Notes on Input Design: From Multi-Sine Design to Data-Driven Procedures

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***Abstract*—We show that a class of optimal input design problems have only discrete spectral measures as solutions. If we fix any finite set of possible frequencies then a randomized version of the resulting convex problem has a unique (sparse) solution with probability 1. We also propose a data-driven approach to optimal input design via virtual off-line estimators that coincide with the optimized PE estimator modulo a negligible error, both for open loop and closed loop systems.**

***Index Terms*—Closed loop systems, data-driven modeling, performance evaluation, system identification.**

I. Introduction

INPUT design for linear stochastic control systems is of fundamental importance in many industrial control systems, see [1] for an excellent survey. Remarkably, input design has become central also in machine learning, see [2]. Considering a family of linear stochastic control systems with input u, parameterized by θ, a central issue of input design is to minimize the expected loss in a performance index $J(\theta)$ due to uncertainty, incurred by replacing the true parameter by its estimate $\hat{\theta}_N$, subject to constraints on the input. A preliminary problem is to optimize the information matrix M, depending linearly on the spectral measure of u. It is readily seen that this is a convex problem with a unique solution, see (16) and Section II. Thus, the input design problem can be reduced to a generalized moment problem of finding a spectral measure $d\Phi^u(\cdot)$ such that it generates M^u via (13) below. This approach has been pursued in [3].

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For practical reasons significant attention has been paid to finding sub-optimal solutions by restricting the search space to a convex set of feasible spectral measures with a density, depending linearly on a parameter η belonging to a compact, convex set in a Euclidean space. A benchmark example is the set of spectral measures of FIR (Finite Impulse Response) processes, with the feasible parameters being the coefficients of the half spectra, constrained by LMI-s (Linear Matrix Inequalities), as first introduced in [4].

In the case of closed loop systems, allowing a LTI (Linear Time Invariant) feedback K, a preliminary convex optimization problem over the pairs of spectral density and cross spectra, (Φ_u, Φ_{ue}), as variables can be formulated, see [5]. Elaborating this idea K itself can be treated as an additional design parameter. A smart way of describing the pairs (Φ_u, Φ_{ue}) is obtained by using the Youla-Kučera parameterization of K, leading to a linear parameterization, see [6] for an advanced exposition. A remarkable feature of closed loop identification is that feedback reduces the asymptotic covariance matrix of the estimate of the signal transfer function, see [7].

In this letter we revisit the problem of multi-sine input design, recently attracting renewed interest, see [8]. Our starting point is that the optimization problem can be redefined as a convex optimization problem over the space of spectral measures $d\Phi^u(.)$ with discrete spectrum subject to energy constraints. In turn, the latter can be approximated with arbitrary prescribed accuracy by a convex design problem over the space of spectral measures $d\Phi^u(.)$ with support on a fixed set of frequencies. We will establish the remarkable fact that the relaxed version of this problem, obtained by adding the energy constraint multiplied by a freely chosen Lagrange-multiplier, has a unique solution w.p.1 (with probability 1) when the weights are chosen randomly according to a probability law that has a density. This sub-optimal spectral measure can be readily realized by a multi-sine.

An additional issue to be considered is what was called the "Achilles' heel of optimal input design" in [1], namely the fact that the optimization problem depends on the true system. This paradox has actually been resolved in [9] for ARMAX systems exited by inputs generated by a FIR filter, with a full-scale technical analysis relying on advanced results on recursive estimation given in [10]. In Section IV we reformulate and extend the basic idea of the above paper. The key tool is the definition of a virtual off-line estimator, having a very accurate characterization. Although we can not compute it in practice, its on-line approximation, obtained along the lines of [11], [12], is computable leading to an adaptive input design method. This data-driven approach is presented first for open loop system. However, it readily generalizes to closed loop systems with a θ-dependent feedback loop, obtained for some optimal control problem, yielding the optimal spectral density for the external excitation. As far as we can see our approach is a digression from the main-stream literature, in which the transfer function of the feedback loop is a design variable for the input design problem itself.

II. Technical Preliminaries

To be specific, we consider a discrete time single input single output linear stochastic control system with input u, external noise e and output y, defined in the range $-\infty < n < +\infty$

$$y = H^u\left(\theta^*, q^{-1}\right)u + H^e\left(\theta^*, q^{-1}\right)e. \tag{1}$$

Here H^u and H^e are rational functions of the backward shift operator q^{-1} of fixed degrees, depending on a parameter θ. We assume $\theta \in D \in \mathbb{R}^p$, where D is an open domain. The true parameter will be denoted by θ^*. The associated transfer functions are obtained when replacing q^{-1} by $e^{-i\omega}$.

Condition 1: The transfer function $H^u(\theta)$ is stable, and $H^e(\theta)$ is stable and inverse stable for all $\theta \in D$. Moreover, they are three-times continuously differentiable functions of θ. In addition, we assume that the input is delayed 1 unit.

The smoothness condition imposed on $H^u(\theta)$ and $H^e(\theta)$ is interpreted via a state-space realization, see also [13].

Condition 2: The input process $u = (u_n)$ and the noise process $e = (e_n)$, with $-\infty < n < +\infty$, are jointly w.s.st. (wide sense stationary) stochastic processes. Moreover, (e_n) is a martingale difference process with respect to (w.r.t.) an increasing family of σ-algebras $(\mathcal{F}_n)$ such that

$$\mathbb{E}\left[e_n|\mathcal{F}_{n-1}\right] = 0 \quad \text{and} \quad \mathbb{E}\left[e_n^2|\mathcal{F}_{n-1}\right] = \sigma^2 \quad \text{a.s.} \tag{2}$$

Finally we assume that u is orthogonal to e, written as $u \perp e$.

For the basic concepts of the theory of w.s.st. processes and system identification see [14], [15], and [16]. It follows that e is the innovation process of $y - H^u(\theta^*, q^{-1})u$.

The spectral distribution measure of u is denoted by $\mathrm{d}\Psi^u(\omega)$, with $-\pi \le \omega \le \pi$. Since it is symmetric, its restriction to $[0, \pi]$, with $\mathrm{d}\Psi^u(\{0\})$ and $\mathrm{d}\Psi^u(\{\pi\})$ halved, is denoted by $\mathrm{d}\Phi^u(\omega)$.

A *multi-sine input* u is defined as

$$u_n := \sum_{k=1}^{t} \sigma_k 2\cos(\varphi_k + \omega_k n) \tag{3}$$

where $0 \le \omega_k \le \pi$ are different, and the random phases φ_k are independent and uniformly distributed in $[-\pi, \pi]$. Thus $\mathrm{d}\Phi^u(\cdot)$ is discrete, assigning the energy σ_k^2 to each frequency.

To fix notations for the description of the off-line PE (prediction error) estimator of θ^* define for any $\theta \in D$ the assumed innovation process $\varepsilon(\theta)$ as the w.s.st. process

$$\varepsilon(\theta) = H^e(\theta)^{-1}\left(y - H^u(\theta)u\right), \tag{4}$$

defined for $-\infty < n < \infty$. Assuming the observations are collected for $1 \le n \le N$ the (idealized) off-line PE method of θ^* is then obtained by minimizing the cost function

$$V_N(\theta) := \frac{1}{2}\sum_{n=1}^{N} \varepsilon_n^2(\theta). \tag{5}$$

The solution will be denoted by $\hat{\theta}_N$. A precise definition, taking into account the possibility of no solution or multiple solutions, can be obtained along the lines of [13]. In practice the w.s.st. process $\varepsilon(\theta)$ is approximated by a process defined via (4) with $u_n = y_n = 0$ for $n \le 0$. The asymptotic cost function associated with the PE method is then

$$W(\theta) := \frac{1}{2}\mathbb{E}\varepsilon_n^2(\theta). \tag{6}$$

The Hessian of $W(\theta)$ for $\theta = \theta^*$ is given by

$$M = M\left(\theta^*\right) := \frac{\partial^2}{\partial\theta^2}W(\theta)_{\theta=\theta^*} = \mathbb{E}\varepsilon_{\theta n}\left(\theta^*\right)\varepsilon_{\theta n}^{\top}\left(\theta^*\right), \tag{7}$$

where $\varepsilon_{\theta n}(\theta)$ denotes the gradient of $\varepsilon_n(\theta)$ w.r.t. θ, considered to be a column vector. The system or θ^* is *locally identifiable* if M is positive definite, written as $M \in \mathbb{R}_+^{p\times p}$.

Condition 3: The system (1) is locally identifiable if the input u is a w.s.st. orthogonal process, independent of e. Equivalently, there exists a multi-sine input (3), independent of e, such that M is non-singular (see Proposition 1).

Precise conditions for the non-singularity of M are given in [17]. The existence of an asymptotic covariance matrix of $\hat{\theta}_N$ is established under a variety of conditions in the literature, see [16]. To ease reference we state two yet unpublished results that can be obtained by straightforward extensions of [13, Th. 2.1]. For the concept of L-mixing, an extremely useful extension of what was defined as exponentially stable processes in [18], see [19].

Theorem 1: Assume Conditions 1, 2 and 3, and let $M(\theta^*)$ be non-singular. In addition, let (u, e) be L-mixing w.r.t. a family of pairs of σ-algebras $(\mathcal{F}_n, \mathcal{F}_n^+)$. Then

$$\hat{\theta}_n - \theta^* = -M\left(\theta^*\right)^{-1}\sum_{n=1}^{N}\varepsilon_{\theta n}\left(\theta^*\right)e_n + r_N \tag{8}$$

where $r_N = O_M(N^{-1})$, indicating that the L_p-norms of r_N decay with rate $O(N^{-1})$ for all $p \ge 1$. It follows that

$$\Sigma_{\theta\theta} := \lim_{n\to\infty} N^{1/2}\left(\hat{\theta}_n - \theta^*\right)\left(\hat{\theta}_n - \theta^*\right)^T = \sigma^2 M\left(\theta^*\right)^{-1}.$$

An extension of this theorem for multi-sine inputs, which are far from being mixing in any sense, can be easily obtained, noting that if u is a multi-sine, independent of e, then the products of processes of the form $D(q^{-1})u$ and $D(q^{-1})e$ satisfy a law of large numbers with controlled rate.

The Hessian M, modulo a constant multiplier, is the information matrix. To capture the effect of u onto $M(\theta^*)$ consider the gradient process $(\varepsilon_{\theta n}(\theta^*))$. Equation (4) gives

$$\varepsilon_\theta\left(\theta^*\right) = -H^e\left(\theta^*\right)^{-1}\left(H_\theta^u\left(\theta^*\right)u + H_\theta^e\left(\theta^*\right)e\right), \quad \text{or} \tag{9}$$

$$\varepsilon_\theta\left(\theta^*\right) = D^u\left(\theta^*\right)u + D^e\left(\theta^*\right)e, \tag{10}$$

with $D^u(\theta^*)$ and $D^e(\theta^*)$ given by

$$D^u\left(\theta^*, e^{-i\omega}\right) = -H^e\left(\theta^*, e^{-i\omega}\right)^{-1}H_\theta^u\left(\theta^*, e^{-i\omega}\right), \tag{11}$$

$$D^e(\theta^*, e^{-i\omega}) = -H^e(\theta^*, e^{-i\omega})^{-1} H^e_\theta(\theta^*, e^{-i\omega}). \quad (12)$$

Taking into account that $u \perp e$, we get $M = M^u + M^e$ where M^u and M^e are given via the expressions:

$$M^u = 2\int_0^\pi \Re\Big(D^u(e^{-i\omega})D^{u\top}(e^{i\omega})\Big)\,\mathrm{d}\Phi^u(\omega), \quad (13)$$

$$M^e = \int_0^\pi D^e(e^{-i\omega})D^{e\top}(e^{i\omega})\,\mathrm{d}\omega \cdot \sigma^2, \quad (14)$$

where the r.h.s. of (13) is a Riemann-Stieltjes integral. Here the transfer functions $D^u(e^{-i\omega}) := D^u(\theta^*, e^{-i\omega})$ and $D^e(e^{-i\omega}) := D^e(\theta^*, e^{-i\omega})$ are explicitly known.

The set of feasible matrices M^u is defined via (13) with $\mathrm{d}\Phi^u(\omega)$ being arbitrary subject to (s.t.) constraint as follows. Let $w(\omega), \omega \in [0,\pi]$ be a bounded, continuous function, for which $w(\omega) \geq w_0 > 0$ for all $\omega \in [0,\pi]$ holds, and impose

$$\int_0^\pi w(\omega)\,\mathrm{d}\Phi^u(\omega) \leq K. \quad (15)$$

The set of spectral distribution measures $\mathrm{d}\Phi^u(\cdot)$, satisfying the above the weighted energy constraint, is a compact convex set in the weak topology. Since the matrices M^u are obtained from $\mathrm{d}\Phi^u(\cdot)$ via a continuous linear operator (13), they also constitute a compact, convex set of symmetric, non-negative definite matrices $\mathcal{M}^u \subset \mathbb{R}^{p\times p}$.

Assume that the performance index $J(\theta)$ is sufficiently smooth in θ. Let its Hessian at $\theta = \theta^*$ be denoted by P. Obviously $P^\top = P \geq 0$. Assume that P is positive definite. The objective of the input design then is to minimize the asymptotic value of the normalized performance degradation $\lim_N N\mathbb{E}(J(\hat\theta_N) - J(\theta^*))$. The *primary input design* problem is then to optimize the information matrix M via

$$\min_{M^u \in \mathcal{M}^u} \mathrm{tr}\Big(\big(M^u + M^e\big)^{-1}P\Big), \quad (16)$$

with $\mathrm{tr}(M^{-1}P)$ defined as $+\infty$, if $M \geq 0$ is singular.

It is easily seen that $M \to M^{-1}$ is strictly convex on $\mathbb{R}^{p\times p}_+$ w.r.t. the usual ordering of symmetric matrices. It follows that $\mathrm{tr}(M^{-1}P)$ is a strictly convex function of $M \in \mathbb{R}^{p\times p}_+$. It is readily seen that the optimization problem (16) has a unique solution in $\mathcal{M}^u$, to be denoted by M^{u*}. The question remains how to construct a spectral distribution measure $\mathrm{d}\Phi^{u*}(\cdot)$ or input u^* that would generate M^{u*}.

III. Multi-Sine Input Design

The matrices $\Re(D^u(e^{-i\omega})D^{u\top}(e^{i\omega}))$, defining M^u via (13), are elements of the vector-space of real, symmetric matrices of dimension $s := p(p+1)/2$. The following result was essentially stated for the case of $w(\cdot) \equiv 1$, in [20, Ch. 3.2, Th. 1]:

Proposition 1: Let $M^u \in \mathcal{M}^u$ be defined in terms of $\mathrm{d}\Phi^u(\cdot)$, satisfying (15) with equality, via (13). Then there exist at most $s+1$ frequencies $0 \leq \omega_k \leq \pi$ and energy levels $\alpha_k \geq 0$, $k = 1,\ldots,s+1$, such that

$$M^u = 2\sum_{k=1}^{s+1} \alpha_k \Re\Big(D^u(e^{-i\omega_k})D^{u\top}(e^{i\omega_k})\Big), \quad (17)$$

$$\sum_{k=1}^{s+1} \alpha_k w(\omega_k) = \int_0^\pi w(\omega)\,\mathrm{d}\Phi^u(\omega) = K. \quad (18)$$

For extremal points of $\mathcal{M}^u$ just s frequencies suffice.

The above fact is a folklore in the literature on input design. The proof is based on Carathéodory's theorem, and is given in [20] for the case $w(\cdot) \equiv 1$,. The general case is readily obtained by introducing the measure $\mathrm{d}\overline{\Phi}^u(\omega) = w(\omega)\,\mathrm{d}\Phi^u(\omega)/K$, and rewriting (13) as

$$M^u = 2\int_0^\pi \frac{K}{w(\omega)} \Re\Big(D^u(e^{-i\omega})D^{u\top}(e^{i\omega})\Big)\,\mathrm{d}\overline{\Phi}^u(\omega).$$

For extremal points of $\mathcal{M}^u$ the discrete representation (17) is not only a possibility but in some cases a must. Adapting the arguments of [20, Ch. MA3, Th. 2]. we get:

Theorem 2: Let $\mathrm{d}\Phi^{u*}(\cdot)$ be a spectral measure defining an extremal point of $\mathcal{M}^u$, denoted by M^{u*}. Then there exists a $p \times p$ matrix Λ^* such that

$$L(\omega) := \mathrm{tr}\Lambda^*\Re\Big(D^u(e^{-i\omega})D^{u\top}(e^{i\omega})\Big) + w(\omega) \geq 0 \quad (19)$$

for all $0 \leq \omega \leq \pi$, and the set of the points of increase of $\mathrm{d}\Phi^{u*}(\cdot)$ is a subset of the solutions of $L(\omega) = 0$.

The theorem is a kind of infinite-dimensional Karush-Kuhn-Tucker condition, see Lemma 1.

Corollary 1: Let $w(\cdot)$ be a piece-wise rational function of $e^{i\omega}$. Then any optimal spectral measure $\mathrm{d}\Phi^{u*}(\cdot)$ is discrete, supported by a finite number of frequencies.

Proof of Theorem 2: Adapting the argument in [20, p. 45] we obtain that M^{u*} is in the boundary of the convex, closed set $\mathcal{M}^u$. Thus, there exists a matrix $\tilde\Lambda$ and a number $c \neq 0$ such that $\mathrm{tr}\tilde\Lambda M^u \leq c$ for any matrix $M^u \in \mathcal{M}^u$ and $\mathrm{tr}\tilde\Lambda M^{u*} = c$. Using the definition (13) of M^u we can write the above condition as

$$\int_0^\pi \Big(\mathrm{tr}\tilde\Lambda\Re\Big(D^u(e^{-i\omega})D^{u\top}(e^{i\omega})\Big) - \frac{c}{K}w(\omega)\Big)\,\mathrm{d}\Phi^u(\omega) \leq 0$$

for any $\mathrm{d}\Phi^u(\cdot)$, for which $\int_0^\pi w(\omega)\,\mathrm{d}\Phi^u(\omega) = K$ holds. Since the function $w(\cdot)$ is strictly positive, it follows that $\mathrm{tr}\tilde\Lambda\Re(D^u(e^{-i\omega})D^{u\top}(e^{i\omega}) - \frac{c}{K}w(\omega) \leq 0$ for $\omega \in [0,\pi]$. Setting $\Lambda = -\frac{K}{c}\tilde\Lambda$ we obtain $\int_0^\pi L(\omega)\,\mathrm{d}\Phi^u(\omega) \geq 0$ for any $\mathrm{d}\Phi^u(\cdot)$, with equality for $\mathrm{d}\Phi^{u*}(\omega)$, implying the claim.

Since $M^{u*} \in \mathcal{M}^u$ is in the boundary of $\mathcal{M}^u$ Proposition 1 implies that it can be generated via a *multi-sine* of at most s terms, see (3). The matrix

$$M^u = 2\sum_{k=1}^{s} \alpha_k \Re\Big(D^u(e^{-i\omega_k})D^{u\top}(e^{i\omega_k})\Big), \quad (20)$$

is linear in the α_k-s, and hence, for a fixed set of ω_k frequencies, the cost function $\mathrm{tr}(M^{-1}P)$ is convex in the α_k-s. Unfortunately, it is a *non-convex* function in the frequencies $\omega_k, k = 1,\ldots,s$.

A sub-optimal solution to this can be obtained by taking a large t, and a set of equidistant frequencies denoted by $\Omega = \{\omega_k : 0 \leq \omega_k \leq \pi, k = 1,\ldots,t\}$, and considering

$$M^u = 2\sum_{k=1}^{t} \alpha_k \Re\Big(D^u(e^{-i\omega_k})D^{u\top}(e^{i\omega_k})\Big). \quad (21)$$

The convex set of matrices M^u generated by (21), s.t. the energy constraint will be denoted by $\mathcal{M}^u(\Omega)$. With this notation our input design problem (16) reduces to the problem with $M^u \in \mathcal{M}^u$ being replaced by $M^u \in \mathcal{M}^u(\Omega)$.

To capture the effect of the approximation replacing $\mathcal{M}^u$ by $\mathcal{M}^u(\Omega)$ in the primary input design problem (16) note that any $M^u \in \mathcal{M}^u$ is defined by a Riemann-Stieltjes integral, given by (13), in which the integrand is continuously differentiable and the total mass of the measure $\mathrm{d}\Phi^u(\cdot)$ on $[0,\pi]$ is bounded, due to the energy constraint (15) and the condition $w(\omega) \geq w_0 > 0$ for all ω. It readily follows that any $M^u \in \mathcal{M}^u$ can be approximated by an $M^{ud} \in \mathcal{M}^u(\Omega)$ with an error of the order $1/t$, inducing an error of the same order of magnitude in approximating the optimal value.

Letting $d_k := D^u(e^{-i\omega_k})$ and $w_k = w(\omega_k)$ we thus get the following convex optimization problem:

$$\min_{\alpha \geq 0} \ \mathrm{tr}\left[\left(2\Re\sum_{k=1}^{t}\alpha_k d_k \overline{d}_k^\top + M_e\right)^{-1} P\right] \tag{22}$$

$$\text{s.t.} \quad \sum_{k=1}^{t}\alpha_k w_k \leq K. \tag{23}$$

The formulation of the the Karush-Kuhn-Tucker condition for (22) – (23) is of didactic interest in light of Theorem 2:

Lemma 1: Let α^* be an optimal solution of (22) – (23). Then there exists a $p \times p$, symmetric negative definite matrix Λ^*, and Lagrange multipliers $\lambda^* \geq 0$ and $\mu_k^* \leq 0$ such that

$$\mathrm{tr}\left[\Lambda^* \Re\left(d_k \overline{d}_k^\top\right)\right] + \lambda^* w_k + \mu_k^* = 0, \tag{24}$$

for all $k = 1, \ldots, t$ and $\alpha_k^* > 0$ implies $\mu_k^* = 0$.

Proof: Let the cost function in (22) be denoted by $F(\alpha)$. Let $\lambda^* \geq 0$ be the Lagrange multiplier corresponding (23) and let $\mu_k^* \leq 0$ be the Lagrange multipliers corresponding to the constraints $\alpha_k \geq 0$. Let

$$M = M(\alpha) = 2\Re\sum_{k=1}^{t}\alpha_k d_k \overline{d}_k^\top + M_e. \tag{25}$$

The gradient of the cost function $F(\alpha)$ is then as follows:

$$\frac{\partial}{\partial \alpha_k}F(\alpha) = 2\mathrm{tr}\left[-M^{-1}\Re\left(d_k \overline{d}_k^\top\right)M^{-1}P\right]. \tag{26}$$

Letting $\Lambda = \Lambda(\alpha) := -2M^{-1}PM^{-1}$ we can write:

$$\frac{\partial}{\partial \alpha_k}F(\alpha) = \mathrm{tr}\left[\Lambda \Re\left(d_k \overline{d}_k^\top\right)\right]. \tag{27}$$

Setting $\Lambda^* = \Lambda(\alpha^*)$, the Karush-Kuhn-Tucker condition, implying also $\mu_k^* \alpha_k^* = 0$, gives the claim. ■

For a fixed a set of equidistant ω_k-s let us consider a *relaxation* of the problem defined in (22)-(23) with $\gamma > 0$,

$$\min_{\alpha \geq 0} \mathrm{tr}\left[\left(2\Re\sum_{k=1}^{t}\alpha_k d_k \overline{d}_k^\top + M_e\right)^{-1} P\right] + \gamma \sum_{k=1}^{t}\alpha_k w(\omega_k).$$

Let α^* be an optimal solution of this *relaxed problem*, and let $\Omega^+ = \{\omega_k : \alpha_k^* > 0\}$. Then the Karush-Kuhn-Tucker conditions, with minor modifications of Lemma 1, imply for $\omega \in \Omega^+$

$$\mathrm{tr}\left[\Re\left(D^u\left(e^{-i\omega}\right)\overline{D}^u\left(e^{i\omega}\right)^\top\right)\Lambda^*\right] + \gamma w(\omega) = 0. \tag{28}$$

Let us introduce the notation for the vectorized matrices

$$\mathrm{vec}\Re\left(D^u\left(e^{-i\omega}\right)\overline{D}^u\left(e^{i\omega}\right)^\top\right) =: \Delta(\omega). \tag{29}$$

Lemma 2: The relaxed optimization problem has a solution such that the vectors $\{\Delta(\omega_k), \omega_k \in \Omega^+\}$ are linearly independent. In particular, $|\Omega^+| \leq s$.

Proof: Let $I^+ = \{k : \alpha_k^* > 0\}$. If the vectors $\{\Delta(\omega_k), \omega_k \in \Omega^+\}$ are linearly dependent, then there exists a nontrivial linear combination $\sum_{k \in I^+} \beta_k \Re(d_k \overline{d}_k^\top) = 0$, where we can assume $|\beta_k| < \alpha_k^*$ for all $k \in I^+$. Adding and subtracting this linear combination from the optimal one that cost function cannot decrease. Thus $\sum_{k \in I^+} \beta_k w(\omega_k) = 0$. Taking $\alpha_k^* + \lambda\beta_k$ for some appropriate λ we can achieve that $\alpha_k^* + \lambda\beta_k = 0$ for some $k = l$ while ensuring $\alpha_k^* + \lambda\beta_k \geq 0$ for all other $k \in I^+$, thus reducing the size of Ω^+. Repeating this procedure will yield the desired optimal solution. ■

Let $I \subset \{1, \ldots, t\}$ and let $\alpha_I^\top := (\alpha_k, k \in I)$ denote the reduced parameter vector. Enforcing $\alpha_k = 0$ for $k \notin I$, let $L_I(\alpha_I)$ be the restricted cost function of the relaxed problem.

Lemma 3: Assume that $\{\Delta(\omega_k), k \in I\}$ are linearly independent. Then the Hessian of $L_I(\alpha_I)$ is positive definite.

The proof is obtained considering the quadratic form induced by the Hessian for $v \in \mathbb{R}^t$, with $G_k = 2\Re d_k \overline{d}_k^\top$

$$2\mathrm{tr}\left(P^{1/2}M^{-1}\left(\sum_{k=1}^{t} v_k G_k\right)M^{-1}\left(\sum_{l=1}^{t} v_l G_l\right)M^{-1}P^{1/2}\right).$$

Restricting summation to $k, l \in I^+$ gives the claim.

Theorem 3: Let the t-dimensional vector with components $w_k := w(\omega_k)$ be chosen randomly according to a distribution having a density in $\mathbb{R}^t$. Then the relaxed problem has a unique solution w.p.1, and the vectorized matrices $\mathrm{vec}\Re(D^u(e^{-i\omega})\overline{D}^u(e^{i\omega})^\top), \omega \in \Omega^+$ are linearly independent w.p.1. In particular, we have $|\Omega^+| \leq s$.

Proof: Let us take an optimal solution α^* with Ω^+ as defined above. Let $I \subset \{1, \ldots, t\}$ be arbitrary and let $\Omega_I := \{\omega_k, k \in I\}$ be the corresponding subset of frequencies. Note that $P(\Omega^+ = \Omega_I) \leq P(\Omega_I \subseteq \Omega^+)$. Express the latter event $\{\Omega_I \subseteq \Omega^+\}$ via the Karush-Kuhn-Tucker condition as

$$\left(\mathrm{vec}\Lambda^*\right)^\top \Delta(\omega) + \gamma w(\omega) = 0 \quad \text{for} \quad \omega \in \Omega_I. \tag{30}$$

Arrange the column-vectors $\{\Delta(\omega_k), k \in I\}$ into a matrix S_I, and define $w_I^\top := (w(\omega_k), k \in I)$. Write (30) as

$$\left(\mathrm{vec}\Lambda^*\right)^\top S_I + \gamma w_I^\top = 0. \tag{31}$$

If $\mathrm{rank} S_I < |I|$, i.e., the vectors $\{\Delta(\omega_k), k \in I\}$ are linearly dependent, then its rows span a proper subspace $L(S_I) \subset \mathbb{R}^{|I|}$. But the marginal distribution of the random vector w_I has a density in $\mathbb{R}^{|I|}$, hence the event $\{w_I \in L(S_I)\}$ has probability 0. Since the number of subsets I is finite, the second claim follows.

To prove unicity, assume the contrary. Then, by convexity, there is an interval of α-s such that the cost function is constant, and optimal along this interval. Consider its midpoint, say $\overline{\alpha}^*$ and let $I := \overline{I}^+ = \{k : \overline{\alpha}_k^* > 0\}$, and $\overline{\Omega}^+ = \{\omega_k : k \in \overline{I}^+\}$. Then by the proven second claim of the theorem the vectors $\{\Delta(\omega_k), k \in I\}$ are linearly independent w.p.1. Hence, by Lemma 3 the Hessian of $L_I(\alpha_I)$ is positive definite. But this is a contradiction, since $L_I(\alpha_I)$ being constant along an interval, its Hessian has a zero eigenvalue. ■

IV. A Data-Driven Approach

A shortcoming of the cited literature on input design is that the optimal spectral measure of the input is determined under the hypothesis that the true system parameter θ^*, is actually known. To bypass this paradox we present the basics of a data-driven method within a fairly general context, recapitulating and extending the basic idea of [9].

The key idea is the construction of a data-driven virtual off-line estimator, approximating the off-line PE estimator of Section II, obtained with optimal input, with accuracy $O_M(N^{-1})$. To be specific, consider a parametric family of inputs

$$u(\eta) = F\Big(\eta, q^{-1}\Big)f, \tag{32}$$

where F is a rational, stable filter, such that $|F|^2$ is linearly parameterized by $\eta \in \mathcal{C} \subset \mathbb{R}^r$, where $\mathcal{C}$ is a closed, convex set. E.g., F may be a FIR filter with η denoting the coefficients of its half-spectra. f is an i.i.d. sequence of random variables, independent of e, with finite moments of all order.

Consider the system dynamics (1) with $u = u(\eta)$, $\eta \in \mathcal{C}$:

$$y(\eta) = H^u\Big(\theta^*, q^{-1}\Big)u(\eta) + H^e\Big(\theta^*, q^{-1}\Big)e. \tag{33}$$

Define, for $\theta \in D$, the assumed innovation process $\varepsilon(\theta, \eta)$

$$\varepsilon(\theta, \eta) = H^e(\theta)^{-1}\big(y(\eta) - H^u(\theta)u(\eta)\big). \tag{34}$$

Renaming the cost function of the PE estimator defined in (5) as $V_N(\theta, \eta)$ let $\hat{\theta}_N(\eta)$ and $M(\theta^*, \eta)$ denote the corresponding off-line PE estimator and information matrix, respectively. Assume that for $\theta \in D$, as true systems parameter, the optimal input design problem has a unique solution $\eta^*(\theta)$, such that $\eta^*(\cdot)$ is three-times continuously differentiable. Then the *virtual* off-line PE estimator of θ^* is obtained by minimizing the cost function

$$V_N(\theta, *) := V_N\big(\theta, \eta^*(\theta)\big) = \frac{1}{2}\sum_{n=1}^{N} \varepsilon_n^2\big(\theta, \eta^*(\theta)\big). \tag{35}$$

The solution is denoted by $\hat{\theta}_N(*)$. This estimator is virtual in the sense that it is not practical, since $V_N(\theta, *)$ can not be evaluated for two different values of θ. Nevertheless, we proceed with its analysis along the lines of Section II.

Noting that $\varepsilon(\theta^*, \eta) = e_n$ for all η, we get $\frac{\partial}{\partial\eta}\varepsilon(\theta^*, \eta) = 0$ for all η. Setting $\varepsilon_{\theta 0,n}(\theta, \eta) := \frac{\partial}{\partial\theta}\varepsilon_n(\theta, \eta)$, we conclude:

$$\frac{\partial}{\partial\theta}\varepsilon_n\big(\theta, \eta^*(\theta)\big) = \varepsilon_{\theta 0,n}\big(\theta, \eta^*(\theta)\big). \tag{36}$$

It follows that for the Hessian of the asymptotic cost function associated with the above virtual PE method, given as,

$$W(\theta, *) := W\big(\theta, \eta^*(\theta)\big) := \frac{1}{2}\mathbb{E}\varepsilon_n^2\big(\theta, \eta^*(\theta)\big), \tag{37}$$

evaluated at θ^*, we have, with self-explanatory notation,

$$M\big(\theta^*, *\big) = M\big(\theta^*, \eta^*(\theta^*)\big). \tag{38}$$

Once again referring to [13, Th. 2.1] we get by its straightforward extension, in analogy with Theorem 1.

Theorem 4: Let $u(\eta)$ be given by (32). Assume Conditions 1, 3, and let $M(\theta^*, *)$ be non-singular. Then

$$\hat{\theta}_n(*) - \theta^* = -M\big(\theta^*, *\big)^{-1}\sum_{n=1}^{N} \varepsilon_{\theta 0,n}\big(\theta^*, \eta^*(\theta^*)\big)e_n + r_N,$$

where $r_N = O_M(N^{-1})$, implying the strong approximation:

$$\hat{\theta}_n(*) = \hat{\theta}_n\big(\eta^*(\theta^*)\big) + O_M\Big(N^{-1}\Big). \tag{39}$$

In particular, the asymptotic covariance matrix of $\hat{\theta}_n(*)$ is

$$\Sigma_{\theta\theta}(*) = \sigma^2 M^*\big(\theta^*, *\big)^{-1} = \sigma^2 M^*\big(\theta^*, \eta^*(\theta^*)\big)^{-1}.$$

Thus the virtual estimator $\hat{\theta}_n(*)$ is optimal from the perspective of input design. The asymptotic estimation problem in the spirit of [12] is defined by the algebraic equation

$$\frac{\partial}{\partial\theta}W(\theta, *) = \mathbb{E}\varepsilon_{\theta 0,n}\big(\theta, \eta^*(\theta)\big)\varepsilon_n\big(\theta, \eta^*(\theta)\big) = 0. \tag{40}$$

Following the ideas of [11], extended in [12], a computable recursive PE estimator $\hat{\hat{\theta}}_N(*)$ can be constructed.

The viability of the proposed approach for data-driven input design has been demonstrated, with all technical details included, in [9] for the case of ARMAX systems exited with inputs u generated by a FIR filter.

To conclude this section we briefly describe the extension of the above approach to *closed loop* systems. Consider a class of linear stochastic control systems with excitation v:

$$y^c\big(\theta'\big) = H^u\Big(\theta^*, q^{-1}\Big)u^c\big(\theta'\big) + H^e\Big(\theta^*, q^{-1}\Big)e \tag{41}$$

$$u^c\big(\theta'\big) = -K\Big(\theta', q^{-1}\Big)y^c\big(\theta'\big) + v. \tag{42}$$

Here v is an external excitation independent of e. We consider the practical scenario when the true parameter θ^* is unknown and we use its tentative value θ' in the feedback loop. The feedback loop or $K(\theta, q^{-1})$ is designed by optimizing a performance criterion for any assumed θ showing up in H^u and H^e. Thus if we had a prior estimate $\hat{\theta}$ of θ^* we would set $\theta' = \hat{\theta}$. Write (41) - (42) as

$$y^c\big(\theta'\big) = H^{cu}\big(\theta^*, \theta'\big)\ v + H^{ce}\big(\theta^*, \theta'\big)e, \tag{43}$$

with $H^{cu}(\theta^*, \theta')$ and $H^{ce}(\theta^*, \theta')$ denoting the closed loop filters. The role of the dither v, being independent of e, is thus identified with that of the input u in open loop identification. Thus, pretending the knowledge of θ^* (and knowing θ') we can proceed with any of the open loop input design methods.

Following (32) assume that v is generated by

$$v(\eta) = F\Big(\eta, q^{-1}\Big)f, \tag{44}$$

yielding the input and output processes $u^c(\theta', \eta)$ and $y^c(\theta', \eta)$. For fixed θ', η the off-line PE estimator of θ^* is obtained via the assumed innovation process defined by

$$\varepsilon^c\big(\theta, \theta', \eta\big) = H^e(\theta)^{-1}\big(y^c\big(\theta', \eta\big) - H^u(\theta)u^c\big(\theta', \eta\big)\big).$$

The off-line PE estimator is then defined as the solution of

$$\min_{\theta \in D}\sum_{n=1}^{N} \varepsilon_n^{c2}\big(\theta, \theta', \eta\big), \tag{45}$$

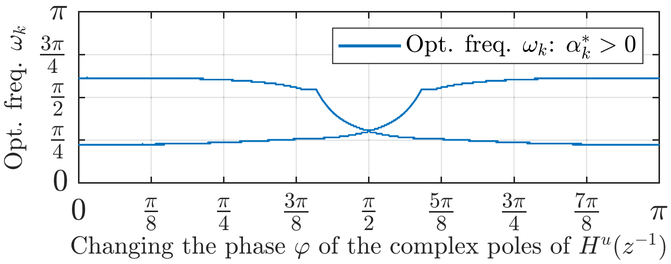

Fig. 1. The two optimal frequencies for $b = 3.0$ and $\omega_c = \frac{3\pi}{4}$.

to be denoted by $\hat{\theta}_N(\theta', \eta)$, which can be analyzed along the lines of Section II. For any θ, θ' let the solution of the input design problem be denoted by $\eta^*(\theta, \theta')$. We redefine the assumed innovation process by *enforcing* $\theta' = \theta$ and $\eta = \eta^*(\theta, \theta')$. Thus we define a virtual closed loop off-line PE estimator, as in [21], which can be considered as the off-line mirror-image of an adaptive control algorithm optimized for the asymptotic covariance matrix of $\hat{\theta}_N(\theta^*, \eta)$, by

$$\min_{\theta \in D} \sum_{n=1}^{N} \varepsilon_n^{c2}\big(\theta, \theta, \eta^*(\theta, \theta)\big). \tag{46}$$

Let the solution be denoted by $\hat{\theta}_N(*, *)$. Noting that $\varepsilon^c(\theta^*, \theta', \eta) = e_n$ for all θ', η, the partial derivatives of $\varepsilon^c(\theta^*, \theta', \eta)$ w.r.t. θ', η are 0, and hence the gradient of the cost function or rather $(\partial/\partial\theta)\varepsilon_n^c(\theta, \theta, \eta^*(\theta, \theta))_{\theta=\theta^*}$ is easily computed. Thus, in analogy with Theorem 4 we get the strong approximation result

$$\hat{\theta}_n(*, *) = \hat{\theta}_n\big(\theta^*, \eta^*(\theta^*)\big) + O_M\Big(N^{-1}\Big), \tag{47}$$

amounting to the fact that $\hat{\theta}_n(*, *)$ is optimal both from control and input design perspective.

V. Experimental Results

We have tested our algorithm for finding the optimal multi-sine on a system modeling a lightly damped oscillator with complex poles $re^{\pm i\varphi}$ and amplification b. Thus we have

$$H^u\Big(z^{-1}\Big) = \frac{b}{1 - 2r\cos(\varphi)z^{-1} + r^2 z^{-2}}. \tag{48}$$

Fixing $r = 0.95$, we let the phase vary uniformly in $[0, \pi]$, while b varied in the interval $[3, 10]$. The transfer function H^e is defined by its stable zeros and poles yielding

$$H^e\Big(z^{-1}\Big) = \frac{1 + c_1 z^{-1} + c_2 z^{-2}}{1 + d_1 z^{-1} + d_2 z^{-2}} = \frac{1 + 0.6z^{-1} - 0.07z^{-2}}{1 - 0.866z^{-1} + 0.25z^{-2}}.$$

Thus we have $p = 7$ parameters: r, φ, b and c_1, c_2, d_1, d_2, implying $s = p(p+1)/2 = 28$. For the weight function $w(\cdot)$ we use a sigmoid-type functions taking their values between 0.1 and 1.0, setting their medians equal to three possible cut-off frequencies ω_c equal to $\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$. We let $t = 5s = 140$. We solved the relaxed problem with $P = I$, and $\gamma = 0.1$ In all experimental scenarios a sparse solution was obtained with a maximum of 4 optimal frequencies. In Fig. 1, we present a typical result with two optimal frequencies assuming a moderate SNR (signal-to-noise ratio) $b = 3.0$, and using a weight function with broad band-pass width: $\omega_c = \frac{3\pi}{4}$.

VI. Discussion

A nice project for future research may be the extension of Theorem 2 to input design problems admitting frequency-wise specifications, introduced in [22]. A second problem of interest may be the clarification if the proposed data-driven approach of Section IV is applicable for multi-sine design. Extension of our results to vector-valued multi-sine design along the lines of [23] may be also of interest. Finally, the authors thank to Hakan Hjalmarsson for inspiring this letter, while visiting HUN-REN SZTAKI in 2021.

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