Enigmas, etc.

Solution to Last Month's Quiz

■ Takashi Ohira

Drawing an auxiliary line from 50 Ω to the revolving point Z, as shown in Figure 1, we find the center angle consistently twice the elevation angle because the triangle 0–50–Z is isosceles.

Because the arc maintains a constant radius of 50 Ω, the R and X coordinates of point Z are represented as

$$\begin{bmatrix} R \\ X \end{bmatrix} = 50 \begin{bmatrix} 1 + \cos 2\theta \\ \sin 2\theta \end{bmatrix}. \tag{1}$$

The angle derivative of (1) yields

$$\begin{bmatrix} dR \\ dX \end{bmatrix} = 100 \begin{bmatrix} -\sin 2\theta \\ \cos 2\theta \end{bmatrix} d\theta. \tag{2}$$

Hence, the Poincaré metric reduces to

$$\begin{aligned} d\Lambda &= \frac{1}{R}\sqrt{dR^2 + dX^2} \\ &= \frac{2}{1+\cos 2\theta} d\theta \\ &= \sec^2\theta \; d\theta. \end{aligned} \tag{3}$$

Integrating (3) along the arc, we obtain its total length as

$$\begin{aligned} \Lambda &= \int_{100}^{Z} d\Lambda \\ &= \int_{0}^{\theta} \sec^2\theta \; d\theta \\ &= \tan\theta. \end{aligned} \tag{4}$$

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Therefore, the correct answer to last month's puzzle is (d).

Physically, the upper semicircle in Figure 1 indicates a simple LR circuit depicted in Figure 2. Imagine that we sweep the stimulus frequency at the port. The circuit's input impedance Z begins from the 0 Ω origin point at dc and then revolves clockwise along the semicircle with frequency increment before finally reaching 100 Ω on the horizon at infinitely high frequency.

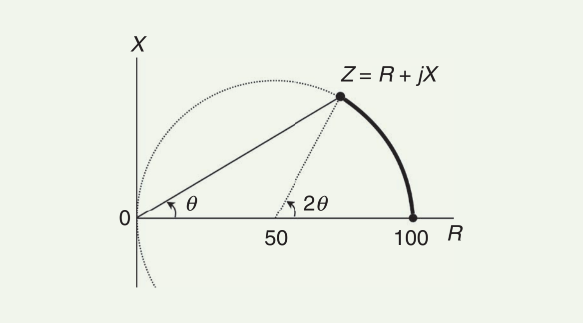

Figure 1. *The circular arc in question.*

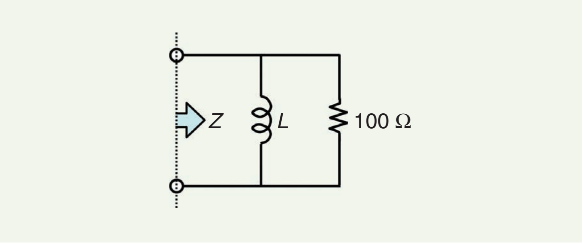

Figure 2. *A simple LR circuit to give a physical insight into the circular locus.*

Another physical aspect of the circular arc is that its Poincaré length (4) signifies the *quality factor*, or Q, of the complex impedance $Z = R + jX$. This can be proven easily because we know that the elevation slope

$$\tan\theta = \frac{X}{R} \tag{5}$$

is identical to Q. To be sure of what Q actually is, see reference [1]. In conclusion, regardless of the circle diameter, the arc length in the Poincaré metric physically implies Q of the parallel LR circuit.

Finally, appending a hyperbolic geometry is worthwhile: a circle osculating the X axis is known as a *horocycle* [2]. A horocycle on the R–X plane generally signifies the input impedance of a linear circuit that involves one variable reactor. The presented LR circuit is a lucid example where the constituent reactance ωL varies linearly with frequency sweep. Thus, this model is familiar to circuit engineers and also instructive to students for understanding the physical meaning of the horocycle.

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with polar coordinates. The Poincaré length of the locus is found to exhibit a modulus exchange factor multiplied by the electrical length of the transmission line, where the modulus exchange factor indicates how far the load resistance is located from the transmission line's characteristic impedance. For a summary, Table 2 provides a comprehensive look at the six loci along with their lengths measured in a Poincaré metric.

We hope that in academia the geometrical approach presented in this article encourages students to intuitively understand impedance and reflectance behavior in undergraduate and graduate classes, from basic circuit theory to advanced microwave technologies. The circuit schemes and their loci demonstrated here should also help mathematics instructors stimulate students' interest with vivid examples of how the Poincaréan study, despite being born two centuries ago, can work for modern practical engineering. Refer to [8] and [9] for further system applications of plane geometry.

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