Configuration-Constrained Tube MPC for Tracking

Filippo Badalamenti[ID], Sampath Kumar Mulagaleti[ID], Alberto Bemporad[ID], *Fellow, IEEE*, Boris Houska[ID], and Mario Eduardo Villanueva[ID]

Abstract—**This paper proposes a novel tube-based Model Predictive Control (MPC) framework for tracking varying setpoint references with linear systems subject to additive and multiplicative uncertainties. The MPC controllers designed using this framework exhibit recursively feasible for changing references, and robust asymptotic stability for piecewise constant references. The framework leverages configuration-constrained polytopes to parameterize the tubes, offering flexibility to optimize their shape. The efficacy of the approach is demonstrated through two numerical examples. The first example illustrates the theoretical results, and the second uses the framework to design a lane-change controller for an autonomous vehicle.**

Index Terms—**Constrained control, linear parameter-varying systems, optimal control, predictive control for linear systems, robust control.**

I. INTRODUCTION

THE INHERENT robustness of Model Predictive Control (MPC) derives from solving optimal control problems in a receding horizon manner based on state feedback, rendering it effective in managing complex control tasks. However, in the presence of significant modelling errors or large disturbances, this may prove inadequate [20], due to the mismatch between predictions and the actual behavior of the controlled process. Tube-based MPC (TMPC) has emerged as a suitable candidate to take an uncertainty model into account. The principle behind TMPC is simple: instead of optimizing over vector-valued system trajectories, one optimizes over Robust Forward Invariant Tubes (RFITs) — together with their associated feedback policies — that are set-valued sequences enclosing all realizations of the uncertain system. Particular

set and control parameterizations lead to different TMPC methods, for example, rigid- [12], [16], homothetic [9], [18], elastic [3], [19], configuration-constrained tube MPC [23] as well as other methods based on ellipsoidal [24] or more general sets [8].

In most practical applications, the control system is used to make the outputs track given set points — typically determined by an external module, such as a steady-state optimizer [14] — that may change over time. For example, in autonomous driving problems, the desired lateral displacement changes as the vehicle switches lanes. The extension of TMPC to reference tracking problems then represents a significant milestone in robust MPC design, addressing challenges arising from uncertain environments and the change in operating set points. Early TMPC approaches for reference tracking were based on a rigid tube formulation and were limited to linear systems with additive disturbances [10]. Subsequent works [6], [15] broadened this concept to include multiplicative uncertainty. All these approaches have in common that they enforce asymptotic stability by using parametric terminal sets, which requires solving optimization problems with additional variables.

Contribution: This paper introduces a variant of Configuration-Constrained Tube MPC (CCTMPC) for varying setpoint reference tracking, based on linear prediction models subject to both additive and multiplicative uncertainty. In contrast to the CCTMPC scheme of [23] which relies on an economic MPC formulation, our approach is developed using a tracking MPC formulation. This enables the synthesis of reference tracking controllers that hinge on the solution of a single Quadratic Program (QP) to guarantee asymptotic stability for piecewise constant references. This extends the applicability of CCTMPC beyond regulation problems.

The use of configuration-constrained polytopes for parameterizing the RFIT results in a controller that exhibits superior properties compared to [6], [10]. Notably, the need for a robust affine feedback law is eliminated, and the resulting RFITs exhibit an improved representational capability, leading to less conservativeness. A particularly relevant feature for the reference tracking problem is the unrestricted online optimization of the shape of the optimal steady-state set around the desired reference. The ability of our approach to handle multiplicative uncertainty, coupled with the removal of the requirement for precomputing a terminal set, renders it apt for designing tracking controllers for practical systems that can

be represented by linear parameter-varying (LPV) models [7]. Furthermore, the scheme can seamlessly incorporate model refinement while preserving asymptotic stability, making it well-suited for the development of adaptive MPC schemes [2].

The paper is organized as follows: Section II introduces the problem formulation. Section III presents the tracking CCTMPC formulation as well as a novel terminal cost and related theoretical guarantees regarding recursive feasibility in the presence of reference changes, robustness, and stability, as summarized in Theorem 1 and Corollary 1. Numerical examples are presented in Section IV. Section V concludes the article.

Notation: The symbol $\mathbb{I}$ represents the identity matrix. For vectors a and b, the symbol (a, b) denotes $[a^\top \; b^\top]^\top$. A block-diagonal matrix formed by matrices $Q_1, \ldots, Q_n$ is denoted by $\mathrm{blkd}(Q_1, \ldots, Q_n)$. The symbol convh denotes the convex hull, and we define $\|z\|_Q^2 := z^\top Q z$.

II. Tube Model Predictive Control

This section introduces the necessary notation for our reference-tracking Tube MPC formulation. Here, we focus on Configuration-Constrained polytopic tube parameterization, as originally introduced in [23].

A. Uncertain Linear Systems

This paper considers uncertain systems of the form

$$x^+ = Ax + Bu + w \quad \text{with} \quad z = Cx + Du, \quad (1)$$

where $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$, and $z \in \mathbb{R}^{n_z}$ denote the state, input and output respectively, and $w \in \mathcal{W}$ an additive disturbance. The set $\mathcal{W}$ is compact and convex. While C and D are given matrices, A and B are only known to satisfy $(A, B) \in \Delta := \mathrm{convh}(\{(A_1, B_1), \ldots, (A_p, B_p)\})$. The system is subject to closed and convex constraints $x \in \mathcal{X}$ and $u \in \mathcal{U}$. Our aim is to design an MPC scheme that robustly steers the output z towards a piecewise constant reference r that may be updated online.

B. Robust Forward Invariant Tubes

Tube MPC methods are based on the construction of Robust Forward Invariant Tubes (RFITs). Let

$$\mathcal{F}(X) := \left\{ X^+ \subseteq \mathbb{R}^n \;\middle|\; \begin{array}{l} \forall\, x \in \mathcal{X}, \exists\, u \in \mathcal{U}: \\ Ax + Bu + w \in X^+, \\ \forall\, (A, B) \in \Delta, w \in \mathcal{W} \end{array} \right\}$$

denote the set of one-step forward reachable sets from a given set $X \subseteq \mathbb{R}^{n_x}$. A sequence $(X_t \subseteq \mathbb{R}^{n_x})_{t \in \mathbb{N}}$ of sets forms an RFIT if for all $t \in \mathbb{N}$, $X_{t+1} \in \mathcal{F}(X_t)$ holds. If $X_s \in \mathcal{F}(X_s)$, then the set X_s is robust control invariant (RCI) since X_s contains its forward reachable set [1].

C. Configuration-Constrained Polytopic Tubes

Our focus is on polytopic RFITs defined as

$$X(y_t) := \{x \in \mathbb{R}^{n_x} \mid Fx \leq y_t\}. \quad (2)$$

Here, $F \in \mathbb{R}^{m \times n_x}$ is a given matrix whose rows encode the facet normals of $X(y_t)$. We enforce *configuration-constraints*

$y_t \in \mathcal{E} := \{y \in \mathbb{R}^m \mid Ey \leq 0\}$ for a given matrix E. Such constraints fix the facial configuration of the parametric polytopes $X(y_t)$ enabling a joint hyperplane-vertex parameterization of the form

$$X(y) = \mathrm{convh}(\{V_j y \in \mathbb{R}^{n_x} \mid j \in \{1, \ldots, v\}\}); \quad (3)$$

see [23, Sec. III.5] for a construction of matrices E and $V_j \in \mathbb{R}^{n_x \times m}$. We define $U_j = e_j \otimes \mathbb{I}_v$, stacking the vertex control inputs as $u_t := (u_{1,t}, \ldots, u_{v,t})$, such that $u_{j,t} = U_j u_t$. Let $\mathbb{S} \subseteq \mathbb{R}^m \times \mathbb{R}^{v \cdot n_u} \times \mathbb{R}^m$ be defined as

$$\mathbb{S} := \left\{ (y, u, y^+) \;\middle|\; \begin{array}{l} \forall\, (i, j) \in \{1, \ldots, p\} \times \{1, \ldots, v\}, \\ F(A_i V_j y + B_i U_j u) + d \leq y^+, \\ Ey \leq 0, \; V_j y \in \mathcal{X}, \; U_j u \in \mathcal{U} \end{array} \right\}, \quad (4)$$

with $d_i := \max\{Y_i w : w \in \mathcal{W}\}$ for all $i \in \{1, \ldots, m\}$.

Proposition 1: Let $(y_t \in \mathbb{R}^m)_{t \in \mathbb{N}}$ be such that for all $t \in \mathbb{N}$, there exists some $u_t \in \mathbb{R}^{v \cdot n_u}$ such that $(y_t, u_t, y_{t+1}) \in \mathbb{S}$. Then, $(X(y_t))_{t \in \mathbb{N}}$ is an RFIT.

Proof: The result follows from [23, Corollary 4]. ∎

Remark 1: The method from [11] identifies m-dimensional faces of the polytope $\{(x, y) \mid Fx - y \leq 0\}$ whose projection onto the y-coordinate space characterizes regions where the vertices of $X(y)$ depend linearly on y. Exploring its relationship with configuration constraints is a future research direction.

D. Optimal Robust Control Invariant Set

We design a tube MPC scheme that computes RFITs which converge to an optimal RCI set for a given reference. By the definition of $\mathbb{S}$, $X(y)$ is an RCI set whenever $(y, u, y) \in \mathbb{S}$ for some $u \in \mathbb{R}^{v \cdot n_u}$. For a given reference r, we define an optimal RCI set as one whose parameters (y, u) minimize a given cost $\ell(y, u, r)$ over $\mathbb{S}$. As an example of such a cost, consider the function

$$\ell_1(y, u) = \sum_{j=1}^{v} \left\| \left((\overline{V} - V_j) y, \; (\overline{U} - U_j) u \right) \right\|_{Q_v}^2,$$

which penalizes the distance of the vertices of $X(y)$ and the vertex control inputs to their centers,

$$\overline{V} y = \frac{1}{v} \sum_{j=1}^{v} V_j y \quad \text{and} \quad \overline{U} u = \frac{1}{v} \sum_{j=1}^{v} U_j u. \quad (5)$$

Now, let $\overline{A}$ and $\overline{B}$ be the average of matrices A_i and B_i over $i = 1, \ldots, p$ respectively. Assuming stabilizability of the system $x^+ = \overline{A}x + \overline{B}u$, let the columns of $M \in \mathbb{R}^{(n_x + n_u) \times n_u}$ be a basis for the nullspace of the matrix $[\overline{A} - \mathbb{I} \; \overline{B}]$. Recalling that every steady state of this system can be expressed as $(x_{ss}, u_{ss}) = M\theta$, the function

$$\ell_2(y, u, r, \theta) := \left\| \begin{bmatrix} \overline{V} & 0 \\ 0 & \overline{U} \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} - M\theta \right\|_{Q_c}^2 + \left\| r - [C \; D] M\theta \right\|_{Q_r}^2$$

penalizes the distance of the center $(\overline{V}y, \overline{U}u)$ to the steady state $M\theta$ and the distance of its output to the reference r. We can now define a function ℓ given by

$$\ell(y, u, r) := \min_{\theta} \left(\ell_1(y, u, r) + \ell_2(y, u, r, \theta) \right),$$

with Q_v, Q_c, and Q_r being positive semidefinite matrices of appropriate dimensions such that ℓ is a convex quadratic function. In conclusion, for a given reference r, the set $X(y_o)$ is an optimal RCI set with vertex control inputs u_o, if (y_o, u_o) is a minimizer of

$$\mathcal{C}_o := \min_{y,u} \; \ell(y, u, r) \quad \text{s.t.} \quad (y, u, y) \in \mathbb{S}. \tag{6}$$

Assumption 1: There exists a pair (y, u) satisfying $(y, u, y) \in \mathbb{S}$, such that $X(y)$ is an RCI set.

Problem Statement: Design a CCTMPC scheme for reference tracking that guarantees (*i*) recursive feasibility for changing references; and (*ii*) robustness, stability, and convergence to the optimal RCI set of the current reference for piecewise constant references.

E. Tube MPC With Stabilizing Initial and Terminal Costs

For a fixed reference r, one can design a CCTMPC scheme based on the optimization problem

$$\min_{\mathbf{y},\mathbf{u}} \quad L_0(y_0, r) + \sum_{k=0}^{N-1} \ell(y_k, u_k, r) + L_N(y_N, r)$$
$$\text{s.t.} \quad Fx \le y_0, \quad y_N \in \mathbb{X}_N, \tag{7}$$
$$(y_k, u_k, y_{k+1}) \in \mathbb{S}, \quad \forall \; k \in \{0, \dots, N-1\}.$$

We use the bold symbols $\mathbf{y}$ and $\mathbf{u}$ to denote the vectors stacking the tube parameters $y_0, \dots, y_N$ and vertex inputs $u_0, \dots, u_N$. The state and reference measurements are x and r, the initial and terminal costs L_0 and L_N, and the terminal region $\mathbb{X}_N$. In [23], a construction of L_0, L_N and $\mathbb{X}_N$ to guarantee asymptotic stability is presented. In general (7) is an economic formulation as the cost, while convex, is not positive definite. While this formulation yields a stabilizing controller for a *fixed* reference r, adapting it to track changing references is challenging since L_0, L_N and $\mathbb{X}_N$ are constructed using the solution of Problem (6) reformulated as

$$\min_{y,u,y^+} \; \ell(y, u, r) \quad \text{s.t.} \quad (y, u, y^+) \in \mathbb{S}, \; y = y^+|\lambda. \tag{8}$$

These ingredients depend on the primal-dual solution $(y_o, u_o, y_o, \lambda_o)$, which in turn depends on the reference r. Therefore, to address the challenges highlighted in the *Problem Statement*, a control scheme would require one module for solving Problem (8) to formulate L_0, L_N and $\mathbb{X}_N$ whenever a change in the reference is detected, and another for solving Problem (7). While such a scheme could operate the system safely, its cumbersome implementation can be unwieldy in practice. We address this using a reference tracking CCTMPC formulation that eliminates the need for reformulating the underlying problem when a reference change occurs.

Remark 2: Elastic Tube-MPC (ETMPC) [19] formulations can also be used to construct RFITs by optimizing y_t. However, such RFITs can be more conservative than those computed using CCTMPC, since CCTMPC uses vertex control laws to induce invariance instead of affine control laws. See [23] for further details.

Remark 3: The methods in [5], [13], etc., can be used to compute matrix F of desired complexity verifying Assumption 1.

III. TUBE MPC FOR REFERENCE TRACKING

We now present a CCTMPC formulation that directly tackles the challenges posed in the *Problem Statement*.

A. Reference-Tracking CCTMPC

Problem (7) necessitates the use of a rotated stage cost due to presence of economic stage cost terms in the objective. Alternatively, we adopt a convex tracking cost $s(y - y_s, u - u_s)$ that vanishes at the origin and remains positive for all other points. For instance, we can define $s(z, v) = \|(z, v)\|_Q^2$ using a positive definite matrix Q. Next, we formulate the optimization problem

$$\min_{\mathbf{y},\mathbf{u},y_s,u_s} \quad \ell(y_s, u_s, r) + \sum_{k=0}^{N-1} s(y_k, y_s, u_k, u_s) + \mathrm{M}(y_N, y_s, u_s)$$
$$\text{s.t.} \quad Fx \le y_0, \quad (y_s, u_s, y_s) \in \mathbb{S}, \quad y_N \in \mathbb{T}(y_s, u_s), \tag{9}$$
$$(y_k, u_k, y_{k+1}) \in \mathbb{S}, \quad \forall \; k \in \{0, \dots, N-1\},$$

where the variables (y_s, u_s) are enforced to represent an RCI set $X(y_s)$, and the tracking cost penalizes the deviation between (y_k, u_k) and (y_s, u_s). We now design a terminal cost $\mathrm{M}(y, y_s, u_s)$ and set $\mathbb{T}(y_s, u_s)$ that guarantee recursive feasibility and asymptotic stability.

B. Terminal Ingredients Design

As a preparation for designing the terminal cost, we define the cost-to-travel function [22]

$$\mathrm{V}(y, y^+, y_s, u_s) := \min_u \quad \|(y - y_s, u - u_s)\|_Q^2 \tag{10}$$
$$\text{s.t.} \quad (y, u, y^+) \in \mathbb{S},$$

for fixed (y_s, u_s). We define $\mathrm{V}(y, y^+, y_s, u_s) = \infty$ if (10) is infeasible. Since the definition of V is based on a tracking term, we can enforce asymptotic stability in the sequel without introducing an initial cost. Denoting

$$\mathcal{V}_N(\mathbf{y}, y_s, u_s) := \sum_{k=0}^{N-1} \mathrm{V}(y_k, y_{k+1}, y_s, u_s),$$

Problem (9) can be equivalently written as

$$\min_{\mathbf{y},y_s,u_s} \quad \ell(y_s, u_s, r) + \mathcal{V}_N(\mathbf{y}, y_s, u_s) + \mathrm{M}(y_N, y_s, u_s)$$
$$\text{s.t.} \quad Fx \le y_0, \quad y_N \in \mathbb{T}(y_s, u_s), \quad (y_s, u_s, y_s) \in \mathbb{S}. \tag{11}$$

For some $\gamma \in [0, 1)$, we define the terminal cost as

$$\mathrm{M}(y, y_s, u_s) := \min_u \quad \|(y - y_s, u - u_s)\|_P^2 \tag{12}$$
$$\text{s.t.} \quad (y, u, \gamma y + (1 - \gamma)y_s) \in \mathbb{S},$$

where P is a positive definite matrix. We show that for any constant $\gamma \in [0, 1)$, $\mathrm{M}(y, y_s, u_s)$ satisfies the Lyapunov descent condition inside the implicit terminal set $\mathbb{T}(y_s, u_s) := \{y \mid \mathrm{M}(y, y_s, u_s) < \infty\}$.

Theorem 1: If Assumption 1 holds, then for any (y_s, u_s) satisfying $(y_s, u_s, y_s) \in \mathbb{S}$, $M(y, y_s, u_s)$ is positive definite at $y = y_s$. Moreover, if the matrix inequality $Q + \gamma^2 P \preceq P$ holds, then for any $\bar{y} \in \mathbb{T}(y_s, u_s)$, functions M and V satisfy the Lyapunov descent condition

$$\min_{y^+} \left\{ M(y^+, y_s, u_s) + V(\bar{y}, y^+, y_s, u_s) \right\} \leq M(\bar{y}, y_s, u_s). \quad (13)$$

Proof: Under Assumption 1, there exist (y_s, u_s) satisfying $(y_s, u_s, y_s) \in \mathbb{S}$. Then, positive definiteness of M holds since the objective of Problem (12) is positive definite at $(\mathbf{0}, \mathbf{0})$. Regarding the descent condition, denoting the minimizer of (12) at $y = \bar{y}$ by $\bar{u}$, we have

$$M(\bar{y}, y_s, u_s) = \|(\bar{y} - y_s, \bar{u} - u_s)\|_P^2. \quad (14)$$

The points $\bar{y}^+ = \gamma \bar{y} + (1 - \gamma) y_s$ and $u = \bar{u}$ are feasible for Problem (10) at $y^+ = \bar{y}^+$. Hence,

$$V(\bar{y}, \bar{y}^+, y_s, u_s) \leq \|(\bar{y} - y_s, \bar{u} - u_s)\|_Q^2. \quad (15)$$

If $Q + \gamma^2 P \preceq P$ holds, then (14) and (15) along with

$$M(\bar{y}^+, y_s, u_s) \leq \gamma^2 \|(\bar{y} - y_s, \bar{u} - u_s)\|_P^2 \quad (16)$$

Simply (13). To show that (16) holds, observe that

$$F(A_i V_j y_s + B_i U_j u_s) + d \leq y_s, \quad (17)$$
$$F(A_i V_j \bar{y} + B_i U_j \bar{u}) + d \leq \gamma \bar{y} + (1 - \gamma) y_s$$

follow from (4). With $\bar{u}^+ := \gamma \bar{u} + (1 - \gamma) u_s$, (17) implies

$$F(A_i V_j \bar{y}^+ + B_i U_j \bar{u}^+) + d \leq \gamma \bar{y}^+ + (1 - \gamma) y_s. \quad (18)$$

Since $u = \bar{u}^+$ is feasible for (12) with $y = \bar{y}^+$, (18) implies $M(\bar{y}^+, y_s, u_s) \leq \|(\bar{y}^+ - y_s, \bar{u}^+ - u_s)\|_P^2$ such that $(\bar{y}^+ - y_s, \bar{u}^+ - u_s) = \gamma(\bar{y} - y_s, \bar{u} - u_s)$ implies (16). ∎

In summary, Problem (9) is implemented as

$$\min_{\mathbf{y}, \mathbf{u}, y_s, u_s} \quad \ell(y_s, u_s, r) + \sum_{k=0}^{N-1} \left\| \begin{bmatrix} y_k - y_s \\ u_k - u_s \end{bmatrix} \right\|_Q^2 + \left\| \begin{bmatrix} y_N - y_s \\ u_N - u_s \end{bmatrix} \right\|_P^2$$

$$\text{s.t.} \quad Fx \leq y_0, \quad (y_s, u_s, y_s) \in \mathbb{S},$$
$$(y_k, u_k, y_{k+1}) \in \mathbb{S}, \quad \forall k \in \{0, \ldots, N-1\},$$
$$(y_N, u_N, \gamma y_N + (1 - \gamma) y_s) \in \mathbb{S}. \quad (19)$$

C. Closed-Loop Control Law

Denoting the parametric minimizers of (19) as $\mathbf{y}^*(x, r)$, $\mathbf{u}^*(x, r)$, $y_s^*(x, r)$ and $u_s^*(x, r)$, the control law is constructed as a convex combination of the vertex control inputs as

$$\mu(x, r) := \sum_{j=1}^{v} \lambda_j(x, r) u_{0,j}^*(x, r), \quad (20)$$

where $\lambda(x, r) \in \mathbb{R}^v$ solves the QP

$$\min_{\lambda \geq 0} \|\lambda\|_2^2 \quad \text{s.t.} \quad x = \sum_{j=1}^{v} \lambda_j V_j y_0^*(x, r), \quad \sum_{j=1}^{v} \lambda_j = 1.$$

Then, the closed loop system associated with dynamics (1) and reference sequence $(r_t)_{t \in \mathbb{N}}$ is $x_{t+1} = A_t x_t + B_t \mu(x_t, r_t) + w_t$, with output $z_t = Cx_t + D\mu(x_t, r_t)$.

We now present recursive feasibility and stability properties of the control scheme. We denote the optimal values of (6) and (19) by $\mathcal{C}_o(r)$ and $\mathcal{C}^*(x, r)$, and the minimizer of (6) by $(y_o(r), u_o(r))$ emphasizing the dependence of the optimal RCI set parameters on the reference r.

Assumption 2: The function ℓ defining the optimal RCI set in Problem (6) is strictly convex.

Corollary 1: Suppose Assumption 2 holds. If $Q \succ 0$, $P \succ 0$, $\gamma \in [0, 1)$ satisfy $Q + \gamma^2 P \preceq P$, then the following statements hold for any reference $(r_t)_{t \in \mathbb{N}}$, independent of $w_t \in \mathcal{W}$ and $(A, B)_t \in \Delta$, $t \in \mathbb{N}$:

(i) For the control law (20) with $(x, r) = (x_t, r_t)$, (19) is feasible $\forall t \in \mathbb{N}$ and $r_t \in \mathbb{R}^{n_z}$ assuming feasibility at $t = 0$;

(ii) For a constant reference $r_t = r$, $\mathcal{C}^*(x_t, r) \geq \mathcal{C}_o(r)$ and $\lim_{t \to \infty} \mathcal{C}^*(x_t, r) = \mathcal{C}_o(r)$. Moreover, $\mathcal{L}(\mathbf{y}^*(x, r), y_s^*(x, r), u_s^*(x, r), r) := \mathcal{C}^*(x, r) - \mathcal{C}_o(r)$ is a strictly descending Lyapunov function, and the minimizers converge to the optimal RCI parameters as $\lim_{t \to \infty} (y_k^*(x_t, r), u_k^*(x_t, r)) = \lim_{t \to \infty} (y_s^*(x_t, r), u_s^*(x_t, r)) = (y_o(r), u_o(r))$, $\forall k \in \{0, \ldots, N\}$.

Proof: (i) Suppose Problem (19) is feasible at some $t \in \mathbb{N}$. Denoting $\tilde{y} := \gamma y_N^*(x_t, r_t) + (1 - \gamma) y_s^*(x_t, r_t)$ and $\tilde{u} := \gamma u_N^*(x_t, r_t) + (1 - \gamma) u_s^*(x_t, r_t)$, Theorem 1 implies $\mathbf{y} = [y_1^*(x_t, r_t), \ldots, y_N^*(x_t, r_t) \tilde{y}]$, $\mathbf{u} = [u_1^*(x_t, r_t), \ldots, u_N^*(x_t, r_t) \tilde{u}]$, $y_s = y_s^*(x_t, r_t)$, $u_s = u_s^*(x_t, r_t)$ are feasible with $x = x_{t+1}$ and any $r = r_{t+1}$, since r affect only the objective function;

(ii) If $y_0^* \neq y_s$, we have from (10) that $V(y_0^*(x_t, r), y_1^*(x_t, r), y_s(x_t, r), u_s(x_t, r)) > 0$. Since (11) and (19) are equivalent, the value of (11) with the feasible solution in (i) is strictly less than $\mathcal{C}^*(x_t, r)$. This implies that the tracking components of (11) are driven to 0, such that $\mathcal{C}^*(x_t, r)$ approaches $\mathcal{C}_o(r)$ from above and $\mathcal{L}$ is a Lyapunov function. The minimizers converge since Assumption 2 implies (6) is strictly convex. ∎

Remark 4: The study of equivalence of the Economic MPC problem (7) and the Tracking MPC problem (9) could be an interesting topic for future research.

IV. NUMERICAL EXAMPLES

We present two case studies to demonstrate the efficacy of the proposed CCTMPC scheme. In both cases, the terminal cost in (12) is computed with $\gamma = 0.95$. A MATLAB library for implementing CCTMPC and running these examples can be found on GitHub.[1]

A. Illustrative Example

We consider the open-loop unstable LTI system

$$x^+ = \begin{bmatrix} 1.1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u + w, \quad z = [1 \ 0] x, \quad (21)$$

with constraints $\mathcal{X} = \{x : (-5, -2) \leq x \leq (5, 3)\}$, $\mathcal{U} = \{u : -1 \leq u \leq 2\}$ and $\mathcal{W} = \{w : |w| \leq (0, 0.5)\}$. To design a tracking CCTMPC controller for this system, we construct matrix F in (2) with $m = v = 12$ as in [23, Remark 3], and define ℓ in (6) with $Q_v = \text{blkd}(10\mathbb{I}_2, 1)$, $Q_c = \text{blkd}(\mathbb{I}_2, 1)$ and $Q_r = 10^2$. We use $Q = \sum_{k=1}^{v} V_k^\top Q_v V_k + 10^{-3} \mathbb{I}_{32}$ and
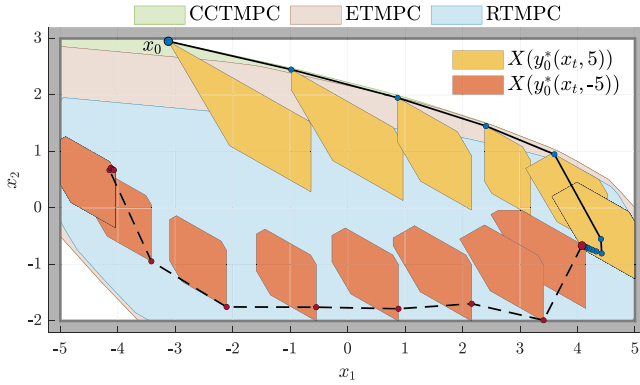
[1] https://github.com/samku/CCTMPC_tracking

Fig. 1. Closed-loop results for system (21). Feasible regions of CCTMPC, RTMPC from [10], and ETMPC from [17], are compared. The reference is $r_t = 5$ for $t \in [0, 14]$ and $r_t = -5$ for $t \geq 15$. Closed-loop tubes and system trajectory are plotted for $x_0 = (-3.12, 2.95)$].

$P = (1 - \gamma^2)^{-1}Q$, where $\mathbf{V}_k := [\overline{V} - V_k \; \overline{U} - U_k]$, such that $Q + \gamma^2 P \preceq P$. In Figure 1, we plot the closed-loop results using this controller for $N = 5$. In Figure 3, the Lyapunov function $\mathcal{L}_t := \mathcal{L}(\mathbf{y}^*(x_t, r_t), y_s^*(x_t, r_t), u_s^*(x_t, r_t), r_t)$ is shown.

Figure 1 compares the feasible regions of our CCTMPC, ETMPC [17] modified for tracking as in [10], and tracking RTMPC [10] controllers. In order to parameterize the RFITs, the ETMPC controller uses the set $X(y)$, and the RTMPC uses the fixed RCI set $X(y_o(0))$. ETMPC induces invariance in the minimal parameterized RPI set with the LQR controller with $Q = \mathbb{I}_2$ and $R = 1$, and both schemes use the maximal positive invariant set [1] corresponding to their respective systems as the terminal set. CCTMPC exhibits a larger feasible region: its Hausdorff distance to the maximal RCI set is 0.0179, compared to 0.2816 for ETMPC and 1.1056 for RTMPC. Hence, we can control the system from states close to the boundary of the MRCI set with a short horizon of $N = 5$ with our tracking CCTMPC controller.

B. Lane Change Maneuver

We now focus on the lateral vehicle dynamics model [13, Sec. V-C],

$$x^+ = \left(A_0 + v^x A_1 + \left(v^x\right)^{-1} A_2\right)x + Bu + B_w w, \quad (22)$$

with $x = (e_y, e_\psi, \dot{y}, \dot{\psi})$, $u = (\delta_s, \mu_b)$ and $z = e_y$.

e_y [m] is the lateral deviation, e_ψ [rad] the angular deviation, $\dot{y}$ [m/s] the lateral velocity, $\dot{\psi}$ [rad/s] the yaw rate, δ_s [rad] the steering angle, and μ_b [N·m] the braking yaw moment. The disturbance is $w \in [0, 100]$, and $v^x \in [40, 65]/3.6$ [m/s] is the longitudinal velocity.

Treating (22) as an LPV system with parameters $(v^x, 1/v^x)$ allows us to capture its behavior as a linear system with multiplicative uncertainty. By defining the parameter set $\mathcal{P}(v_t^x) := \{(v^x, 1/v^x) : v^x \in [v_t^x, 65]/3.6\}$, we can derive the matrices (A_i, B_i) that characterize the system's multiplicative uncertainty using the vertices of $\hat{\mathcal{P}} \supset \mathcal{P}(40)$, where

$$\hat{\mathcal{P}} := \text{convh}\left\{\begin{bmatrix} 11.111 \\ 0.090 \end{bmatrix}, \begin{bmatrix} 18.056 \\ 0.055 \end{bmatrix}, \begin{bmatrix} 11.111 \\ 0.086 \end{bmatrix}, \begin{bmatrix} 17.217 \\ 0.055 \end{bmatrix}\right\}.$$
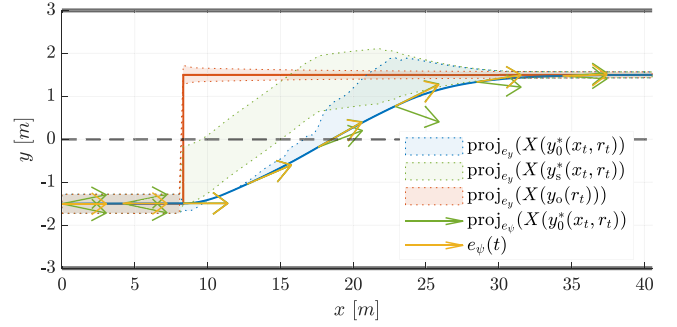

Fig. 2. Closed-loop spatial evolution of system (22).

System constraints are $\mathcal{U} = \{u : |u| \leq (10\pi/180, 2)\}$ and $\mathcal{X} = \{x : |x| \leq (3, 4, 20\pi/180, 3)\}$. Matrix F in (2) is computed using the approach in [13, Sec. IV-D]; therefore, $F = \hat{S}T^{-1}$, where $\hat{S} = [\mathbf{I}^4 \; -\mathbf{I}^4]^\top$ and

$$T = \begin{bmatrix} 0.0712 & -0.2667 & 0.2667 & 0.0712 \\ 2.5077 & 0 & 0 & -1.4923 \\ -0.0718 & -0.0024 & -0.1472 & -0.0011 \\ 0 & 1.1729 & 1.8271 & 0 \end{bmatrix},$$

resulting in $m = 8$ and $v = 16$. The cost ℓ is defined with $Q_v = \text{blkd}(10^{-3}\mathbb{I}_4, 10^{-2}\mathbb{I}_2)$, $Q_c = \text{blkd}(10^2\mathbb{I}_4, 10^{-2}\mathbb{I}_2)$ and $Q_r = 10$, while we set $Q = \text{blkd}(10^{-3}\mathbb{I}_8, 10^{-2}\mathbb{I}_{32})$ and $P = (1 - \gamma^2)^{-1}Q$. For simulation, we simplify the "merge into faster lane" maneuver [21], where a vehicle shifts from a slow lane ($r = -1.5$, minimum speed 40/3.6 [m/s]) to a fast lane ($r = 1.5$, minimum speed 60/3.6 [m/s]), during which the parameter v_x is non-decreasing, resulting in a decreasing multiplicative uncertainty over time, as $\mathcal{P}(v_{t_2}^x) \subseteq \mathcal{P}(v_{t_1}^x)$ for $t_2 \geq t_1$. The tracking CCTMPC scheme can seamlessly account for this uncertainty reduction: by recomputing the matrices (A_i, B_i) at each time step using the vertices of $\mathcal{P}(v_t^x) \subset \hat{\mathcal{P}}$, we still ensure stability.

In Figure 2 we illustrate the closed-loop dynamics using the aforementioned online model adaptation, where the CCTMPC controller is formulated with $N = 3$. To visualize the set dynamics, we project $X(y_0^*(x_t, r_t))$ along e_y (blue tube) and e_ψ (angle between green arrows) states, in accordance with the spatial evolution of the system. We also show the projection of $X(y_s^*(x_t, r_t))$ along e_y (green tube) as well as the projection of $X(y_o(r_t))$ for e_y (red tube). The yellow arrow denotes $e_\psi(t)$ — sampled every 15 time steps — which consistently resides within the corresponding polytope projection. By explicitly accounting for reduction in multiplicative uncertainty upon moving into the fast lane, we reduce the uncertainty in the tubes as can be observed from a reduction in their width. The Lyapunov function $\mathcal{L}_t$ is shown in Figure 3.

C. Computational Aspects

The CCTMPC scheme has $(N + 1)v[m(1 + p) + n_{\mathcal{X}} + n_{\mathcal{U}}] + m$ inequality constraints and $(N + 1)(m + v \cdot n_u)$ optimization variables, where $n_{\mathcal{X}}$ and $n_{\mathcal{U}}$ are the number of halfspaces defining polytopes $\mathcal{X}$ and $\mathcal{U}$. As highlighted in [23] [Remark 2], the matrix E is typically sparse and contains redundant rows, which can be reduced using an LP solver.
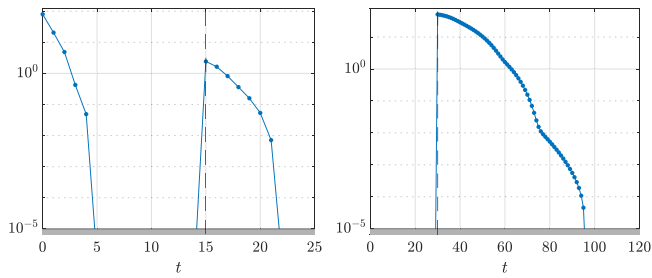
Trade-off between complexity of the RFIT (hence reduction in conservativeness over prediction steps) and computational burden must be taken into account in real world applications, which can be mitigated by means of high-performance QP solvers [4].

V. CONCLUSION

This paper has introduced a novel CCTMPC formulation for tracking piecewise constant reference signals. The MPC scheme is based on the implementation of a single convex QP that simultaneously computes an RFIT and a target invariant set. Moreover, Theorem 1 and Corollary 1 have established the fact that the RFIT converges to the optimal RCI set for each constant reference, while maintaining recursive feasibility and stability guarantees in the presence of changing references. Finally, our numerical results have not only illustrated the practical applicability of the proposed CCTMPC scheme, but also its benefits compared to existing rigid- and elastic-tube MPC methods for reference tracking.

REFERENCES

[1] F. Blanchini and S. Miani, *Set-Theoretic Methods in Control*. Boston, MA, USA: Birkhäuser, 2015.
[2] M. Bujarbaruah, X. Zhang, H. E. Tseng, and F. Borrelli, "Adaptive MPC for autonomous lane keeping," 2018, *arXiv:1806.04335*.
[3] J. Fleming, B. Kouvaritakis, and M. Cannon, "Robust tube MPC for linear systems with multiplicative uncertainty," *IEEE Trans. Autom. Control*, vol. 60, no. 4, pp. 1087–1092, Apr. 2015.
[4] G. Frison and M. Diehl, "HPIPM: A high-performance quadratic programming framework for model predictive control," *IFAC-PapersOnLine*, vol. 53, no. 2, pp. 6563–6569, 2020.
[5] A. Gupta, H. Köroğlu, and P. Falcone, "Computation of low-complexity control-invariant sets for systems with uncertain parameter dependence," *Automatica*, vol. 101, pp. 330–337, Mar. 2019.
[6] J. Hanema, M. Lazar, and R. Tóth, "Tube-based LPV constant output reference tracking MPC with error bound," *IFAC-Pap.*, vol. 50, no. 1, pp. 8612–8617, 2017.
[7] S. M. Hashemi, H. S. Abbas, and H. Werner, "Low-complexity linear parameter-varying modeling and control of a robotic manipulator," *Control Eng. Pract.*, vol. 20, no. 3, pp. 248–257, 2012.
[8] B. Houska and M. E. Villanueva, "Robust optimization for MPC," in *Handbook of Model Predictive Control*, S. V. Raković and W. Levine, Eds. Cham, Switzerland: Birkhäuser, 2019, pp. 413–443.
[9] W. Langson, I. Chryssochoos, S. V. Raković, and D. Q. Mayne, "Robust model predictive control using tubes," *Automatica*, vol. 40, no. 1, pp. 125–133, 2004.
[10] D. Limon, I. Alvarado, T. Alamo, and E. F. Camacho, "Robust tube-based MPC for tracking of constrained linear systems with additive disturbances," *J. Process Control*, vol. 20, no. 3, pp. 248–260, 2010.
[11] V. Loechner and D. K. Wilde, "Parameterized polyhedra and their vertices," *Int. J. Parallel Program.*, vol. 25, no. 6, pp. 525–549, 1997.
[12] D. Q. Mayne, M. M. Seron, and S. V. Raković, "Robust model predictive control of constrained linear systems with bounded disturbances," *Automatica*, vol. 41, no. 2, pp. 219–224, 2005.
[13] S. K. Mulagaleti, M. Mejari, and A. Bemporad, "Parameter dependent robust control invariant sets for LPV systems with bounded parameter variation rate," 2023, *arXiv:2309.02384*.
[14] K. R. Muske, "Steady-state target optimization in linear model predictive control," in *Proc. Am. Control Conf. (Cat. No.97CH36041)*, 1997, pp. 3597–3601.
[15] T. Peschke and D. Görges, "Robust tube-based tracking MPC for linear systems with multiplicative uncertainty," in *Proc. IEEE 58th Conf. Decis. Control (CDC)*, 2019, ,pp. 457–462.
[16] S. V. Raković, "The implicit rigid tube model predictive control," *Automatica*, vol. 157, Nov. 2023, Art. no. 111234.
[17] S. V. Raković, B. Kouvaritakis, and M. Cannon, "Equi-normalization and exact scaling dynamics in homothetic tube model predictive control," *Syst. Control Lett.*, vol. 62, no. 2, pp. 209–217, 2013.
[18] S. V. Raković, B. Kouvaritakis, R. Findeisen, and M. Cannon, "Homothetic tube model predictive control," *Automatica*, vol. 48, no. 8, pp. 1631–1638, 2012.
[19] S. V. Raković, W. S. Levine, and B. Açikmese, "Elastic tube model predictive control," in *Proc. Am. Control Conf. (ACC)*, 2016, pp. 3594–3599.
[20] J. B. Rawlings and D. Q. Mayne, *Model Predictive Control: Theory and Design*. Madison, WI, USA: Nob Hill Publ., 2009.
[21] G. Schildbach and F. Borrelli, "Scenario model predictive control for lane change assistance on highways," in *Proc. IEEE Intell. Veh. Symp. (IV)*, 2015, pp. 611–616.
[22] M. E. Villanueva, E. De Lazzari, M. A. Müller, and B. Houska, "A set-theoretic generalization of dissipativity with applications in tube MPC," *Automatica*, vol. 122, Dec. 2020, Art. no. 109179.
[23] M. E. Villanueva, M. A. Müller, and B. Houska, "Configuration-constrained tube MPC," *Automatica*, vol. 163, May 2024, Art. no. 111543.
[24] M. E. Villanueva, R. Quirynen, M. Diehl, B. Chachuat, and B. Houska, "Robust MPC via min–max differential inequalities," *Automatica*, vol. 77, pp. 311–321, Mar. 2017.