Stochastic Model Predictive Control With Minimal Constraint Violation Probability for Time-Variant Chance Constraints

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***Abstract*—Despite the effectiveness of Robust and Stochastic Model Predictive Control, not all scenarios require robust trajectories or permit constraint violations. Achieving a balance between safety and performance is crucial. A Model Predictive Control approach is proposed that provides an optimal control law by minimizing the probability of constraint violations for time-variant constraints while aiming at achieving a performance criterion. Either the constraints are satisfied robustly or with minimal probability of constraint violation. Further, the proposed method switches between minimizing the performance criterion and minimizing the constraint violation probability whenever either does not meet the requirements anymore. Recursive feasibility and stability of the method are proved. The approach is evaluated for overtaking of an autonomous vehicle in a simulation study.**

***Index Terms*—Predictive control for linear systems, robust control, stochastic optimal control.**

I. Introduction

IN RECENT years, Model Predictive Control (MPC) has emerged as a powerful technique for controlling dynamic systems under constraints [1]. However, traditional MPC approaches encounter limitations when applied to disturbed systems, such as loss of feasibility. In such cases, Robust Model Predictive Control (RMPC) and Stochastic Model Predictive Control (SMPC) offer promising solutions to address disturbances and uncertainties.

RMPC takes into account a worst-case disturbance, ensuring that the resulting inputs guarantee constraint satisfaction for all possible disturbances [2]. SMPC considers the stochastic properties of disturbances in various ways. One common approach is using chance constraints, where the Constraint Violation Probability (CVP) is bounded by a threshold [3]. This allows it to get closer to the boundaries than RMPC but with a small probability of constraint violation.

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However, not all applications require a robust trajectory or unnecessary high CVP for the sake of cost minimization. A desirable outcome involves a balance between safety and performance. On the one hand, if a CVP of zero is achieved while guaranteeing adequate performance, there is no need to tolerate any constraint violation, not even to a small probability. A performance criterion indicates that solutions guarantee adequate performance, i.e., the state is close enough to the reference. On the other hand, if it is only possible to meet the performance criterion by violating a constraint, this should be done with the lowest possible CVP. Consider, for instance, an overtaking maneuver in autonomous driving. The CVP translates into a collision probability. It is preferable to choose a trajectory with a collision probability of zero. However, if overtaking is only possible with a non-zero collision probability, the trajectory with the smallest possible collision probability must be chosen. Moreover, overtaking should not be attempted if the collision probability exceeds a predefined maximum value.

In addition to approaches that rely on chance constraints, there are other interpretations of the stochastic properties in SMPC. For instance, with [4], a covariance steering-based SMPC approach is introduced, which steers the mean and the covariance of the system state to meet a prescribed target covariance. However, the approach accepts a constraint violation even if robust trajectories are feasible. In contrast, the collision probability with obstacles is minimized in [5]. If there is a robust solution, it is preferred, but the work focuses only on grid-based path planning, i.e., not on MPC. Having the requirements of the application to mobile robotics with a changing environment in mind, the controller needs to account for changes in the constraint. For such a time-variant constraint, obtaining guarantees for recursive feasibility and stability is a challenge. In [6], the authors propose a method capable of handling time-variant constraints, yet it assumes the reference to be stabilized and stationary without considering disturbances. Furthermore, [7] combines time-variant constraints with non-linear MPC. However, the uncertainty is only taken into account in the constraints, while the dynamic system is assumed to be deterministic.

The previously mentioned approaches consider either a bounded CVP or robust constraints, i.e., constraints with a zero CVP. Our work focuses on an SMPC approach that yields a trajectory with minimal CVP. This concept was introduced in [8], [9]. These prior works primarily focus on a small class of constraints, namely norm constraints. Therefore, [10] extends the concept to consider general linear constraints, though only for stable systems. Our approach further extends the idea of Constraint Violation Probability Minimization (CVPM) and combines it with a state feedback control law [11]. Recursive feasibility is achieved by constraining the first prediction step, as proposed in [12].

This letter proposes an MPC approach that addresses stochastic time-variant constraints and hard time-invariant constraints. The novel approach blends between RMPC and SMPC, as we propose to minimize the CVP as long as a performance criterion is satisfied. This performance criterion defines the achievement of the control objective.

Two scenarios can occur when minimizing the CVP while meeting the performance criterion. In the first scenario, the CVP is zero, denoted as the robust case, while in the second scenario, it remains greater than zero, referred to as the probabilistic case. The desired case is the robust case because then, the robust constraint and the performance criterion are satisfied. However, the probabilistic case is applied if the time-variant constraint cannot be satisfied robustly. In this case, a trajectory is found with the minimal CVP as long as the performance criterion is satisfied. Hence, the objective is to minimize the CVP instead of the cost function. Additionally, a chance constraint is applied such that only a maximum allowed CVP is possible. Based on the chance constraint, a third case arises: the restricted case. It is used if the performance criterion is infeasible under the chance constraint. In the restricted case, the cost function is used as an objective to eventually fulfill the performance criterion, while the chance constraint limits the CVP to the maximum value. Thus, the method switches between two objectives: cost function minimization and CVP minimization. An alternative would be a weighted sum of the cost function and CVP. The advantage of our novel switching-based approach is that there is no need for extensive parameter tuning. To maintain feasibility of the Optimal Control Problem (OCP) in the subsequent time step, we consider a time-variant terminal constraint, which is used as first-step constraint.

In summary, the contributions of this letter are as follows: 1) We propose an MPC method where the resulting trajectory has a minimal CVP, although it is nevertheless bound. 2) The method handles time-variant constraints, i.e., with chance constraints, while recursive feasibility is ensured. 3) The method compromises between performance as defined by the cost and performance as defined by the chance constraints.

This letter continues with Section II, introducing the preliminaries to the method proposed in Section III. Recursive feasibility and stability are discussed in Section IV. A simulation example of an overtaking maneuver is provided in Section V, and this letter is concluded with Section VI.

Notation: Vectors are written as bold and italic lowercase letters, matrices are denoted as bold and non-italic uppercase letters, and sets are denoted with calligraphic font, e.g., $\boldsymbol{x}$, $\mathbf{A}$, and $\mathcal{X}$, respectively. The variable $\boldsymbol{x}_k$ is the state at time k, and $\boldsymbol{x}_{i|k}$ is the prediction i steps ahead at time k. Predicted sequences are denoted as uppercase vectors, i.e., $\boldsymbol{X}_k = [\boldsymbol{x}_{0|k}^\top \ \cdots \ \boldsymbol{x}_{N|k}^\top]^\top$, $\boldsymbol{U}_k = [\boldsymbol{u}_{0|k}^\top \ \cdots \ \boldsymbol{u}_{N-1|k}^\top]^\top$.

The weighted norm is $\|\boldsymbol{x}\|_{\mathbf{A}}^2 = \boldsymbol{x}^\top \mathbf{A}\boldsymbol{x}$, the Minkowski sum is $\mathcal{A} \oplus \mathcal{B} = \{\boldsymbol{a} + \boldsymbol{b} | \boldsymbol{a} \in \mathcal{A}, \boldsymbol{b} \in \mathcal{B}\}$ and the Pontryagin difference is $\mathcal{A} \ominus \mathcal{B} = \{\boldsymbol{x} | \boldsymbol{x} + \boldsymbol{b} \in \mathcal{A}, \forall \boldsymbol{b} \in \mathcal{B}\}$. The function diag : $\mathbb{R}^{n\times n} \to \mathbb{R}^n$ maps diagonal elements of a matrix into a column vector and $\sqrt{\cdot}$ is an element-wise square root.

II. Problem Setup

In the following, an SMPC formulation for a time-variant constraint is introduced, which is the starting point for the proposed method in Section III.

A. System

The linear, discrete-time control system

$$\boldsymbol{x}_{k+1} = \mathbf{A}\boldsymbol{x}_k + \mathbf{B}\boldsymbol{u}_k + \boldsymbol{w}_k \tag{1}$$

is considered with state $\boldsymbol{x}_k \in \mathcal{X} \subseteq \mathbb{R}^n$ and bounded input $\boldsymbol{u}_k \in \mathcal{U} \subseteq \mathbb{R}^m$ at time step k, where $\mathbf{A}, \mathbf{B}$ have appropriate dimensions. The proposed approach uses the state set $\mathcal{X}$ and input set $\mathcal{U}$ as a time-invariant hard constraint.

Assumption 1 (Disturbance): The disturbance $\boldsymbol{w}_k$ is a realization of a random variable with a truncated zero-mean Gaussian distribution with covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{w}}$. The distribution set $\mathcal{W}$ is the support of the distribution, i.e., $\boldsymbol{w}_k \in \mathcal{W} \subseteq \mathbb{R}^n$.

Utilizing the feedback law from [11]

$$\boldsymbol{u}_k = -\mathbf{K}\boldsymbol{x}_k + \boldsymbol{v}_k \tag{2}$$

results in a stabilized system with auxiliary input $\boldsymbol{v}_k \in \mathbb{R}^m$, where the feedback matrix $\mathbf{K}$ is chosen such that $\mathbf{A}_{\mathrm{c}} = \mathbf{A} - \mathbf{B}\mathbf{K}$ is stable.

B. Constraints and Constraint Violation Probability

For the OCP to remain recursively feasible, the definition of a robust control invariant set is introduced [13].

Definition 1: A set $\mathcal{X}_{\mathrm{RCI}} \subseteq \mathcal{X}$ is *robust control invariant* if

$$\exists \boldsymbol{u} \in \mathcal{U} : \forall \boldsymbol{x} \in \mathcal{X}_{\mathrm{RCI}}, \boldsymbol{w} \in \mathcal{W} : \mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{u} + \boldsymbol{w} \in \mathcal{X}_{\mathrm{RCI}}. \tag{3}$$

We define the set of admissible states $\mathcal{X}_{\mathrm{Adm}} \subseteq \mathcal{X}$ as a robust control invariant set, i.e., for trajectories with initial state in $\mathcal{X}_{\mathrm{Adm}}$, there is a guarantee that a control input exists such that the subsequent state is also in $\mathcal{X}_{\mathrm{Adm}}$ [13].

In addition to the hard constraint, the proposed method considers the time-variant constraint

$$\boldsymbol{x}_k \in \mathcal{X}_{\mathrm{TV},k} \quad \forall k \tag{4}$$

where the time-variant set $\mathcal{X}_{\mathrm{TV},k} = \{\boldsymbol{x} | \mathbf{H}_k \boldsymbol{x} \leq \boldsymbol{h}_k\}$ is a polytope. Based on this constraint, the Constraint Violation Probability (CVP) for the prediction $\boldsymbol{x}_{i|k}$ is defined as

$$\Pr\left(\boldsymbol{x}_{i|k} \notin \mathcal{X}_{\mathrm{TV},i+k}\right), \tag{5}$$

where k denotes the current time step, and i denotes the time instants in the prediction horizon.

C. SMPC With Time-Variant Constraint

In MPC, the measurement of the current state $\boldsymbol{x}_k$ serves as the initial state of the OCP. The cost function

$$J(\boldsymbol{X}_k, \boldsymbol{U}_k) = \sum_{i=0}^{N} l_{i+k}\left(\boldsymbol{x}_{i|k}, \boldsymbol{u}_{i|k}\right) + \left\|\boldsymbol{x}_{N|k} - \boldsymbol{x}_{\mathrm{ref},N+k}\right\|_{\mathbf{Q}_{\mathrm{f}}}^2 \tag{6a}$$

$$l_{i+k}\left(\boldsymbol{x}_{i|k}, \boldsymbol{u}_{i|k}\right) = \left\|\boldsymbol{x}_{i|k} - \boldsymbol{x}_{\mathrm{ref},i+k}\right\|_{\mathbf{Q}}^2 + \left\|\boldsymbol{u}_{i|k} - \boldsymbol{u}_{\mathrm{ref},i+k}\right\|_{\mathbf{R}}^2 \tag{6b}$$

is defined over the horizon N and penalizes the deviation from the reference with positive definite weight matrices $\mathbf{Q} \in \mathbb{R}^{n\times n}$, $\mathbf{R} \in \mathbb{R}^{m\times m}$. The terminal weight $\mathbf{Q}_\mathrm{f} \in \mathbb{R}^{n\times n}$ is the solution of the discrete-time Riccati equation.

Assumption 2: The reference trajectory is reachable and thus $\boldsymbol{x}_{\mathrm{ref},k}$ and $\boldsymbol{u}_{\mathrm{ref},k}$ are a solution of (1) with $\boldsymbol{w}_k = \mathbf{0}$.

With the chance constraint, i.e., a CVP bounded by β, a stochastic OCP handling time-variant constraints is given as

$$\boldsymbol{V}_k^* = \arg\min_{\boldsymbol{V}_k} J(\boldsymbol{X}_k, \boldsymbol{U}_k) \tag{7a}$$

$$\text{s.t. } \boldsymbol{x}_{i+1|k} = \mathbf{A}\boldsymbol{x}_{i|k} + \mathbf{B}\boldsymbol{u}_{i|k} \qquad \forall i \in [0, N-1] \tag{7b}$$

$$\boldsymbol{u}_{i|k} = -\mathbf{K}\boldsymbol{x}_{i|k} + \boldsymbol{v}_{i|k} \in \mathcal{U} \qquad \forall i \in [0, N-1] \tag{7c}$$

$$\Pr\left(\boldsymbol{x}_{i|k} \notin \mathcal{X}_{\mathrm{TV},i+k}\right) \leq \beta,\ \boldsymbol{x}_{i|k} \in \mathcal{X}_{\mathrm{Adm}} \qquad \forall i \in [1, N] \tag{7d}$$

$$\boldsymbol{x}_{0|k} = \boldsymbol{x}_k. \tag{7e}$$

with $\boldsymbol{V}_k =^* [\boldsymbol{v}_{0|k}^{*}{}^\top \cdots \boldsymbol{v}_{N-1|k}^{*}{}^\top]^\top$. As in the OCP in (7a), in this letter, we propose an SMPC method for time-variant constraints that bounds the CVP with a chance constraint such as (7d). In addition to these requirements, our aim is to find a trajectory that is sufficiently close to the reference while the CVP defined in (5) is minimal within the prediction horizon.

III. Method

In this section, we introduce the method by starting with the description of the performance criterion. The performance criterion is applied to each predicted state and input to check whether the prediction is close enough to the reference.

Definition 2 (Performance Criterion): The performance criterion is fulfilled for the state $\boldsymbol{x}_{i|k}$ and input $\boldsymbol{u}_{i|k}$ if the distance to the reference is less than or equal to α, i.e.,

$$\boldsymbol{x}_{i|k} \in \mathcal{X}_{\mathrm{PC},i+k} = \left\{\boldsymbol{x} \mid \left\|\boldsymbol{x} - \boldsymbol{x}_{\mathrm{ref},i+k}\right\|_{\mathbf{Q}}^2 \leq \alpha\right\} \tag{8a}$$

$$\boldsymbol{u}_{i|k} \in \mathcal{U}_{\mathrm{PC},i+k} = \left\{\boldsymbol{u} \mid \left\|\boldsymbol{u} - \boldsymbol{u}_{\mathrm{ref},i+k}\right\|_{\mathbf{R}}^2 \leq \alpha\right\}. \tag{8b}$$

The tuning parameter α is adapted to the respective application. Achieving a minimal CVP while satisfying the performance criterion requires a case differentiation. In the following, the three cases and conditions for their applicability are introduced. Details about the evaluation are presented in Section III-D. In each time step, based on the measured state $\boldsymbol{x}_k$, it is determined which case the OCP is feasible for the prediction $\boldsymbol{X}_k$ and $\boldsymbol{U}_k$.

Definition 3 (Robust Case): There exists a feasible input sequence $\boldsymbol{U}_k$, such that the time-variant constraint (4) is satisfied robustly and the performance criterion (8) is fulfilled, i.e., $\Pr(\boldsymbol{x}_{i|k} \notin \mathcal{X}_{\mathrm{TV},i+k}) = 0$, $\boldsymbol{x}_{i|k} \in \mathcal{X}_{\mathrm{PC},i+k}$, and $\boldsymbol{u}_{i|k} \in \mathcal{U}_{\mathrm{PC},i+k}$, $\forall i \in [1, N]$.

Definition 4 (Probabilistic Case): The robust case is not applicable and there exists a feasible input sequence $\boldsymbol{U}_k$, such that the CVP is bounded by β and the performance criterion (8) is fulfilled, i.e., $0 < \Pr(\boldsymbol{x}_{i|k} \notin \mathcal{X}_{\mathrm{TV},i+k}) \leq \beta$, $\boldsymbol{x}_{i|k} \in \mathcal{X}_{\mathrm{PC},i+k}$, and $\boldsymbol{u}_{i|k} \in \mathcal{U}_{\mathrm{PC},i+k}$, $\forall i \in [1, N]$.

Definition 5 (Restricted Case): For all feasible input sequences $\boldsymbol{U}_k$, either the CVP is greater than β, or the performance criterion (8) is not feasible, i.e., $\exists i \in [1, N]$: $\Pr(\boldsymbol{x}_{i|k} \notin \mathcal{X}_{\mathrm{TV},i+k}) \geq \beta$, or $\boldsymbol{x}_{i|k} \notin \mathcal{X}_{\mathrm{PC},i+k}$, or $\boldsymbol{u}_{i|k} \notin \mathcal{U}_{\mathrm{PC},i+k}$.

Due to recursive feasibility, the existence of a solution with CVP that is less than β is guaranteed. Without performance criterion, the probabilistic case would only minimize the CVP, disregarding the cost function. Including the performance criterion leads to solutions that represent a trade-off between safety and performance, i.e., remaining close to the minimum of the cost function with minimum CVP.

The robust case has the highest priority; if it is not feasible, the probabilistic case is applied. If none of these cases apply, the restricted case is applied. This is always possible due to recursive feasibility. The decision is made by checking if the system dynamics, input, and state constraints result in an empty set by solving a linear program [14].

A. Robust Constraint

The robust constraint is used in the robust case to ensure that the time-variant constraint is satisfied robustly. For each prediction at time i based on time step k, we transform the time-variant constraint into a tightened constraint [11],

$$\mathcal{X}_{\mathrm{R},i|k} = \mathcal{X}_{\mathrm{TV},i+k} \ominus \sum_{j=0}^{i-1} \mathbf{A}_\mathrm{c}{}^j \mathcal{W}, \tag{9}$$

such that a trajectory starting at time k never violates the original time-variant constraint, although there are disturbances. In other words, for states $\boldsymbol{x}_{i|k} \in \mathcal{X}_{\mathrm{R},i|k}$, the CVP is zero.

B. Chance Constraint

In the probabilistic and the restricted case, the CVP is bounded by β, i.e., the chance constraint (7d) is utilized. Handling the chance constraint deterministically requires a reformulation based on the probability distribution of the predicted states, utilizing the steady-state solution of the covariance propagation $\boldsymbol{\Sigma}_{\boldsymbol{x}} = \mathbf{A}_\mathrm{c}\boldsymbol{\Sigma}_{\boldsymbol{x}}\mathbf{A}_\mathrm{c}{}^\top + \boldsymbol{\Sigma}_{\boldsymbol{w}}$. By using the inverse error function $\mathrm{erf}^{-1}(\cdot)$ the tightened sets $\mathcal{X}_{\mathrm{C},i|k}$ for the chance constraint are given as [15]

$$\mathcal{X}_{\mathrm{C},i|k} = \left\{\boldsymbol{x} \mid \mathbf{H}_{i+k}\boldsymbol{x} \leq \boldsymbol{h}_{i+k} - \boldsymbol{\gamma}_{i+k}\right\} \tag{10}$$

with $\boldsymbol{\gamma}_{i+k} = \mathrm{erf}^{-1}(2\beta - 1)\sqrt{2\,\mathrm{diag}(\mathbf{H}_{i+k}\boldsymbol{\Sigma}_{\boldsymbol{x}}\mathbf{H}_{i+k}^\top)}$.

Assumption 3 (Probability Bound): The design parameter β is chosen such that $\mathcal{X}_{\mathrm{R},i|k} \subseteq \mathcal{X}_{\mathrm{C},i|k}$.

C. First-Step Constraint

In order to prove recursive feasibility in Section IV-A, we introduce a first-step constraint, compare [12]. The first-step constraint is applied in all three cases. The idea behind it is to project the terminal constraint, which is usually applied to obtain recursive feasibility, to the first predicted state $\boldsymbol{x}_{1|k}$.

In a time-invariant setup, a fixed terminal set is used. However, the terminal sets must be time-variant in the proposed time-variant situation. Therefore, we define a trajectory of an unforced system that is always constraint admissible. The sequence of terminal sets is arranged along the trajectory such that if a state is in a terminal set of time step k, an input exists such that the next state is in the terminal set of time step $k+1$. For this purpose, we make an assumption on the time-variant constraints $\mathcal{X}_{\mathrm{TV},k}$:

Assumption 4 (Proximity Set): Given the time-variant sets $\mathcal{X}_{\mathrm{TV},k}$, there exists an auxiliary initial state $\tilde{\boldsymbol{x}}_0$, and there exists a robust control invariant proximity set $\tilde{\mathcal{X}} \subset \mathbb{R}^n$ such that all states in the proximity $\tilde{\mathcal{X}}$ around the solution of the unforced system $\tilde{\boldsymbol{x}}_{k+1} = \mathbf{A}\tilde{\boldsymbol{x}}_k$ are in $\mathcal{X}_{\mathrm{TV},k}$, i.e., $\tilde{\boldsymbol{x}}_k \oplus \tilde{\mathcal{X}} \subseteq$

$\mathcal{X}_{\mathrm{TV},k}$ $\forall k$. This allows for the construction of a terminal set $\tilde{\boldsymbol{x}}_{k+N} \oplus \tilde{\mathcal{X}}$ for the terminal state $\boldsymbol{x}_{N|k}$.

An implication of Ass. 4 is that the time-variant constraint set $\mathcal{X}_{\mathrm{TV},k}$ needs to be large enough to contain the robust control invariant set $\tilde{\mathcal{X}}$. In the overtaking example, Ass. 4 is reasonable because it guarantees a path to pass the vehicle.

We define a time-variant first-step constraint applied on the first prediction step at time k, i.e., $\boldsymbol{x}_{1|k}$ as follows

$$\mathcal{X}_{\mathrm{FS},k} = \left\{ \boldsymbol{x}_{1|k} \middle| \begin{array}{l} \forall \boldsymbol{w}_k \in \mathcal{W} : \exists \boldsymbol{U}_k: \\ \boldsymbol{x}_{N|k} + \sum_{i=0}^{N} \mathbf{A}^{N-i} \boldsymbol{w}_i \in \tilde{\boldsymbol{x}}_{k+N} \oplus \tilde{\mathcal{X}} \end{array} \right\}. \quad (11)$$

From (11), we conclude that for all $\boldsymbol{x}_{1|k} \in \mathcal{X}_{\mathrm{FS},k}$, a trajectory exists, such that the terminal state is in the terminal set. However, in the OCP, the constraint is only applied to the first predicted step and not to the terminal state. Therefore, the predicted trajectory does not end in the terminal set, resulting in less constrained solutions. The auxiliary states $\tilde{\boldsymbol{x}}_k$ for all considered k can be determined online for the given scenario, for example, for a single overtaking maneuver. The linearity of (1) allows the shape of $\mathcal{X}_{\mathrm{FS},k}$ to be precomputed by only considering $\tilde{\mathcal{X}}$ as a terminal set. The utilized auxiliary state $\tilde{\boldsymbol{x}}_{k+N}$ is then taken into account by shifting the precomputed set.

D. Proposed Structure

The components introduced above are used to outline the proposed approach. First, the current state $\boldsymbol{x}_k$ is measured, and the appropriate case is determined based on this measurement. Then, the feasibility of the robust case is checked by solving a linear program with the constraints of Def. 3. If not applicable, the probabilistic case is checked for feasibility based on Def. 4 and applied if suitable. Otherwise, the restricted case is used.

All three cases consider the system dynamics, admissible states, feedback law, input constraints, the first-step constraint, and the initial state, i.e.,

$$\boldsymbol{x}_{i+1|k} = \mathbf{A}\boldsymbol{x}_{i|k} + \mathbf{B}\boldsymbol{u}_{i|k} \in \mathcal{X}_{\mathrm{Adm}} \quad \forall i \in [0, N-1] \quad (12a)$$

$$\boldsymbol{u}_{i|k} = -\mathbf{K}\boldsymbol{x}_{i|k} + \boldsymbol{v}_{i|k} \in \mathcal{U} \quad \forall i \in [0, N-1] \quad (12b)$$

$$\boldsymbol{x}_{1|k} \in \mathcal{X}_{\mathrm{FS},k}, \quad \boldsymbol{x}_{0|k} = \boldsymbol{x}_k. \quad (12c)$$

in the OCP. The three OCPs are introduced in the following.

1) Robust Case: If the conditions of Def. 3 hold, the performance criterion and the robust constraint are fulfilled. Therefore, the resulting system behavior meets our expectations, i.e., adequate performance while ensuring safety. The OCP considers the cost function as the objective and achieves even better performance while considering robust constraint set $\mathcal{X}_{\mathrm{R},i|k}$. The OCP is given as

$$\boldsymbol{V}_k^* = \arg\min_{\boldsymbol{V}_k} J(\boldsymbol{X}_k, \boldsymbol{U}_k) \quad (13a)$$

$$\text{s.t.} \quad \boldsymbol{x}_{i|k} \in \mathcal{X}_{\mathrm{R},i|k} \cap \mathcal{X}_{\mathrm{PC},i+k} \forall i \in [1, N] \quad (13b)$$

$$\boldsymbol{u}_{i|k} \in \mathcal{U}_{\mathrm{PC},i+k} \forall i \in [0, N-1] \quad (13c)$$

$$\text{and (12a)} - \text{(12c)}. \quad (13d)$$

2) Probabilistic Case: According to Def. 4, the robust constraint is not feasible. In that situation, we aim to gain the smallest possible CVP of the state sequence while still fulfilling the performance criterion. As a scalar cost, the joint CVP of the state sequence is used, which makes the OCP solvable by an approximation (see Rem. 1). The minimization of the CVP for all prediction steps results in the OCP

$$\boldsymbol{V}_k^* = \arg\min_{\boldsymbol{V}_k} \Pr\left(\boldsymbol{x}_{i|k} \notin \mathcal{X}_{\mathrm{TV},i+k} \ \forall i \in [1, N]\right) \quad (14a)$$

$$\text{s.t.} \quad \boldsymbol{x}_{i|k} \in \mathcal{X}_{\mathrm{C},i|k} \cap \mathcal{X}_{\mathrm{PC},i+k} \ \forall i \in [1, N] \quad (14b)$$

$$\boldsymbol{u}_{i|k} \in \mathcal{U}_{\mathrm{PC},i+k} \ \forall i \in [0, N-1] \quad (14c)$$

$$\text{and (12a)} - \text{(12c)}. \quad (14d)$$

Even though the joint CVP of the state sequence is minimal, the CVP in one specific prediction time step can be higher than β. Therefore, in addition to the performance criterion, which is represented as $\mathcal{X}_{\mathrm{PC},i+k}$ and $\mathcal{U}_{\mathrm{PC},i+k}$, the chance constraint set $\mathcal{X}_{\mathrm{C},i|k}$ constrains the OCP solution resulting in a CVP less than β for each prediction step.

3) Restricted Case: If Def. 5 applies, it is not possible to satisfy the performance criterion together with the chance constraint (10). In this case, the satisfaction of the chance constraint is prioritized over the performance criterion, and minimization of the cost function aims to fulfill the performance criterion (8) again, yielding

$$\boldsymbol{V}_k^* = \arg\min_{\boldsymbol{V}_k} J(\boldsymbol{X}_k, \boldsymbol{U}_k) \quad (15a)$$

$$\text{s.t.} \quad \boldsymbol{x}_{i|k} \in \mathcal{X}_{\mathrm{C},i+k} \ \forall i \in [1, N] \quad (15b)$$

$$\text{and (12a)} - \text{(12c)}. \quad (15c)$$

In order to achieve this, the CVP is not minimized but only bound to the maximum value β by the chance constraint.

From the optimal solution $\boldsymbol{V}_k^*$ of the OCP, i.e., either (13a), (14a), or (15a), the first element $\boldsymbol{v}_k = \boldsymbol{v}_{0|k}$ is applied to the system by utilizing the feedback law (2).

Remark 1 (Probability Minimization [10]): In the probabilistic case, the CVP of the state sequence in (14a) is optimized. A numerical integration combined with an optimization is not practicable in an online approach. Since only the optimizing variable, but not the probability itself, is relevant, the approximation of the optimization problem suggested in [10] is utilized, i.e., (14a) is approximated by

$$\arg\min_{\boldsymbol{V}_k, \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_N} \sum_{i=1}^{N} \left\| \boldsymbol{x}_{i|k} - \boldsymbol{\xi}_i \right\|^2_{\boldsymbol{\Sigma}_x^{-1}} \ \text{ s.t. } \ \boldsymbol{\xi}_i \in \mathcal{X}_{\mathrm{R},i|k}. \quad (16)$$

In (16), the integration of the probability is approximated by discretization of the integral with a single partition. The integration variables $\boldsymbol{\xi}_i$ become optimization variables in (16).

IV. Properties

A. Recursive Feasibility

Recursive feasibility of the whole switching approach is determined by recursive feasibility of the restricted case only because the constraints of the robust case and probabilistic case are subsets of the constraints in (15a). The reasoning is based on three steps. First, it is shown that a time-variant terminal constraint exists such that the OCP is recursively feasible. Then, we prove that the first-step constraint based on the terminal constraint ensures recursive feasibility, i.e., recursive feasibility of an OCP with the constraints (12a). Finally, it is shown that together with (12a), the chance constraint results in a CVP less than β in the subsequent step. We start by showing the existence of a time-variant robust invariant set.

Lemma 1 (Time-Variant Robust Invariant Set): If Ass. 4 holds, and the state $\boldsymbol{x}_k$ is in the set $\tilde{\boldsymbol{x}}_k \oplus \tilde{\mathcal{X}}$ for some k, an input $\boldsymbol{u}_k \in \mathcal{U}$ exists, such that the subsequent state $\boldsymbol{x}_{k+1}$ is in $\tilde{\boldsymbol{x}}_{k+1} \oplus \tilde{\mathcal{X}}$ for all possible disturbances $\boldsymbol{w}_k \in \mathcal{W}$, i.e.,

$$\forall \boldsymbol{w}_k \in \mathcal{W} : \exists \boldsymbol{u}_k \in \mathcal{U} : \boldsymbol{x}_k \in \tilde{\mathcal{X}} \oplus \tilde{\boldsymbol{x}}_k \Rightarrow \boldsymbol{x}_{k+1} \in \tilde{\mathcal{X}} \oplus \tilde{\boldsymbol{x}}_{k+1} \quad (17)$$

Proof: For all k, we write the state $\boldsymbol{x}_k$ as the sum of the auxiliary state and $\boldsymbol{x} \in \tilde{\mathcal{X}}$, i.e., $\boldsymbol{x}_k = \tilde{\boldsymbol{x}}_k + \boldsymbol{x}$. Thus, the subsequent state is $\boldsymbol{x}_{k+1} = \mathbf{A}\tilde{\boldsymbol{x}}_k + \mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{u}_k + \boldsymbol{w}_k$. By Def. 1, there exists a $\boldsymbol{u}_k \in \mathcal{U}$ such that for all $\boldsymbol{w}_k \in \mathcal{W}$, it holds that $\mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{u}_k + \boldsymbol{w}_k \in \tilde{\mathcal{X}}$. By Ass. 4, it follows that $\boldsymbol{x}_{k+1} \in \mathbf{A}\tilde{\boldsymbol{x}}_k \oplus \tilde{\mathcal{X}} = \tilde{\boldsymbol{x}}_{k+1} \oplus \tilde{\mathcal{X}}$. ∎

For the following proofs, a feasible solution of the OCP (15a) is assumed to exist and written as $\boldsymbol{X}_k^*$. We construct the candidate solution as a shifted version of $\boldsymbol{X}_k^*$, i.e.,

$$\boldsymbol{X}_{k+1} = \begin{bmatrix} \boldsymbol{x}_{0|k+1} \\ \vdots \\ \boldsymbol{x}_{N-1|k+1} \\ \boldsymbol{x}_{N|k+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_{1|k}^* + \boldsymbol{w}_k \\ \vdots \\ \boldsymbol{x}_{N|k}^* + \mathbf{A}_\mathrm{c}{}^{N-1}\boldsymbol{w}_k \\ \boldsymbol{x}_{N|k+1} \end{bmatrix}, \quad (18)$$

where $\boldsymbol{x}_{N|k+1}$ is specified later. With the candidate solution, recursive feasibility with the first-step constraint is presented.

Lemma 2 (Feasibility Under First-Step Constraint): If Lem. 1 holds, an OCP with the constraints (12a), which contains the first-step constraint, is recursively feasible, i.e.,

$$\boldsymbol{x}_{1|k} \in \mathcal{X}_{\mathrm{FS},k} \implies \boldsymbol{x}_{1|k+1} \in \mathcal{X}_{\mathrm{FS},k+1}. \quad (19)$$

Proof: We assume $\boldsymbol{x}_{1|k}^* \in \mathcal{X}_{\mathrm{FS},k}$. For candidate solution (18), the initial state of the subsequent OCP is $\boldsymbol{x}_{0|k+1} = \boldsymbol{x}_{1|k}^* + \boldsymbol{w}_k$. According to (11), for all $\boldsymbol{w}_k \in \mathcal{W}$, a trajectory exists with the initial state $\boldsymbol{x}_{0|k+1}$ such that $\boldsymbol{x}_{N-1|k+1} \in \tilde{\mathcal{X}} \oplus \tilde{\boldsymbol{x}}_{k+N}$. By Lem. 1, an input $\boldsymbol{u}_{N-1|k+1}$ exists such that the terminal state satisfies $\boldsymbol{x}_{N|k+1} \in \tilde{\mathcal{X}} \oplus \tilde{\boldsymbol{x}}_{k+N+1}$. Therefore, again by definition of first-step constraint, we conclude $\boldsymbol{x}_{1|k+1} \in \mathcal{X}_{\mathrm{FS},k+1}$. ∎

In addition to the constraints (12a), the OCP (15a) must be considered, i.e., the chance constraint is added to the OCP.

Lemma 3 (Feasibility Under Chance Constraint): The solution (18) is feasible for (15a) with CVP of at most β, i.e.,

$$\boldsymbol{x}_{i|k}^* \in \mathcal{X}_{\mathrm{C},i|k} \implies \Pr\left(\boldsymbol{x}_{i-1|k+1} \notin \mathcal{X}_{\mathrm{TV},i+k}\right) \leq \beta. \quad (20)$$

Proof: Using the candidate solution in (20) yields

$$\boldsymbol{x}_{i|k}^* \in \mathcal{X}_{\mathrm{C},i|k} \implies \Pr\left(\boldsymbol{x}_{i|k}^* + \mathbf{A}_\mathrm{c}{}^{i-1}\boldsymbol{w}_k \notin \mathcal{X}_{\mathrm{TV},i+k}\right) \leq \beta, \quad (21)$$

which must be proven. The OCP solution $\boldsymbol{X}_k^*$ is deterministic. Therefore, the covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{x},i-1}$ of the state $\boldsymbol{x}_{i-1|k+1}$ results from $\mathbf{A}_\mathrm{c}{}^{i-1}\boldsymbol{w}_k$ and, due to the uncertainty propagation, is given as $\boldsymbol{\Sigma}_{\boldsymbol{x},i-1} = \mathbf{A}_\mathrm{c}{}^{i-1}\boldsymbol{\Sigma}_{\boldsymbol{w}}\mathbf{A}_\mathrm{c}{}^{\top i-1}$. The constraint tightening (10) results in a smaller value $\boldsymbol{\gamma}_{i+k}^*$ for $\boldsymbol{\Sigma}_{\boldsymbol{x},i-1}$, i.e., $\boldsymbol{\gamma}_{i+k}^* < \boldsymbol{\gamma}_{i+k}$. This is shown by writing the covariance propagation as $\boldsymbol{\Sigma}_{\boldsymbol{x}} = \sum_{j=0}^{\infty} \mathbf{A}_\mathrm{c}{}^{j}\boldsymbol{\Sigma}_{\boldsymbol{w}}\mathbf{A}_\mathrm{c}{}^{\top j}$, which contains $\boldsymbol{\Sigma}_{\boldsymbol{x},i-1}$ as a summand. Therefore, the mean of $\boldsymbol{x}_{i-1|k+1}$, i.e., $\boldsymbol{x}_{i|k}^*$, is in a set with $\boldsymbol{\gamma}_{i+k}^*$, which shows that the CVP is less than or equal β for $\boldsymbol{\Sigma}_{\boldsymbol{x},i-1}$. The terminal state $\boldsymbol{x}_{N|k+1}$ is in the robust control invariant terminal set, resulting in a zero CVP. Additionally, Lem. 2 proves recursive feasibility with respect to the first-step constraint while all states are admissible for the hard constraint by satisfying $\mathcal{X}_{\mathrm{Adm}}$. In conclusion, the OCP (15a) is recursively feasible. ∎

With these results, we can formulate the theorem:

Theorem 1 (Recursive Feasibility): Under Ass. 1, Ass. 2, Ass. 3, and Ass. 4 with Lem. 1, Lem. 2, and Lem. 3 hold, the method proposed in Section III-D is recursively feasible.

Proof: The least bounded case is the restricted case, i.e., $\mathcal{X}_{\mathrm{R},i|k} \subseteq \mathcal{X}_{\mathrm{C},i|k}$. This means that if the robust case or the probabilistic case is not feasible, the restricted case is applied. Consequently, it is enough to prove recursive feasibility of the restricted case, as shown in Lem. 3. ∎

B. Stability

As shown in the following, the proposed method stabilizes the system in the robust and restricted case. In the probabilistic case, the aim is to minimize the CVP, therefore the boundedness of the solution will be shown.

Theorem 2 (Stability): Given the system (1) with the proposed control scheme in the robust or restricted case, the reference trajectory is stabilized. In the probabilistic case, the trajectory remains bounded.

Proof: The proof of stability can be done with standard methods of MPC based on the terminal cost in (6a) and Ass. 2. With bounded inputs, disturbances, and a bounded terminal state due to Ass. 4, all possible states of the trajectory are bounded in the restricted case. ∎

V. Results

The method is evaluated in an overtaking scenario on a highway. First, the vehicle dynamics and the constraints for obstacle avoidance are defined. Then, the implementation of the method is discussed. MPT3 [16] is used to calculate the sets, and the OCPs are solved by CasADi [17].

A. System and Obstacle

The controlled vehicle is modeled by a linearized bicycle model [18], [19], where a discrete time step $T = 0.1\,\mathrm{s}$ is considered. The system and input matrices are

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0.1 \\ 0 & 1 & 1.5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0.005 \\ 0.996 & 0 \\ 0.33 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad (22)$$

where the elements of $\boldsymbol{x}_k$ represent the x- and y-position of the center, angle of direction and velocity. The input $\boldsymbol{u}_k$ combines the commanded acceleration and steering angle. The disturbance has zero mean and the covariance matrix

$$\boldsymbol{\Sigma}_{\boldsymbol{w}} = 10^{-3} \cdot \begin{bmatrix} 1.66 & 0 & 0 & -0.49 \\ 0 & 0.3 & -0.01 & 0 \\ 0 & -0.01 & 0.1 & 0 \\ -0.49 & 0 & 0 & 0.26 \end{bmatrix}, \quad (23)$$

while it is bounded by a polytope $\mathcal{W}$ around the ellipsoid at 3-σ. The state set $\mathcal{X}$ contains all states such that the external dimensions of a vehicle are on a 2-lane highway with lane width of 3.5 m, and the input set $\mathcal{U}$ limits the steering angle to 0.5 rad and the acceleration to 11.5 $\frac{\mathrm{m}}{\mathrm{s}^2}$.

In the shown scenario, the controlled vehicle overtakes a target vehicle driving on the right lane with 12 $\frac{\mathrm{m}}{\mathrm{s}}$. The time-variant set $\mathcal{X}_{\mathrm{TV},k}$ is a half-space of the state space, such that the target vehicle is avoided for each time step k. The approach is tested with an introductory overtaking scenario, where the prediction of the target vehicle is given. In an experiment, the prediction of other traffic participants can be estimated, e.g., on the basis of probabilistic learning methods [20]. A robust control invariant set $\tilde{\mathcal{X}} \subset \{\boldsymbol{x} \mid \mathrm{abs}(\boldsymbol{x}) \leq [1, 0.2, \infty, \infty]^\top\}$ is

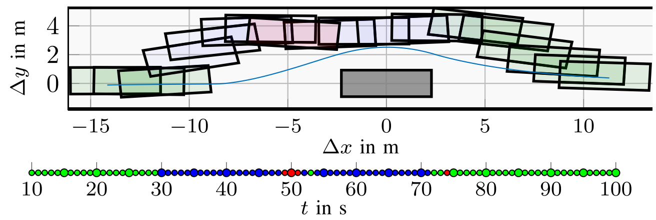

Fig. 1. Overtaking maneuver of a vehicle controlled with the proposed method. The robust case is shown in green, the probabilistic case in blue, and the restricted case in red. The plot is centered around the target vehicle (gray). The timeline shows the applied case in each time step. Larger dots correspond to samples in the upper plot. The blue line represents the vehicle center trajectory under SMPC with risk β=0.05.

used. The initial state of the sequence of auxiliary states is $\tilde{\boldsymbol{x}}_0 = [0, 1.75, 0, 15]^\top$, i.e., the auxiliary sequence represents states on the left lane with a constant speed of 15 $\frac{\text{m}}{\text{s}}$.

B. Overtaking Maneuver

Fig. 1 shows the overtaking maneuver centered on the target vehicle (gray), which is moving at 12 $\frac{\text{m}}{\text{s}}$. The controlled car is depicted every 5 time steps, i.e., the time between each visualization is 0.5 s. The control goal is to drive in the right lane. The reference trajectory is, therefore, set to that lane. The diagonal elements of $\mathbf{Q}$ and $\mathbf{R}$ are $[0.5, 0.03, 10, 1]$ and $[1, 0.1]$, respectively. The bound of the cost function and probability are $\alpha = 1$ and $\beta = 0.05$.

At the beginning and end of the scenario, the controlled vehicle is far away from the target vehicle; therefore, the robust constraint and the performance criterion (8) are feasible. The robust case is applied and shown as green boxes. During overtaking, the probabilistic case is applied (blue). The robust constraints are not feasible, but the chance constraint (10) is feasible. The constraints (14b) and (14c) satisfy the performance criterion. The trajectory is optimized by minimizing the CVP, resulting in a trajectory close to the left boundary of the lane with minimal collision probability. On some occasions, the application of the restricted case is necessary. The situation is marked in Fig. 1 by a red box. In that situation, the performance criterion is not feasible. The cost is minimized to satisfy the performance criterion again while allowing a maximal CVP of β. The cases used in each time step of the simulation are shown in the timeline in Fig. 1.

In this scenario, RMPC results in no overtaking maneuver, and the vehicle remains behind the obstacle. While overtaking the CVP would not be zero, which is unacceptable in RMPC. SMPC would lead to a trajectory closer to the obstacle, as depicted in Fig. 1 with a blue line. For SMPC, the CVP is β during the overtaking maneuver. However, this is not beneficial in this situation, as more distance to the obstacle is safer.

VI. Conclusion

In conclusion, a novel method for handling disturbances is presented, which works in safe situations similar to RMPC and is intended to regain safety in a critical situation. Ensuring that the CVP never exceeds a predefined maximum value is essential. The method presented is suitable not only for autonomous driving applications but also for applications where safe behavior is preferred but where a certain amount of risk must be accepted to achieve the desired behavior, such as meeting market demands in process control.

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