Lambert W Function Applications in Electrical Insulation Engineering

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***Abstract*— Mathematical modelling is a fundamental instrument in the development of theoretical validation for experimental results and in their interpretation. The gist of this paper is the application of a special multi-valued function, called Lambert W function, to several aspects of electrical engineering. This multi-valued function serves the purpose of inverting a vast gamut of expressions involving exponential functions, and can be used in many models concerning electrical insulation issues, yielding lower computational times as opposed to other methods solving nonlinear problems. In particular, this paper presents the analytical calculation of parameters for capacitor field and equipotential lines, Paschen's law inversion, a model for electric field distribution in power cables, the inversion of Schottky and Dissado, Montanari, Mazzanti model for cable life estimation.**

***Index Terms*— Electric field distribution, Gas discharges, Lifetime modelling, Mathematical methods.**

I. Introduction

MODELLING of physical-chemical behavior of materials is a key step for reducing the effort in experimental testing and guaranteeing efficient and reliable application of electrical equipment in real conditions. In this framework, mathematical modelling aims at providing the most accurate approach, even though it yields a plethora of approximations from real experimental outcomes. Nonetheless, available mathematical models for electrical engineering applications are characterized by transcendental equations, which are usually very difficult to solve analytically [1, 2].

Lambert W function can be used to solve problems involving the sum or the product between a linear and an exponential term exactly [3]. This behavior is quite recurrent in several fields of science, from condensed matter physics [4] to enzyme catalytic reaction kinetics [5]. Computational numerical implementation of Lambert W function is possible. In addition, it helps in the development of more accurate models in a considerable number of cases. This function has already been used in several fields connected to electrical engineering: the main use is for the exact resolution of the equivalent circuit of first-generation photovoltaic cells [6]. In this case, the contemporary presence of a diode, characterized by an exponential relationship among the current and the voltage, and resistances, linear components, renders Lambert W function the most suitable mathematical tool for this scope [7]. This led to more accurate estimations of parameters connected to their performance, such as the fill factor. In some particular cases, it is also possible to implement a similar solution for third generation photovoltaic cells, but the result is extremely more complicated due to the higher complexity of the equivalent circuit. Metals exhibit a simple linear relationship between their thermal and electronic conductivities, namely Wiedemann-Franz law. However, experimental results can diverge from this ideal behavior. Recent research [8] shows that expressions involving a generalization of Lambert W function correctly describe the deviation from linearity and optimize the figure of merit Z_T, a parameter strictly related to the efficiency of thermoelectric materials.

With reference to electrical insulation engineering, several important employments of Lambert W function have been attained. One of the main topics which can be addressed is space charge effect on electrical insulating materials and semiconductors, previously studied analytically both with [9] and without Lambert W function employment [10]. Even though this topic has seen a significant interest especially in the last decades due to its impact on HVDC applications, to the Authors' knowledge, the most complete theoretical exposition of space charge was developed by Lampert and Mark in the 1970s [11]. The validation of this approach has been also confirmed by a recent work from Guedes et al. [12]. Under the assumption of an exponential trap distribution in the sample, the model uses a generalized Lambert W function to give the expression for the current flowing through a material which

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respects Child's law. This latter represents the typical square law current-voltage characteristic of an insulating material. An additional application is related to the electric field distribution in a coaxial geometry, characteristic of the electrical insulation of HVDC cables, was investigated by Boggs [13]. In particular, it was shown that under the hypothesis of an inhomogeneous field due to accumulation of space charge, the electric field distribution can be calculated by means of Lambert W function. Finally, an analysis of Schottky emission found the expression for the temperature required to have a certain value of current density [14]. This could be a useful instrument in case of planar geometry design for electrical insulation applications.

The aim of this paper is to further expand the application of the Lambert W function to other electrical applications which have not been totally covered in literature. In addition, comparison with conventional approaches is reported to validate the proposed method, highlighting a reduction in computational times. Furthermore, the closed form expression for the solution can be analyzed in a simpler way once the basics of Lambert W function are understood. In this article, the law will be applied to three main areas: gas discharges, electric field distribution and charge transport in solid insulating samples, lifetime modelling. The former topic is covered by Rogowski profile [15] parameters calculation and by Paschen's law inversion [16], the second by Eoll theory [17-18], the latter by Dissado, Montanari and Mazzanti (DMM) model [19] for aging owing to space charge.

II. Fundamentals of Lambert W Function

Let's suppose x is a real variable and define the function:

$$f(x) = xe^x \tag{1}$$

The Lambert W function W(x) is defined as the inverse of the function $f(x)$ in the interval [-1/e,+∞). Therefore, if we write:

$$xe^x = a \tag{2}$$

the solution to (2) is:

$$x = W(a) \tag{3}$$

This definition already shows the fact that the presence of both a linear and an exponential function require the introduction of a special function in order to invert it. Nonetheless, this definition is ambiguous by itself, as W(x) can assume two different values in the interval [-1/e,0[, as shown in Fig. 1. For this reason, it is necessary to distinguish these two possibilities as two branches of the function defined in the [-1/e, +∞) and in [-1/e,0), which are called 0 branch and -1 branch, respectively.

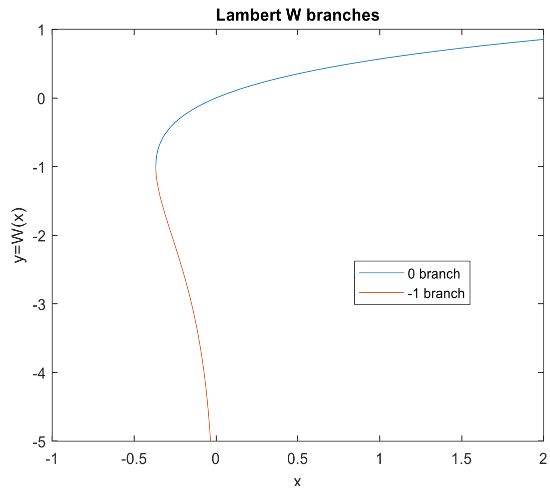

Fig. 1 Lambert W function branches

This separation of the two branches is fundamental in the inversion of functions which have a non-monotonic behavior, as the absence of its implementation would lead to the outcome of a double-valued Lambert W function, as will be shown in its application to Paschen's law. The Lambert W function can be generalized to solve a more general equation, which is (4):

$$x = a + be^{cx}. \tag{4}$$

Its solution is:

$$x_k = a - \frac{1}{c}W_k(-bce^{ac}). \tag{5}$$

In (4) and (5) a, b and c are real parameters, and k can assume the values 0 and -1, representing the 0 and -1 branches, respectively.

Routines for the calculation of $W(x)$ are provided in several numerical software packages, including MATLAB, as a function called *lambertw*, and Python.

Compared to the symbolic solving method in MATLAB, *vpasolve*, the numeric (approximate) *lambertw* MATLAB function requires considerably smaller computational times, especially for high numerousness of the analyzed array. In particular, when used for solving (2) for an array a of 1000 elements, the time is reduced from around forty seconds for *vpasolve* to almost ten milliseconds for *lambertw*. This divergence further increases when considering more complex problems, thus assessing Lambert W function method as less time consuming.

III. Numerical Solution

To numerically evaluate $W(a)$, i.e., the Lambert function of a generic real number a, one must solve a transcendental equation. $W(a)$ is indeed the solution of (6):

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$$xe^x - a = 0. \tag{6}$$

Fortunately, calculating the zeros of the former expression is a problem that can be solved by many root-finding algorithms. One may, for example, use the well-known Newton's method (also known as Newton-Raphson algorithm), which is based on an expansion of the unknown function $f(x) = xe^x - a$. The approximate solution is updated from the iteration k to the next one $k+1$ with:

$$x^{k+1} = x^k - \frac{f(x^k)}{f^{'}(x^k)}, \tag{7}$$

where:

$$f'(x) = e^x\,(1+x) \tag{8}$$

The criteria for stopping the iterative procedure may be based on how close to zero is $f(x^k)$ or, more often, on the change of the tentative solution $x^{k+1} - x^k$. The order of convergence of the method is quadratic [20]; this means that – at least when the initial guess x^0 is reasonably close to the real root – Newton's method is considerably faster than linearly converging methods, e.g., the simple bisection method.

Let's now consider the second-order derivative of $f(x)$:

$$f^{''}(x) = e^x\,(2+x). \tag{9}$$

The simplicity with which this expression can be computed suggests the use of a higher-order convergence method. Applying Newton's method (7) to the function $g(x) = \frac{f(x)}{\sqrt{|f'(x)|}}$ instead of $f(x)$ leads to Halley's method [21]:

$$x^{k+1} = x^k - \frac{f(x^k)}{f^{'}(x^k)}\left[1 - \frac{f(x^k)}{f^{'}(x^k)}\,\frac{f^{''}(x^k)}{2f^{'}(x^k)}\right]^{-1} \tag{10}$$

Halley's method has a cubic convergence rate, and is used in several codes for the evaluation of the Lambert W function, including the built-in MATLAB, MAPLE and Mathematica functions [21-22], the GNU Scientific Library (GSL) [23] and Istvan Mezo's C++ routine [24]. It is worth realizing that both Newton's method and Halley's method are examples of Householder's methods (of order 1 and 2, respectively). $f(x)$ is infinitely continuously derivable, and its differentiation is inexpensive given that: $f^n(x) = e^x\,(n+x)$. Therefore, one may also consider using methods with orders higher than two to get faster rates of convergence. Veberič has shown in [25] that the iterative method proposed by Fritsch in [26], which is fourth-order accurate, grants substantial advantages in terms of computation time over Halley's method. Finally, it should be noted that the performance of all mentioned iterative schemes depends on the quality of the initial guess; Strategies to determine the initial guess for (10) and for the Fritsch method can be found in [21] and [26], respectively.

IV. Applications

In the following subsections, Lambert W functions will be utilized to find variables which are implicit in the equations modelling field lines in a capacitor, Paschen's law, Eoll theory for HVDC cables, Schottky emission and DMM lifetime modelling. The final expressions containing Lambert W functions, except for when explicitly said, are on the x axis of the plots representing them.

A. Electric field distribution in a capacitor

Other valuable attempts at finding an expression for electric field outside a capacitor were made. In particular, in [27] an expression was found, but the electric field lines were not drawn. In order to get a smoother geometric shape, Rogowski [15] used the Maxwell solution of the electric fields associated with a finite flat plane above an infinite flat plane.

This theoretical approach managed to link field lines to the spatial positions x and y in the case of air insulation between two electrodes. This correlation can be represented by the

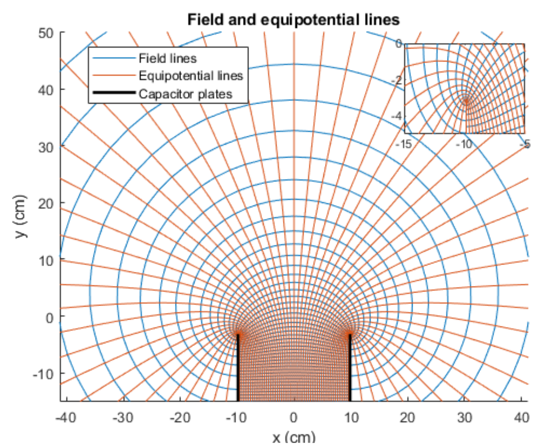

Fig. 2 Plane plate capacitor field and equipotential lines

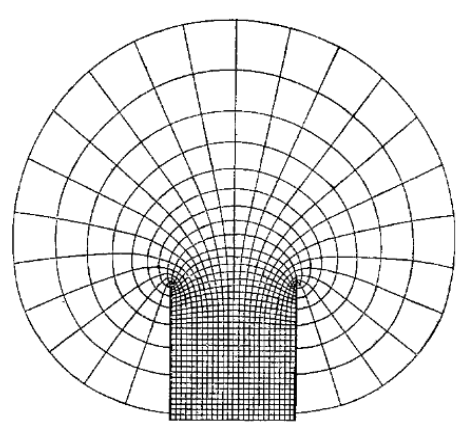

Fig. 3 Rogowski field lines (after [14])

following equations:

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$$x = A(\varphi + e^{\varphi}\cos(\psi)) \tag{11}$$

$$y = A(\psi + e^{\varphi}\sin(\psi)) \tag{12}$$

where A is d/π, with d the distance between the electrodes, φ defines an electrical field line and ψ, proportional to the potential and in the range [0,π]. The electrodes are usually obtained following the profile given by:

$$\psi = \pi/2 \tag{13}$$

In this case, equation (11) is simply a linear expression. Nonetheless, it may be important to determine the potential and the field intensity in other points of the air domain, where ψ assumes different values. In this way, edge effects are taken into account. Assigning different values to ψ, it is possible to obtain the following expression for φ:

$$\varphi = \frac{x}{A} - W\left(\cos(\psi)\, e^{\frac{x}{A}}\right) \tag{14}$$

After φ has been found in this manner, the corresponding value of x and y can be calculated through (11-12). Also, the potential V and electric field E are:

$$E(\varphi,\psi) = \frac{V_0}{d\sqrt{1 + e^{2\varphi} - 2e^{\varphi}\cos(\psi)}} \tag{15}$$

$$V = \frac{V_0}{\pi}\psi \tag{16}$$

The field and equipotential lines corresponding to 10 kV applied to an interelectrodic distance of 20 cm are plotted in Fig. 2, and are quite in agreement with Fig. 2 of [14], here reported as Fig. 3, representing field and equipotential lines in Maxwell theory for a capacitor. The insight also shows the classical denser concentration of field lines near the edges.

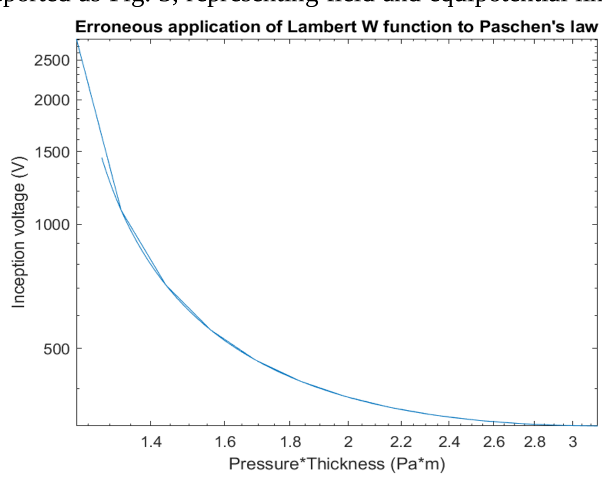

Fig. 4 Paschen's law inversion without distinction of the two branches

B. Paschen's law inversion

Paschen's law expresses the relationship between the breakdown voltage VB of a gaseous medium and the pressure p applied to it in a specific geometrical configuration. This law, which is widely used in the design of high pressure switches, relates the applied electrical stress with the increase in kinetic energy of charge carriers in the inter-electrodic region. This causes the rise in the probability of inelastic collisions with neighboring neutrals. These ones lead to the release of new electrons, which may trigger the ionization of other surrounding particles. As a consequence, an exponential avalanche process described by Townsend first coefficient will take place. In particular this last parameter is inversely proportional to the free mean path of a charge carrier, which in turn is inversely proportional to the pressure. Thus, after some simple mathematical considerations, it is possible to obtain Paschen's law:

$$V_B = \frac{Bpd}{\ln(Apd) - \ln\left(\ln\left(1 + \frac{1}{\gamma}\right)\right)} \tag{17}$$

where d is the interelectrode distance (or gap length), A and B are constant coefficients which depend upon the employed gas, and γ, the second Townsend coefficient, which depends on both the gas and the cathode. The Lambert W function can be used to solve the equation in the following way, whose details are shown in Appendix A:

$$pd = a - W\left(-\frac{B}{V_B}e^{a}\right) \tag{18}$$

where:

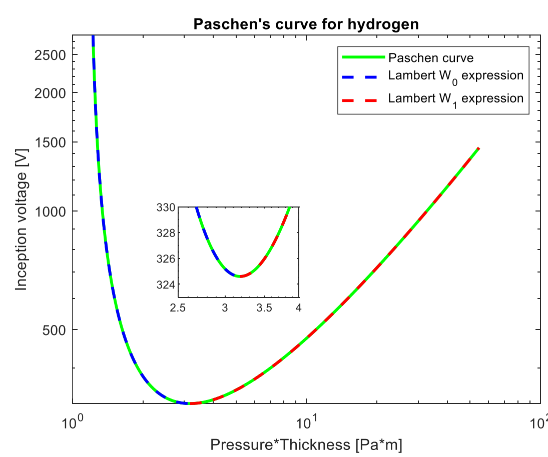

Fig. 5 Paschen's law inversion with distinction of the two branches

$$a = -ln(A) + ln\left(ln\left(1 + \frac{1}{\gamma}\right)\right) \tag{19}$$

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As known, Paschen curve plots have a minimum V_B, and if values lower than this value are inserted in (18), it does not return real values, but complex ones; another consequence is that two possible pressure values are available for each acceptable breakdown voltage value. This leads to a possible ambiguity in the implementation of (18), as shown in Fig. 4.

To solve this issue, it is sufficient to distinguish between the two branches of the Lambert W function, as exposed in Section II: the 0 branch and the -1 branch provide the solutions below and above the pd value corresponding to the Paschen's minimum:

$$pd = \begin{cases} pd_0 = a - W_0\left(-\frac{B}{V_B}e^a\right) \text{ below minimum (a)} \\ pd_{-1} = a - W_{-1}\left(-\frac{B}{V_B}e^a\right) \text{ after minimum } (b) \end{cases} \quad (20)$$

Equation (20a) is assigned to the part of the plot with lower pd values because the expression contains a 0 branch, which by itself is higher than the -1 branch (see Fig. 1), with changed sign. The parameters used in the simulation are those of hydrogen, namely A=3.6, B=102, γ=0.015.

Fig. 5 reports the comparison between the conventional Paschen law expressed by (17) and the two branches of the curve obtained by applying Lambert W function, as per (20). It is evident that the two curves are practically superposed throughout the *pd* region considered, exception given for the minimum (insight in Fig. 5), which is excluded from the calculation due to the change of the expression from (20a) to (20b). More importantly, this approach, which could be used for several applications, such as GIS design, avoids approximated methods (e.g., look-up tables) for the determination of the correct p value. In case Paschen's law is inverted for a high numerousness of V_B values, Lambert W function considerably diminishes computational times.

C. Electric field distribution in cables

In the backdrop of recent developments of electrical energy transmission, DC cables compete with AC ones. The apparent economic shortcomings, represented by the need of converters for interfacing with AC grids, can be strongly abated in case of extremely high voltage transmission over long distances. Among the most critical aspects which influence the operating lifetime of a cable, there is the concurrent effect of electrical field and temperature on electrical conductivity in the insulating layer.

In this context, several theories lay the groundwork for an analytical approach to assess the possible effects of this conjoint applied stress. More specifically, Eoll [17-18] developed a theory which models the electric field distribution in a coaxial cylinder capacitor filled with a polymer, which is a typical geometry for HVDC cable insulation. It starts from the

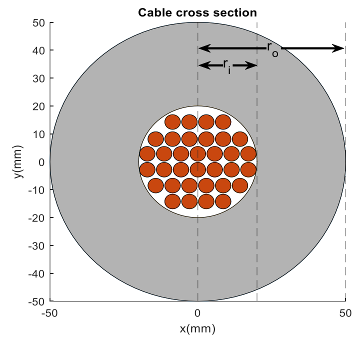

Fig. 6 HVDC cable cross section

TABLE I
PARAMETERS FOR SIMULATION OF HVDC CABLE

a	b	r_o	r_i	σ_0	ΔT	$T(r_o)$
K^{-1}	m/V	mm	mm	S/m	°C	°C
0.15	10^{-8}	50	20	10^{-17}	15	20

nonlinear behavior of electrical conductivity with temperature and electric field:

$$\sigma(E,T) = \sigma_0 e^{a(T-T_0)+b(E-E_0)} \quad (21)$$

where σ_0 is the intrinsic electrical conductivity of the material in a given thermal condition, T is the temperature, T_0 is a reference temperature (usually 0 or 20°C), E_0 is a reference electric field (usually 0 kV/mm) and a and b are parameters dependent on the investigated material. Developing other equations which describe the thermal losses, exposed in [16-17], equation (22) is obtained:

$$\frac{I}{2\pi r E(r)\sigma_0} = e^{aT(r_o)+bE(r)}\left(\frac{r_o}{r}\right)^A \; with \; A = \frac{a\Delta T}{\ln\left(\frac{r_o}{r_i}\right)} \quad (22)$$

where E(r) is the electric field at radial position r, $T(r_0)$ the temperature at outer radius, r_o the outer radius, r_i the inner radius, r the generic radius, ρ_0 the resistivity at T_0, T_0 the reference temperature, ΔT the temperature difference between the extremes, *I* the current flowing through the insulation.

A cross section with the geometrical parameters listed in Table I is shown in Fig. 6.

At this point, Eoll employed an approximation for the exponential term with E(r), namely

$$e^{-bE(r)} \approx \left(\frac{eE(r)}{E_m}\right)^{-bE_m} \quad (23)$$

leading to the equation (24):

$$E(r) = \frac{\delta U_0\left(\frac{r}{r_0}\right)^{\delta-1}}{r_0\left(1-\left(\frac{r_i}{r_o}\right)^{\delta}\right)}, \delta = \frac{\frac{a\Delta T}{\ln\left(\frac{r_o}{r_i}\right)} + \frac{bU_0}{r_o - r_i}}{1 + \frac{bU_0}{r_o - r_i}} \quad (24)$$

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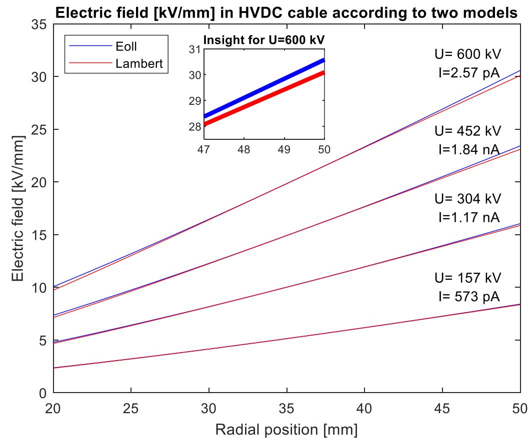

Fig. 7 Electric field in Eoll and Lambert models

Equation (24) achieves a good approximation, especially when the electric field does not drift significantly from the average electric field inside the cable. On the contrary, important deviations from the real electric field distribution may arise by the employment of (24) where the electric field is not constant e.g., in the presence of space charge in the insulation bulk. In these circumstances, a more accurate solution may be given by the deployment of Lambert W function, as explained further in Appendix B:

$$E(r) = \frac{W(c(r)b)}{b} \tag{25}$$

where:

$$\mathrm{c(r)} = \frac{e^{-aT(r_o)}I}{2\pi r\sigma_0}\left(\frac{r}{r_o}\right)^A \tag{26}$$

and a power law has not been solved yet, to the best of the authors' knowledge.

Nonetheless, Lambert W function does not get a closed form expression for the voltage. This happens because the integration of the composite function of a Lambert W function authors' knowledge. Thus, an approximation is introduced by performing numerical integration. In Fig. 7, the electric field distribution is shown, whereas Fig. 8 shows the maximum percentage error in the calculation of the electric field when Eoll's model is used instead of Lambert's one. As shown in Fig. 7, Eoll's approximation is extremely accurate to evaluate the electric field distribution in the middle of the cable, but slightly differs from the exact value near the inner and outer radial positions. Furthermore, Fig. 8 displays an increasing difference between Eoll's and Lambert's models when the nominal voltage increases. From this figure, it may be stated that, though for voltages lower than 100 kV Eoll's solution does not bring

TABLE III

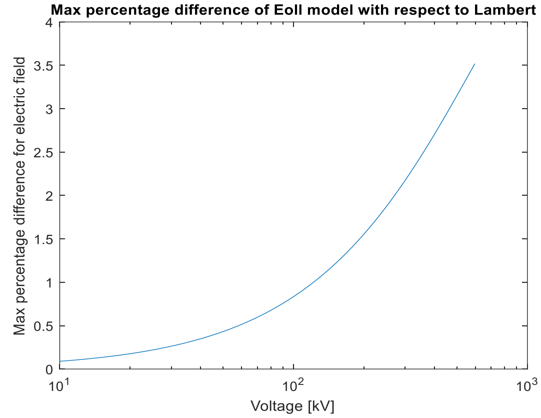

Fig. 8 Percentage difference of Eoll model for maximum electric field as a function of rated voltage

significant errors in the calculation of electric field, for voltages higher than this value the error follows an exponential behaviour. On the contrary, Lambert W permits a quite accurate evaluation of electric field, introducing an almost negligible error in the computation of the voltage by numerical integration. This result is particularly of interest given the current HVDC cables, whose design voltage is nowadays higher than 525 kV.

D. DMM first model

Space charge is a known cause of degradation in DC applications. the space charge effect on the performance of a cable system [19, 28-30] was widely studied by Dissado, Montanari and Mazzanti. In this paper, only the formula in [19] is taken into consideration, as the expression for lifetime is extremely simpler to analyze from a mathematical point of view. The formula is:

$$L(E,T) = \frac{K_E(T)e^{-\xi(T)(E-E_t)^{4b}}}{(E-E_t)^{\mu}} \tag{27}$$

where μ and b are characteristic constants of the material, ξ(T) and K_E(T) are parameters dependent on the temperature, E_t is the threshold electric field for charge injection. Through the Lambert W function application, one can find the value of E when T and all other parameters are known. The following formula holds:

$$E = e^{-\frac{\left(e^{a-W\left(\frac{4b}{\mu}e^{a}\right)}+A\right)}{\mu}} + E_t, a = \ln\big(\xi(T)\big) - \frac{4Ab}{\mu},$$
$$A = \ln\left(\frac{K_E}{L}\right) \tag{28}$$

TABLE II

PARAMETERS FOR SIMULATION OF DMM MODEL

K_E s(kV/mm)$^{\mu}$	ξ (kV/mm)$^{-4b}$	E_t kV/mm	μ	B
8.58·10^6	1.1·10^{-6}	300	0.45	1.29

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ADVANTAGES AND LIMITATIONS OF LAMBERT W FUNCTION APPLICATIONS

Applications	Advantages	Limitations
Rogowski	Explicit expression considering edge effects for parallel plate capacitor can be achieved	If only the behavior between the plates is required in the analysis, classical field expression is much simpler
Paschen	Pressure conditions can be obtained with limited computational efforts	Complication due to Lambert W branches
HVDC cables	Improvement of Eoll expression for electric field	Errors for numerical integration to get voltage
DMM	Closed form expression for electric field estimation for high voltage applications	The required parameters are not deeply studied in literature

where μ and b are characteristic constants of the material, ξ(T) and $K_E(T)$ are parameters dependent on the temperature, E_t is the threshold electric field for charge injection. Through the Lambert W function application, one can find the value of E when T and all other parameters are known. The following formula holds:

$$E = e^{-\frac{\left(e^{a-W\left(\frac{4b}{\mu}e^{a}\right)}+A\right)}{\mu}} + E_t, a = \ln\big(\xi(T)\big) - \frac{4Ab}{\mu},$$
$$A = \ln\left(\frac{K_E}{L}\right) \quad (28)$$

This expression is a viable alternative in electric field estimation for the design of a high voltage component to reach a predefined value of lifetime L, even though the usage of this DMM model has not been so covered in literature. Full derivation is realized in Appendix C. Parameters from Table II lead to the plot in Fig. 9.

V. CONCLUSIONS

In this paper, it was shown that some mathematical expressions easily solvable by means of Lambert W function are almost ubiquitous in different branches of science, especially in the electrical ones. Therefore, a thorough application of the Lambert W function tool to several existing models in the electrical insulation engineering field was carried out. This led to a revamping of previous models on gas insulation, solid insulating samples and lifetime modelling, achieving closed form expressions for the mathematical variables arising from these models. This resulted in proper compatibility with conventional approaches, which was even improved in some cases, e.g. Eoll. Further work may include the exploitation of more advanced formulations of Lambert W function to solve much more general and complex problems with equations resembling (4), and a review on the current research involving Lambert W function applications.

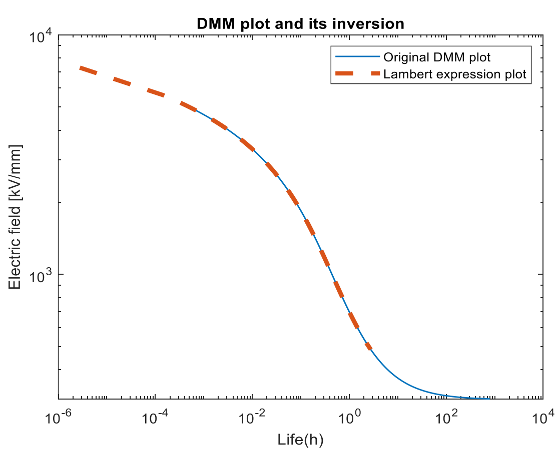

Fig. 9 Plot of (27) and (28) by using parameters reported in Table II

APPENDIX A - LAMBERT APPLICATION TO PASCHEN'S LAW

Starting from (17), we can obtain:

$$V_B(\ln(A) + \ln(pd) - \ln(\ln(1 + 1/\gamma)\,) = Bpd \quad (A.1)$$

by performing variable substitutions:

$$a = (-\ln(A) + \ln\left(\ln\left(1 + \frac{1}{\gamma}\right)\right), \qquad x = \ln(pd) \quad (A.2)$$

we get to

$$V_B(x - a) = Be^x, x = \frac{B}{V_B}e^x + a \quad (A.3)$$

And applying eqs. (4-5), the result is (20).

APPENDIX B - LAMBERT APPLICATION TO HVDC CABLES

By isolating the terms with E(r), it is possible to simplify (22) to:

$$E(r)e^{bE(r)} = \frac{Ie^{-aT(r_o)}}{2\pi r\sigma_0}\left(\frac{r}{r_o}\right)^A \quad (A.4)$$

By posing the right-hand side equal to c(r) (as in (26)):

$$E(r)e^{bE(r)} = c(r), xe^x = bc(r) \quad (A.5)$$

here x=bE. Final expression in (A.6) can be solved by means of

APPENDIX C - LAMBERT W FUNCTION APPLICATION TO DMM

MODEL

After applying natural logarithm to both sides of (27), we obtain:

$$-\ln\left(\frac{L}{K_E}\right) - \mu \ln(E - E_t) = \xi(T)(E - E_t)^{4b} \qquad (A.6)$$

and applying logarithm once again, we obtain:

$$\ln(A - \mu y) = \ln\big(\xi(T)\big) + 4by, A = -\ln\left(\frac{L}{K_E}\right),$$
$$x = E - E_t, y = \ln(x) \qquad (A.7)$$

We can further proceed by means of the following substitution:

$$z = \ln\big(\xi(T)\big) + \frac{4bA}{\mu} - \frac{4be^z}{\mu}, z = \ln(A - \mu y) \qquad (A.8)$$

The formula now is in form (4). Thus, it can be solved as:

$$z = \ln\big(\xi(T)\big) + \frac{4bA}{\mu} - W\left(\frac{4b}{\mu} e^{\ln(\xi(T)) + \frac{4bA}{\mu}}\right) \qquad (A.9)$$

which is equal to (28).

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