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Influence of Attribute Granulation on Three-Way Concept Lattices

Jun Long, Yinan Li, and Zhan Yang*

Abstract: In formal concept analysis based applications, controlling the structure of concept lattice is of vital importance, especially for big data, and is achieved via clarifying the granularity of attributes. Existing approaches for solving this issue are within the framework of classical formal concept analysis, which focuses on positive attributes. However, experiments have demonstrated that both positive and negative attributes exert comparable influence on knowledge discovery. Thus, it is essential to explore the granularity of attributes in positive and negative perspectives altogether. As a solution, we investigate this problem within the framework of three-way concept analysis. Specifically, we present zoom-in and zoom-out algorithms to obtain more particular and abstract three-way concepts, separately. Furthermore, we provide illustrative examples to show the practical significance of this study.

Key words: granularity of attributes; three-Way Concept Analysis (3WCA); three-way concept lattice

1 Introduction

Recent years have witnessed the increasing interest towards Granular Computing (GrC)[1, 2]. At present, GrC has proved to be an effective tool for solving complex problems by modeling and manipulating the discussed universe at different levels of granulations. As an example, when characterizing the color of a vehicle, “red” and “blue” are attributes at the coarser granularity level, while “light red”, “deep red”, “light blue”, and “deep blue” are attributes at the finer granularity level. On one hand, coarser attributes may lead invisibility of some interesting patterns, such as a deep red seat, a light blue cover, and a deep blue plate. On the other hand, too specific attributes will bring too many patterns, which may be a waste of time in choosing our favorite colors. In a word, properly manipulating granularity levels plays a vital role in applications[3, 4].

It is worth noting that on most occasions, only pointing out the common features shared by the target set is far from enough. For instance, in our daily life, when going to a restaurant, we not only tell the waiter or waitress what kinds of food we prefer, but also the ingredients we do not want. Sometimes, the latter will determine whether you can have a good dinner, especially you have some taboos. Actually, the equal importance of positive and negative attributes has been stressed in many researches, such as bipolar fuzzy graph representation[5], association rule mining[6], conflict analysis[7], etc. Therefore, manipulating the granularity of attributes both in positive and negative perspectives is necessary. However, the existing studies focus on positive attributes but leave negative ones aside. Thus, the focus of this paper is to carry out a systematic study of the granularity of attributes both in positive and negative perspectives.

Formal Concept Analysis (FCA), the key tool for GrC proposed by Wille[8], has attracted increasing popularity across various domains[9–20]. The initial information of FCA is formal context, which is composed of objects and attributes among binary relations. The basic structure obtained from formal

• Jun Long, Yinan Li, and Zhan Yang are with Big Data Institute, Central South University, Changsha 410083, China. E-mail: jlong@csu.edu.cn; liyinan@csu.edu.cn; zyang22@csu.edu.cn.
∗ To whom correspondence should be addressed.
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context is concept lattice, ordering a collection of particular couples, which are known as so-called formal concepts. When the objects are fixed, the choice of attributes decides the structure of the concept lattice[21−28]. As reducing the complexity of lattice construction plays a crucial role in FCA, by using a granularity tree, Shao et al.[29] investigated the connection between extent and intent of the concept in the granularity pre/post-transformation stages of attributes.

Rough Set Theory(RST) was proposed by Pawlak[30], which is related to and complementary with FCA[31−36]. The granularity of attributes has also attracted the increasing interest of researchers from the community of RST[34−36]. Formal concepts, as well as rough sets, describe the target set by solely considering the attributes they have. Sometimes, it is far from enough to show all details of the target. Three-Way Decisions (3WD), the efficient architecture for taking decisions, proposed by Yao[37−39], exhibits great merits in many research fields, such as large-scale and multi-view clustering[40, 41]. By incorporating this theory with FCA, Qi et al.[42, 43] proposed three-Way Concept Analysis (3WCA), which enables us to simultaneously exploit both positive and negative attributes. Moreover, inspired by possibility theory, three-way object (property)-oriented concept[44, 45] and dual concept[46] are proposed to fit more specific applications. At present, 3WCA has attracted growing interests across various domains. For instance, Shivhare and Aswani[47] described a cognitive memory process based on 3WCA. In order to manage incomplete information, Zhi and Li[48] constructed a kind of approximate three-way concept lattices by employing the spirit of granule description to support approximate decision rule extraction. Wei et al.[10] extracted positive and negative rules based on three-way concept lattices. Campagner et al.[49] presented a three-way-in/out architecture to deal with the ambiguity of data.

Up to now, the granularity of attributes in FCA has been studied on the basis of classical concepts and object (property)-oriented concepts[26, 27, 29, 50, 51], but none of them discusses the granularity of attributes both in positive and negative views. To tackle this issue, we resort to 3WCA and present our solutions.

The rest is organized as follows. Section 2 reviews some basic notions in 3WCA. Section 3 describes a framework for the granularity of attributes in 3WCA. Section 4 presents zoom-in and zoom-out algorithms to change the granularity level of attributes and update the related three-way concept lattices. Section 5 demonstrates the influence of attribute granulation on three-way and formal concept lattices via experiments. Finally, a conclusion is provided in Section 6.

2 Preliminary

Let $K=(G, M, I)$ be a formal context. Concretely, G is a nonempty set of objects, M is a nonempty set of attributes, and I denotes the relationship between G and M. Besides, we use $I(x, y)=1$ (or $I(x, y)=0$) to express that the object x contains (or does not contain) the attribute y.

For $X \in 2^G$ and $A \in 2^M$, a pair of associated positive operators $*: 2^G \to 2^M$ and $*: 2^M \to 2^G$ are defined as

$$X^* = \{y \in M \mid \forall x \in X, I(x, y)=1\},$$
$$A^* = \{x \in G \mid \forall y \in A, I(x, y)=1\} \tag{1}$$

In addition, a pair of associated negative operators $\bar{*}: 2^G \to 2^M$ and $\bar{*}: 2^M \to 2^G$ are defined as

$$X^{\bar{*}} = \{y \in M \mid \forall x \in X, I(x, y)=0\},$$
$$A^{\bar{*}} = \{x \in G \mid \forall y \in A, I(x, y)=0\} \tag{2}$$

Definition 1[42, 43] Let $K=(G, M, I)$ be a formal context, $X \in 2^G$ and $(A, B) \in 2^M \times 2^M$. A pair of associated three-way operators $\gtrdot: 2^G \to 2^M \times 2^M$ and $\lessdot: 2^M \times 2^M \to 2^G$ are defined as

$$X^{\gtrdot} = (X^*, X^{\bar{*}}) \text{ and } (A, B)^{\lessdot} = A^* \cap B^{\bar{*}} \tag{3}$$

Moreover, if $X^{\gtrdot} = (A, B)$ and $(A, B)^{\lessdot} = X$, then $(X, (A, B))$ is an object-induced three-way concept.

Then, all concepts contained in K form a complete lattice, which is the object-induced three-way concept lattice of K and denoted by $OEL(K)$. By using the duality principle, attribute-induced three-way concept lattice can be obtained according to Ref. [42].

3 Granularity of Attributes in 3WCA

The changing of granularity of attributes reflects human cognitive nature to some extent. On one hand, attributes of coarse granularity can be refined to reveal some more interesting details. On the other hand, a set of attributes can also be abstracted to achieve a higher level of thinking. The manipulation of granularity is just the essence of granular computing to solve complex problems.

Granule transformation frequently appears in our daily life. For instance, there is a software development team, which is composed of four engineers. The members of this team are described by their abilities, i.e., programming, software testing, and algorithm analysis. Besides, their shortcomings are also considered. For simplicity, we denote these abilities by a, b, and c, respectively, and denote shortcoming by d. The details of these four engineers are shown in Table 1.

However, on some occasions, Table 1 cannot provide enough detailed information, and brings obstacles in finding some useful embedded patterns. For instance, at a specific software development stage, it needs engineers who are specialized in Java language with a strong sense of responsibility. Then, it is apparent that the granularity of the related attributes needs to be adjusted. That is, the ability of programming needs to be refined to Java and Python; and the shortcoming is substituted by indecisiveness and carelessness. For simplicity, a_1 and a_2 are used to represent the abililies of mastering Java and Python language, respectively; d_1 and d_2 are used to denote carelessness and indecisiveness, respectively. Then we obtain Table 2.

From Table 2, we can derive that Engineers 1 and 2 are the qualified ones for the requirements of this software development stage. Actually, it can be verified that ({1, 2}, ({a_1}, {a_2, d_2})) is a three-way concept embedded in Table 2, which contains the needed information. For the sake of convenience, we collectively list the three-way concepts of Tables 1 and 2 in Table 3. It is clear that Table 2 contains more concepts with finer granularity.

However, attributes with finer granularity may not always be necessary, and may bring obstacles in some situations. For instance, engineers who master one kind of programming language with no obvious shortcomings can be engaged in after-sale service. Then, adjusting a set of finer attributes about programming skills to a coarser attribute will make sense on this occasion. With respect to this example, attributes a_1 and a_2 need to be coarsened to attribute a, and attributes d_1 and d_2 need to be coarsened to attribute d. Then, we can see that Engineer 1 is a suitable candidate for this job, indicated by the three-way concept ({1}, ({a, c}, {b, d})).

To sum up, there are two related facets in the adjusting of the granularity of attributes. One is changing a coarse attribute to a set of finer attributes and the other is the reverse.

In the above discussion, it is apparent that we manipulate the granularity of attributes in the settings

Table 1 Four software engineers and their abilities.

Engineer	Ability			
	a	b	c	d
1	1	0	1	0
2	1	1	0	1
3	1	0	1	1
4	0	1	0	1

Table 2 Four software engineers and their specific abilities.

Engineer	Ability					
	a_1	a_2	b	c	d_1	d_2
1	1	0	0	1	0	0
2	1	0	1	0	1	0
3	0	1	0	1	0	1
4	0	0	1	0	0	1

Table 3 Object-induced three-way concepts of Tables 1 and 2.

Number	Object-induced three-way concepts of Table 1	Object-induced three-way concepts of Table 2
1	$(\{1,2,3,4\},(\varnothing,\varnothing))$	$(\{1,2,3,4\},(\varnothing,\varnothing))$
2	$(\{1,2,3\},(\{a\},\varnothing))$	$(\{1,2,4\},(\varnothing,\{a_2\}))$
3	$(\{2,3,4\},(\{d\},\varnothing))$	$(\{1,3,4\},(\varnothing,\{d_1\}))$
4	$(\{1,3\},(\{a,c\},\{b\}))$	$(\{1,2\},(\{a_1\},\{a_2,d_2\}))$
5	$(\{2,3\},(\{a,d\},\varnothing))$	$(\{1,4\},(\varnothing,\{a_2,d_1\}))$
6	$(\{2,4\},(\{b,d\},\{c\}))$	$(\{1,3\},(\{c\},\{b,d_1\}))$
7	$(\{1\},(\{a,c\},\{b,d\}))$	$(\{2,4\},(\{b\},\{a_2,c\}))$
8	$(\{2\},(\{a,b,d\},\{c\}))$	$(\{3,4\},(\{d_2\},\{a_1,d_1\}))$
9	$(\{3\},(\{a,c,d\},\{b\}))$	$(\{1\},(\{a_1,c\},\{a_2,b,d_1,d_2\}))$
10	$(\{4\},(\{b,d\},\{a,c\}))$	$(\{2\},(\{a_1b,d_1\},\{a_2,c,d_2\}))$
11	$(\varnothing,(\{a,b,c,d\},\{a,b,c,d\}))$	$(\{3\},(\{a_2,c,d_2\},\{a_1,b,d_1\}))$
12	–	$(\{4\},(\{b,d_2\},\{a_1,a_2,c,d_1\}))$
13	–	$(\varnothing,(\{a_1,a_2,b,c,d_1,d_2\},\{a_1,a_2,b,c,d_1,d_2\}))$

of 3WCA, other than classical FCA. If we choose classical FCA, some interesting patterns cannot be obtained. In Table 4, we collectively list the formal concepts of Tables 1 and 2. It is clear that carrying analysis on the basis of 3WCA can present more details than on the basis of classical FCA. Actually, as three-way concepts are not simply the union of concepts derived from the original formal contexts and their complements, they can provide more embedded useful patterns than separately considering them one after the other. Hence, it is also known that 3WCA and classical FCA are not mutually reducible. Although the granularity of attributes has been investigated in classical FCA[26, 50, 51], the obtained results can not be simply extended to 3WCA, and the studies in the settings of 3WCA are still open, interesting, and important issues.

In what follows, Theorem 1 indicates the relationships among three-way concepts while adjusting the granularity of attributes.

Let $\Delta = \{X | (X, (A, B)) \in OEL(K)\}$. For $X \subseteq G$ and $X \notin \Delta$, we set forth an object set,

$$t(X) = \{Y | Y \in \Delta,\ Y \subset X,\ \forall\, Y' \in \Delta, Y' \supset Y \Rightarrow Y' \not\subset X\} \tag{4}$$

Moreover, for $X \subseteq G$, we define a cut object set,

$$p(X) = \begin{cases} X, & X \in \Delta; \\ t(X), & \text{otherwise} \end{cases} \tag{5}$$

Theorem 1 Let $K = (G, M, I)$ be a formal context, and $C_1, C_2 \subseteq M$. If $C_1 \leqslant C_2$, then for $(X, (A, B)) \in OEL(G, M_{C_2}, I_{C_2})$, there exists unique concept $\Sigma = \{(X_k, (A_k, B_k)) | X_k \in p(X)\}$, such that $\bigcup_{(X_k, (A_k, B_k)) \in \Sigma} X_k = X$.

Proof Let $(X, (A, B)) \in OEL(G, M_{C_2}, I_{C_2})$. The claim is evident if $X = \varnothing$. Suppose $X \neq \varnothing$. Put $A = \{e_1, e_2, \ldots, e_p\}$ and $B = \{h_1, h_2, \ldots, h_q\}$, according to the property of a three-way concept, it is clear that the attributes contained in A and B are pairwise disjoint. Denote by $\vec{e_r} = \{e_{r_1}, e_{r_2}, \ldots, e_{r_m}\}$ the collection of all attributes which refine $e_r \in A$ and denote by $\vec{h_s} = \{h_{s_1}, h_{s_2}, \ldots, h_{s_n}\}$ the collection of all attributes which refine $h_s \in B$. Then, for any $e_{r_j} \in \vec{e_r}$, we have $e^*_{r_j} \subseteq e^*_r$ and $e^*_r = e^*_{r_1} \cup e^*_{r_2} \cup \cdots \cup e^*_{r_m}$. Similarly, for any $h_{s_j} \in \vec{h_s}$, we have $h^*_{s_j} \subseteq h^*_s$ and $h^*_s = h^*_{s_1} \cup h^*_{s_2} \cup \cdots \cup h^*_{s_n}$. Let $\langle e_{1_{r_1}}, \ldots, e_{p_{r_p}}, h_{1_{s_1}}, \ldots, h_{q_{s_q}} \rangle$ be a possible choice of attributes in C_1, which refine the attributes in $A \cup B$. For convenience, let $\{e_{1_{r_1}}, e_{1_{r_2}}, \ldots, e_{p_{r_p}}\} = E$ and $\{h_{1_{s_1}}, h_{1_{s_2}}, \ldots, h_{q_{s_q}}\} = F$. Then, $(X_k, (A_k, B_k)) = (E^* \cap F^{\bar{*}}, ((E^* \cap F^{\bar{*}})^*, (E^* \cap F^{\bar{*}})^{\bar{*}}))$ is a three-way concept of $OEL(G, M_{C_1}, I_{C_1})$ generated by refined attributes. As E is a refinement of A and F is a refinement of B, it follows that $X_k \subseteq X$.

On one hand, if $x \in X$, then we have $x \in e^*_r$ for every $r \in \{1, 2, \ldots, p\}$ and $x \in h^*_s$ for every $s \in \{1, 2, \ldots, q\}$. As $e^*_{i_{r_i}}$ and $h^*_{j_{s_j}}$ form two partitions of e_i and h_j, there must be at least one X_k, such that $x \in X_k$. On the other hand, based on Eq. (5) and the structure of three-way concept lattices, all elements of X_k are unique. In all, Theorem 1 is proved. ■

If $C_1 \leqslant C_2$, then shifting from three-way concept lattice $OEL(G, M_{C_2}, I_{C_2})$ to three-way concept lattice $OEL(G, M_{C_1}, I_{C_1})$, and vice versa, are called zooming in and zooming out, respectively. In the subsequent sections, we respectively describe a zoom-in algorithm and a zoom-out algorithm in 3WCA.

Table 4 Formal concepts of Tables 1 and 2.

Number	Formal concept of Table 1	Formal concept of Table 2
1	$(\{1,2,3,4\},\varnothing)$	$(\{1,2,3,4\},\varnothing)$
2	$(\{1,2,3\},\{a\})$	$(\{1,2\},\{a_1\})$
3	$(\{2,3,4\},\{d\})$	$(\{1,3\},\{c\})$
4	$(\{1,3\},\{a,c\})$	$(\{2,4\},\{b\})$
5	$(\{2,3\},\{a,d\})$	$(\{3,4\},\{d_2\})$
6	$(\{2,4\},\{b,d\})$	$(\{1\},\{a_1,c\})$
7	$(\{2\},\{a,b,d\})$	$(\{2\},\{a_1,b,d_1\})$
8	$(\{3\},\{a,c,d\})$	$(\{3\},\{a_2,c,d_2\})$
9	$(\varnothing,\{a,b,c,d\})$	$(\{4\},\{b,d_2\})$
10	–	$(\varnothing,(\{a_1,a_2,b,c,d_1,d_2\})$

4 Algorithm for Manipulating Granularity of Attributes in 3WCA

In this section, we present zoom-in and zoom-out algorithms to manipulate the granularity level of attributes within the framework of 3WCA.

4.1 Zoom-in algorithm

Within the framework of 3WCA, the zoom-in is performed by Algorithm 1. Generally speaking, in Algorithm 1, we traverse the three-way concept lattice via the ascending order to the cardinalities of the extent of each concept, make modifications, and at the same time generate new concepts or not.

It is clear that the number of concepts visited in Algorithm 1 is the number of concepts of $OEL(K)$. Besides, note that Algorithm 1 is a particular case, i.e., only one attribute is refined, but the general case is accessed via iterated applications of this case. In what

Algorithm 1 Zoom-in

input: $OEL\ (G,M,I)$ and $\delta=\{g_i \mid i\in N^*\}$ be the refinement of $g\in M$ // N^* is a set of positive integers
output: Updated $OEL\ (K')$ after refining the attribute g

Initialize $OEL\ (K')=\varnothing$
Deem $OEL\ (K)$ as an undirected graph and traverse the concept lattice from the minimal concept of $OEL\ (K)$ in the ascending order to the cardinalities of extents of concepts
for each concept $(X,\ (A,\ B))$ **do**
    Let $\varPi$ be a set of three-way concepts and initialize $\varPi=\{(X,\ (A-\{g\},B-\{g\}))\}$;
    for each $g_i\in\delta$ **do**
        if there exists $(Y,\ (C,\ D))$ of $\varPi$, such that $Y=X\cap g_i^*$ **then**
            modify $(Y,\ (C,\ D))$ into $(Y,\ (C\cup g_i,D))$;
        else
            add $(X\cap g_i^*,((A-\{g\})\cup\{g_i\},B-\{g\}))$ into $\varPi$.
        end
        if there exists $(Z,(E,F))$ of $\varPi$, such that $Z=X\cap g_i^{\bar{*}}$ **then**
            modify $(Z,(E,F))$ into $(Z,(E,F\cup g_i))$;
        else
            add $(X\cap g_i^{\bar{*}},(A-\{g\},(B-\{g\})\cup\{g_i\}))$ into Π.
        end
    end
    for each $(X_i,(A_i,B_i))\in\varPi$ **do**
        if there exists a concept $(Y,(A_i,B_i))$ with $Y\subseteq X_i$ **then**
            delete $(Y,(A_i,B_i))$.
    end
    Add $\varPi$ into $OEL\ (K')$ and establish covering relations.
end

follows, we prove the correctness of Algorithm 1.

Theorem 2 Algorithm zoom-in is correct.

Proof To begin with, let K and K' be the coarser and finer formal context, respectively. Moreover, concept-forming operators denoted by $\langle^{\gtrdot},\ ^{\lessdot},\ ^{*},\ ^{\bar{*}}\rangle$ and $\langle^{\gtrdot'},\ ^{\lessdot'},\ ^{*'},\ ^{\bar{*}'}\rangle$.

On one hand, we need to verify that $\forall\,(X,\ (A,\ B))\in OEL\,(K')$ is a three-way concept of K'. Obviously, $(A,\ B)$ is equal to $((A_1-\{g\})\cup(\bigcup_{g_i\in\alpha}g_i),\ (B_1-\{g\})\cup(\bigcup_{g_j\in\beta}g_j))$ for some intent $(A_1,\ B_1)$ in K, and $\alpha,\beta\subseteq\delta$. Assume the corresponding extent to $(A_1,\ B_1)$ in K is X_1. As the only change made to the context is the refining of attribute g, it is clear that X does not have common attribute g and derives commonly possessed positive and negative attributes in α and β, respectively. Then, we can conclude that $X_1\cap(\bigcap_{g_i\in\alpha}g_i^*)$ is a set of objects which possesses an attribute set $(A_1-\{g\})\cup(\bigcup_{g_i\in\alpha}g_i)$, and $X_1\cap(\bigcap_{g_j\in\beta}g_j^{\bar{*}})$ is a set of objects, which possesses an attribute set $(B_1-\{g\})\cup(\bigcup_{g_j\in\beta}g_j)$. Then, we can conclude that $(A_1-\{g\})\cup(\bigcup_{g_i\in\alpha}g_i)\cap(\bigcap_{g_j\in\beta}g_j^{\bar{*}})$ is a set of objects which possesses both $(A_1-\{g\})\cup(\bigcup_{g_i\in\alpha}g_i)$ and $(B_1-\{g\})\cup(\bigcup_{g_j\in\beta}g_j)$. Ergo, $(A,\ B)=((A_1-\{g\})\cup(\bigcup_{g_i\in\alpha}g_i)\cap(\bigcap_{g_j\in\beta}g_j^{\bar{*}}))^{\gtrdot'}$ is an intent of K'. Let $(A_1,\ B_1)$ be the smallest intent, for which $(A,\ B)$ is equal to $((A_1-\{g\})\cup(\bigcup_{g_i\in\alpha}g_i),\ (B_1-\{g\})\cup(\bigcup_{g_j\in\beta}g_j))$ for some intent $(A_1,\ B_1)$ in K and $\alpha,\beta\subseteq\delta$. Then, $X_1\cap(\bigcap_{g_i\in\alpha}g_i^*)\cap(\bigcap_{g_j\in\beta}g_j^{\bar{*}})$ is the corresponding biggest extent, which implies that any object possessed attribute sets $(A_1-\{g\})\cup(\bigcup_{g_i\in\alpha}g_i)$ and $(B_1-\{g\})\cup(\bigcup_{g_j\in\beta}g_j)$ falls into $X_1\cap(\bigcap_{g_i\in\alpha}g_i^*)\cap(\bigcap_{g_j\in\alpha}g_j^{\bar{*}})$. Thus, $X=X_1\cap(\bigcap_{g_i\in\alpha}g_i^*)\cap(\bigcap_{g_j\in\beta}g_j^{\bar{*}})$ is an extent in K'.

On the other hand, we verify that all three-way concepts in K' are in $OEL\,(K')$. Based on Algorithm 1, any three-way concept $(X,\ (A,\ B))$ of K' is generated by $((A-\alpha,\ B-\beta)^{\lessdot},\ (A-\alpha,\ B-\beta)^{\lessdot\gtrdot})$, $(((A-\alpha)\cup\{g\},\ B-\beta)^{\lessdot},\ ((A-\alpha)\cup\{g\},\ B-\beta)^{\lessdot\gtrdot})$, or $((A-\alpha,\ (B-\beta)\cup\{g\})^{\lessdot},\ (A-\alpha,\ (B-\beta)\cup\{g\})^{\lessdot\gtrdot})$ in $OEL\,(K)$ with $\alpha,\beta\subseteq\delta$. Hence, we can conclude that all three-way concepts in K' are in $OEL\,(K')$.

In all, Theorem 2 is proved. ■

Example 1 In Table 5, formal context $K=(G,M,I)$ demonstrates five travelers and cities in which they booked hotels. Specifically, G is an object set consisted of five travelers and M is an attribute set consisted of five cities. Furthermore, a, b, c, d, and e denote Changsha, Wuhan, Guangzhou, Suzhou, and Shenzhen, respectively.

Figure 1 depicts the three-way concept lattice $OEL\,(K)$, by which common interests of these travelers can be analyzed. For instance, concept $(\{1,3\},(\{a,e\},\{d\}))$ manifests that both Travelers 1 and 3 want to visit Changsha and Shenzhen, and meanwhile neither of them has an interest in Suzhou.

However, as there are three regions of Wuhan, it is necessary to get more details about their interests. After

Table 5 Formal context $K=(G,M,I)$ of Example 1.

Traveler	City				
	a	*b*	*c*	*d*	*e*
1	1	0	1	0	1
2	0	1	0	1	0
3	1	1	0	0	1
4	1	1	1	1	0
5	0	1	1	1	0

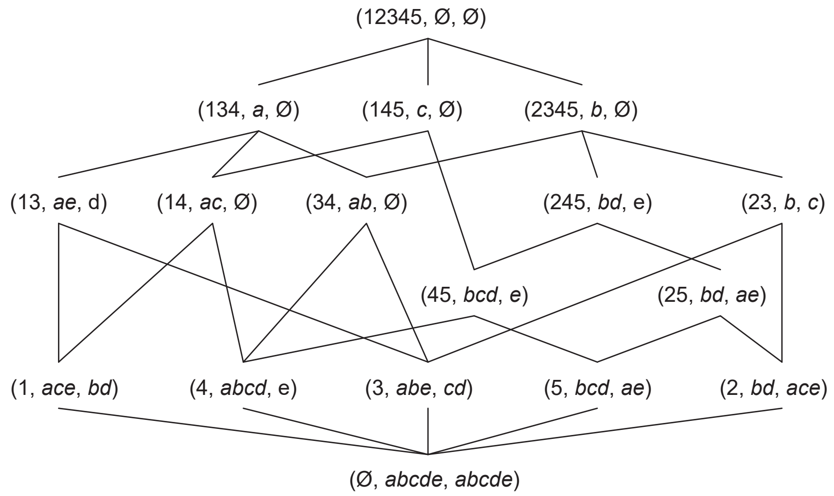

Fig. 1 Three-way concept lattice $OEL(K)$ of Example 1. For convenience, as an example, we use (245, *bd*, *e*) instead of ({2, 4, 5},({*b*, *d*}, {*e*})).

further investigation, two regions are more attractive for them, i.e., Wuchang and Hankou, denoted by b_1 and b_2 respectively. More concretely, Travelers 2 and 4 choose Wuchang, and Travelers 3 and 5 favor Hankou. Then, we obtain the updated formal context K' in Table 6, which describes the refined information. Based on Algorithm 1, we obtain the updated three-way concept lattice depicted in Fig. 2, by which we can derive more details of the common interests of travelers. For instance, by the concept ({3, 5}, ({b_2}, {b_1})) of $OEL(K')$ shows that both Travelers 3 and 5 want to visit Hankou, and meanwhile neither of them has an interest in Wuchang.

Table 6 Updated formal context K' of Example 1.

Traveler	City					
	a	b_1	b_2	c	d	e
1	1	0	0	1	0	1
2	0	1	0	0	1	0
3	1	0	1	0	0	1
4	1	1	0	1	1	0
5	0	0	1	1	1	0

For the sake of better understanding of Algorithm 1, we list the sequence of the visited concepts and their corresponding actions in Table 7.

4.2 Zoom-out algorithm

Correspondingly, the zoom-out is presented in Algorithm 2. In this algorithm, the input is $OEL(G, M, I)$, g be the abstraction of $\delta = \{g_i \in M \mid i \in N^*\}$, and the output is the updated $OEL(K')$ after abstracting δ. Generally, we traverse the three-way concept lattice via the descending order to the cardinalities of the extent of each concept, and at the same time generate new concepts when necessary.

It is clear that the number of concepts visited in Algorithm 2 is the number of concepts of $OEL(G, M, I)$. Besides, note that Algorithm 2 is a particular case, i.e., several related attributes are upgraded to one coarser attribute, but the general case

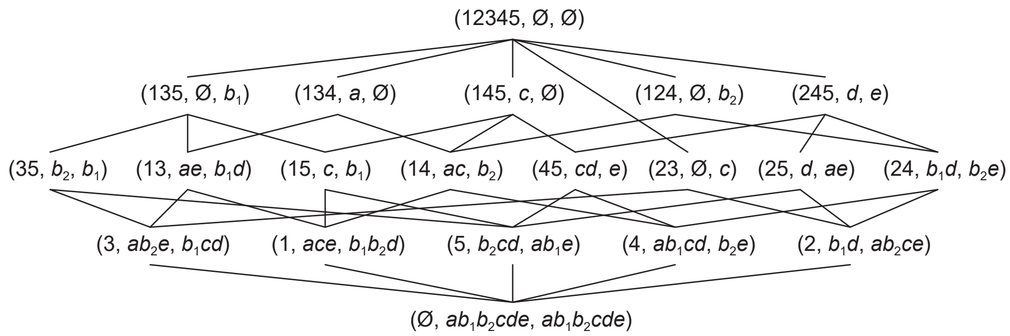

Fig. 2 Updated three-way concept lattice $OEL(K')$ of Example 1. For convenience, as an example, we use (245, *bd*, *e*) instead of ({2, 4, 5},({*b*, *d*}, {*e*})).

Table 7 Summary of zoom-in process of Example 1.

Number	Visited concept	Action
1	$(\varnothing,(\{a,b,c,d,e\},\{a,b,c,d,e\}))$	Obtain $(\varnothing,(\{a,b_1,b_2,c,d,e\},\{a,b_1,b_2,c,d,e\}))$
2	$(\{1\},(\{a,c,e\},\{b,d\}))$	Obtain $(\{1\},(\{a,c,e\},\{b_1,b_2,d\}))$
3	$(\{4\},(\{a,b,c,d\},\{e\}))$	Obtain $(\{4\},(\{a,b_1,c,d\},\{b_2,e\}))$
4	$(\{3\},(\{a,b,e\},\{c,d\}))$	Obtain $(\{3\},(\{a,b_2,e\},\{b_1,c,d\}))$
5	$(\{5\},(\{b,c,d\},\{a,e\}))$	Obtain $(\{5\},(\{b_2,c,d\},\{a,b_1,e\}))$
6	$(\{2\},(\{b,d\},\{a,c,e\}))$	Obtain $(\{2\},(\{b_1,d\},\{a,b_2,c,e\}))$
7	$(\{1,3\},(\{a,e\},\{d\}))$	Obtain $(\{1,3\},(\{a,e\},\{b_1,d\}))$
8	$(\{1,4\},(\{a,c\},\varnothing))$	Obtain $(\{1,4\},(\{a,c\},\{b_2\}))$
9	$(\{3,4\},(\{a,b\},\varnothing))$	Obtain $(\{3,4\},(\{a\},\varnothing))$
10	$(\{4,5\},(\{b,c,d\},\{e\}))$	Obtain $(\{4,5\},(\{c,d\},\{e\}))$
11	$(\{2,5\},(\{b,d\},\{a,e\}))$	Obtain $(\{2,5\},(\{d\},\{a,e\}))$
12	$(\{2,3\},(\{b\},\{c\}))$	Obtain $(\{2,3\},(\varnothing,\{c\}))$
13	$(\{2,4,5\},(\{b,d\},\{e\}))$	Obtain $(\{2,4,5\},(\{d\},\{e\}))$ and $(\{2,4\},(\{b_1,d\},\{b_2,e\}))$
14	$(\{1,3,4\},(\{a\},\varnothing))$	Unchanged, delete $(\{3,4\},(\{a\},\varnothing))$
15	$(\{1,4,5\},(\{c\},\varnothing))$	Unchanged, generate a concept $(\{1,5\},(\{c\},\{b_1\}))$
16	$(\{2,3,4,5\},(\{b\},\varnothing))$	Obtain $(\{2,3,4,5\},(\varnothing,\varnothing))$ and $(\{3,5\},(\{b_2\},\{b_1\}))$
17	$(\{1,2,3,4,5\},(\varnothing,\varnothing))$	Unchanged, delete $(\{2,3,4,5\},(\varnothing,\varnothing))$, obtain $(\{1,3,5\},(\varnothing,\{b_1\}))$, $(\{1,2,4\},(\varnothing,\{b_2\}))$

is accessed via iterated applications of this case.

Theorem 3 Algorithm zoom-out is correct.

Proof Suppose the existing concept is $(X, (A, B))$, and the proof can be separated into two independent halves.

Consider the case for which there does not exist $g_i \in A$ and $g_i \in B$.

(a) If $X \not\subseteq g^*$ and $X \not\subseteq \overline{g^*}$, then it follows that $(X, (A, B))$ remains unchanged, and is a concept of $OEL(K')$.

(b) If $X \subseteq g^*$, then X obtains a coarser attribute g, which follows that $(X, (A \cup \{g\}, B))$ is a concept of $OEL(K')$. Besides, in line with the proof of Theorem 2, we can prove that if $X \cap g^* \notin OEL(G, M, I)$, then $(X \cap g^*, (A \cup \{g\}, B))$ is a concept of $OEL(K')$.

(c) If $X \subseteq \overline{g^*}$, then X obtains a negative coarser attribute g, which implies that $(X, (A, B \cup \{g\}))$ is a concept of $OEL(K')$. Besides, in line with the proof of Theorem 2, we can prove that if $X \cap \overline{g^*} \notin OEL(G, M, I)$, then $(X \cap \overline{g^*}, (A, B \cup \{g\}))$ is a concept of $OEL(K')$.

Consider the case for which there exists $g_i \in A$ or $g_j \in B$.

There are four independent cases, and each case can be proved analogously as Theorem 2, and thus we skip here.

In addition, by considering that the algorithm visits all concepts of $OEL(K)$ and tries all possibilities to generate a new concept, we can conclude that all concepts of $OEL(K')$ can be obtained.

In all, Theorem 3 is proved. ■

Example 2 In Table 8, $K = (G, M, I)$ demonstrates five high school students and their interested courses. Specifically, G is an object set composed of five students and M is an attribute set composed of four courses. Moreover, a refers to mathematics; b denotes chemistry; c_1 and c_2 respectively stand for C and Java programming. Their common interests can be analyzed by $OEL(K)$ depicted in Fig. 3. For instance, concept $(\{4, 5\}, (\{a\}, \{b\}))$ implies that both Students 4 and 5 favor chemistry, and meanwhile neither of them has an interest in C or Java programming.

Table 8 Formal context $K = (G, M, I)$ of Example 2.

Student	Course			
	a	b	c_1	c_2
1	1	0	1	0
2	0	0	0	1
3	1	0	0	1
4	0	1	0	0
5	1	1	0	0

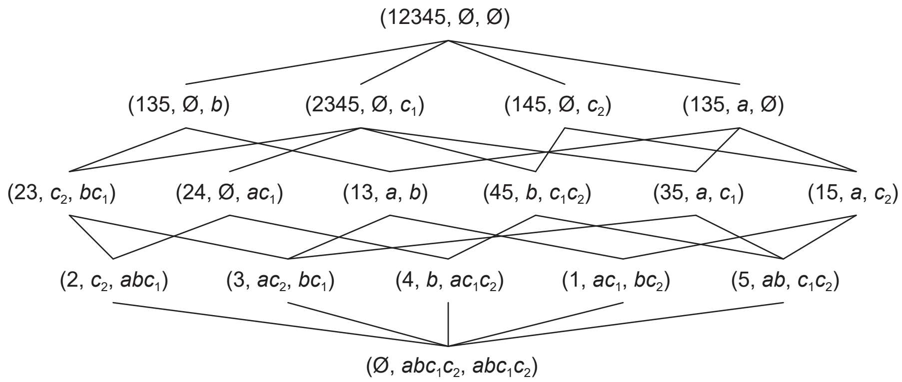

Fig. 3 Three-way concept lattice $OEL(K)$ of Example 2. For convenience, as an example, we use (245, bd, e) instead of ({2, 4, 5},({b, d}, {e})).

However, sometimes there is a need to learn who has an interest in computer programming. Then, C and Java programming should be upgraded to computer programming, and the updated formal context K' in Table 9 is obtained. According to Algorithm 2, we obtain the updated $OEL(K')$ depicted in Fig. 4, by which we can analyze the common interests of the students from a macroscopic viewpoint. For instance, concept ({4, 5}, ({b}, {c})) of $OEL(K')$ shows that both Students 4 and 5 favor chemistry, and meanwhile neither of them like computer programming.

For the sake of better understanding of Algorithm 2, we list the sequence of the visited concepts and their corresponding actions in Table 10.

5 Experimental Evaluation

This section demonstrates the influence of attribute granulation on three-way concept lattices via experiments. Besides, we compare these three-way concept lattices with formal concept lattices. All experiments were conducted on a single Intel Xeon Silver 4210 CPU@2.2 GHz with 64 GB RAM.

In order to effectively test the influence of attribute granulation on three-way and formal concept lattices, we use artificial data sets in the experiments, by which we can control the size of formal contexts. In fact, the zoom-in algorithm has been already proposed by Belohlavek[26], we can directly use it in our experiments.

The general process is as follows.

Step 1: Initialize formal context K in 6×15, 9×25, 12×35, and 15×45 matrices with values randomly from 1 and 0.

Step 2: Respectively build $OEL(K)$ and formal concept lattice $L(K)$, and track their number of concepts.

Step 3: Respectively perform Algorithm 2 to update $OEL(K)$ and $L(K)$.

Step 4: Repeat Step 3 and track the number of concepts in each stage.

Concretely, we generate 4 Groups of datasets, each Group consists of 3 datasets in the same size. Datasets 1−3 possess 6 objects and 15 attributes in Group 1. Datasets 4−6 have 9 objects and 25 attributes in Group 2. Datasets 7−9 include 12 objects and 35 attributes in Group 3. And Group 4 consists of Datasets 10−12 with 15 objects and 45 attributes.

Table 11 and Fig. 5 record our experimental results. In Fig. 5, the abscissa demonstrates the number of attributes that have been split, and the ordinate demonstrates the number of concepts.

It is manifest that repeatedly performing Algorithm 2 on $OEL(K)$, the number of three-way concepts keeps increasing, and the larger the number of objects, the more significant the growth trend is. By contrast, repeatedly performing Algorithm 2 on $L(K)$, the number of formal concepts keeps decreasing, and the larger the number of objects, the more significant the decrease trend is.

The above findings can be explained as follows. For

Table 9 Updated formal context K' of Example 2.

Student	Course		
	a	b	c
1	1	0	1
2	0	0	1
3	1	0	1
4	0	1	0
5	1	1	0

Algorithm 2 Zoom-out

input: $OEL\ (G, M, I)$ and g be the abstraction of $\delta = \{g_i \in M \mid i \in N^*\}$
output: The updated $OEL\ (K')$ after abstracting g
Initialize $OEL(K') = \varnothing$
Sort $OEL\ (G, M, I)$ into a queue $Q =$ $Q = \{(X_1, (A_1, B_1)), (X_2, (A_2, B_2)), \ldots, (X_k, (A_k, B_k))\}$, such that $|X_m| \geqslant |X_n|$ for $1 \leqslant m < n \leqslant k$
while $Q \neq \varnothing$ **do**
  Fetch the head from the queue Q and denote it as $(X, (A, B))$;
  if there does not exist $g_i \in A$ and $g_i \in B$ **then**
    if $X \nsubseteq g^*$ and $X \nsubseteq \overline{g^*}$ **then**
      add $(X, (A, B))$ into $OEL\ (K')$;
    end
    if $X \subseteq g^*$ **then**
      add $(X, (A \cup \{g\}, B))$ into $OEL\ (K')$;
    else
      add $(X, (A, B \cup \{g\}))$ into $OEL\ (K')$.
    end
    if $X \subseteq \overline{g^*}$ **then**
      add $(X, (A, B \cup \{g\}))$ into $OEL\ (K')$;
    else
      add $(X \cap \overline{g^*}, (A, B \cup \{g\}))$ into $OEL\ (K')$.
    end
  end
  if there exists $g_i \in A$ or $g_j \in B$ **then**
    if $X \nsubseteq g^*$, $X \nsubseteq \overline{g^*}$, and $(A - \{g_i\}, B - \{g_j\}) \notin OEL\ (K')$ **then**
      add $(X, (A - \{g_i\}, B - \{g_j\}))$ into $OEL\ (K')$.
    end
    if $X \subseteq g^*$, $X \nsubseteq \overline{g^*}$, and $((A - \{g_i\}) \cup \{g\}, B - \{g_j\}) \notin OEL\ (K')$ **then**
      add $(X, ((A - \{g_i\}) \cup \{g\}, B - \{g_j\}))$ into $OEL(K')$.
    end
    if $X \subseteq \overline{g^*}$, $X \nsubseteq g^*$, and $(A - \{g_i\}, (B - \{g_j\}) \cup \{g\}) \notin OEL\ (K')$ **then**
      add $(X, (A - \{g_i\}, (B - \{g_j\}) \cup \{g\}))$ into $OEL(K')$.
    end
    if $X \subseteq \overline{g^*}$ and $X \subseteq g^*$ **then**
      add $(X, ((A - \{g_i\}) \cup \{g\}, (B - \{g_j\}) \cup \{g\}))$ into $OEL\ (K')$.
    end
  end
end
return: $OEL\ (K')$.

$K = (G, M, I)$, fill rate $R = \dfrac{|\{I(x, y) \mid I(x, y) \neq 0\}|}{|G| \times |M|}$ can be defined to measure the sparsity of K. Studies have shown that the smaller R, the fewer the formal concepts a formal context has[52]. As repeatedly splitting attributes will continually reduce the fill rate R, the number of formal concepts will keep decreasing.

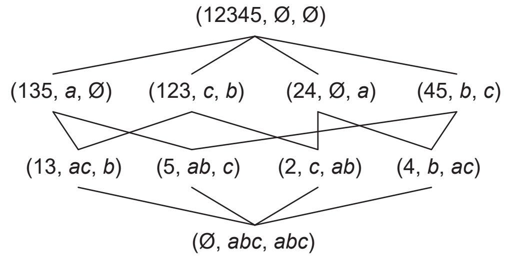

Fig. 4 Updated three-way concept lattice *OEL*(*K*′) of Example 2. For convenience, as an example, we use (245, *bd*, *e*) instead of ({2, 4, 5},({*b*, *d*}, {*e*})).

On the other hand, when the fill rate R of K decreases, the fill rate R' of the complement increases. As a three-way concept lattice embodies the information of both original context and complement, the number of three-way concepts will keep increasing.

The above findings also show that although there are relationships between three-way and classical concept lattices, there still exist many differences. In other words, three-way concept lattice has unique features, and is worthy of our in-depth study.

6 Conclusion

In this paper, we investigate the problem of granularity of attributes within the framework of 3WCA, which enables us to consider both positive and negative attributes altogether. Specifically, we present a zoom-in algorithm to get more particular three-way concepts, and a zoom-out algorithm to derive more abstract three-way concepts, which also reveal that the granularity of attributes influences the structure of three-way concepts extracted from data.

By comparing the algorithms proposed in this paper and the ones for classical FCA, we can see that within the framework of 3WCA, both positive and negative attributes are simultaneously considered, which provides deeper insight into the original formal context with more detailed information. In addition, if we only consider positive attributes, the proposed approach will degrade to fit the classical FCA. In other words, the methods already proposed for classical FCA can be regarded as a particular case of the ones proposed for 3WCA.

For this topic, although some important results have been obtained, there are some challenges. For example,

Table 10 Summary of zoom-out process of Example 2.

Number	Visited concept	Action
1	$(\{1,2,3,4,5\},(\varnothing,\varnothing))$	Add $(\{1,2,3,4,5\},(\varnothing,\varnothing))$ into $OEL(K')$
2	$(\{2,3,4,5\},(\varnothing,\{c_1\}))$	Do nothing
3	$(\{1,2,3\},(\varnothing,\{b\}))$	Add $(\{1,2,3\},(\{c\},\{b\}))$ into $OEL(K')$
4	$(\{1,4,5\},(\varnothing,\{c_2\}))$	Do nothing
5	$(\{1,3,5\},(\{a\},\varnothing))$	Add $(\{1,3,5\},(\{a\},\varnothing))$ into $OEL(K')$
6	$(\{2,3\},(\{c_2\},\{b,c_1\}))$	Do nothing
7	$(\{2,4\},(\varnothing,\{a,c_1\}))$	Add $(\{2,4\},(\varnothing,\{a\}))$ into $OEL(K')$
8	$(\{1,3\},(\{a\},\{b\}))$	Add $(\{1,3\},(\{a,c\},\{b\}))$ into $OEL(K')$
9	$(\{4,5\},(\{b\},\{c_1,c_2\}))$	Add $(\{4,5\},(\{b\},\{c\}))$ into $OEL(K')$
10	$(\{3,5\},(\{a\},\{c_1\}))$	Do nothing
11	$(\{1,5\},(\{a\},\{c_2\}))$	Do nothing
12	$(\{2\},(\{c_2\},\{a,b,c_1\}))$	Add $(\{2\},(\{c\},\{a,b\}))$ into $OEL(K')$
13	$(\{3\},(\{a,c_2\},\{b,c_1\}))$	Do nothing
14	$(\{4\},(\{b\},\{a,c_1,c_2\}))$	Add $(\{4\},(\{b\},\{a,c\}))$ into $OEL(K')$
15	$(\{1\},(\{a,c_1\},\{b,c_2\}))$	Do nothing
16	$(\{5\},(\{a,b\},\{c_1,c_2\}))$	Add $(\{5\},(\{a,b\},\{c\}))$ into $OEL(K')$
17	$(\varnothing,(\{a,b,c_1,c_2\},\{a,b,c_1,c_2\}))$	Add $(\varnothing,(\{a,b,c\},\{a,b,c\}))$ into $OEL(K')$

Table 11 Number of split attributes of formal and three-way concepts.

Dataset	Number of split attributes of formal concepts										Number of split attributes of three-way concepts									
	0	5	10	15	20	25	30	35	40	45	0	5	10	15	20	25	30	35	40	45
1	23	25	20	15	–	–	–	–	–	–	42	60	64	64	–	–	–	–	–	–
2	28	25	20	16	–	–	–	–	–	–	43	54	56	58	–	–	–	–	–	–
3	33	30	28	24	–	–	–	–	–	–	49	54	57	64	–	–	–	–	–	–
4	90	89	75	62	52	49	–	–	–	–	227	281	317	380	458	476	–	–	–	–
5	141	110	92	77	62	50	–	–	–	–	208	247	336	375	424	468	–	–	–	–
6	157	126	99	83	79	68	–	–	–	–	219	290	345	372	424	462	–	–	–	–
7	342	305	238	198	177	146	124	126	–	–	783	1314	1749	2322	2840	3137	3274	3454	–	–
8	475	381	310	248	216	172	143	128	–	–	830	943	1321	1621	2113	2443	2629	2953	–	–
9	575	515	423	332	265	218	171	155	–	–	923	1197	1437	1813	2180	2549	2748	2800	–	–
10	953	853	661	527	455	383	334	301	282	242	2856	4339	5246	7302	9540	12 293	14 335	15 788	19 990	22 093
11	1135	911	772	692	588	498	405	346	316	271	3006	3819	5177	6980	9560	12 280	14 097	16 527	18 489	20 495
12	1302	1094	934	833	660	568	435	354	320	279	2869	4126	6255	9003	10 418	13 090	15 065	16 194	17 885	19 449

automatic detection of granularity levels to reveal interesting patterns via three-way concept lattices is a promising study. Besides, calculating concept lattices is an NP-hard problem, and the execution time exhibits exponential increase with the addition of cases. It is a challenge to introduce the concept reduction algorithm to obtain the number of concepts after attribute granulation efficiently in the big data application scenarios. Additionally, the extension of the proposed method to cater data with multi-valued attributes in machine learning[53] and medical big data analytics[54], is another interesting topic. Therefore, more efforts should be made on this topic in the future.

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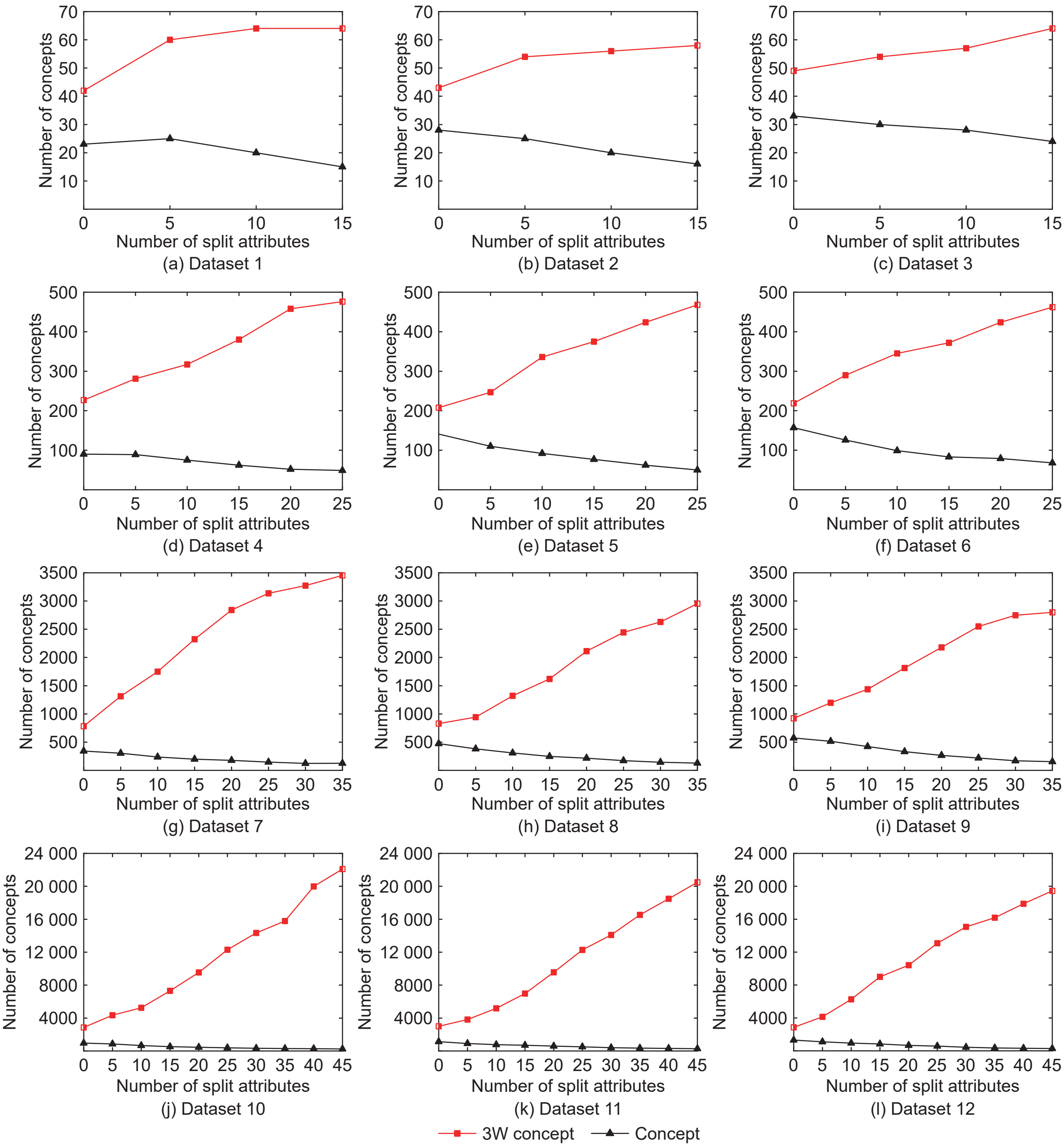

Fig. 5 Experimental results with Datasets 1–12.

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Jun Long received the BEng degree in software engineering from Changsha Railway Campus, China in 1996, and the MEng and PhD degrees in computer science from Central South University, China in 2004 and 2011, respectively. He is currently a professor at Big Data Institute, Central South University, China. His major research interests include machine learning, computer vision, and natural language processing.

Zhan Yang received the PhD degree in computer science and technology from Central South University, China in 2020. He is currently an associate professor at Big Data Institute, Central South University, China. His research interests include multimedia retrieval, computer vision, and pattern recognition.

Yinan Li received the BEng and MEng degrees in computer science and technology from Henan Polytechnic University, China in 2019 and 2022, respectively. He is currently a PhD candidate at Big Data Institute, Central South University, China. His current research interests include granular computing and cross-modal retrieval.