(t) + \boldsymbol{\mathcal{D}}^{\gamma}(t)\bar{\mathbf{v}}(t) + \boldsymbol{\mathcal{D}}^{1-\gamma}(t)\bar{\mathbf{y}}(t-1), \quad (18)$$

with $\boldsymbol{\Gamma}_p = \mathbf{I}_4 \otimes \boldsymbol{\Delta}_p$, $\boldsymbol{\mathcal{D}}^{\gamma}(t) = \boldsymbol{\mathcal{F}}_p\,\mathrm{diag}(\boldsymbol{\gamma}^r(t))\boldsymbol{\mathcal{T}}_p^{-1}$, $\boldsymbol{\mathcal{D}}^{1-\gamma}(t) = \boldsymbol{\mathcal{F}}_p\,\mathrm{diag}(\mathbf{1}_{4p} - \boldsymbol{\gamma}^r(t))\boldsymbol{\mathcal{T}}_p^{-1}$, with $\boldsymbol{\mathcal{T}}_p$ defined in [19] and $\boldsymbol{\mathcal{F}}_p = [1, \mathrm{i}, \mathrm{j}, \mathrm{k}] \otimes \mathbf{I}_p$. Then, from (18), it follows that

$$\hat{\mathbf{y}}(t|t-1) = \breve{\boldsymbol{\mathcal{D}}}^{\gamma}(t)\boldsymbol{\Gamma}_p\hat{\bar{\boldsymbol{\eta}}}(t|t-1) + \breve{\boldsymbol{\mathcal{D}}}^{1-\gamma}(t)\bar{\mathbf{y}}(t-1), \quad (19)$$

with $\breve{\boldsymbol{\mathcal{D}}}^{\kappa}(t) = E[\boldsymbol{\mathcal{D}}^{\kappa}(t)]$, for $\kappa = \gamma, 1-\gamma$. Hence, since $\boldsymbol{\varepsilon}(t) = \mathbf{y}(t) - \hat{\mathbf{y}}(t|t-1)$, (7) is obtained from (19) under properness conditions.

From (18) and (19), the following expression is found:

$$\boldsymbol{\varepsilon}(t) = \bar{\boldsymbol{\mathcal{D}}}^{\gamma}(t)\boldsymbol{\Gamma}_p\bar{\boldsymbol{\eta}}(t) + \boldsymbol{\mathcal{D}}^{\gamma}(t)\bar{\mathbf{v}}(t) + \bar{\boldsymbol{\mathcal{D}}}^{1-\gamma}(t)\bar{\mathbf{y}}(t-1) + \breve{\boldsymbol{\mathcal{D}}}^{\gamma}(t)\boldsymbol{\Gamma}_p\tilde{\bar{\boldsymbol{\eta}}}(t|t-1), \quad (20)$$

with $\bar{\boldsymbol{\mathcal{D}}}^{\kappa}(t) = \boldsymbol{\mathcal{D}}^{\kappa}(t) - \breve{\boldsymbol{\mathcal{D}}}^{\kappa}(t)$, for $\kappa = \gamma, 1-\gamma$, and $\tilde{\bar{\boldsymbol{\eta}}}(t|t-1) = \bar{\boldsymbol{\eta}}(t) - \hat{\bar{\boldsymbol{\eta}}}(t|t-1)$. Then, to derive (8), $\bar{\boldsymbol{\theta}}(t)$ is computed from (20) and the resultant expression is used to deduce $\boldsymbol{\theta}(t) = E[\boldsymbol{\eta}(t)\boldsymbol{\varepsilon}^{\mathtt{H}_{3-2\nu}}(t)]$ by applying properness. Moreover, (9) for $\boldsymbol{\Omega}(t) = E[\boldsymbol{\varepsilon}(t)\boldsymbol{\varepsilon}^{\mathtt{H}_{3-2\nu}}(t)]$ is derived from (20), after some calculations. Expressions (10)–(12) are easily devised from (2) and (18) by characterizing for properness.

In addition, (13) for $\mathbf{P}(t|t) = E[\tilde{\boldsymbol{\eta}}(t|t)\tilde{\boldsymbol{\eta}}^{\mathtt{H}_{3-2\nu}}(t|t)]$ and (14) for $\mathbf{P}(t+1|t) = E[\tilde{\boldsymbol{\eta}}(t+1|t)\tilde{\boldsymbol{\eta}}^{\mathtt{H}_{3-2\nu}}(t+1|t)]$, with $\tilde{\boldsymbol{\eta}}(s|t) = \boldsymbol{\eta}(s) - \hat{\boldsymbol{\eta}}(s|t-1)$, for $s = t, t+1$, are immediately derived from (2), (5) and (6). Finally, (15) is deduced from Corollary 1 in [19]. ■

V. Numerical Results

The aim is to illustrate the improvement of the proposed estimators over the ones obtained by using quaternion processing [14], under first-order properness. The quaternion filter is denoted by $\hat{\mathbf{x}}^{\mathtt{QSL}}(t|t)$, and its associated error by $\epsilon^{\mathtt{QSL}}(t)$.

A. Example 1: $P^1_{\beta_0}$ Processing From Simulated Data

Let $x \in \mathbb{Q}_\beta$ be a $P^1_{\beta_0}$WSM(2) signal satisfying (3) for $\mathbf{K}^{(1)}(t) = [0.9, -0.2]$, and $\mathbf{w}^{(1)}(t) \triangleq w(t)$ is a β-quaternion white noise with $\boldsymbol{\Upsilon}_{\mathbf{w}^r}(t, s) = \boldsymbol{\mathcal{G}}\delta_{ts}$, and $\boldsymbol{\mathcal{G}}$ is of the form:

$$\boldsymbol{\mathcal{G}} = \begin{pmatrix} |\beta_0|\varrho_1 & |\beta_0|\varrho_2 & \varrho_3 & \varrho_4 \\ -|\beta_0|\varrho_2 & |\beta_0|\varrho_1 & \varrho_4 & -\varrho_3 \\ \varrho_3 & \varrho_4 & \varrho_1 & \varrho_2 \\ \varrho_4 & -\varrho_3 & -\varrho_2 & \varrho_1 \end{pmatrix} \quad (21)$$

Likewise, the autocorrelation matrix of the real vector of $\boldsymbol{\eta}(1) = [x(1), x(0)]^{\mathtt{T}}$ satisfies that the submatrices $\boldsymbol{\Upsilon}_{\mathbf{x}^r}(1)$, $\boldsymbol{\Upsilon}_{\mathbf{x}^r}(0)$, and $\boldsymbol{\Upsilon}_{\mathbf{x}^r}(1, 0)$ have the form of the matrix $\boldsymbol{\mathcal{G}}$ given in (21), with the following values for each of them:

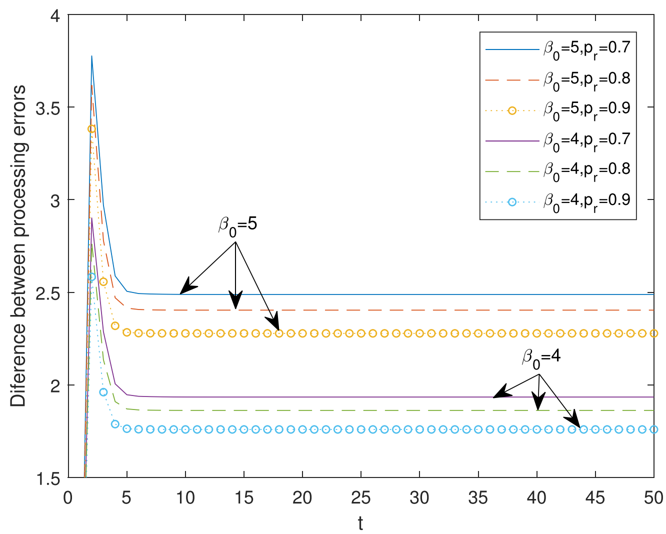

Fig. 1. Differences $D^{P^1_{\beta_0}}(t)$ by setting $\beta_0 = 4, 5$ and $p_{\rm r} = 0.7, 0.8, 0.9$.

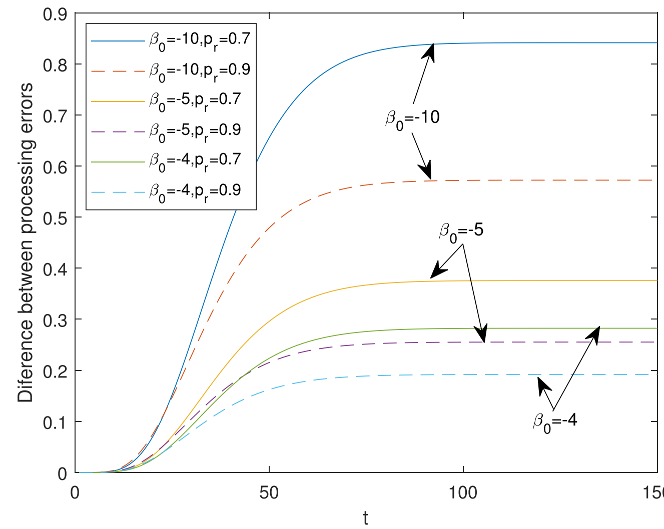

Fig. 2. Differences $D^{N^1_{\beta_0}}(t)$ for $\beta_0 = -4, -5, -10$ and $p_{\rm r} = 0.7, 0.9$.

- $\boldsymbol{\Upsilon}_{\mathbf{x}^r}(1) = \boldsymbol{\Upsilon}_{\mathbf{x}^r}(0)$: $\varrho_1 = 4$, $\varrho_2 = \varrho_4 = 0$, $\varrho_3 = -1$,
- $\boldsymbol{\Upsilon}_{\mathbf{x}^r}(1, 0)$: $\varrho_1 = 0.4$, $\varrho_2 = -0.1$, $\varrho_3 = -0.5$, $\varrho_4 = 0.6$.

The observation equation is given by (4), where p_μ, for $\mu = {\rm r, i, j, k}$, are constant in time, and $v(t)$ is a β-quaternion white noise with $\boldsymbol{\Upsilon}_{\mathbf{v}^r}(t,s) = \boldsymbol{\mathcal{G}}\delta_{ts}$, and $\boldsymbol{\mathcal{G}}$ given in (21).

To guarantee that the signal is $P^1_{\beta_0}$WSM(2), with $\beta_0 = 4, 5$, the values $\varrho_1 = 1$, $\varrho_2 = 0$, $\varrho_3 = 0.5$, $\varrho_4 = 0.1$ and $\varrho_1 = 3$, $\varrho_2 = 0$, $\varrho_3 = 0.5$, $\varrho_4 = 0.1$ in (21) have been taken for $\boldsymbol{\Upsilon}_{\mathbf{w}^r}(t,s)$ and $\boldsymbol{\Upsilon}_{\mathbf{v}^r}(t,s)$, respectively. Moreover, to ensure that the signal and the observation are jointly $P^1_{\beta_0}$, $p_{\rm r} = p_{\rm i} = p_{\rm j} = p_{\rm k}$ is considered.

Then, to illustrate the effectiveness of the estimators proposed, the β-quaternion filtering errors in (15), $\epsilon^{P^1_{\beta_0}}(t)$, are compared with the quaternion ones, $\epsilon^{\texttt{QSL}}(t)$. Specifically, the differences between them, that is, $D^{P^1_{\beta_0}}(t) = \epsilon^{\texttt{QSL}}(t) - \epsilon^{P^1_{\beta_0}}(t)$, have been computed and displayed in Fig. 1, for $p_{\rm r} = 0.7, 0.8, 0.9$ and $\beta_0 = 4, 5$. As observed, since the differences are positive, the fact that better results are obtained with the estimators proposed than with those obtained with quaternion processing is concluded. Moreover, in these particular cases, these differences are smaller as the probabilities increase, and also as $|\beta_0|$ decreases.

B. Example 2: $N^1_{\beta_0}$ Processing in a Real-Life Application Model

Consider now the discrete-time model equivalent to the general equation of motion [25]:

$$\mathbf{x}(t+1) = \begin{pmatrix} 1 & \Delta T \\ 0 & 1 \end{pmatrix} \mathbf{x}(t) + \begin{bmatrix} \Delta T^2/2 \\ \Delta T \end{bmatrix} w(t),$$

with $\mathbf{x}(t) = [\varphi(t), \phi(t)]^{\rm T}$, where φ denotes the variable of interest, ϕ delineating its range of variation, and w being the system input. This equation is useful when applied to a broad spectrum of practical scenarios that covers many areas including bearings-only and rotation tracking. Note that this model represents a two-dimensional WSM(1) signal. It is assumed that the initial condition $\mathbf{x}(0) = \mathbf{0}_{2\times 1}$, and $\Delta T = 0.04$ represents the sampling interval. Moreover, the same hypotheses about $w(t)$ as in Example 1 are considered.

(4) models the observations, where $\mathbf{v}(t) = [v_1(t), v_2(t)]^{\rm T}$ is a β-quaternion white noise, with $\boldsymbol{\Upsilon}_{\mathbf{v}^r_l}(t,s) = \boldsymbol{\Upsilon}_{\mathbf{v}^r}(t,s)$ as in Example 1, for $l = 1, 2$, and $p_{l\mu} = p_\mu$, for $l = 1, 2$, $\mu = {\rm r, i, j, k}$. To ensure the $N^1_{\beta_0}$ scenario, with $\beta_0 = -4, -5, -10$, the following values have been taken in (21): $\varrho_1 = 4$, $\varrho_2 = \varrho_3 = \varrho_4 = 0$, for $\boldsymbol{\Upsilon}_{\mathbf{w}^r}(t,s)$, and $\varrho_1 = 3$, $\varrho_2 = \varrho_3 = \varrho_4 = 0$, for $\boldsymbol{\Upsilon}_{\mathbf{v}^r}(t,s)$.

By setting different values of $p_{\rm r}$, the differences between the errors obtained from a $N^1_{\beta_0}$ processing (15), and its counterparts in the quaternion domain are calculated: $D^{N^1_{\beta_0}}(t) = \epsilon^{\texttt{QSL}}(t) - \epsilon^{N^1_{\beta_0}}(t)$. The results for $p_{\rm r} = 0.7, 0.9$ are displayed in Fig. 2. Similar conclusions to those drawn from Fig. 1 can also be made from this graph. This real-life example illustrates how the β-quaternion processing provides better results than those obtained with quaternion one under properness conditions.Figs. 1 and 2 have been displayed for different time instants, to also show the error convergence. Note that properness has no effect on the convergence (see e.g. [26]).

VI. Conclusion

The main contributions presented in this letter are: firstly, the representation of β-quaternion wide-sense stationary signals in the form of a state equation, which is a key factor as it enables the problem of estimating a state-space model to be studied; secondly, a recursive filtering algorithm is proposed under first-order properness conditions, which leads to the optimal estimators, and this is achieved by considerably reducing the computational load. Moreover, the fact that the estimators proposed perform better than those obtained with a quaternion processing is numerically illustrated.

In [19], another properness of the β-quaternions has been introduced, called the second-order one. Using the same methodology, Algorithm 1 can be easily extended to a second-order proper setting, reducing the computational load by half (details omitted for brevity).

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