Stabilization of a Limit Cycle for Discrete-Time Switched Nonlinear Systems

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***Abstract*—This letter studies global exponential stabilisation of a limit cycle of interest for discrete-time switched nonlinear systems, in which the subsystems may have different equilibria. As a first step, a set of candidate limit cycles is determined according to a criterion related to the steady-state behavior of the system trajectories. Afterwards, a state-dependent switching function, based on sufficient conditions derived from a time-periodic Lyapunov function, is proposed to ensure global exponential stability of the limit cycle and a guaranteed performance level for the overall system. A class of polynomial switched systems is used to illustrate the main results. For this class, new LMI conditions are obtained that ensure local exponential stability of the limit cycle, inside a polyhedral set given by the designer. An ellipsoidal set of maximum volume is determined such that any trajectory starting inside it does not leave the polyhedron. The main features of this methodology are illustrated by academic examples.**

***Index Terms*—Switched nonlinear systems, discrete-time domain, limit cycles, global exponential stability.**

I. Introduction

SWITCHED systems are formed by a finite set of subsystems and a switching function (or rule) responsible for activating one of them at each instant of time. It characterizes a subclass of hybrid systems, where each subsystem represents a continuous dynamic that interacts with a discrete dynamic represented by the switching rule. This rule may lead the system to behave differently from the isolated subsystems. The intrinsic characteristics of this subclass of hybrid systems have been explored in several areas of science and are responsible for phenomena not encountered in non-switched systems, [14]. For instance, the switching function can be designed to ensure stability and suitable performance for the overall system, even when all subsystems are unstable, or to govern the state trajectories to an equilibrium point that is not common to any of the subsystems. See [1], [4], [12] for some practical application examples in the area of power electronics and [14], [16], [21] for some basic references on switched and hybrid systems.

Naturally, switched linear systems are simpler and have attracted the attention of the scientific community. For this class of systems, there exist several results dealing with asymptotic stability of the origin, see, e.g., [3], [10], [13]. However, the generalization of these results to cope with switched affine systems is not immediate. These differ from linear systems by the presence of affine terms in their dynamic equation allowing the subsystems to have distinct equilibria. Generally, the equilibrium point of interest is not common to any of the subsystems, which requires the implementation of an arbitrarily fast switching frequency to ensure asymptotic stability. References [18], [19] deal with global asymptotic stability of an equilibrium for switched affine systems in the continuous-time domain.

However, when the switching frequency is constrained to respect some upper bound, asymptotic stability is not always possible to attain. This is the case for discrete-time systems in which the switching frequency is upper bounded by the sampling rate. In this case, the literature provides sufficient conditions to ensure practical stability, whereby the state trajectories are driven to an attractive set containing the equilibrium point of interest (see [8], [17]) or to a limit cycle, see [2], [9], [20]. The advantage of the limit cycle approach is that the steady-state is well-defined by the asymptotic stability of the chosen limit cycle, whilst in the case of practical stability, there is no information on the state trajectories when they are inside the residual set, making the steady-state bounded but completely unknown.

For the general class of switched nonlinear systems there are only a few results dealing with the control design of a stabilizing switching rule. Making an analogy with the two previously mentioned classes, we can split switched nonlinear systems into two important subclasses. In the first, the origin is the equilibrium point of interest, common to all the subsystems. In

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the second, the subsystems may have distinct equilibria, which generally do not coincide with the desired equilibrium. For the first subclass, [6] and, more recently, [7] have proposed sufficient conditions for global asymptotic stability of the origin and a guaranteed cost of performance for continuous and discrete-time switched nonlinear systems, respectively. To the best of the authors' knowledge, the study of the second subclass is still open.

In this letter our goal is to study this second subclass by dealing with the asymptotic stabilisation of a desired limit cycle for discrete-time switched nonlinear systems, thus providing a generalization of [9] that treats switched affine systems, exclusively. As a first step, a set of candidate limit cycles, that satisfies the criterion of interest related to the steady-state response, is determined. Afterwards, a state-dependent switching function is proposed to guarantee global exponential stability of a limit cycle, within the family of candidates, and a desired level of performance. These results are then particularized to cope with a class of polynomial switched systems and sufficient conditions expressed in terms of linear matrix inequalities (LMIs) are obtained that ensure local exponential stability of the desired limit cycle, inside a polyhedral set defined by the designer. This LMI-based methodology has been inspired by the recent results proposed in [7]. An ellipsoidal set of maximum volume is also determined, such that any state trajectory starting inside it does not leave the polyhedron of interest. The main features of the proposed theory are illustrated through two academic examples.

Notation: The notation used throughout is standard. For real vectors or matrices $'$ denotes their transpose. For symmetric matrices $\bullet$ denotes each of their symmetric blocks. The sets of real and natural numbers, including zero, are denoted by $\mathbb{R}$ and $\mathbb{N}$, respectively. The set $\mathbb{K} = \{1, \dots, N\}$ is composed of the N first positive natural numbers. For any symmetric matrix, $X >$ (resp., $<$) 0 denotes a positive (resp., negative) definite matrix. The square L_2 norm of $z[n],\ n \in \mathbb{N}$, is $\|z\|_2^2 = \sum_{n\in\mathbb{N}} \|z[n]\|^2$, where $\|z\|^2 = z'z$ is the square of the Euclidean norm. The modulo operator is defined by $c = a \mod b$, where c is the remainder of the Euclidean division between the integers a and b. For a positive $\kappa \in \mathbb{N}$, the function $k(n) = n \mod \kappa$. For a sequence $x[n]$, the one step ahead operator is $x^+ = x[n+1],\ \forall n \in \mathbb{N}$.

II. Problem Statement

Consider the discrete-time switched nonlinear system described by the equation

$$x^+ = f_\sigma(x),\ \ x[0] = x_0 \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the state vector, and $\sigma : \mathbb{N} \to \mathbb{K}$ is the switching function that selects at each instant of time one of the N available subsystems $f_\sigma \in \{f_1, \dots, f_N\}$. In our context, $f_i,\ \forall i \in \mathbb{K}$, are continuous functions and the associated subsystems may have distinct equilibrium points $x_{ei},\ \forall i \in \mathbb{K}$, such that

$$x_{ei} = f_i(x_{ei}), \tag{2}$$

and $x_{ei} \neq x_{ej}$ for some $j \neq i \in \mathbb{K}$. Moreover, each subsystem may have multiple equilibria. In general, they do not coincide with the point of interest chosen by the designer. This problem is clearly more intricate than the stabilisation problem for the case in which all subsystems share a common equilibrium as in [7] and, in this sense, this letter is a generalization of [7].

Making a parallel with the switched affine system $x^+ = A_\sigma x + b_\sigma$, which is a particular case of (1), observe that exponential stability of a desired equilibrium point that does not coincide with those equilibria of the subsystems is impossible to ensure, since it would require an arbitrarily high switching frequency, which does not occur in the discrete-time domain, because it is always upper bounded by the sampling rate, see [8] and [17] for detail. This is not the case for switched linear systems, where $b_i = 0,\ \forall i \in \mathbb{K}$, since the origin is a common equilibrium point for all subsystems.

For switched affine systems the literature studies this problem by considering different attractors, instead of a single point, for instance, invariant sets (see [8], [15]), or limit cycles (see [9], [20]). The advantage of dealing with limit cycles in comparison with invariant sets is the possibility of ensuring exponential stability of the cycle, and guaranteed performance indexes. If an invariant set is considered, only practical stability is taken into account, whereby the state trajectories are driven to an attractive set, as small as possible, but nothing can be concluded about these trajectories when they are inside such a set.

Assume that the system (1) admits a periodic solution x_e with period $\kappa > 0$, associated to a periodic switching function $\sigma[n] = c[k(n)] \in \mathbb{K},\ \ n \in \mathbb{N}$, where $\{c[n]\}_{n=0}^{\kappa-1}$ is a sequence of length κ denoted simply by c. The corresponding candidate limit cycle is given as

$$\mathcal{X}_e(c) = \{x_e[k(n)] : n \in \mathbb{N}\}, \tag{3}$$

where the fundamental period $\{x_e[n]\}_{n=0}^{\kappa-1}$ satisfies

$$\begin{aligned} x_e[1] &= f_{c[0]}(x_e[0]), \\ x_e[2] &= f_{c[1]}(x_e[1]), \\ &\ \vdots \\ x_e[0] &= f_{c[\kappa-1]}(x_e[\kappa-1]). \end{aligned} \tag{4}$$

Therefore, we suppose that the system of nonlinear equations (4), to be solved by an appropriate nonlinear equation solver, provides a candidate limit cycle associated to the periodic switching sequence c. See [20] for a discussion on this matter in the particular framework of switched affine systems.

For each $\kappa > 0$, define the set $\mathfrak{C}(\kappa) = \mathbb{K}^\kappa$ obtained from the Cartesian product of $\mathbb{K}$ with itself κ times. This set contains N^κ sequences $c = \{c[n]\}_{n=0}^{\kappa-1}$, that allow obtaining a family of candidate limit cycles:

$$\mathfrak{X} = \{\mathcal{X}_e(c) : c \in \mathfrak{C}(\kappa)\}. \tag{5}$$

Let the subset $\mathfrak{X}_s \subset \mathfrak{X}$ be formed by the candidates (5) that satisfy some criterion for the steady-state trajectories, as for instance, a maximum ripple criterion. In this case, given a reference point x_*, we have

$$\mathfrak{X}_s = \left\{\mathcal{X}_e \in \mathfrak{X} : \max_{n\in[0,\kappa)} \|\Gamma(x_e[n] - x_*)\| \leq 1\right\}, \tag{6}$$

where Γ is selected by the designer. As an example, choosing $\Gamma = (1/\bar{r})I$, with $\bar{r} > 0$, this subset contains all the limit cycles whose ripple values, related to the reference point x_*, do not exceed the magnitude $\bar{r}$, see [9] for detail. For completeness, $\mathfrak{C}_s \subset \mathfrak{C}(\kappa)$ is the corresponding subset of sequences c associated to $\mathfrak{X}_s$.

Adopting the auxiliary state variable $\xi = x - x_e$ with $x_e \in \mathcal{X}_e(c)$ given in (3), we obtain from (1) the equivalent switched nonlinear system given by

$$\xi^+ = g_{c[k(n)]\sigma}(\xi), \ \ \xi[0] = \xi_0, \tag{7}$$

$$z = h_\sigma(\xi), \tag{8}$$

where the function $g_{c[k(n)]\sigma}$ is defined as

$$g_{c[k(n)]\sigma}(\xi) = f_\sigma(\xi + x_e) - f_{c[k(n)]}(x_e), \tag{9}$$

since, from (3)-(4), $x_e^+ = f_{c[k(n)]}(x_e)$. Moreover, $z \in \mathbb{R}^{n_z}$ is the controlled output, such that $h_i(0) = 0, \ \forall i \in \mathbb{K}$, which is used to establish a guaranteed level of performance for the overall system. Note that global exponential stability of the origin $\xi = 0$ of (7) implies the same for the associated limit cycle $\mathcal{X}_e(c)$ of (1), since $\xi \to 0$ whenever $x[n] \to x_e[k(n)]$.

Our main goal is to determine a state-dependent switching function $\sigma[n] = u(\xi, n): \mathbb{R}^{n_x} \times \mathbb{N} \to \mathbb{K}$ to ensure global exponential stability of a limit cycle $\mathcal{X}_e(c)$ of interest and a suitable upper bound for the L_2 norm of the controlled output $\|z\|_2^2$. The next section provides our main results.

III. Main Results

Consider the time-varying Lyapunov function candidate

$$v(\xi, n) = V(\xi, k(n)), \tag{10}$$

for all $n \in \mathbb{N}$, where V is a time-periodic, radially unbounded, and positive definite function, such that $V(0, k(n)) = 0$ and $V(\xi, 0) = V(\xi, \kappa)$. The next theorem provides, for a given sequence $c \in \mathfrak{C}(\kappa)$, conditions that ensure global exponential stability of the limit cycle $\mathcal{X}_e(c) \in \mathfrak{X}_s$ and a guaranteed cost of performance.

Theorem 1: Let a sequence $c \in \mathfrak{C}_s$ associated to the limit cycle $\mathcal{X}_e(c) \in \mathfrak{X}_s$ of period $\kappa > 0$ be given. Define $q_i(\xi) = h_i(\xi)' h_i(\xi)$ and assume that there exists a time-periodic, radially unbounded, and positive definite function V, as well as positive scalars a and b satisfying, for all $\xi \in \mathbb{R}^{n_x}$ and $n \in [0, \kappa)$, the inequalities

$$a\|\xi\|^2 \le V(\xi, k(n)) \le b\|\xi\|^2 \tag{11}$$

and, for all $\xi \ne 0$ and $n \in [0, \kappa)$, the inequalities

$$V\big(g_{c[n]c[n]}(\xi), k(n+1)\big) - V(\xi, k(n)) + q_{c[n]}(\xi) < -\epsilon V(\xi, k(n)), \tag{12}$$

with $\epsilon > 0$ and arbitrarily small. Then, the state-dependent switching function $\sigma[n] = u(\xi, n)$ with

$$u(\xi, n) = \arg\min_{i \in \mathbb{K}} V\big(g_{c[k(n)]i}(\xi), k(n+1)\big) + q_i(\xi) \tag{13}$$

ensures global exponential stability of the origin $\xi = 0$ of the system (7)-(8) or, equivalently, global exponential stability of the limit cycle $\mathcal{X}_e(c)$ of the system (1). Moreover, the inequality

$$\|z\|_2^2 < V(\xi[0], 0) \tag{14}$$

holds for $\xi[0] \ne 0$.

Proof: Assume that inequalities (11)-(12) are satisfied. Using the switching function $\sigma[n] = u(\xi, n)$ given in (13) and the Lyapunov function (10), along an arbitrary trajectory of the system, we have, for all $n \in [0, \kappa)$ and $\xi \ne 0$,

$$\begin{aligned}\Delta v &= V\big(\xi^+, k(n+1)\big) - V(\xi, k(n)) + q_\sigma(\xi) - z'z \\ &= \min_{i \in \mathbb{K}} \Big(V\big(g_{c[n]i}(\xi), k(n+1)\big) + q_i(\xi)\Big) - V(\xi, k(n)) - z'z \\ &\le V\big(g_{c[n]c[n]}(\xi), k(n+1)\big) - V(\xi, k(n)) + q_{c[n]}(\xi) - z'z \\ &< -z'z,\end{aligned} \tag{15}$$

where the second equality is due to the switching function (13) and the first inequality is a consequence of the minimum operator. The last inequality is due to (12). The time-periodicity of the Lyapunov function $v(\xi, n) = V(\xi, k(n))$ guarantees that $\Delta v(\xi, n) < 0$ for all $n \in \mathbb{N}$ and $\xi \ne 0$, as a consequence of (15). Hence, with $\epsilon > 0$, arbitrarily small, together with (11), we have

$$\|\xi[n]\|^2 \le (b/a)\mu^n \|\xi[0]\|^2 \tag{16}$$

where $\mu = 1 - \epsilon \in (0, 1)$, indicating that the origin $\xi = 0$ is globally exponentially stable. Now calculating the telescoping sum of both sides of $\Delta v < -z'z$ for all $n \in \mathbb{N}$ we obtain (14), which concludes the proof. ∎

Note that using Theorem 1 we can guarantee a suitable level of performance for the steady-state and the transient response. Indeed, the steady-state requirements are accomplished when the subset of candidate limit cycles $\mathfrak{X}_s$ is determined. For a given limit cycle $\mathcal{X}_e(c) \in \mathfrak{X}_s$ the transient response is optimized by minimizing the right-hand side of (14). We can also choose the limit cycle inside $\mathfrak{X}_s$ that provides the best guaranteed performance. In this case, the switching rule $\sigma[n] = u(\xi, n)$ of Theorem 1 must be determined by solving, for a given initial condition $\xi[0]$, the optimization problem

$$\min_{c \in \mathfrak{C}_s} \inf_{V > 0} V(\xi[0], 0) \tag{17}$$

subject to (11)-(12). The resulting limit cycle $\mathcal{X}_e(c_*)$ is the one, among the family $\mathfrak{X}_s$, that provides the best guaranteed performance. Observe that, in this context, uniqueness of solution of (4) is not an important issue whenever the stability conditions of Theorem 1 are fulfilled.

Although Theorem 1 provides conditions for global exponential stability of $\xi = 0$, they may be satisfied only in a pre-specified domain $\xi \in \mathbb{X}$ containing the origin, leading to a local stability result. This is the case for the polynomial switched systems to be treated in the next subsection, where conditions expressed in terms of LMIs are provided to ensure local exponential stability of a limit cycle $\mathcal{X}_e(c) \in \mathfrak{X}_s$.

A. Degree Two Polynomial Systems

In this subsection our main result is applied to the class of degree two polynomial systems. Consider the system (1) with the structure

$$f_\sigma(x) = F_\sigma(x)\ x + b_\sigma \tag{18}$$

where the matrix functions $F_i(x), \ \forall i \in \mathbb{K}$, are affine with respect to the state variable x, that is:

$$F_i(x) = F_{0i} + \sum_{j=1}^{n_x} F_{ji} x_j, \tag{19}$$

where x_j is the j-th component of the vector x and $F_{ji} \in \mathbb{R}^{n_x \times n_x}$ are constant matrices for all $j \in [0, n_x]$ and $i \in \mathbb{K}$. Note that the system (1) with (18)-(19) is a polynomial system, having the state of dimension n_x, with dynamics described by polynomials of degree two.

We can define a controlled output z and apply the change of variable $\xi = x - x_e$ to obtain the equivalent system (7)-(8) with

$$g_{c[k(n)]\sigma}(\xi) = A_\sigma(\xi, x_e)\xi + \ell_{c[k(n)]\sigma}(x_e), \quad (20)$$
$$h_\sigma(\xi) = C_\sigma(\xi)\xi, \quad (21)$$

in which

$$A_i(\xi, x_e) = M_i(x_e) + F_{0i} + \sum_{j=1}^{n_x} F_{ji}\big(\xi_j + x_{ej}\big), \quad (22)$$

with $M_i(x_e) = [F_{1i}x_e \ \cdots \ F_{n_x i}x_e]$, and

$$C_i(\xi) = C_{0i} + \sum_{j=1}^{n_x} C_{ji}\xi_j, \quad (23)$$
$$\ell_{si}(x_e) = f_i(x_e) - f_s(x_e), \quad (24)$$

where ξ_j and x_{ej} are the j-th components of ξ and x_e, respectively, and $(i, s) \in \mathbb{K} \times \mathbb{K}$.

Inspired by the numerical method developed in [7], our goal is to ensure local exponential stability of a limit cycle $\mathcal{X}_e(c) \in \mathfrak{X}_s$ of interest, considering that the state variable ξ is constrained to a convex polyhedral set, provided by the designer. This set is defined by the convex combination of N_v vertices ξ_r, that is $\mathbb{X} = \text{co}\{\xi_r\}_{r=1}^{N_v}$, and can be rewritten, alternatively, as

$$\mathbb{X} = \{\xi \mid a_m'\xi \le 1, \ m \in [1, 2n_x]\} \quad (25)$$

in which a_m defines the m-th hyperplane, see [5, Section 8.4.2].

Adopting the Lyapunov function (10) with

$$V(\xi, k(n)) = \xi' P[k(n)]\xi \quad (26)$$

for $n \in \mathbb{N}$ and $P[k(n)] > 0$, the next corollary particularizes Theorem 1 to cope with this class of nonlinear systems and provides sufficient conditions for local exponential stability, as well as an ellipsoidal set $\mathcal{E}_0 \subset \mathbb{X}$ of maximum volume, such that for all $\xi_0 \in \mathcal{E}_0$, we have that $\xi[n] \in \mathbb{X}$ for any $n \in \mathbb{N}$.

Corollary 1: Let a sequence $c \in \mathfrak{C}_s$ associated to the limit cycle $\mathcal{X}_e(c) \in \mathfrak{X}_s$ of period $\kappa > 0$ be given, as well as vectors a_m, $\forall m \in [1, 2n_x]$, that define the convex polyhedral set $\mathbb{X}$ in (25). The solution of the convex optimization problem

$$\min_{S[n]>0} \ \log \ \det S[0]^{-1} \quad (27)$$

subject to

$$a_m' S[n] a_m < 1 \quad (28)$$
$$\begin{bmatrix} S[k(n)] & \bullet & \bullet \\ A_{c[n]}(\xi_r, x_e[n])S[k(n)] & S[k(n+1)] & \bullet \\ C_{c[n]}(\xi_r)S[k(n)] & 0 & I \end{bmatrix} > 0 \quad (29)$$

for all $m \in [1, 2n_x]$, $r \in [1, N_v]$, and $n \in [0, \kappa)$, ensures that the state-dependent switching function $\sigma[n] = u(\xi, n)$ with

$$u(\xi, n) = \arg\min_{i \in \mathbb{K}} \begin{bmatrix} \xi \\ 1 \end{bmatrix}' \begin{bmatrix} A_i'P^+A_i + Q_i & \bullet \\ \ell_{c[k(n)]i}'P^+A_i & \rho_{c[k(n)]i} \end{bmatrix} \begin{bmatrix} \xi \\ 1 \end{bmatrix}' \quad (30)$$

in which matrices $A_i = A_i(\xi, x_e[k(n)])$, $P[k(n)] = S[k(n)]^{-1}$, $P^+ = P[k(n+1)]$, $Q_i = C_i(\xi)'C_i(\xi)$ and the scalar $\rho_{c[k(n)]i} = \ell_{c[k(n)]i}'P^+\ell_{c[k(n)]i}$, makes the origin $\xi = 0$ of system (7)-(8), defined by the functions (20)-(21), locally exponentially stable or, equivalently, the limit cycle $\mathcal{X}_e(c) \in \mathfrak{X}$ of the system (1) defined by (18)-(19) locally exponentially stable. Moreover, the set

$$\mathcal{E}_0 = \{\xi \mid \xi' S[0]^{-1}\xi \le 1\} \quad (31)$$

inscribed in $\mathbb{X}$ is the ellipsoid with maximum volume with the property that for all $\xi[0] \in \mathcal{E}_0$, we have $\xi[n] \in \mathbb{X}$ for all $n \in \mathbb{N}$ and the inequality $\|z\|_2^2 < \xi[0]'P[0]\xi[0]$ holds.

Proof: The first part of the proof consists in showing that the conditions of this corollary ensure that the ones of Theorem 1 are valid for all $\xi \in \mathbb{X}$. Firstly, note that the choice of the Lyapunov function (10) with the time-periodic quadratic function (26) implies that the inequalities (11) are satisfied. Moreover, for this class of systems, inequality (12) becomes, for $\xi \neq 0$:

$$\xi'\Big(A_{c[n]}(\xi, x_e)'P[k(\ n+1)]A_{c[n]}(\xi, x_e) - P[k(n)] + Q_{c[n]}\Big)\xi < 0. \quad (32)$$

This inequality is always satisfied if the term inside the parentheses is negative definite for all $\xi \in \mathbb{X}$. This is accomplished if there exist matrices $S[k(n)] = P[k(n)]^{-1} > 0$ satisfying

$$\begin{bmatrix} S[k(n)] & \bullet & \bullet \\ A_{c[n]}(\xi, x_e)S[k(n)] & S[k(n+1)] & \bullet \\ C_{c[n]}(\xi)S[k(n)] & 0 & I \end{bmatrix} > 0, \quad (33)$$

for all $\xi \neq 0$, such that $\xi \in \mathbb{X}$ for all $n \in [0, \kappa)$. Indeed, multiplying both sides of (33) by $\text{diag}(S[k(n)]^{-1}, I, I)$ and performing the Schur Complement with respect to the two last rows and columns, we obtain the term inside the parentheses in (32). Since the matrices A_i and C_i are affine with respect to ξ (see (22) and (23)), to check the feasibility of (33) for all $\xi \in \mathbb{X}$ is equivalent to check feasibility at the N_v vertices of $\mathbb{X}$, leading to (29). The switching function (30) follows directly from (13).

The last part of the proof consists in showing that for all $\xi[0] \in \mathcal{E}_0$, we have that $\xi[n] \in \mathbb{X}$ for all $n \in \mathbb{N}$. Indeed, note that according to [5, Sec. 8.4.2], the inequality (28) ensures that the level sets of the Lyapunov function are inside the polyhedron $\mathbb{X}$ defined in (25), that is $\{\xi \mid \xi'P[n]\xi \le 1\} \subset \mathbb{X}$ for all $n \in \mathbb{N}$. Moreover, the optimization problem (27)-(29) maximizes the volume of the ellipsoid $\mathcal{E}_0$ inscribed in $\mathbb{X}$. Hence, the fact that $\xi[n] \in \{\xi \mid \xi'P[n]\xi \le 1\}$ and $\Delta v < 0$ implies that $\xi[n+1] \in \{\xi \mid \xi'P[n+1]\xi \le 1\}$ and as a consequence $\xi[n+1] \in \mathbb{X}$ for all $n \in \mathbb{N}$, indicating that once $\xi[0] \in \mathcal{E}_0$, the trajectory $\xi[n]$ does not leave the set $\mathbb{X}$ for all $n \in \mathbb{N}$ as stated. The proof is complete. ∎

This corollary provides a state-dependent switching function based on conditions expressed in terms of LMIs. The objective function is nonlinear, but is convex, and well-suited to be optimized via the Frank-Wolfe algorithm [11]. As before, instead of providing a given sequence $c \in \mathfrak{C}_s$, we can determine the one that optimizes the objective function

$$\min_{c\in\mathfrak{C}_s} \min_{S[n]>0} \ \log\ \det S[0]^{-1}, \tag{34}$$

subject to (28)-(29). Moreover, note that these conditions take into account the guaranteed cost $\|z\|_2^2 < \xi[0]' P[0]\xi[0]$ valid for $\xi[0] \neq 0$ and, therefore, a larger set $\mathcal{E}_0$ may be obtained by considering only stability by setting $C_i(\xi) = \epsilon I,\ \forall i \in \mathbb{K}$, with $\epsilon > 0$ arbitrarily small.

IV. Illustrative Examples

This subsection provides two academic examples inspired by [7, Examples 4 and 5]. All the calculations have been performed in MATLAB - R2017a using the LMI Solver routines in an Apple computer with operating system MAC OS version 10.13.6.

Example 1: Consider the second order switched nonlinear system (1) given by

$$f_1(x) = \begin{bmatrix} 1.3x_1 - 0.8x_2 - 0.02x_1^2 + 1 \\ 0.4x_1 + 0.8x_2 + 1 \end{bmatrix}, \tag{35}$$

$$f_2(x) = \begin{bmatrix} x_1 + 0.2x_2 - 1 \\ 0.4x_1 + 0.2x_2 + 0.02x_2^2 - 1 \end{bmatrix}, \tag{36}$$

which has the structure of (18)-(19). Note that the equilibrium points of the first and second subsystems are $x_{eq}^1 = [-2.40\ 0.21]'$ and $x_{eq}^2 = [11.25\ 5.00]'$, respectively. Our goal is to drive the trajectory of the first state component to a value around 4 with a maximum ripple of 1, that is, $x_1 \in [3, 5]$. This value is not close to the equilibrium of any of the subsystems. For $\kappa = 6$, we have considered the criterion (6) with $\Gamma = [1\ 0]$ and $x_* = [4\ ?]'$, where the symbol "?" indicates that the second component is not important, and solving (4), we have obtained 6 candidate limit cycles composing the set $\mathfrak{X}_s \subset \mathfrak{X}$. Defining the controlled output (21), with $h_1(\xi) = h_2(\xi) = \epsilon\xi$ and $\epsilon = 0.001$, and taking into account the region $\xi \in \mathbb{X} = [-5\ 5] \times [-5\ 5]$, which is alternatively defined in (25) by the hyperplanes $a_1' = [-0.2\ 0]$, $a_2' = [0.2\ 0]$, $a_3' = [0\ -0.2]$ and $a_4' = [0\ 0.2]$, we have solved the optimization problem (34), subject to the conditions (28)-(29) of Corollary 1, and obtained

$$P[0] = \begin{bmatrix} 0.0565 & 0.0070 \\ 0.0070 & 0.0409 \end{bmatrix} \tag{37}$$

associated to $c = \{2, 2, 1, 1, 2, 1\}$ and the limit cycle

$$\{x_e[n]\}_{n=0}^5 = \left\{ \begin{bmatrix} 3.8912 \\ 3.9839 \end{bmatrix}, \begin{bmatrix} 3.6879 \\ 1.6707 \end{bmatrix}, \begin{bmatrix} 3.0221 \\ 0.8651 \end{bmatrix}, \begin{bmatrix} 4.0539 \\ 2.9009 \end{bmatrix}, \begin{bmatrix} 3.6207 \\ 4.9423 \end{bmatrix}, \begin{bmatrix} 3.6091 \\ 1.9253 \end{bmatrix} \right\}. \tag{38}$$

The top plot in Figure 1 displays the state trajectory ξ evolving from $\xi_0 = [-2.95\ -3]' \in \mathcal{E}_0$, where it is possible to note that $\xi \in \mathbb{X}$ for all $n \in \mathbb{N}$. Finally, the bottom plot shows the corresponding switching function. Note that it does not coincide with the periodic switching sequence $\sigma[n] = c[k(n)]$.

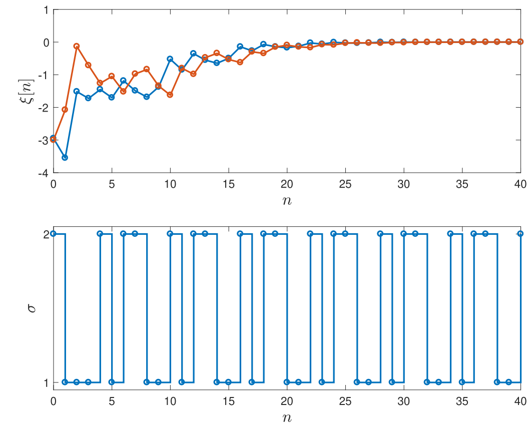

Fig. 1. State trajectory ξ (top); switching function (bottom).

Example 2: Consider the third order switched nonlinear system defined as

$$f_1(x) = \begin{bmatrix} 1.2x_1 + 0.02x_1^2 - 0.2 \\ -x_1 + 0.8x_2 - 0.1x_2^2 + 1.3 \\ 0.5x_3 + 1 \end{bmatrix}, \tag{39}$$

$$f_2(x) = \begin{bmatrix} 0.7x_1 + 0.5 \\ -0.6x_2 - 2x_3 - 0.02x_3^2 + 7 \\ -1.2x_3 + 0.02x_1x_2 + 4 \end{bmatrix}. \tag{40}$$

The equilibrium points of the isolated subsystems are $x_{eq}^1 = [0.92\ 1.20\ 2.00]'$ and $x_{eq}^2 = [1.67\ 2.02\ 1.85]'$, respectively. We have considered the criterion (6) with $\Gamma = [0\ 0\ 1]$ and $x_* = [?\ \ ?\ \ 1.5]'$ to drive the third component of the state inside the interval $x_3 \in [0.5, 2.5]$ in steady-state. For $\kappa = 5$, solving (4), we have obtained 17 candidate limit cycles composing the set $\mathfrak{X}_s \subset \mathfrak{X}$. Moreover, considering the region $\mathbb{X} \equiv [-5\ 5] \times [-5\ 5] \times [-5\ 5]$, and $h_1(\xi) = h_2(\xi) = h_3(\xi) = \epsilon\xi$ with $\epsilon = 0.001$, the solution of the optimization problem (34), subject to (28)-(29), has provided

$$P[0] = \begin{bmatrix} 0.2241 & -0.0221 & 0.0321 \\ -0.0221 & 0.0434 & -0.0108 \\ 0.0321 & -0.0108 & 0.1301 \end{bmatrix} \tag{41}$$

associated to $c = \{1, 2, 1, 2, 2\}$ and the limit cycle

$$\{x_e[n]\}_{n=0}^4 = \left\{ \begin{bmatrix} 2.0632 \\ 1.5138 \\ 1.9249 \end{bmatrix}, \begin{bmatrix} 2.3610 \\ 0.2187 \\ 1.9625 \end{bmatrix}, \begin{bmatrix} 2.1527 \\ 2.8669 \\ 1.6554 \end{bmatrix}, \begin{bmatrix} 2.4759 \\ 0.6189 \\ 1.8277 \end{bmatrix}, \begin{bmatrix} 2.2331 \\ 2.9064 \\ 1.8374 \end{bmatrix} \right\}. \tag{42}$$

The top of Figure 2 displays the state trajectory ξ evolving from $\xi_0 = [-0.7\ 1.4\ 2.5]' \in \mathcal{E}_0$. Note that this trajectory does not leave $\mathbb{X}$ and converges to the origin $\xi = 0$. The middle plot shows x_3, indicating that the steady-state response belongs to the interval $[0.5, 2.5]$. Finally the bottom plot shows the ellipsoid $\mathcal{E}_0$ expressed in terms of x, the state trajectory x and the corresponding limit cycle.

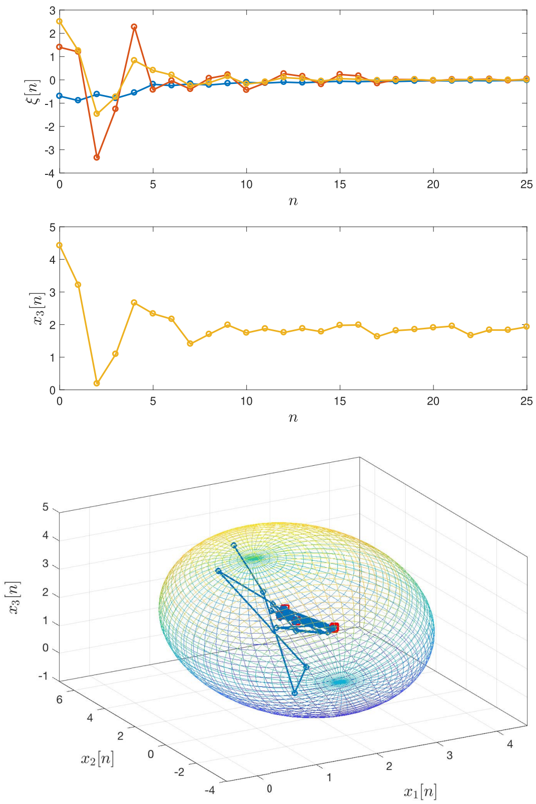

Fig. 2. State trajectory ξ (top); state x_3 (middle); ellipsoid $\mathcal{E}_0$ expressed in terms of x, as well as the trajectory x together with the corresponding limit cycle (bottom).

V. Conclusion

In this letter we have studied the stabilization of a limit cycle of interest for discrete-time switched nonlinear systems. After determining the set of candidate limit cycles, by means of a criterion related to the steady-state response, we have provided sufficient conditions for the design of a state-dependent switching function able to ensure global exponential stability of the desired limit cycle. For a particular class of switched polynomial systems we have derived a new control methodology based on LMI conditions to guarantee local exponential stability of the limit cycle. Two academic examples shown the validity of the proposed theory. An interesting and challenging subject for future work is to deal with robust asymptotic stabilisation of limit cycles for discrete-time switched nonlinear systems, when the dynamic matrix is subject to polytopic uncertainties. This theme is a challenge even for switched affine systems that have a simpler structure.

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