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Negativizability for Nonlinear Estimation in Cyber-Physical Systems

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***Abstract*— This paper introduces a novel fully distributed estimation scheme for nonlinear continuous-time dynamics over directed and strongly connected graphs. Leveraging on the assumption of local *negativizability*, the proposed approach performs the estimation of the interdependent subsystems of a cyber-physical system, despite the presence of nonlinear dependencies on the dynamics. This transforms the intricate task of nonlinear state estimation by each agent into more manageable local negativizability problems for the design of the estimation gains. A pivotal aspect of the approach is that each agent should be aware of an upper bound on the Lipschitz constant of the overall nonlinear function that characterizes the dynamics. To face this issue, we developed a novel distributed methodology for the estimation of the global Lipschitz constant, starting from the local observations of the system's nonlinearities. The effectiveness of the proposed scheme is numerically demonstrated through simulations.**

***Index Terms*— Cyber-Physical Systems, Distributed State Estimation, Negativizability, Nonlinear Systems**

I. Introduction

In recent years, distributed state estimation has become a focal point of research for multi-agent and cyber-physical systems, aiming to estimate states using local observers based on partial measurements and neighbors' estimates [1], [2]. In this context, nonlinearity is a significant challenge, as it can degrade performance if not properly managed. Researchers have addressed this through nonlinear state estimation under bounded noises [3], [4], particularly when noise statistics are unavailable [5]. Studies have also explored event-triggered robust state estimation [6] and methods for systems with time-varying delays and random nonlinearities [7]. While several techniques assume local observability, others have considered jointly observable nonlinear networks with known stochastic properties of nonlinear changes [8] or assuming the nonlinearities are Lipschitz functions (which implies that its growth is at most linear) [9], [10]. In particular, these works offer strategies for broader nonlinear systems [9] and enhance robustness through sensor redundancy [10]. Notice that all the aforementioned approaches have common characteristics: they consider the analysis of the full dynamical system as a *monolithic* entity, partially measured by a set of distributed sensors that collectively aim to obtain the state estimate of the entire system. This framework, in addition to requiring the agents to have global knowledge of information concerning the system, such as the entire dynamics, is often inefficient and implausible in a purely distributed context, such as a cyber-physical system [11], [12]. In this context, the physical part of the system is divided into subsystems, each of which is associated with a single agent on the cyber layer, while the network is not a mere transmission model but takes on a role also to indicate the physical interconnections between subsystems. Therefore, the objective of each agent in an *interdependent* setting is to estimate the state of its associated subsystem starting from local information about the dynamics while handling complications due to coupling with other subsystems. Although this setting has been explored for linear systems [13]–[15], and has found practical applications in the literature, such as for the problem of multiarea state estimation (e.g., see [1] and references therein), to the best of our knowledge, it remains unexplored for nonlinear systems. In this paper, we present a novel fully distributed estimation scheme for Cyber-Physical Systems (CPS)s characterized by nonlinear continuous-time dynamics over directed graphs. Unlike the other approaches in the literature [3]–[10], our scheme leverages on a novel *interdependent* setting that is fully integrable with the interconnected structure of CPSs, and whose main features are: (i) the physical level is divided into interdependent subsystems, each associated with an agent at the cyber layer that is in charge of estimating the state of its related physical part in spite of the nonlinear couplings. (ii) each local observer is assumed to have only partial knowledge of the dynamics, limited to the blocks related to its own subsystem; (iii) under the assumption of negativizability (discussed in Section II-A) subsystems are able to locally compute their own estimation gains while collectively carrying out the estimation tasks, despite the additional complexity represented by the nonlinear couplings and the limited knowledge of the system. Compared with the negativizability problem presented in [13], the novelty lies in the distributed handling of the different nonlinearities experienced at the level of each subsystem. This requires, on one side, that the presence of nonlinear couplings be incorporated into the application of the negativizability property, and, on the other side, the development of a novel fully distributed algorithm to estimate an upper bound on the Lipschitz constant, based on knowledge of the local nonlinearities that are experienced at each CPS.

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II. Preliminaries

We denote vectors with boldface lowercase letters and matrices with uppercase letters. We refer to the (i, j)-th entry of a matrix A by A_{ij}. We represent by $\mathbf{0}_n$ and $\mathbf{1}_n$ vectors with n entries, all equal to zero and to one, respectively and we use $0_{n\times m}$ to denote the $n \times m$ matrix with all entries equal to zero, while I_n denotes the $n \times n$ identity matrix. We use $\mathtt{ker}(A)$ to denote the kernel of a matrix A and $\mathtt{span}(\boldsymbol{x})$ to denote the span of $\boldsymbol{x}$. We use $\sigma(A)$ to denote the spectrum of A, i.e., the set of its eigenvalues. Given N matrices $A_i \in \mathbb{R}^{n_i\times m_i}$, we use $\mathtt{blkdiag}(A_i)$ to denote a block diagonal matrix having A_i as its i-th diagonal block. Given a matrix $A \in \mathbb{R}^{n\times n}$, we use $\mathtt{diag}(A)$ to denote the $n \times n$ diagonal matrix such that $\mathtt{diag}_{ii}(A) = A_{ii}$. We use $\|\cdot\|$ to denote the Euclidean norm. A matrix $A \in \mathbb{R}^{n\times n}$ is *positive semidefinite* (respectively, *negative semidefinite*) if for all $\boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{x} \neq \mathbf{0}_n$ it holds $\boldsymbol{x}^T(A + A^T)\boldsymbol{x} \geq 0$ (respectively, $\boldsymbol{x}^T(A + A^T)\boldsymbol{x} \leq 0$); if the inequality is tight, then A is said *positive definite* (respectively, *negative definite*). We use $A \succ 0$ (resp. $A \succeq 0$) to specify that a matrix A is positive definite (resp. positive semidefinite); similarly, we use $A \prec 0$ and $A \preceq 0$ to denote negative definiteness and semidefiniteness of A, respectively. Let $\mathbb{B}(c)$ and $\mathbb{B}[c]$ denote the open and closed ball centered in the origin of radius c in $\mathbb{R}^n$. A function $\boldsymbol{f}(\boldsymbol{x}, t)$ is Lipschitz in its first argument in a set $\mathbb{X}$ if, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{X}$ and for all times $t \geq 0$, it holds $\|f(\boldsymbol{x}, t) - f(\boldsymbol{y}, t)\| \leq \ell(t)\|\boldsymbol{x} - \boldsymbol{y}\|$, for some Lipschitz constant $\ell(t) > 0$. Let $G = \{V, E\}$ be a *graph* with n nodes $V = \{v_1, v_2, \dots, v_n\}$ and e edges $E \subseteq V \times V$, where $(v_i, v_j) \in E$ captures the existence of a link from node v_i to node v_j. A graph is said to be *undirected* if the existence of an edge $(v_i, v_j) \in E$ implies the existence of $(v_j, v_i) \in E$, while it is said to be *directed* otherwise. In this paper, we consider graphs that are, in general, directed. A graph is *strongly connected* if each node can be reached from each other node via some edges, respecting their orientation. Let the in-neighborhood $\mathcal{N}_i^{\mathtt{in}}$ of a node v_i be the set of nodes v_j such that $(v_j, v_i) \in E$. Moreover, let the out-neighborhood $\mathcal{N}_i^{\mathtt{out}}$ of a node v_i be the set of nodes v_j such that $(v_i, v_j) \in E$.

A. Negativizability

In [13] the pair $(\mathcal{A}, \mathcal{C})$, where $\mathcal{A} \in \mathbb{R}^{n\times n}$ and $\mathcal{C} \in \mathbb{R}^{q\times n}$, is said to be *negativizable* if it is possible to find a matrix $K \in \mathbb{R}^{n\times q}$ such that $\mathcal{A} - \mathcal{K}\mathcal{C}$ is negative definite. Interestingly, a necessary and sufficient negativizability condition is derived in [13] by noting that the problem can equivalently be expressed as a *semidefinite programming* (SDP) problem, for which the existence of a solution can be decided based on the semidefinite version of the Farkas Lemma (see Lemma 6.3.3, p. 153 in [16]). This implies that the problem can be numerically solved by resorting to an SDP solver (the interested reader is referred to [13] for more details). Relying on the fact that a negative definite matrix is also Hurwitz stable and exploiting a block version of the Gershgorin Circle Criterion, in [13] the negativizability property is exploited to solve the distributed and decoupled state estimation and control problem of a linear LTI system. Specifically, consider block partitioned matrices $\mathcal{A}, \mathcal{C}$ in the form

$$\mathcal{A} = \begin{bmatrix} A_{11} & \dots & A_{1N} \\ \vdots & \ddots & \vdots \\ A_{N1} & \dots & A_{NN} \end{bmatrix}, \quad \mathcal{C} = \mathtt{blkdiag}(C_i), \tag{1}$$

where $A_{ij} \in \mathbb{R}^{n_i\times n_j}$, and $C_i \in \mathbb{R}^{q_i\times n_i}$.

The following proposition [13] shows that, solving local negativizability problems on the pairs

$$(A_{ii} + \sum_{j\neq i} \|A_{ij}\| I_{n_i} + \beta I_{n_i},\, C_i), \tag{2}$$

for some $\beta > 0$, i.e., identifying the corresponding gains $K_i \in \mathbb{R}^{n_i\times q_i}$ such that $A_{ii}+\sum_{j\neq i} \|A_{ij}\| I_{n_i}+\beta I_{n_i}-K_iC_i$ is negative definite for all $i \in \{1, \dots, N\}$, results in an asymptotically stable matrix $\mathcal{A} - \mathcal{K}\mathcal{C}$, with $\mathcal{K} = \mathtt{blkdiag}(K_i)$.

Proposition 1: Let the pairs in Eq. (2) be negativizable for all $i \in \{1, \dots, N\}$ for some $\beta > 0$, and let $K_i \in \mathbb{R}^{n_i\times q_i}$ be a gain matrix that solves the negativizability problem for the i-th pair. Then, defining $\mathcal{K} = \mathtt{blkdiag}(K_i)$, we have that $\mathcal{A} - \mathcal{K}\mathcal{C}$ is negative definite.

Remark 1: The agents do not need to coordinate in order to consider a global parameter β. In fact, in the proof of Proposition 1 in [13], the role of β is to guarantee that the real part of the eigenvalues of $\mathcal{A} - \mathcal{K}\mathcal{C}$ are more negative than $-\beta$. Therefore, if each agent computes its gain K_i considering a $\beta_i > 0$, using the same argument as in [13] it can be shown that $\mathcal{A} - \mathcal{K}\mathcal{C}$ is negative definite with eigenvalues that have real part smaller than $-\min_{i=1,\dots,n} \beta_i < 0$. In the following, we assume that each agent chooses its own β_i independently and we use β to denote $\min_{i=1,\dots,n} \beta_i$.

III. Distributed Nonlinear State Estimation

In this section, we aim to exploit the negativizability concept to provide a useful tool for distributed state estimation of nonlinear systems dynamics. In particular, as shown in [13], by fulfilling the negativizability assumption, the agents in a network are able to implement the overall estimation strategy by means of local gains. Let us consider a Cyber-Physical system composed of N interdependent and non-overlapping subsystems, which interact with each other in accordance with the directed and strongly connected graph $G = \{V, E\}$, both at the physical and cyber level. Specifically, let us assume that the physical layer of the i-th subsystem is characterized by a nonlinear dynamics that can be represented as follows

$$\begin{cases} \dot{\boldsymbol{x}}_i(t) = A_{ii}\boldsymbol{x}_i(t) + \sum_{j\in\mathcal{N}_i^{\mathtt{in}}} A_{ij}\boldsymbol{x}_j(t) + \boldsymbol{f}_i(\overline{\boldsymbol{x}}_i(t), t), \\ \boldsymbol{y}_i(t) = C_i\boldsymbol{x}_i(t), \end{cases} \tag{3}$$

where $\boldsymbol{x}_i(t)$ is the state of the i-th subsystem, $A_{ii} \in \mathbb{R}^{n_i\times n_i}$, $A_{ij} \in \mathbb{R}^{n_i\times n_j}$, $C_i \in \mathbb{R}^{q_i\times n_i}$, and $\overline{\boldsymbol{x}}_i(t)$ is the stack of the vectors $\boldsymbol{x}_j(t) \in \mathbb{R}^{n_j}$ corresponding to all agents $j \in \mathcal{N}_i^{\mathtt{in}} \cup \{i\}$, and thus it holds $\overline{\boldsymbol{x}}_i(t) \in \mathbb{R}^{\overline{n}_i}$, where $\overline{n}_i = \sum_{j\in\mathcal{N}_i^{\mathtt{in}}\cup\{i\}} n_j$. In other words, $\boldsymbol{f}_i(\cdot, t)$ is computed only over the state of the in-neighbors of agent i and the state of agent i itself[1]. From this perspective, within the physical layer, an edge (v_j, v_i) exists whenever the

[1]The structure of $\boldsymbol{f}_i(\cdot, t)$ is quite general and includes, as a particular case, the class of functions in the form $\boldsymbol{f}_i(\cdot, t) = \sum_{j\in\mathcal{N}_i^{\mathtt{in}}} \boldsymbol{f}_{ij}(\boldsymbol{x}_i, \boldsymbol{x}_j, t)$, where $\boldsymbol{f}_{ij}(\boldsymbol{x}_i, \boldsymbol{x}_j, t)$ models the pairwise coupling between the i-th and j-th subsystems. An example of application of this class of functions in the context of distributed consensus can be found in [17].

i-th subsystem is influenced by the j-th one. Defining $n = \sum_{i=1}^{N} n_i, \quad q = \sum_{i=1}^{N} q_i$ and $\boldsymbol{x}(t) \in \mathbb{R}^n$, $\boldsymbol{y}(t) \in \mathbb{R}^q$ as the stack of the vectors $\boldsymbol{x}_i(t)$, $\boldsymbol{y}_i(t)$, respectively, the overall dynamics can be expressed (for the sake of the analysis) in a compact form as

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \mathcal{A}\boldsymbol{x}(t) + \boldsymbol{f}(\boldsymbol{x}(t), t), \\ \boldsymbol{y}(t) = \mathcal{C}\boldsymbol{x}(t), \end{cases} \tag{4}$$

where $\mathcal{A}$ and $\mathcal{C}$ are as in Eq. (1) and $\boldsymbol{f}(\boldsymbol{x}(t), t)$ is the stack of the functions $\boldsymbol{f}_i(\overline{\boldsymbol{x}}_i(t), t)$.

In the considered framework we assume that, for the physical layer, each subsystem is regulated by the corresponding one at the cyber layer. Notably, both layers are interconnected according to the same graph topology G. Within the cyber layer, each subsystem j functions as an *agent* and can transmit information to an agent i if there is an edge (v_j, v_i) in the graph. In this view, a coupling on the dynamics that affects two subsystems in the physical layer corresponds to a flow of information between agents at the cyber layer. The goal of each agent is to estimate the state of its own physical subsystem, based on knowledge of the physical coupling with its in-neighbors. We point out that this setting fundamentally differs from other distributed nonlinear state estimation approaches [3]–[10] which aim at the estimation of a complete and *monolithic* dynamical system, while in our model the subsystems composing the CPS are able to locally compute their gains while collectively carrying out the estimation tasks, despite the additional complexity represented by the nonlinear couplings. The limited knowledge of the agents in the estimation process is expressed by the following assumption.

Assumption 1: Each agent i has only knowledge of the blocks of the dynamical system matrix regarding its own piece of dynamics A_{ii} and the piece of dynamics that describes the interaction with its incoming neighbors A_{ij}, where $j \in \mathcal{N}_i^{\mathtt{in}}$. Moreover, each agent possesses knowledge of its local output matrix C_i and of the nonlinear function $\boldsymbol{f}_i(\cdot)$ associated with its own subsystem, which depends on the state of agent i itself and its in-neighbors.

In the given cyber-physical context, we demonstrate how the negativizability assumption ensures the effectiveness of a distributed nonlinear state estimation scheme that relies on local gains. Specifically, we will examine a scenario in which each agent has the aim to estimate its individual state using a distributed Luenberger-type observer, building the estimation dynamics as follows.

$$\begin{aligned} \dot{\boldsymbol{z}}_i(t) = & A_{ii}\boldsymbol{z}_i(t) + \sum_{j \in \mathcal{N}_i^{\mathtt{in}}} A_{ij}\boldsymbol{z}_j(t) + K_i(\boldsymbol{y}_i(t) - C_i\boldsymbol{z}_i(t)) \\ & + \boldsymbol{f}_i(\overline{\boldsymbol{z}}_i(t), t) \end{aligned} \tag{5}$$

where $\overline{\boldsymbol{z}}_i(t)$ is the stack of the terms $\boldsymbol{z}_j(t)$ for all $j \in \mathcal{N}_i^{\mathtt{in}} \cup \{i\}$ and $K_i \in \mathbb{R}^{n_i \times q_i}$. Interestingly, according to the above equation, the feedback is local and only involves the state and estimates locally available at node i; however, the different observers are coupled by the fact that each observer exhibits a dependency on the estimates of its in-neighbors. Notice that the previous observer dynamics can be expressed in a compact form by using $\boldsymbol{z}(t) \in \mathbb{R}^n$ to represent the collection of vectors $\boldsymbol{z}_i(t)$ for all agents. The overall observer dynamics is given by $\dot{\boldsymbol{z}}(t) = \mathcal{A}\boldsymbol{z}(t) + \mathcal{K}(\boldsymbol{y}(t) - \mathcal{C}\boldsymbol{z}(t)) + \boldsymbol{f}(\boldsymbol{z}(t), t)$, where $\mathcal{K} = \mathtt{blkdiag}(K_i) \in \mathbb{R}^{n \times q}$.

Let us now define the estimation error as $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \boldsymbol{z}(t) \in \mathbb{R}^n$, where $\boldsymbol{e}(t)$ is the stack of all the terms $\boldsymbol{e}_i(t) = \boldsymbol{x}_i(t) - \boldsymbol{z}_i(t) \in \mathbb{R}^{n_i}$. The error dynamics is given by

$$\dot{\boldsymbol{e}}(t) = (\mathcal{A} - \mathcal{K}\mathcal{C})\,\boldsymbol{e}(t) + \boldsymbol{f}(\boldsymbol{x}(t), t) - \boldsymbol{f}(\boldsymbol{z}(t), t). \tag{6}$$

Starting from the definition of the error dynamics, it is possible to analyze the convergence of the proposed nonlinear estimation methodology.

IV. Convergence Analysis

This section is devoted to establishing the convergence of the proposed estimation scheme. Specifically, we first show that convergence is achieved under the assumption that $\boldsymbol{f}(\cdot)$ is Lipschitz in all $\mathbb{R}^n$ and that nonlinear state estimation problem amounts to solving N local negativizability problems. Then, under the hypothesis that the state of the system being estimated is bounded, we establish an analogous result also when $\boldsymbol{f}$ is Lipschitz only in a neighborhood of the origin; this latter result will prove useful when the overall Lipschitz constant is unknown, and the agents need to collectively estimate it in a distributed fashion. To this aim, let us introduce the following assumption.

Assumption 2: The pairs

$$(A_{ii} + \sum_{j \neq i} \|A\|_{ij}\, I_{n_i} + \bar{\ell}\, I_{n_i},\, C_i) \tag{7}$$

are negativizable for all $i \in \{1, \ldots, N\}$ and let $\widehat{K}_i \in \mathbb{R}^{n_i \times q_i}$ be a matrix that solves the negativizability problem for the i-th pair in Eq. (7).

We are now in position to prove convergence.

Theorem 1: Suppose $\boldsymbol{f}(\boldsymbol{x}, t)$ is Lipschitz in its first argument in all $\mathbb{R}^n$, with $\ell(t) \leq \bar{\ell}$ for some $\bar{\ell}$. Moreover, suppose that Assumption 2 holds; then, choosing $K_i = \widehat{K}_i$ for all i, the origin of Eq. (6) is globally asymptotically stable.

Proof: In order to prove the statement, let us consider the Lyapunov candidate function $V(\boldsymbol{e}(t)) = \frac{1}{2}\|\boldsymbol{e}(t)\|^2$. The above function is positive for all $t \geq 0$ and $\boldsymbol{e}(t) \neq \mathbf{0}_n$, while it is zero for $\boldsymbol{e}(t) = \mathbf{0}_n$. Since we assumed $\boldsymbol{f}(\cdot, t)$ is Lipschitz in its first argument, we have that (omitting the time dependency of $\boldsymbol{e}(t)$, $\dot{\boldsymbol{e}}(t)$, $\boldsymbol{x}(t)$ and $\boldsymbol{z}(t)$ for the sake of brevity), for all $t \geq 0$ it holds $\dot{V}(\boldsymbol{e}) = \boldsymbol{e}^T\dot{\boldsymbol{e}} = \boldsymbol{e}^T(\mathcal{A} - \mathcal{K}\mathcal{C})\boldsymbol{e} + \boldsymbol{e}^T(f(\boldsymbol{x}, t) - f(\boldsymbol{z}, t))$, and $\boldsymbol{e}^T(f(\boldsymbol{x}, t) - f(\boldsymbol{z}, t)) \leq \bar{\ell}\|\boldsymbol{e}\|^2$, since $\boldsymbol{e} = \boldsymbol{x} - \boldsymbol{z}$; therefore,

$$\dot{V}(\boldsymbol{e}) \leq \boldsymbol{e}^T(\mathcal{A} - \mathcal{K}\mathcal{C} + \bar{\ell}\, I_n)\boldsymbol{e} < 0, \tag{8}$$

where the last inequality follows from Proposition 1 ($\mathcal{K} = \mathtt{blkdiag}(K_i)$ for suitable gains K_i, which exist by Assumption 2). Therefore, the origin of Eq. (6) is globally asymptotically stable. This completes our proof. ■

Remark 2: A class of CPSs that satisfy Assumption 2 is represented by those CPSs that, in the absence of coupling, can be made asymptotically stable and monotonically

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converging by means of suitable estimation gains K_i. In this case, introducing a nonlinear coupling with Lipschitz constant bounded by $\overline{\ell} = \min_i \rho_i$, where ρ_i is the largest real part among the eigenvalues of $A_{ii} - K_i C_i$, the same gains K_i still guarantee convergence of the estimation process.

Since, as discussed later in the paper, existing approaches to compute the Lipschitz constant are typically able to compute a constant only with respect to a bounded subset of the domain of the function, we now extend our analysis to the case where $\boldsymbol{f}(\cdot)$ is Lipschitz in the first argument in a neighborhood of the origin. Specifically, the next results show that asymptotic converge of the error to zero is guaranteed when the state is bounded in $\mathbb{B}[\eta]$ and $\boldsymbol{f}(\boldsymbol{x}, t)$ is Lipschitz in its first argument in a ball twice as big.

***Corollary** 1:* Suppose $\|\boldsymbol{x}(t)\| \leq \eta$ for all $t \geq 0$ and assume $\boldsymbol{z}(0) = \boldsymbol{0}_n$; moreover, let $\boldsymbol{f}(\boldsymbol{x}, t)$ be Lipschitz in $\boldsymbol{x}$ for all $t \geq 0$ and for all $\boldsymbol{x} \in \mathbb{B}[2\eta]$, with Lipschitz constant $\ell(t) \leq \overline{\ell}$ for some $\overline{\ell}$. Finally, suppose that Assumption 2 holds; then, choosing $K_i = \widehat{K}_i$ for all agents i, the error $\boldsymbol{e}(t)$ asymptotically converges to zero.

Proof: In order to prove the result we observe that, while both $\boldsymbol{x}(t)$ and $\boldsymbol{z}(t)$ are in $\mathbb{B}[2\eta]$, being $\boldsymbol{f}(\cdot, 0)$ Lipschitz in $\mathbb{B}[2\eta]$, their distance in norm is monotonically decreasing according to Eq. (8). At this point we observe that $\|\boldsymbol{e}(0)\| = \|\boldsymbol{x}(0) - \boldsymbol{z}(0)\| = \|\boldsymbol{x}(0)\| \leq \eta$; therefore, since $\|\boldsymbol{e}(t)\|$ is monotonically decreasing while both $\boldsymbol{x}(t)$ and $\boldsymbol{z}(t)$ are in $\mathbb{B}[2\eta]$, we have that in the worst case where $\|\boldsymbol{x}(t)\| = \eta$ it holds $\|\boldsymbol{x}(t) - \boldsymbol{z}(t)\| \leq \eta$, which implies that $\|\boldsymbol{z}(t)\| \leq 2\eta$. In other words, both $\boldsymbol{x}(t)$ and $\boldsymbol{z}(t)$ remain by construction in $\mathbb{B}[2\eta]$ at all times t, and thus Eq. (8) holds at all times $t \geq 0$; this implies asymptotic convergence of the error to zero. The proof is complete. ∎

Remark 3: The above results establish convergence to zero of the estimation error, but do not allow to reach conclusions on the convergence rate. However, if Assumption 2 holds when $\overline{\ell}$ is replaced with $\widetilde{\ell} > \overline{\ell}$ in Eq. (7) then, since we have that $\mathcal{A} - \mathcal{KC} \prec -\widetilde{\ell} I_n$, we conclude that $\dot{V}(\boldsymbol{e}) \leq -(\widetilde{\ell} - \overline{\ell})\|\boldsymbol{e}\|^2$. Hence, we are guaranteed that $V(t)$ converges to zero at least as fast as $\overline{V}(t) = e^{-(\widetilde{\ell} - \overline{\ell})t}\|\boldsymbol{e}(0)\|^2$.

V. Distributed Lipschitz Constant Estimation

In Section IV, we have proved that each agent is able to perform the estimation of the nonlinear subsystem in a purely distributed manner, solving local negativizability problems. However, our approach relies on the fact that the agents know an upper-bound $\overline{\ell}$ on the Lipschitz constant of the overall nonlinear function $\boldsymbol{f}(\cdot)$. To address this requirement, this section is devoted to developing an upper bound on the overall Lipschitz constant which can be computed in a distributed manner. It is based on the upper bounds on the local Lipschitz constants $\overline{\ell}_i$ associated to the components $\boldsymbol{f}_i(\cdot)$ of the overall nonlinear function $\boldsymbol{f}(\cdot)$, which can be computed locally by each agent. To this end, let us first present some ancillary results.

***Proposition** 2 (Ex. 3 in [18], p. 356):* Let $\mathbb{S}$ be an open and convex subset of $\mathbb{R}^n$ and let $\boldsymbol{g} : \mathbb{S} \to \mathbb{R}^m$ be differentiable at each point of $\mathbb{S}$. Then for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{S}$ there is a $\boldsymbol{z} \in \mathbb{S}$ in the form $\boldsymbol{z} = \tau\boldsymbol{x} + (1-\tau)\boldsymbol{y}$ for some $\tau \in [0,1]$, such that $\|\boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{y})\| \leq \sum_{i=1}^m \|\nabla g_i(\boldsymbol{z})\| \|\boldsymbol{x} - \boldsymbol{y}\|$, where $\nabla g_i(\boldsymbol{z})$ is the gradient of the i-th component of $\boldsymbol{g}(\cdot)$, evaluated at $\boldsymbol{z}$.

***Corollary** 2:* Let the assumptions of Proposition 2 hold true and, further to that, suppose $\mathbb{S}$ is boudned. Then $\boldsymbol{g}(\cdot)$ is Lipschitz in $\mathbb{S}$ with Lipschitz constant $\ell = \sup_{\boldsymbol{w} \in \mathbb{S}} q(\boldsymbol{w})$, where $q(\boldsymbol{w}) = \sum_{i=1}^m \|\nabla g_i(\boldsymbol{w})\|$.

Proof: The proof follows noting that, $\forall \boldsymbol{z} \in \mathbb{S}$ it holds $\sum_{i=1}^m \|\nabla g_i(\boldsymbol{z})\| \leq \sup_{\boldsymbol{w} \in \mathbb{S}} q(\boldsymbol{w})$; hence we have that $\|\boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{y})\| \leq \sup_{\boldsymbol{w} \in \mathbb{S}} q(\boldsymbol{w}) \|\boldsymbol{x} - \boldsymbol{y}\|$. ∎

Let us now apply the above technique to our problem setting.

Theorem 2: Suppose that, for all $t \geq 0$ the local nonlinear functions $\boldsymbol{f}_i(\boldsymbol{w}, t)$ are differentiable at all $\boldsymbol{w} \in \mathbb{S}_i$, with $\mathbb{S}_i \subseteq \mathbb{R}^{\overline{n}_i}$ an open, bounded and convex set. Then, $\boldsymbol{f}_i(\boldsymbol{w}, t)$ is Lipschitz in its first argument in for all $t \geq 0$ and all $\boldsymbol{w} \in \mathbb{S}_i$, i.e., for all $t \geq 0$ and all $\overline{\boldsymbol{x}}_i, \overline{\boldsymbol{y}}_i \in \mathbb{S}_i$ it holds $\|\boldsymbol{f}_i(\overline{\boldsymbol{x}}_i, t) - \boldsymbol{f}_i(\overline{\boldsymbol{y}}_i, t)\| \leq \ell_i(t)\|\overline{\boldsymbol{x}}_i - \overline{\boldsymbol{y}}_i\|$, for some $\ell_i(t) > 0$. Moreover, it holds

$$\ell_i(t) \leq \sup_t \left\{ \underbrace{\sup_{\boldsymbol{w} \in \mathbb{S}_i} \left\{ \sum_{j=1}^{n_i} \|\nabla f_{ij}(\boldsymbol{w}, t)\| \right\}}_{\zeta(t)} \right\} = \overline{\ell}_i,$$

where $\nabla f_{ij}(\cdot)$ is the gradient of the j-th component of $\boldsymbol{f}_i(\cdot)$.

Proof: We observe that $\mathbb{S}_i$ is open, bounded, and convex and that, for all $t \geq 0$, the function $\boldsymbol{f}_i(\boldsymbol{w}, t)$ is differentiable at all $\boldsymbol{w} \in \mathbb{S}_i$. Hence, for all $t' \geq 0$, the function $\boldsymbol{f}_i(\cdot, t')$ satisfies the assumptions of Corollary 2 and thus $\ell_i(t') \leq \zeta(t')$. Let us consider the directional derivative of $f_i(\boldsymbol{w}, t)$ with respect to $\boldsymbol{w} \in \mathcal{S}_i$ along a given unit-length vector $\boldsymbol{v} \in \mathbb{R}^{n_i}$ for any $t \geq 0$, i.e.,

$$\nabla_{\boldsymbol{v}} f(\boldsymbol{w}, t) = \lim_{h \to 0} \frac{f(\boldsymbol{w} + h\boldsymbol{v}, t) - f(\boldsymbol{w}, t)}{h} \tag{9}$$

Since $f(\boldsymbol{w}, t)$ is differentiable at $\boldsymbol{w}$ we have that $\nabla_{\boldsymbol{v}} \boldsymbol{f}(\boldsymbol{w}, t) = \nabla \boldsymbol{f}(\boldsymbol{w}, t)^\top \boldsymbol{v}$. At this point, since f is Lipschitz in $\boldsymbol{w}$, we have that

$$\|\nabla \boldsymbol{f}(\boldsymbol{w}, t)^\top \boldsymbol{v}\| = \lim_{h \to 0} \frac{\|f(\boldsymbol{w} + h\boldsymbol{v}, t) - f(\boldsymbol{w}, t)\|}{|h|} \leq \bar{\ell}\|\boldsymbol{v}\|.$$

We conclude that the gradient has bounded entries by choosing $\boldsymbol{v}$ as the i-th vector in the canonical basis of $\mathbb{R}^n$. Therefore $\zeta(t)$ is bounded for all $t \geq 0$. The proof follows taking the sup of $\zeta(t)$ over all time instants $t \geq 0$. ∎

Let us now show that, based on the local knowledge each agent has of its own Lipschitz constant $\overline{\ell}_i$, it is possible to derive an upper bound on the Lipschitz constant $\overline{\ell}$ of the overall nonlinear function $\boldsymbol{f}(\boldsymbol{x}(t), t)$.

Theorem 3: Let the assumptions of Theorem 2 hold true for all $i \in \{1, \ldots, n\}$ and considering the set $\mathbb{S}_i = (-\gamma, \gamma)^{\overline{n}_i}$, with $\gamma > 0$. Moreover, suppose that the graph G underlying the agents' interaction is directed and strongly connected. Then, for all $t \geq 0$, the overall nonlinear function $\boldsymbol{f}(\boldsymbol{x}, t)$ is Lipschitz (in its first argument) in $(-\gamma, \gamma)^n$, i.e., in the open hypercube in

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$\mathbb{R}^n$ with side 2γ, centered in the origin. Finally, for all $t \geq 0$ and all $\boldsymbol{x} \in (-\gamma,\gamma)^n$, the Lipschitz constant $\overline{\ell}$ of the overall nonlinear function $\boldsymbol{f}(\boldsymbol{x}(t),t)$ satisfies $\overline{\ell} \leq \sqrt{(\max_{i=1,\dots,N}\{|\mathcal{N}_i^{\text{out}}|\}+1)\max_{i=1,\dots,N}\{\overline{\ell}_i^2\}}$.

Proof: Since, by Theorem 2, the local functions $\boldsymbol{f}_i(\cdot)$ are Lipschitz in their first argument in $\mathbb{S}_i$, we have that, for all the state vectors $\overline{\boldsymbol{x}}_i$ and for all t, it holds

$$\|\boldsymbol{f}_i(\overline{\boldsymbol{x}}_i,t)-\boldsymbol{f}_i(\overline{\boldsymbol{y}}_i,t)\|^2 \leq \overline{\ell}_i^2\|\overline{\boldsymbol{x}}_i-\overline{\boldsymbol{y}}_i\|^2 = \overline{\ell}_i^2 \sum_{j\in\mathcal{N}_i^{\text{in}}\cup\{i\}} \|\boldsymbol{x}_j-\boldsymbol{y}_j\|^2,$$

where the last equality holds since, by definition, $\overline{\boldsymbol{x}}_i$ and $\overline{\boldsymbol{y}}_i$ are the stack of the terms $\boldsymbol{x}_j$ and $\boldsymbol{y}_j$ for all $j \in \mathcal{N}_i^{\text{in}} \cup \{i\}$. Then, summing the above equation for all i yields

$$\begin{aligned}\sum_{i=1}^{N}\|\boldsymbol{f}_i(\overline{\boldsymbol{x}}_i,t)-\boldsymbol{f}_i(\overline{\boldsymbol{y}}_i,t)\|^2 &\leq \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i^{\text{in}}\cup\{i\}}\overline{\ell}_i^2\|\boldsymbol{x}_j-\boldsymbol{y}_j\|^2\\ &= \sum_{i=1}^{N}\underbrace{\sum_{j\in\mathcal{N}_i^{\text{out}}\cup\{i\}}\overline{\ell}_j^2}_{\omega_i^2}\|\boldsymbol{x}_i-\boldsymbol{y}_i\|^2,\end{aligned}$$

with $\omega_i > 0$. Notice that the fact G is strongly connected guarantees that all terms $\|\boldsymbol{x}_i-\boldsymbol{y}_i\|^2$ appear at least once in the above sum. Therefore, since we assumed all $\overline{\boldsymbol{x}}_i$ and $\overline{\boldsymbol{y}}_i \in \mathbb{S}_i$ (i.e., all entries are in the range $(-\gamma,\gamma)$), we have that, by construction, it holds $\sum_{i=1}^{N}\|\boldsymbol{f}_i(\overline{\boldsymbol{x}}_i,t)-\boldsymbol{f}_i(\overline{\boldsymbol{y}}_i,t)\|^2 = \|\boldsymbol{f}(\boldsymbol{x},t)-\boldsymbol{f}(\boldsymbol{y},t)\|^2$, and

$$\begin{aligned}\sum_{i=1}^{N}\omega_i^2\|\boldsymbol{x}_i-\boldsymbol{y}_i\|^2 &= \|\mathtt{blkdiag}(\omega_i I_{n_i})(\boldsymbol{x}-\boldsymbol{y})\|^2\\ &\leq \|\mathtt{blkdiag}(\omega_i I_{n_i})\|^2\|\boldsymbol{x}-\boldsymbol{y}\|^2 = \max_i(\omega_i^2)\|\boldsymbol{x}-\boldsymbol{y}\|^2;\end{aligned}$$

then we observe that any choice of $\boldsymbol{x},\boldsymbol{y} \in (-\gamma,\gamma)^n$ corresponds to some choice of the vectors $\overline{\boldsymbol{x}}_i, \overline{\boldsymbol{y}}_i \in \mathbb{S}_i$ for all agents i. Moreover, for all $\boldsymbol{x},\boldsymbol{y} \in (-\gamma,\gamma)^n$ and all $t \geq 0$ it holds $\|f(\boldsymbol{x},t)-f(\boldsymbol{y},t)\| \leq \sqrt{\max_i(\omega_i^2)}\|\boldsymbol{x}-\boldsymbol{y}\|$. In conclusion, we consider that

$$\omega_i^2 = \sum_{j\in\mathcal{N}_i^{\text{out}}\cup\{i\}}\overline{\ell}_j^2 \leq (|\mathcal{N}_i^{\text{out}}|+1)\max_{i=1,\dots,N}\{\overline{\ell}_i^2\}.$$

This completes our proof. ■

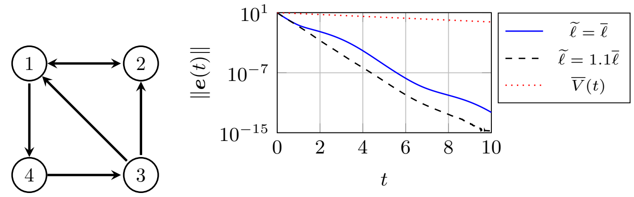
Notice that this upper bound can be computed in a distributed way. In fact, the out-degree can be computed in finite time via algorithms available at the state of the art (e.g., see [19]). As for $\max_{i=1,\dots,N}\{\overline{\ell}_i^2\}$, it is sufficient to run a max-consensus algorithm where the initial state of each agent is set to $\overline{\ell}_i^2$; such an algorithm is guaranteed to converge in finite-time [20]. The overall procedure is summarized in Algorithm 1.

Algorithm 1 Distributed computation of $\overline{\ell}$

1) Distributed computation of $|\mathcal{N}_i^{\text{out}}|$;
2) Distributed computation of $\max_{i=1,\dots,N}\{|\mathcal{N}_i^{\text{out}}|\}$;
3) Local computation of $\overline{\ell}_i$ as in Theorem 2;
4) Distributed computation of $\max_{i=1,\dots,N}\{\overline{\ell}_i^2\}$;
5) Local computation of $\overline{\ell}$ as in Theorem 3.

VI. Simulations

In this section, we show the effectiveness of the proposed approach via a simulation scenario. Specifically, we consider a distributed nonlinear estimation problem involving four subsystems, interconnected according to the graph topology reported in Fig. 1a. The overall dynamical matrix $\mathcal{A}$ and the nonlinear function $\boldsymbol{f}(\boldsymbol{x}(t),t)$ are reported below and



(a) **(b)**

Fig. 1: (a) Topology of the CPS considered in the example; (b) temporal evolution of the Euclidean norm of the error $\boldsymbol{e}(t)$ between the overall state vector $\boldsymbol{x}(t)$ and estimated state vector $\boldsymbol{z}(t)$ (blue solid line), along with the one obtained using the strategy in Remark 3 (black dashed line) and its corresponding upper bound (red dotted line).

are block-partitioned in a way that is consistent with the dynamics of each subsystem

$$\mathcal{A} = \left[\begin{array}{cc|cc|ccc|cc} 0.2 & 0.8 & 0.9 & 1 & 0.3 & 0.05 & 0.07 & 0 & 0\\ -0.8 & -10 & -2 & 0.7 & 0.2 & 0.1 & 0.1 & 0 & 0\\ \hline -0.1 & 0.2 & -5 & 0.1 & 0.2 & 0.2 & 0.1 & 0 & 0\\ 0.01 & 0.01 & -1 & -3.5 & -0.1 & 0.2 & 0.06 & 0 & 0\\ \hline 0 & 0 & 0 & 0 & -8 & -2 & -1 & 0.8 & -0.3\\ 0 & 0 & 0 & 0 & -1 & -1 & 5 & 0.6 & 0.5\\ 0 & 0 & 0 & 0 & 2 & -5 & -6 & 0.01 & 0.02\\ \hline 0.02 & 0.03 & 0 & 0 & 0 & 0 & 0 & -2 & -0.5\\ 0.01 & 0.03 & 0 & 0 & 0 & 0 & 0 & 0.5 & -5 \end{array}\right],$$

$$\boldsymbol{f}(\boldsymbol{x}(t),t) = \left[\begin{array}{c} 1.3\sqrt{x_1^2+5}\,cos(5t) - 0.2\sqrt{x_3^2+5}\\ 0.9sin(10t)\\ \hline 0.4cos(t)\\ -0.1\sqrt{x_2^2+5} - 0.8\sqrt{x_4^2+5}\\ \hline 0.2\sqrt{x_5^2+2} - 0.4\sqrt{x_7^2+4}\\ 1.2sin(x_9)\\ 0.6cos(3t)\\ \hline 1.3sin(t)\\ 0 \end{array}\right],$$

Moreover, the output matrices C_i that constitute the overall output matrix $\mathcal{C}$ are $C_1 = [1,0]$, $C_2 = [1,1]$, $C_3 = [0,2,1]$, and $C_4 = [1,-1]$, while the initial conditions $\boldsymbol{x}_i(0) = \boldsymbol{x}_{i,0}$ of each subsystems are $\boldsymbol{x}_{1,0} = [4.95,-3.83]^\top$, $\boldsymbol{x}_{2,0} = [4.50,-4.66]^\top$, $\boldsymbol{x}_{3,0} = [-0.61,-1.18]^\top$, and $\boldsymbol{x}_{4,0} = [2.95,-3.13]^\top$. Firstly, we let the agents compute their local Lipschitz constants. According to Theorem 2 it holds $\overline{\ell_1} = 1.3153$, $\overline{\ell_2} = 0.8062$, $\overline{\ell_3} = 1.6472$, $\overline{\ell_4} = 0$; interestingly, the last Lipschitz constant is zero as the nonlinear function only depends on time. Based on the above local Lipschitz constants, the agents identify an upper bound on the overall Lipschitz constant following the procedure expressed by Algorithm 1; as a result, the upper bound $\overline{\ell} = 2.853$ is obtained.At this point, we implement the proposed local state estimation scheme. To this end, we

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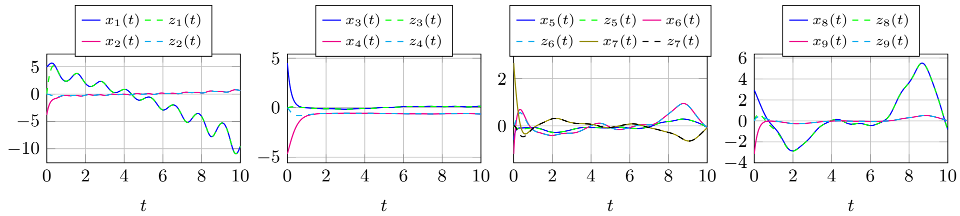

Fig. 2: Temporal evolution for the state variables $\boldsymbol{x}_i(t)$ (solid lines) and the estimated states $\boldsymbol{z}_i(t)$ (dashed lines), considering the example discussed in Section VI. Each subplot reports the evolution of the variables associated with a given subsystem.

consider the negativizability problems for the pairs $(\widehat{A}_i, C_i)$ with $\widehat{A}_i = A_{ii} + \sum_{j \neq i} \|A_{ij}\| I_{n_i} + \bar{\ell} I_{n_i}$, and we use an SDP solver for Matlab, namely *fminsdp* [21] to solve a SDP problem similar to the one in [13], where F_0 contains the value of $\bar{\ell}$. The resulting gain matrices K_i for the local observers are: $K_1 = [6.497, 0]^\top$, $K_2 = [-0.445, 1.306]^\top$, $K_3 = [-0.358, 1.873, -0.818]^\top$, and $K_4 = [1.028, 0.716]^\top$. For all the subsystems, the negativizability problems are successfully solved. Fig. 2 reports the temporal evolution of the states $\boldsymbol{x}_i(t)$ associated with each subsystem, and the related observer state $\boldsymbol{z}_i(t)$. As noted by the figure, the satisfaction of the negativizability assumptions guarantees that the vectors $\boldsymbol{z}_i(t)$ asymptotically coincide with the actual states $\boldsymbol{x}_i(t)$; in other words, the estimation error $\boldsymbol{e}(t)$ asymptotically converges to zero (see the blue curve in Fig. 1b) despite the presence of the nonlinear term in the dynamics. To conclude, Fig. 1b also provides an application of the strategy outlined in Remark 3 to the example discussed in this section. Specifically, the black dashed curve shows the convergence of the error when $\tilde{\ell} = 1.1\bar{\ell}$ (i.e., $\tilde{\ell} - \bar{\ell} = 0.2853$). In this case, the local negativizability problems all admit a solution. Interestingly, the red dotted line in Fig. 1b shows the convergence of the theoretical upper bound $\overline{V}(t)$, which is quite conservative.

VII. Conclusions

In this paper, we have presented a new methodology for distributed estimation of interconnected nonlinear subsystems that, based on solving local negativizability problems at the edge of each agent, allows for estimation gains design in a fully distributed manner. We also demonstrated the feasibility of a novel algorithm for estimating an upper bound on the global Lipschitz constant to the network that is also purely distributed and functional to the proposed estimation scheme. Through simulation, we demonstrated the effectiveness of the approach on an example of a CPS with four agents. Possible future directions of the methodology include the application of negativizability to Kalman filtering and optimal control problems and the extension to the case of subsystems featuring overlapping states.

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