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Optimal system, reductions and conservation laws of a nonlinear Damped Klein-Gordon-Fock equation

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ABSTRACT A detailed Lie symmetry analysis of the nonlinear damped Klein-Gordon Fock equation: $u_{tt} + \alpha(u)\, u_t = u_{xx} + f(u)$ is addressed in this paper. Applying the Lie symmetry method, a comprehensive Lie group classification is performed for the arbitrary smooth functions $\alpha(u)$ and $f(u)$ present in the equation, leading to two distinct cases. Additionally, for each case an optimal system of one-dimensional subalgebras is derived, which is a minimal set of all the linearly independent symmetry generators without redundant symmetries. Using the similarity transformation method, the above-mentioned partial differential equation is reduced into a set of ordinary differential equations. In certain cases, several exact invariant solutions encompassing the travelling wave solutions and soliton waves are obtained. The graphs of the soliton solutions and traveling wave solutions are also presented. Finally, the conservation laws are identified via the partial Noether approach, leading to distinct cases with several subcases. The derived conservation laws provide valuable tools for examining the dynamics and stability of physical systems, making this research suitable to a range of scientific studies.

INDEX TERMS Conservation laws, invariant solutions, Lie symmetry classification, mathematical model, optimal system

I. INTRODUCTION

Nonlinear partial differential equations find wide applications in modeling various phenomena in engineering, mathematics, quantum mechanics, and related fields. Some well-known partial differential equations (Pdes) are the Blasius equation, heat equation, Laplace equation, and the wave equation. Several studies have been undertaken in finding solutions of Pdes, such as the Navier-Stokes equation, nonlinear diffusion equation, and the Klein-Gordon-Fock equation in quantum theory, as well as others [1], [2].

In this paper, we consider a damped Klein-Gordon Fock (Kgf) equation, formulated as

$$u_{tt} + \alpha(u)\, u_t = u_{xx} + f(u), \tag{1}$$

here, the damping term $\alpha(u)\, u_t$ and the source function $f(u)$ imparts nonlinearity into the equation giving rise to solitons and complex wave dynamics. Moreover, damping leads to a gradual reduction in the amplitude of the wave due to energy dissipation and it is used to model the effects of viscosity, friction, or other dissipative processes that removes energy from a system, resulting in eventual reduction or stabilization of the system. The inclusion of a damping in Pdes has significant implications for the behaviour of the solutions, for instance, it can lead to stable and well-behaved solutions in some cases, while in some others, it may cause oscillations to diminish over time. The combined effects of nonlinearity and damping can result in interesting wave dynamics, including the formation of solitons, wave steepening, wave breaking, or other types of nonlinear wave patterns.

Equation (1) represents a second-order equation in both space and time, comes under the class of nonlinear evolution equations. It models the dynamics of scalar particles in quantum theory. This equation has extensive applications in non-linear dynamics, wave propagation, and quantum theory [3]. Also, the nonlinear damped Kgf equation is significant in applied mathematics as it acts as a test tool for the devel-

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opment of semi-analytical and analytical methods, to study the stability, uniqueness and existence of solutions, as well as construction of numerical methods to understand the more complex nonlinear Pdes [4].

Obtaining solutions of nonlinear differential equations holds paramount importance in nonlinear science, and the Lie symmetry method is a powerful algebraic technique for this purpose. It not only deals with the general class of nonlinear equations but also efficiently identifies reductions, invariant solutions, and simpler forms of these equations. The Lie symmetry method and prolongation formulas are proposed in [5], [6], for conducting symmetry analysis of differential equations. The application of Lie symmetry method to the nonlinear damped Kgf equation allows for a construction of its Lie symmetries. These symmetries lead to the reduction of independent variables or reduces the complexity of the equation, which in turn helps identifying the invariant solutions, and derive conservation laws. These invariant solutions and conservation laws offer complete insight into the properties and dynamics described by the equation.

Various tools, involving numerical and analytical techniques, have been developed to solve such nonlinear Pdes [7]. Liu *et al.* proposed an appropriate and efficient method for obtaining approximate solutions to problems related to wave propagation. The proposed method generates algebraic results that remain robust against discretizations [8]. Employing the Nucci's reduction and new extended direct algebraic methods Faridi *et al.* derived the solitary waves and exact solutions of Kuralay equation [9]. Subsequently, Hosseini *et al.* applied the Lie symmetry method to obtain the reductions and exact solutions of Kodama equation, alongside performing bifurcation and sensitivity analyses on its derived dynamical system [10]. Moreover, Hosseini *et al.* studied the generalized Kadomtsev–Petviashvili equation and obtained its positive multi-complexiton solution [11]. To study the wave propagation in high-frequency, Huang proposed the Gaussian Airy beam approximation [12], [13]. By employing the $(\phi'/\phi,\, 1/\phi)-$expansion method Dey *et al.* derived the soliton solutions of a generalized $(3+1)-$dimensional shallow water-like equation [14]. Additionally, Akbar *et al.* [15] discussed the dynamics of solitons of the perturbed nonlinear Schrodinger equation using the generalized Kudryashov scheme.

The symmetry group properties of a nonlinear wave equation with arbitrary function was, e.g., carried out by Ames *et al.* [16]. The group theoretic approach for the classification of nonlinear differential equations was earlier presented by Ovsiannikov [5]. The method of Lie symmetry rooted in group theory is quite systematic and effective for solving linear as well as nonlinear differential equations and works without making early guesses and approximations. The method is feasible for reducing the dimension of the differential equations. For a detailed exposure to this method the reader is referred to the well-known books by Ovsiannikov [5], Bluman and Kumei [6], Ibragimov [17] and Olver [18]. The soliton solutions of the 3-dimensional nonlinear Klein-Fock equation was proposed by Tajiri *et al.* [19]. They reduced it to $2D$ Klein Gordon equation and subsequently to Odes by similarity transformations to seek solition solutions. The exact solution as well as the complete Lie symmetry analysis of the damped wave equation was presented by Usamah *et al.* [20].

Moreover, conservation laws and optimal systems play central roles in Lie symmetry analysis as they are helpful in performing reductions of the differential equations. Furthermore, conservation laws are effective in order to achieve accuracy and existence of numerical solutions of nonlinear and linear Pdes.

Symmetries are also applicable in Finance (see, e.g., Mahomed *et al.* [21]). For nonlinear wave equations, the optimal system, reductions and conservation laws were recently provided by Raza *et al.* [22]. The relation between symmetries and conservation laws was studied via the port-representation by Nishida *et al.* [23]. Recent applications of group symmetry theory are found in machine learning, for instance, [24].

Other important aspects of Lie group analysis in differential equations is that of symmetry classification, optimal system, and reductions. The reader is referred to, e.g., Azad *et al.* [25] on the Kgf equation

$$u_{tt} = u_{xx} + f(u). \tag{2}$$

Azad *et al.* performed the Lie group classification, obtained optimal system and reductions of Equation (2). The present paper deals with the Lie group classification, optimal system, reductions, identification of some exact invariant solutions including soliton waves as well as travelling wave solutions, and determination of conservation laws for Equation (1), which features nonlinear damping and is a generalized version of Equation (2). In addition, the presence of the damping term changes the dynamics, overall analysis, symmetries, and solutions found in [25]. The considered Equation (1) can serve as a test tool in applied mathematical research for evaluating both analytical and numerical methods used for solving Pdes. Furthermore, by means of Lie symmetry analysis, Khalique *et al.* [26] analysed the nonlinear Kgf equation and obtained stationary solutions. To obtain the optimal system and invariant solutions, we have used the approaches presented in the known works [5], [6] and [17].

The outline of this paper is as follows: In Section 2, we obtain the Lie symmetry group classification of the damped Kgf Equation (1). Section 3 gives the adjoint table and the optimal systems. Also, the reduction of the damped Kgf equation to ordinary differential equations through optimal systems is given in the subsequent section 4. We also find invariant solutions by solving the reduced Odes. Section 5

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provides examples of soliton solutions of Equation (1). Additionally, this sections includes the graphical representation of these solutions. Section 6 includes the conserved quantities of the considered damped Kgf equation obtained via the partial Noether approach.

II. SYMMETRY GROUP CLASSIFICATION

This section finds the principal Lie symmetries and performs a group classification to identify particular forms of the arbitrary functions $\alpha(u)$ and $f(u)$. To derive the Lie symmetries of Equation (1), the one-parameter Lie group of point transformations in (u,t,x), is given by

$$\bar{x} = x + \varepsilon\,\xi^1(u,t,x) + O(\varepsilon^2),$$
$$\bar{t} = t + \varepsilon\,\xi^2(u,t,x) + O(\varepsilon^2),$$
$$\bar{u} = u + \varepsilon\,\varphi(u,t,x) + O(\varepsilon^2),$$

here, ε denotes the parameter in the group transformation. The operator of the transformation is represented as follows

$$\chi = \xi^1(u,t,x)\,\frac{\partial}{\partial x} + \xi^2(u,t,x)\,\frac{\partial}{\partial t} + \varphi(u,t,x)\,\frac{\partial}{\partial u}. \quad (3)$$

We require the second-order prolongation as the Equation (1) is of second order. This is well-known from the sources [5], [6] and given by

$$\chi^{[2]} = \chi + \varphi^x\frac{\partial}{\partial u_x} + \varphi^t\frac{\partial}{\partial u_t} + \varphi^{xt}\frac{\partial}{\partial u_{xt}} + \varphi^{xx}\frac{\partial}{\partial u_{xx}} + \varphi^{tt}\frac{\partial}{\partial u_{tt}}, \quad (4)$$

where

$$\varphi^x = \mathcal{D}_x\,\varphi - u_x\,\mathcal{D}_x\,\xi^1 - u_t\,\mathcal{D}_x\,\xi^2,$$
$$\varphi^t = \mathcal{D}_t\,\varphi - u_x\,\mathcal{D}_t\,\xi^1 - u_t\,\mathcal{D}_t\,\xi^2,$$
$$\varphi^{xt} = \mathcal{D}_t\,\varphi^x - u_{xx}\,\mathcal{D}_t\,\xi^1 - u_{xt}\,\mathcal{D}_t\,\xi^2,$$
$$\varphi^{xx} = \mathcal{D}_x\,\varphi^x - u_{xx}\,\mathcal{D}_x\,\xi^1 - u_{xt}\,\mathcal{D}_x\,\xi^2,$$
$$\varphi^{tt} = \mathcal{D}_t\,\varphi^t - u_{xt}\,\mathcal{D}_t\,\xi^1 - u_{tt}\,\mathcal{D}_t\,\xi^2,$$

and $\mathcal{D}_i$ is the total derivative operator:

$$\mathcal{D}_i = \frac{\partial}{\partial x^i} + u_i\frac{\partial}{\partial u} + \ldots, \quad (x^1,x^2) = (x,t).$$

According to the Lie symmetry method, the local group of transformations is a Lie group that leaves Equation (1) invariant, if it satisfies the following invariance condition

$$\chi^{[2]}\left(u_{tt} + \alpha(u)\,u_t - f(u) - u_{xx}\right)\Big|_{(u_{tt} = -\alpha(u)\,u_t + f(u) + u_{xx})} = 0. \quad (5)$$

The expansion of Equation (5) and comparison of the coefficients of independent derivatives of u lead to the following system of determining Pdes

$$\varphi_{uu} = 0, \quad (6)$$
$$\xi^1_t\,\alpha(u) + (2\varphi_{xu} - \xi^1_{xx}) + \xi^1_{tt} = 0, \quad (7)$$
$$\varphi\,\alpha_u + \xi^2_t\,\alpha(u) + (2\varphi_{tu} - \xi^2_{tt}) + \xi^2_{xx} = 0, \quad (8)$$
$$\xi^2_x - \xi^1_t = 0, \quad (9)$$
$$\xi^1_x - \xi^2_t = 0, \quad (10)$$
$$-\varphi f_u + \alpha(u)\,\varphi_t + (\varphi_u - 2\xi^2_t)\,f(u) + \varphi_{tt} - \varphi_{xx} = 0. \quad (11)$$

Equation (6) implies that

$$\varphi = m(x,t)\,u + n(x,t).$$

Further, Equations (9) and (10) lead to

$$\xi^1_{tt} = \xi^1_{xx}, \qquad \xi^2_{tt} = \xi^2_{xx}.$$

After some manipulations, we obtain

$$\xi^1_t\,\alpha(u) + 2\,m_x = 0, \quad (12)$$
$$(m(x,t)\,u + n(x,t))\,\alpha_u + \xi^2_t\,\alpha(u) + 2\,m_t = 0. \quad (13)$$

If α and f are arbitrary in u, then Equations (12) and (13) result in

$$m = 0, \qquad n = 0,$$
$$\xi^2_t = 0, \qquad \xi^1_t = 0.$$

Consequently, Equations (9) and (10) imply that

$$\xi^2 = C_1, \qquad \xi^1 = C_2.$$

Thus, for arbitrary $\alpha(u)$ and $f(u)$, we have the two dimensional Lie algebra, which is the principal Lie algebra of Equation (1), spanned by

$$\chi_1 = \frac{\partial}{\partial t}, \qquad \chi_2 = \frac{\partial}{\partial x}, \qquad (\alpha \neq 0).$$

Now for the extension of the principal Lie algebra, we have

$$\xi^1 = C_1\,x + C_2,$$
$$\xi^2 = C_1\,t + C_3.$$

From Equation (13), we obtain

$$\big(m(t)\,u + n(x,t)\big)\,\alpha_u + C_1\,\alpha + 2\,m_t = 0. \quad (14)$$

The following two cases arise for which the principal algebra extends.

Case 1: $m \neq 0,\ k \neq 0,\ 1.$

This case yields the following form of the function $\alpha(u)$

$$\alpha(u) = B_1\,(u + k_1)^k,$$

which subsequently leads to $f(u)$ of the form

$$f(u) = D_1\,(u + k_1)^{1+2k}.$$

The additional symmetry generator of Equation (1) for these forms of the functions is

$$\chi_3 = -k\,x\,\frac{\partial}{\partial x} - k\,t\,\frac{\partial}{\partial t} + (u + k_1)\,\frac{\partial}{\partial u}.$$

Case 2: $m = 0,\ k_2 \neq 0.$

In this case, we obtain different forms of the functions, $\alpha(u)$ and $f(u)$

$$\alpha(u) = E_1\,e^{k_2\,u}, \qquad f(u) = F_1\,e^{2\,k_2\,u}, \qquad k_2 \neq 0.$$

Consequently, the principal algebra extends to 3-dimensions, along with the following symmetry generator

$$\chi_3 = -k_2\,x\,\frac{\partial}{\partial x} - k_2\,t\,\frac{\partial}{\partial t} + \frac{\partial}{\partial u}.$$

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A. EQUIVALENCE TRANSFORMATION

The equivalence transformations of an equation leaves the family invariant [17]. Equation (1) has the following equivalence transformations

$$\bar{t} = a_1 t + a_2, \quad \bar{x} = a_1 x + b_1, \quad \bar{u} = c_1 u + c_2, \quad (15)$$

where a_1, a_2, b_1, c_1, and c_2 are the arbitrary constants. We show this to be the case. Indeed

$$\mathcal{D}_i = \mathcal{D}_i f^j \bar{\mathcal{D}}_{\bar{j}}, \qquad \bar{x}^i = f^i(x),$$

where $\bar{x}^1 = \bar{t} = f^1$, $\bar{x}^2 = \bar{x} = f^2$ and $\mathcal{D}_i$ is the total derivative transform on f^i. For the above transforms in $(x, t)-$space, we have

$$\mathcal{D}_x = \mathcal{D}_x f^1 \bar{\mathcal{D}}_{\bar{t}} + \mathcal{D}_x f^2 \bar{\mathcal{D}}_{\bar{x}},$$

$$\mathcal{D}_t = \mathcal{D}_t f^1 \bar{\mathcal{D}}_{\bar{t}} + \mathcal{D}_t f^2 \bar{\mathcal{D}}_{\bar{x}}.$$

After some manipulations, we deduce

$$\bar{\mathcal{D}}_{\bar{x}} = \frac{1}{a_1} \mathcal{D}_x,$$

$$\bar{\mathcal{D}}_{\bar{t}} = \frac{1}{a_1} \mathcal{D}_t.$$

Implication of the above two on $\bar{u}$ results in

$$\bar{u}_{\bar{t}} = \frac{c_1}{a_1} u_t, \qquad \bar{u}_{\bar{t}\bar{t}} = \frac{c_1}{a_1^2} u_{tt},$$

$$\bar{u}_{\bar{x}} = \frac{c_1}{a_1} u_x, \qquad \bar{u}_{\bar{x}\bar{x}} = \frac{c_1}{a_1^2} u_{xx}.$$

Equation (1) under the equivalence transformations becomes

$$\bar{u}_{\bar{t}\bar{t}} + \bar{\alpha}(\bar{u})\, \bar{u}_{\bar{t}} = \bar{f}(\bar{u}) + \bar{u}_{\bar{x}\bar{x}},$$

subject to the following equivalence conditions

$$f(u) = \frac{a_1^2}{c_1} \bar{f}(\bar{u}), \quad (16)$$

and

$$\alpha(u) = a_1 \bar{\alpha}(\bar{u}). \quad (17)$$

By means of the equivalence transformations one can simplify $\alpha(u)$ and $f(u)$ for both cases.

Case 1:

Here, if we set $a_1 = c_1 = 1$ and $c_2 = k_1$, then $\bar{\alpha} = B_1 \bar{u}^k$ and $\bar{f} = D_1 \bar{u}^{1+2k}$.

Case 2:

In this case, if we take $a_1 = 1$, $c_1 = k_2$, and $c_2 = 0$, then $\bar{\alpha} = E_1 e^{\bar{u}}$ and $\bar{f} = \tilde{F}_1 e^{2\bar{u}}$.

Therefore, one can take $k_1 = 0$ and $k_2 = 1$ in the extended algebras.

III. OPTIMAL SYSTEM OF ONE-DIMENSIONAL SUBALGEBRAS

To reduce the effort for finding invariant solutions, we look for the optimal system of one-dimensional subalgebras. For this, we partition the symmetry generators into dissimilar classes and find the complete set of invariants via the optimal system for reduction purposes. This section represents the commutator table, adjoint table of the above extended symmetry algebras, and the optimal system of one-dimensional subalgebras for each case.

A. COMMUTATOR TABLE AND ADJOINT REPRESENTATION

The commutation relations for cases **1** and **2** are represented in Table 1, which are subsequently used in finding the adjoint representation of the generators in the extended three-dimensional algebra.

Table 1 Commutation relations for Cases **1** and **2**

$[\chi_i, \chi_j]$	χ_1	χ_2	χ_3
χ_1	0	0	$-k\,\chi_1$
χ_2	0	0	$-k\,\chi_2$
χ_3	k χ_1	$k\,\chi_2$	0

The adjoint table for the extended symmetry algebras is represented by

Table 2 Adjoint Table for Cases **1** and **2**

$Ad(e^{\chi_i})\chi_j$	χ_1	χ_2	χ_3
χ_1	χ_1	χ_2	$\chi_3 + \varepsilon k\,\chi_1$
χ_2	χ_1	χ_2	$\chi_3 + \varepsilon k\,\chi_2$
χ_3	$e^{-k\varepsilon}\chi_1$	$e^{-k\varepsilon}\chi_2$	χ_3

The adjoint map used in the computation of the adjoint table is given by

$$Ad(e^{\chi_i})\,\chi_j = \chi_j - \varepsilon\,[\chi_i, \chi_j] + \frac{\varepsilon^2}{2!}\,[\chi_i, [\chi_i, \chi_j]] + \ldots. \quad (18)$$

Since, the adjoint table and the commutator table of the extended symmetry algebras overlap for the cases **1** and **2**, we represent them by Table 2 and Table 1, respectively. However, the symmetry χ_3 differs for each case of the adjoint representation, and in the commutator table, k can be replaced by k_2 for case **2**.

B. OPTIMAL SYSTEM OF ONE-DIMENSIONAL SUBALGEBRAS

To find the optimal system of one-dimensional subalgebras, we take the generic element $\chi \in \Gamma_3$, defined by

$$\chi = \upsilon_1 \chi_1 + \upsilon_2 \chi_2 + \upsilon_3 \chi_3, \quad (19)$$

which we need to simplify by adjoint maps. Next, by using Table 2, we invoke the adjoint action representation (18) on

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the generic element defined in Equation (19) and obtain the following optimal system of one-dimensional subalgebras

$$\begin{aligned}\chi^1 &= \chi_3 \pm \chi_2,\\ \chi^2 &= \chi_3,\\ \chi^3 &= \chi_2,\\ \chi^4 &= \chi_1 + \upsilon_2\, \chi_2.\end{aligned}$$

Case 1:

The optimal system of subalgebras is

$$\begin{aligned}\chi^1 &= (\pm 1 - k\,x)\,\frac{\partial}{\partial x} - k\,t\,\frac{\partial}{\partial t} + (u + k_1)\,\frac{\partial}{\partial u},\\ \chi^2 &= -k\,x\,\frac{\partial}{\partial x} - k\,t\,\frac{\partial}{\partial t} + (u+k_1)\,\frac{\partial}{\partial u},\\ \chi^3 &= \frac{\partial}{\partial x},\\ \chi^4 &= \frac{\partial}{\partial t} + \upsilon_2\,\frac{\partial}{\partial x}.\end{aligned}$$

Case 2:

The optimal system of subalgebras is as follows

$$\begin{aligned}\chi^1 &= (\pm 1 - k_2\,x)\,\frac{\partial}{\partial x} - k_2\,t\,\frac{\partial}{\partial t} + \frac{\partial}{\partial u},\\ \chi^2 &= -k_2\,x\,\frac{\partial}{\partial x} - k_2\,t\,\frac{\partial}{\partial t} + \frac{\partial}{\partial u},\\ \chi^3 &= \frac{\partial}{\partial x},\\ \chi^4 &= \frac{\partial}{\partial t} + \upsilon_2\,\frac{\partial}{\partial x}.\end{aligned}$$

IV. REDUCTIONS TO ORDINARY DIFFERENTIAL EQUATIONS

After obtaining the Lie point symmetries, an essential step is to reduce the partial differential equation into a simpler form through the derived symmetry generators, typically by deriving the similarity variables associated with those symmetry generators. The solution of the reduced Des results in the invariant solutions.

This section outlines the reductions of Equation (1) through the principal symmetry algebras and the corresponding one-dimensional subalgebras resulted from the optimal system.

A. REDUCTIONS FOR ARBITRARY FUNCTIONS $\alpha(U)$ AND $F(U)$

We begin by taking the time translation symmetry generator, given by

$$\chi_1 = \frac{\partial}{\partial t},$$

by the method of characteristics, the characteristic equation associated with this symmetry generator is written as

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{0},$$

which gives the following similarity variables, $x = \gamma$ and $u = \omega(\gamma)$. Corresponding to these similarity variables, Equation (1) is reduced into the following form

$$\omega_{\gamma\gamma} + f(\omega) = 0.$$

Now, we consider the translation in x symmetry generator, given by

$$\chi_2 = \frac{\partial}{\partial x},$$

corresponding to this symmetry, we have

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0},$$

which yields $t = \gamma$ and $u = \omega(\gamma)$. Hence, we have the following reduced Ode

$$\omega_{\gamma\gamma} + \alpha(\omega)\,\omega_\gamma = f(\omega).$$

B. REDUCTIONS FOR CASE 1

By considering the symmetry generator χ_1, the following reduced form of Equation (1) is obtained

$$\omega_{\gamma\gamma} + D_1\,(\omega + k_1)^{1+2k} = 0,$$

which has exact solution of the form

$$c_2 \pm \gamma = \int \Big[c_1 - 2\int D_1\,(k_1 + \omega)^{1+2k} d\omega\Big]^{-\frac{1}{2}} d\omega.$$

The reduced form of Equation (1) for the translation in x generator, *i.e.*, $\chi_2 = \frac{\partial}{\partial x}$, is given as follows

$$\omega_{\gamma\gamma} + \omega_\gamma\,B_1\,(\omega + k_1)^k - D_1\,(\omega + k_1)^{1+2k} = 0,$$

having solution of the form

$$[(\gamma + c_2)^2, \omega(\gamma)] = \int_1^{\omega(\gamma)} \left(\frac{1}{\sqrt{c_1 + 2\int_1^{K_2} (-B_1\,[k_1 + K_1^k + D_1\,[k_1 + K_1^{1+2k})dK_1}}\right) dK_2.$$

Also, the symmetry generator

$$\chi_3 = -k\,x\,\frac{\partial}{\partial x} - k\,t\,\frac{\partial}{\partial t} + (u + k_1)\,\frac{\partial}{\partial u},$$

results in the following reduction of Equation (1)

$$(\gamma^2 - 1)\,\omega_{\gamma\gamma} + \omega_\gamma\big(2\frac{1+k}{k}\,\gamma - B_1\,\omega^k\big) + \omega\big(D_1\,\omega^{2k} + \frac{1+k}{k^2}\big) = 0.$$

1) Reductions by optimal system of subalgebras

The symmetry generator

$$\chi^1 = (\pm 1 - k\,x)\,\frac{\partial}{\partial x} - k\,t\,\frac{\partial}{\partial t} + (u + k_1)\,\frac{\partial}{\partial u},$$

reduces Equation (1) into the following Ode

$$\begin{aligned}\gamma^2(\gamma^2 - 1)\,\omega_{\gamma\gamma} + \gamma\,\omega_\gamma\big(2\,\gamma^2 + \frac{2}{k} - B_1\,\omega^k\big) +&\\ \omega\big(D_1\,\omega^{2k} + \frac{B_1}{k}\,\omega^k - \frac{1+k}{k^2}\,\gamma^4\big) &= 0,\end{aligned}$$

which provides similarity solutions of the equation.

The reduced form of Equation (1) associated with the symmetry generator

$$\chi^4 = \frac{\partial}{\partial t} + \upsilon_2\,\frac{\partial}{\partial x},$$

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is given by

$$(\upsilon_2^2-1)\,\omega'' + B_1\,(\omega+\upsilon_2\,k_1)^k\,\omega' - D_1\,(\omega+\upsilon_2\,k_1)^{1+2k} = 0,$$

which gives rise to the traveling wave solutions. The special cases

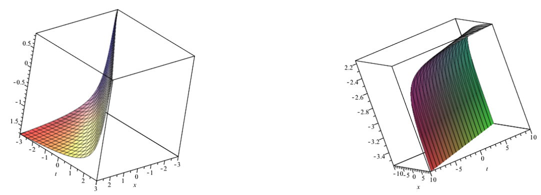
$$\chi^4 = \frac{\partial}{\partial t} \pm \frac{\partial}{\partial x},$$

yield the following similarity solutions of Equation (1)

$$u_1(t,x) = \left(\frac{B_1}{D_1\,k\,(-t+x+c_1)}\right)^{k^{-1}} - k_1,$$

$$u_2(t,x) = \left(-\frac{B_1}{D_1\,k\,(t+x+c_1)}\right)^{k^{-1}} - k_1.$$

These solutions can be visualized graphically as follows:



(a) $u_1(t,x) = \sqrt{\frac{1}{2(x-t)}} - 2$ (b) $u_2(t,x) = -\sqrt{\frac{1}{2(x+t)}} - 2$

FIGURE 1: $3D$ graphs of u_1 and u_2 with parameters $B = D = 2,\ C_1 = 0,$ and $k = k_1 = 2$.

These solutions in Fig. 1 illustrate a wave propagation in a nonlinear damped medium. The singular behavior of solution u_1 in Fig. 1a is due to the term $\frac{1}{2(x-t)}$, it is clear that as t approaches x, the solution u_1 tends to infinity, reflecting that the solution is undefined at this point. Moreover, as x increases, the damping effects become more dominant and the energy in the system dissipates, while keeping the wave-like profile. While, in Fig. 1b, one can see the flattening of the wave, this is due to the dominant dissipative effects and for increase in $x+t$.

C. REDUCTIONS FOR CASE 2

For the time translation symmetry generator, $\chi_1 = \frac{\partial}{\partial t}$, we determine the following reduced form of Equation (1)

$$\omega_{\gamma\gamma} + F_1\,e^{2k_2\,\omega} = 0,$$

having solution

$$\omega(\gamma) = \frac{1}{k_2}\,\log\left[\pm\frac{\sqrt{k_2\,c_1\,(1-\tanh[k_2^2\,\sqrt{c_1\,(\gamma+c_2)^2}]^2)}}{\sqrt{F_1}}\right].$$

For translation in x, $\chi_2 = \frac{\partial}{\partial x}$, we have

$$\omega_{\gamma\gamma} + E_1\,e^{k_2\,\omega}\,\omega_\gamma - F_1\,e^{2k_2\,\omega} = 0,$$

having solution of the form

$$\omega(\gamma) = \frac{1}{\sigma}\,\ln[-\frac{\sigma}{F_1}\,(c_1 + \delta^2\,\ln|s| - \delta s)],$$

where $\delta = F_1/E_1$ and

$$\gamma = -\frac{\delta}{\sigma}\int s^{-1}\,(c_1+\delta^2\ln|s| - \delta s)^{-1} ds + c_2.$$

Similarly, associated with the symmetry generator

$$\chi_3 = -k_2\,x\,\frac{\partial}{\partial x} - k_2\,t\,\frac{\partial}{\partial t} + \frac{\partial}{\partial u},$$

the reduction of Equation (1) is given by

$$\gamma^2\,(1-\gamma^2)\left(\omega\,\omega_{\gamma\gamma}-\omega_\gamma^2\right) + \omega\,\omega_\gamma\left(E_1\,\omega-2\gamma\right)\gamma^2 + \omega^2\left(F_1\,\omega^2\,k_2+1\right) = 0.$$

1) Reductions by optimal system of subalgebras

Associated with the symmetry generator

$$\chi^1 = (\pm 1 - k_2\,x)\,\frac{\partial}{\partial x} - k_2\,t\,\frac{\partial}{\partial t} + \frac{\partial}{\partial u},$$

we obtain the following reduction of Equation (1)

$$\gamma^2\,(1-\gamma^2)\left(\omega\,\omega_{\gamma\gamma}-\omega_\gamma^2\right) + \omega\,\omega_\gamma\left(E_1\,\omega-2\gamma\right)\gamma^2 + \omega^2\left(F_1\,\omega^2\,k_2+1\right) = 0.$$

Similarly, the symmetry generator

$$\chi^4 = \frac{\partial}{\partial t} + \upsilon_2\,\frac{\partial}{\partial x},$$

leads to the following reduction

$$(\upsilon_2^2-1)\,\omega_{\gamma\gamma} + E_1\,\upsilon_2\,e^{\frac{k_2}{\upsilon_2}\,\omega}\,\omega_\gamma - +F_1\,\upsilon_2\,e^{2\frac{k_2}{\upsilon_2}\,\omega} = 0,$$

which yields travelling wave solutions. The special cases

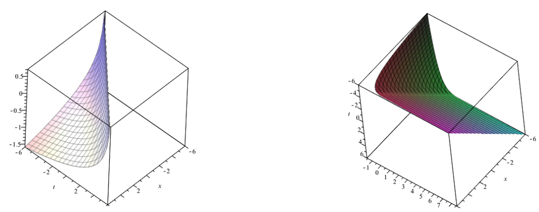
$$\chi^4 = \frac{\partial}{\partial t} \pm \frac{\partial}{\partial x},$$

yield the following similarity solutions of damped Kgf Equation (1)

$$u_3(t,x) = \frac{1}{k_2}\,\ln\left(\frac{E_1}{F_1\,k_2\,(-t+x+C_1)}\right),$$

$$u_4(t,x) = \frac{1}{k_2}\,\ln\left(-\frac{E_1}{F_1\,k_2\,(t+x+C_1)}\right).$$

Graphically, these solutions can be represented as follows:



(a) $u_3(t,x) = \frac{1}{2}\ln\frac{1}{2(x-t)}$ (b) $u_4(t,x) = \frac{1}{2}\ln(-\frac{1}{2}(x+t))$

FIGURE 2: $3D$ graphs of u_3 and u_4 with parameters $E_1 = F_1 = 2,\ C_1 = 0,$ and $k_2 = 2$.

Fig. 2 reflect how the solutions u_3 and u_4 evolves over space and time. The vertical asymptote in the Figure 2a is due to the presence of the term $x - t$ in the solution. It can be seen that, near $x = t$ the solution breaks down causing the

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amplitude of the wave to be undefined, reflecting phenomena like shock or singularity in the system. Moreover, Fig. 2b represents the solution profile spreading logarithmically and being undefined as $x + t \to 0$.

V. EXAMPLES OF SOLITON SOLUTIONS

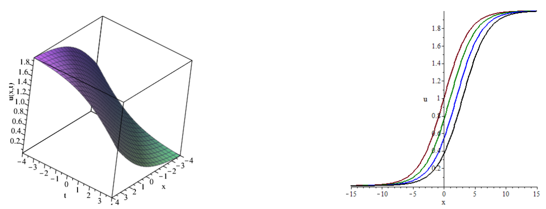
This section presents the examples of soliton solutions for Equation (1) along with their graphical representations. The balance between nonlinear and dispersion effect results in stable and localized wave packets that maintain their shape and speed over distance and time; these wave packets form solitons.

Example 1: $\alpha(u) = c_3$ and $f(u) = -c_1 u + c_2 u^2$.
Given $\alpha(u) = c_3$ and $f(u) = -c_1 u + c_2 u^2$, Equation (1) admits the following soliton solutions with parameters $c_1 = 2,\ c_2 = 1,$ and $c_3 = 4$

$$u_5(x,t) = 1 + \tanh\big(\frac{1}{4}(x-t) - C\big),$$

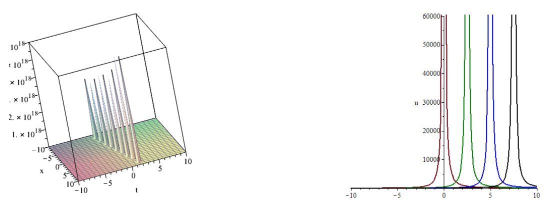
$$u_6(x,t) = 6\coth\big(\frac{5}{24}t - \frac{1}{24}x - C\big)\Big(\coth\big(\frac{5}{24}t - \frac{1}{24}x - C\big) - 2\Big) + 6.$$

Graphically, these solutions can be seen as:



(a) $u_5(x,t) = 1 + \tanh\big(\frac{1}{4}(x-t)\big)$ (b) Solution profiles of u_5 for $t = 0,\ 1,\ 2,\ 3$

FIGURE 3: $3D$ and $2D$ plots of u_5



(a) $u_6(x,t) = 6\coth\big(\frac{5}{24}t - \frac{1}{24}x\big)\Big(\coth\big(\frac{5}{24}t - \frac{1}{24}x\big) - 2\Big) + 6$ (b) Solution profiles of u_6 for $t = 0,\ 1,\ 2,\ 3$

FIGURE 4: $3D$ and $2D$ plots of u_6

Fig. 3 represents a traveling wave that evolves diagonally over different values of time t. It can be seen in Fig. 3b that as t increases the wave moves towards the positive x-axis with the width of $1/4$. The graph shows a wave traveling to the right with the stable amplitude, illustrating the damped nature of the considered model. The presence of the $\tanh$ function in the solution introduces nonlinearity allowing the wave to preserve its speed and shape. Fig. 3b shows that: at $t = 0$ the center of the wave lies at $x = 0$, at $t = 1$ its center is at $x = 1$, similarly for $t = 2$ the wave shifts to right having center at $x = 2$, and so on, indicating proportionality between x and t. Additionally, the wave reflects a soliton nature as it is traveling with a constant speed and shape.

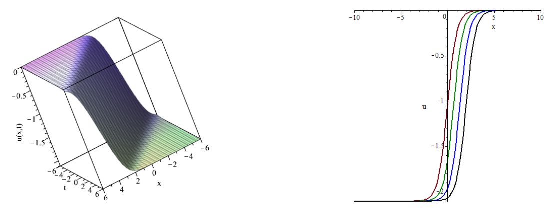
Fig. 4 represents a damped oscillatory wave traveling towards the positive x-axis with the increase in time t. Initially the wave reflects sharp changes that spreads out but due to the damping effect, the wavefront gradually loses amplitude, allowing the wave to flatten out.

Example 2: $\alpha(u) = c_3$ and $f(u) = c_1 u + c_2 u^3$.
With $\alpha(u) = c_3$ and $f(u) = c_1 u + c_2 u^3$, Equation (1) yields the following soliton solutions for the parameters $c_1 = \frac{40}{9},\ c_2 = -1,$ and $c_3 = 5$

$$u_7(x,t) = \frac{\sqrt{10}}{3}\big(-1 + \tanh(x - \frac{2}{3}t - C)\big),$$

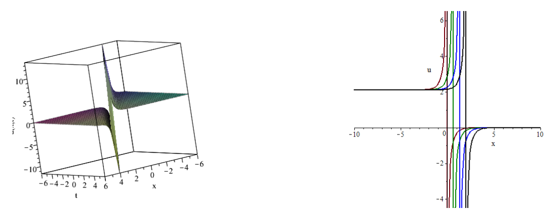
$$u_8(x,t) = \frac{\sqrt{10}}{3}\big(1 + \coth(\frac{2}{3}t - x - C)\big).$$

These solutions can be graphically represented as:



(a) $u_7(x,t) = \frac{\sqrt{10}}{3}\big(-1 + \tanh(x - \frac{2}{3}t)\big)$ (b) Solution profiles of u_7 for $t = 0,\ 1,\ 2,\ 3$

FIGURE 5: $3D$ and $2D$ plots of u_7



(a) $u_8(x,t) = \frac{\sqrt{10}}{3}\big(1 + \coth(\frac{2}{3}t - x)\big)$ (b) Solution profiles of u_8 for $t = 0,\ 1,\ 2,\ 3$

FIGURE 6: $3D$ and $2D$ plots of u_8

Fig. 5 illustrates a solitary wave that travels through a damped medium. Fig. 5b shows that as t increases the wave shifts along the positive x-axis while preserving its shape and speed over different time values, due to the factor $\frac{\sqrt{10}}{3}$ that keeps the amplitude of the wave bounded and finite.

Fig. 6 represents sharp peaks in the wave behavior due the nature of $\coth$ function. It can be observed from the graph that the solution depicts asymptotic behavior. Moreover, as time progresses the influence of damping become more significant and it stabilizes the amplitude of the wave. Also, the graph shows vertical asymptotes for $t = 1,\ 2,$ and 3 at $x = \frac{2}{3},\ \frac{4}{3},$ and 2, respectively.

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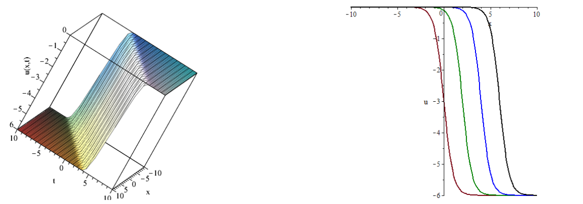
Example 3: $\alpha(u) = u$ and $f(u) = -c_1 u + c_2 u^2$.
Equation (1) yields the following soliton solutions for $\alpha(u) = c_3 u$ and $f(u) = -c_1 u + c_2 u^2$

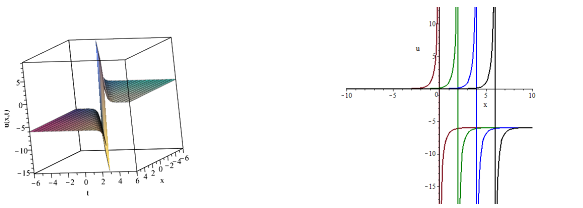
$$\begin{aligned} u_9(x,t) &= -3\left(1 + \tanh(x - 2t + C)\right), \\ u_{10}(x,t) &= -3\left(1 + \coth(x - 2t + C)\right). \end{aligned}$$

where $c_1 = -12$ and $c_2 = 2$. These solutions can be visualized graphically as:



(a) $u_9(x,t) = -3\,(1 + \tanh(x - 2t))$ (b) Solution profiles of u_9 for $t = 0, 1, 2, 3$

FIGURE 7: $3D$ and $2D$ plots of u_9



(a) $u_{10}(x,t) = -3\,(1 + \coth(x - 2t))$ (b) Solution profiles of u_{10} for $t = 0, 1, 2, 3$

FIGURE 8: $3D$ and $2D$ plots of u_{10}

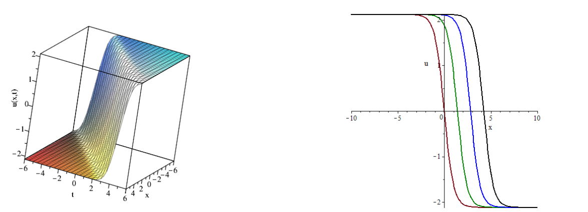
Fig. 7 represents a traveling wave that moves rightward keeping a consistent shape and speed with the width of 2. While the minus sign indicates negative amplitude of the wave.

Fig. 8 represents vertical asymptotes in the wave behavior due the nature of coth function.

Example 4: $\alpha(u) = u$ and $f(u) = c_1 u + c_2 u^3$.
Equation (1) provides the following soliton solutions for $\alpha(u) = c_3 u$ and $f(u) = c_1 u + c_2 u^3$

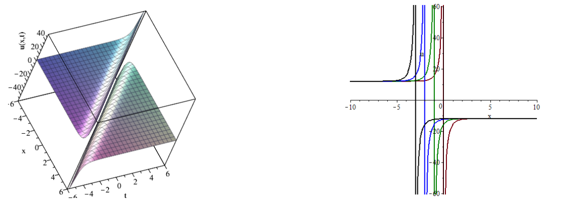
$$\begin{aligned} u_{11}(x,t) &= -\frac{3}{\sqrt{2}} \tanh\left(x - \sqrt{2}t + C\right), \\ u_{12}(x,t) &= 12\left(\coth(-t - x + C)\right). \end{aligned}$$

where $c_1 = 1$ and $c_2 = -\frac{2}{9}$. Graphically, we have:



(a) $u_{11}(x,t) = -\frac{3}{\sqrt{2}} \tanh\left(x - \sqrt{2}t\right)$ (b) Solution profiles of u_{11} for $t = 0, 1, 2, 3$

FIGURE 9: $3D$ and $2D$ plots of u_{11}



(a) $u_{12}(x,t) = 12\,(\coth(-t - x))$ (b) Solution profiles of u_{12} for $t = 0, 1, 2, 3$

FIGURE 10: $3D$ and $2D$ profiles of u_{12}

Fig. 9 represents a wave that retains its shape and shift rightwards with constant speed of $\sqrt{2}$, characterizing a soliton solution. While the negative sign indicates amplitude of the wave increases downwards.

Fig. 10 represents singularity in the wave behavior due the presence of coth function. The graph depicts how the wave maintains its speed while traveling through the medium and its amplitude diminishes over time, reflecting the characteristics of a damping medium.

VI. CONSERVATION LAWS

This section deals with the conservation laws of the damped Klein-Gordon Fock equation through the partial Lagrangian approach. The partial Lagrangian works even when the usual Lagrangian for a system does not exist. For instance, this is the case for scalar evolution equations, including the simple classical heat equation. To find the conserved currents of the equation under normal consideration with usual Lagrangian, one utilizes the classical Noether's theorem [27]. However, in the absence of usual Lagrangian or in the existence of partial Lagrangian only, the partial Noether's approach becomes an effective technique in finding the conserved quantities with the existence of the partial Lagrangian. In order to pursue conservation laws here, we invoke the partial Noether's approach [28].

The partial Lagrangian of Equation (1) is given as

$$\mathcal{L} = \frac{1}{2} u_t^2 - \frac{1}{2} u_x^2 + \int f(u)\, du, \tag{20}$$

where

$$\frac{\partial \mathcal{L}}{\partial u} = f(u) + u_{xx} - u_{tt} = \alpha(u)\, u_t.$$

The operator defined in Equation (3) associated with the partial Lagrangian (20), is called a partial Noether symmetry operator of Equation (1) if the following condition is satisfied

$$\chi^{[1]} \mathcal{L} + (\mathcal{D}_t \xi^2 + \mathcal{D}_x \xi^1) \mathcal{L} = \mathcal{W} \frac{\partial \mathcal{L}}{\partial u} + \mathcal{D}_t \mathcal{B}^1 + \mathcal{D}_x \mathcal{B}^2, \tag{21}$$

where

$$\mathcal{W} = \varphi - \xi^2 u_t - \xi^1 u_x,$$

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$\mathcal{B}^1$ & $\mathcal{B}^2$ are gauge terms depending on (x, t, u). From Equation (21), we arrive at the following set of equations with $\xi_u^1 = 0 = \xi_u^2$,

$$\varphi_u - \frac{1}{2}\xi_t^2 + \frac{1}{2}\xi_x^1 + \alpha(u)\,\xi^2 = 0, \quad (22)$$

$$-\varphi_u + \frac{1}{2}\xi_x^1 - \frac{1}{2}\xi_t^2 = 0, \quad (23)$$

$$\xi_x^2 - \xi_t^1 + \alpha(u)\,\xi^1 = 0, \quad (24)$$

$$\varphi_t - \mathcal{B}_u^1 - \alpha(u)\,\varphi = 0, \quad (25)$$

$$\varphi_x + \mathcal{B}_u^2 = 0, \quad (26)$$

$$\varphi f + \xi_t^2 \int f(u)\,du + \xi_x^1 \int f(u)\,du - \mathcal{B}_t^1 - \mathcal{B}_x^2 = 0. \quad (27)$$

From Equation (23), we have

$$\varphi = \frac{1}{2}(\xi_x^1 - \xi_t^2)\,u + \mathcal{A}(t,x). \quad (28)$$

Now, we consider different cases for $\alpha(u)$ and $f(u)$.

Case 1: $\alpha(u)$ is arbitrary function of u, i.e., $\alpha(u) \neq Const$.

From Equations (22) and (24), we find, $\xi^1 = 0 = \xi^2$. Thus, Equation (28), implies that

$$\varphi = \mathcal{A}(t,x). \quad (29)$$

Subsequently, from Equations (25) and (26), we obtain the following gauge terms

$$\mathcal{B}^1 = \mathcal{A}_t\,u - \mathcal{A}\int \alpha(u)\,du + \mathcal{H}^2(t,x),$$

and

$$\mathcal{B}^2 = -\mathcal{A}_x\,u + \mathcal{H}^1(t,x),$$

with the following conserved vector components

$$\mathcal{T}^t = \mathcal{A}_t\,u - \mathcal{A}\int \alpha(u)\,du + \mathcal{H}^2(t,x) - \varphi\,u_t, \quad (30)$$

$$\mathcal{T}^x = -\mathcal{A}_x\,u + \mathcal{H}^1(t,x) + \varphi\,u_x, \quad (31)$$

subject to the condition

$$\mathcal{A}f(u) = \mathcal{A}_{tt}\,u - \mathcal{A}_t \int \alpha(u)\,du + \mathcal{H}_t^2 - \mathcal{A}_{xx}\,u + \mathcal{H}_x^1. \quad (32)$$

Subcase 1.1: If $f(u)$, $\alpha(u)$, and u are not related.

This case implies that

$$\mathcal{A} = 0,$$
$$\mathcal{H}_x^1 + \mathcal{H}_t^2 = 0.$$

So, no operator is obtained in this case.

Subcase 1.2: If $f(u) = 0$ and $\alpha(u)$ is an arbitrary function of u.

In this case, the following components of the conserved vectors are obtained

$$\mathcal{T}^t = -(\mathcal{A}_1 + \mathcal{A}_2\,x)\int \alpha(u)\,du + \mathcal{H}^2 - (\mathcal{A}_1 + \mathcal{A}_2\,x)\,u_t,$$
$$\mathcal{T}^x = -\mathcal{A}_2\,u + \mathcal{H}^1 + (\mathcal{A}_1 + \mathcal{A}_2\,x)\,u_x.$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are constants. So, we have the following conserved vectors

$$(\mathcal{T}_1^t, \mathcal{T}_1^x) = \left(-\int \alpha(u)\,du + \mathcal{H}^2 - u_t\,,\; u_x + \mathcal{H}^1\right),$$
$$(\mathcal{T}_2^t, \mathcal{T}_2^x) = \left(-x\int \alpha(u)\,du + \mathcal{H}^2 - x\,u_t\,,\; x\,u_x + \mathcal{H}^1 - u\right).$$

Subcase 1.3: If $f(u) = f_1 + f_2\,u$, and $\alpha(u)$ is not linear function in u.

For $f_2 > 0$, the following components are determined

$$\mathcal{T}^t = -(\mathcal{A}_1\cos\sqrt{f_2}\,x + \mathcal{A}_2\sin\sqrt{f_2}\,x)\,(\int \alpha(u)\,du + u_t) + \mathcal{H}^2,$$

$$\mathcal{T}^x = -\sqrt{f_2}\,(\mathcal{A}_2\cos\sqrt{f_2}\,x - \mathcal{A}_1\sin\sqrt{f_2}\,x)\,u + (\mathcal{A}_1\cos\sqrt{f_2}\,x + \mathcal{A}_2\sin\sqrt{f_2}\,x)\,u_x + \frac{f_1}{\sqrt{f_2}}\,(\mathcal{A}_1\sin\sqrt{f_2}\,x - \mathcal{A}_2\cos\sqrt{f_2}\,x).$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are constants.

Now for $f_2 < 0$, we obtain the following components of conserved quantities

$$\mathcal{T}^t = -(\mathcal{A}_1\,e^{\sqrt{f_2}\,x} + \mathcal{A}_2\,e^{-\sqrt{f_2}\,x})\,(\int \alpha(u)\,du + u_t) + \mathcal{H}^2,$$
$$\mathcal{T}^x = \mathcal{A}_1\,e^{\sqrt{f_2}\,x}\,(-\sqrt{f_2}\,u + \frac{f_1}{\sqrt{f_2}} + u_x) + \mathcal{A}_2\,e^{-\sqrt{f_2}\,x}\,(\sqrt{f_2}\,u - \frac{f_1}{\sqrt{f_2}} + u_x).$$

So, for the constants $\mathcal{A}_1$ and $\mathcal{A}_2$, there are two independent conserved quantities, i.e., $\mathcal{T}_1 = (\mathcal{T}_1^t\,,\,\mathcal{T}_1^x)$ and $\mathcal{T}_2 = (\mathcal{T}_2^t\,,\,\mathcal{T}_2^x)$ for $(\mathcal{A}_1 = 1\,,\,\mathcal{A}_2 = 0)$ and $(\mathcal{A}_2 = 1,\,\mathcal{A}_1 = 0)$, respectively.

Subcase 1.4: If $f(u) = k_1 \int \alpha(u)\,du$.

In this case, we derive the following conserved components

$$\mathcal{T}^t = -e^{-k_1\,t}\,(\mathcal{A}_1\,e^{k_1\,x} + \mathcal{A}_2\,e^{-k_1\,x})\,[k_1\,u + u_t + \int \alpha(u)\,du] + \mathcal{H}^2,$$
$$\mathcal{T}^x = -e^{-k_1 t}(\mathcal{A}_1 e^{k_1\,x} - \mathcal{A}_2 e^{-k_1\,x})k_1 u + \mathcal{H}^1 + e^{-k_1\,t}(\mathcal{A}_1 e^{k_1\,x} + \mathcal{A}_2\,e^{-k_1\,x})\,u_x.$$

Hence, for both constants $\mathcal{A}_1$ and $\mathcal{A}_2$, we find two independent conserved quantities given as $\mathcal{T}_1 = (\mathcal{T}_1^t\,,\,\mathcal{T}_1^x)$ and $\mathcal{T}_2 = (\mathcal{T}_2^t\,,\,\mathcal{T}_2^x)$ for $(\mathcal{A}_1 = 1\,,\,\mathcal{A}_2 = 0)$ as well as $(\mathcal{A}_2 = 1,\,\mathcal{A}_1 = 0)$, respectively.

Case 2: $\alpha(u) = \alpha$ is constant.

In this case, we have the following gauge terms

$$\mathcal{B}^1 = \frac{1}{4}(\xi_{xt}^1 - \xi_{tt}^2)\,u^2 + (\mathcal{A}_t - \alpha\,\mathcal{A})\,u - \frac{1}{4}(\xi_x^1 - \xi_t^2)\,u^2\,\alpha + \mathcal{H}^2(t,x), \quad (33)$$

$$\mathcal{B}^2 = -\frac{1}{4}(\xi_{xx}^1 - \xi_{tx}^2)\,u^2 - u\,\mathcal{A}_x + \mathcal{H}^1(t,x). \quad (34)$$

Now, Equation (27) becomes

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$$[\frac{1}{2}(\xi_x^1-\xi_t^2)\,u+\mathcal{A}]f+(\xi_t^2+\xi_x^1)\int f(u)\,du-\mathcal{B}_t^1-\mathcal{B}_x^2=0. \quad (35)$$

Substituting the values of $\mathcal{B}_t^1$ and $\mathcal{B}_x^2$ in Equation (34) and then differentiating thrice $w.r.t.\ u$, we arrive at

$$f_{uuu}\,(\frac{1}{2}(\xi_x^1-\xi_t^2)\,u+\mathcal{A})+(\frac{5}{2}\xi_x^1-\frac{1}{2}\xi_t^2)f_{uu}=0. \quad (36)$$

Here, we have different subcases.

Subcase 2.1: If $f_{uu}\neq 0$.

Differentiating Equation (36) $w.r.t.\ u$, we get

$$\frac{1}{2}(\xi_x^1-\xi_t^2)\,(u\frac{f_{uuu}}{f_{uu}})_u+(\frac{f_{uuu}}{f_{uu}})_u\,\mathcal{A}=0. \quad (37)$$

Subsubcase 2.1.1: If $\frac{f_{uuu}}{f_{uu}}=m_1$ and $m_1\neq 0$.

In this case, we determine the following conserved components

$$\mathcal{T}^t=c_1\,e^{\alpha t}\,u_xu_t+\mathcal{H}^2,$$
$$\mathcal{T}^x=-c_1\,e^{\alpha t}\,(\frac{1}{2}u_x^2+\frac{1}{2}u_t^2+\int f(u)du)+\mathcal{H}^1.$$

Subsubcase 2.1.2: If $\frac{f_{uuu}}{f_{uu}}=m_1$ and $m_1=0$.

The case $m_1=0$ implies that $f(u)=\frac{1}{2}m_1u^2+m_2u+m_3$, providing the following conserved currents

$$\mathcal{T}^t=\mathcal{H}^2+c_3\,u_x\,u_t+\left(c_1\,e^{\frac{\alpha}{4}(x+5t)}+c_2\,e^{-\frac{\alpha}{4}(x+5t)}\right)$$
$$\left\{-\frac{1}{16}\alpha^2u^2-\frac{4\alpha^2}{(16m_2+\alpha^2)}m_3\,u+\alpha\,(\frac{1}{2}u+\frac{16}{(16m_2+\alpha^2)\,m_3}\,u_t+u_t^2\right\}$$
$$\left(c_1\,e^{\frac{\alpha}{4}(x+5t)}-c_2\,e^{-\frac{\alpha}{4}(x+5t)}\right)u_x\,u_t+(c_3\,\sin\sqrt{m_2}\,x+c_4\,\cos\sqrt{m_2}\,x)$$
$$\{u_t+\alpha u\}-\left(c_1\,e^{\frac{\alpha}{4}(x+5t)}+c_2\,e^{-\frac{\alpha}{4}(x+5t)}\right)(\frac{1}{2}(u_t^2-u_x^2)+\int f(u)du),$$

$$\mathcal{T}^x=\mathcal{H}^1-c_3\,u_x^2+\left(c_1\,e^{\frac{\alpha}{4}(x+5t)}-c_2\,e^{-\frac{\alpha}{4}(x+5t)}\right)$$
$$\left\{\frac{1}{16}\alpha^2u^2+\frac{4\alpha^2}{(16m_2+\alpha^2)}m_3\,u-\alpha\,(\frac{1}{2}u+\frac{16}{(16m_2+\alpha^2)\,m_3}\,u_x-u_x^2\right\}-$$
$$\left(c_1\,e^{\frac{\alpha}{4}(x+5t)}-c_2\,e^{-\frac{\alpha}{4}(x+5t)}\right)u_x\,u_t-(c_3\,\sin\sqrt{m_2}\,x+c_4\,\cos\sqrt{m_2}\,x)$$
$$u_x-(c_3\,\cos\sqrt{m_2}\,x-c_4\,\sin\sqrt{m_2}\,x)\sqrt{m_2}\,u-$$
$$\left(c_1\,e^{\frac{\alpha}{4}(x+5t)}-c_2\,e^{-\frac{\alpha}{4}(x+5t)}+c_3e^{\alpha t}\right)(\frac{1}{2}(u_t^2-u_x^2)+\int f(u)du).$$

where the constants c_1, c_2, c_3, and c_4 result in four independent conserved quantities, i.e., $\mathcal{T}_1=(\mathcal{T}_1^t,\mathcal{T}_1^x)$, $\mathcal{T}_2=(\mathcal{T}_2^t,\mathcal{T}_2^x)$, $\mathcal{T}_3=(\mathcal{T}_3^t,\mathcal{T}_3^x)$ and $\mathcal{T}_4=(\mathcal{T}_4^t,\mathcal{T}_4^x)$.

Subcase 2.2: $f_{uu}=0$.

Here, we deduce $f(u)=m_1\,u+m_2$, and inserting into Equation (36), the following equation arises

$$(2\,m_1+\frac{\alpha^2}{2})\,(\xi_t^2-\alpha\,\xi^2)=0.$$

Subsubcase 2.2.1: If $\xi_t^2-\alpha\,\xi^2=0$.

For this case, the conserved currents are

$$\mathcal{T}^t=\mathcal{H}^2-(c_1\,\cos\sqrt{m_1}x+c_2\,\sin\sqrt{m_1}x)\,(\alpha u+u_t)+(c_3\,t+c_5)\,e^{\alpha t}\,u_xu_t$$
$$+(c_3\,x+c_4)\,e^{\alpha t}\left(\frac{\alpha}{2}u\,u_t+\frac{\alpha}{2}\frac{m_2}{m_1}u_t+\frac{1}{2}u_t^2+\frac{1}{2}u_x^2-\int f(u)du\right),$$

$$\mathcal{T}^x=\mathcal{H}^1+(\frac{1}{4}u+\frac{1}{2}\frac{m_2}{m_1})\,c_3\,\alpha\,u\,e^{\alpha t}+(c_1\,\cos\sqrt{m_1}x+c_2\,\sin\sqrt{m_1}x)\,u_x-$$
$$(c_3\,x+c_4)\,e^{\alpha t}\left(\frac{\alpha}{2}u\,u_x+u_tu_x+\frac{\alpha}{2}\frac{m_2}{m_1}u_x\right)-(c_3\,t+c_5)\,e^{\alpha t}\,(\frac{1}{2}u_t^2+\frac{1}{2}u_x^2+$$
$$\int f(u)du)-\sqrt{m_1}\,u\,(c_2\,\cos\sqrt{m_1}x-c_1\sin\sqrt{m_1}x).$$

Subsubcase 2.2.2: $2\,m_1+\frac{\alpha^2}{2}=0$.

The subsequent conserved quantities are derived for this case

$$\mathcal{T}^t=\mathcal{H}^2-(c_1\,\sin\sqrt{m_1}x+c_2\,\cos\sqrt{m_1}x)\,(\alpha u+u_t)+h_3\,e^{\alpha t}\,u_xu_t+$$
$$(h_1+h_2)\,e^{\alpha t}\left(\frac{\alpha}{2}u\,u_t+\frac{1}{2}u_t^2+\frac{1}{2}u_x^2-\int f(u)du+\frac{\alpha}{2}\frac{m_2}{m_1}u_t-u_xu_t\right),$$

$$\mathcal{T}^x=\mathcal{H}^1-(c_1\,\cos\sqrt{m_1}x-c_2\,\sin\sqrt{m_1}x)\sqrt{m_1}\,u-h_3\,e^{\alpha t}\,(\frac{1}{2}(u_x^2+u_t^2)+$$
$$\int f(u)du)-(h_1+h_2)\,e^{\alpha t}\left(\frac{\alpha}{2}u\,u_x+u_tu_x+\frac{\alpha}{2}\frac{m_2}{m_1}u_x-\frac{1}{2}u_x^2-\frac{1}{2}u_t^2-\int f(u)du\right)$$
$$+u_x\,(c_2\,\cos\sqrt{m_1}x+c_1\,\sin\sqrt{m_1}x).$$

Subsubcase 2.2.3: $\xi^2=0=\xi^1$.

Equation (28) in this case becomes

$$\varphi=\mathcal{A}(t,x),$$

substituting it back into the determining system and additional equations, we find the following Pde in $\mathcal{A}$

$$\mathcal{A}_{tt}-\alpha\mathcal{A}_t-\mathcal{A}_{xx}-m_1\mathcal{A}=0.$$

The solution of this equation further provide three subcases for the construction of conservation laws:

$$\mathcal{D}=\frac{\alpha}{2}\pm\sqrt{\alpha^2-4\lambda}.$$

Subcase 2.2.3.1: If $\alpha^2-4\lambda=0$.

We obtain the following conserved vectors

$$\mathcal{T}^t=\mathcal{H}^2+\left(-\alpha\,u(c_2t+c_3)-(c_3+c_2t)\,u_t\right)e^{x\sqrt{m_1-\lambda}+\frac{\alpha}{2}t}+$$
$$\left(-\alpha\,u(c_4t+c_1)-(c_1+c_4t)\,u_t\right)e^{-x\sqrt{m_1-\lambda}+\frac{\alpha}{2}t},$$

$$\mathcal{T}^x=\mathcal{H}^1+\left(-\sqrt{m_1-\lambda}\,u\,(c_2t+c_3)+(c_3+c_2t)\,u_x\right)e^{x\sqrt{m_1-\lambda}+\frac{\alpha}{2}t}+$$
$$\left(-\sqrt{m_1-\lambda}\,u\,(c_1-c_4t)+(c_1+c_4t)\,u_x\right)e^{-x\sqrt{m_1-\lambda}+\frac{\alpha}{2}t}.$$

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Subcase 2.2.3.2: If $\alpha^2 - 4\lambda > 0$.
Let $\alpha^2 - 4\lambda = a$, we obtain the following conserved vectors

$$\mathcal{T}^t = \mathcal{H}^2 + \left(\left((a-\frac{\alpha}{2})u - u_t\right) c_3\right) \mathrm{e}^{x\sqrt{m_1-\lambda} + (\frac{\alpha}{2}+a)\,t} + \left(\left((a-\frac{\alpha}{2})u - u_t\right) c_1\right) \mathrm{e}^{-x\sqrt{m_1-\lambda}+(\frac{\alpha}{2}+a)\,t} - \left(\left((a+\frac{\alpha}{2})u + u_t\right) c_2\right) \mathrm{e}^{x\sqrt{m_1-\lambda}+(\frac{\alpha}{2}-a)\,t} - \left(\left((a+\frac{\alpha}{2})u + u_t\right) c_4\right) \mathrm{e}^{-x\sqrt{m_1-\lambda}+(\frac{\alpha}{2}-a)\,t},$$

$$\mathcal{T}^x = \mathcal{H}^1 + \left(\left((-\sqrt{m_1-\lambda})u + u_x\right) c_3\right) \mathrm{e}^{x\sqrt{m_1-\lambda} + (\frac{\alpha}{2}+a)\,t} + \left(\left((\sqrt{m_1-\lambda})u + u_x\right) c_1\right) \mathrm{e}^{-x\sqrt{m_1-\lambda}+(\frac{\alpha}{2}+a)\,t} - \left(\left((\sqrt{m_1-\lambda})u - u_x\right) c_2\right) \mathrm{e}^{x\sqrt{m_1-\lambda}+(\frac{\alpha}{2}-a)\,t} + \left(\left((\sqrt{m_1-\lambda})u + u_x\right) c_4\right) \mathrm{e}^{-x\sqrt{m_1-\lambda}+(\frac{\alpha}{2}-a)\,t}.$$

Subcase 2.2.3.3: If $\alpha^2 - 4\lambda < 0$.
For $\alpha^2 - 4\lambda = a$, we obtain the following conserved vectors

$$\mathcal{T}^t = \mathcal{H}^2 - \left(\cos at(\alpha u + u_t)\, c_3 + \sin at(\alpha u + u_t)\, c_2\right) \mathrm{e}^{x\sqrt{m_1-\lambda} + \frac{\alpha}{2}\,t} - \left(\cos at(\alpha u + u_t)\, c_1 + \sin at(\alpha u + u_t)\, c_4\right) \mathrm{e}^{-x\sqrt{m_1-\lambda} + \frac{\alpha}{2}\,t},$$

$$\mathcal{T}^x = \mathcal{H}^1 - \left(\cos at(\sqrt{m_1-\lambda} u - u_x)\, c_3 - \sin at(\alpha u - u_x)\, c_2\right) \mathrm{e}^{x\sqrt{m_1-\lambda} + \frac{\alpha}{2}\,t} + \left(\cos at(\alpha u + u_x)\, c_1 + \sin at(\alpha u + u_x)\, c_4\right) \mathrm{e}^{-x\sqrt{m_1-\lambda} + \frac{\alpha}{2}\,t}.$$

VII. CONCLUSION

We have studied the damped nonhomogeneous Klein-Gordon-Fock equation from the Lie symmetry and conservation law standpoints. We first performed the Lie symmetry analysis for arbitrary nonlinear damping function $\alpha(u)$ as well as nonhomogeneous function $f(u)$ in order to obtain reductions. Next, we classified the damping and nonhomogeneous functions and found the extended Lie symmetries and the optimal system of one-dimensional subalgebras for different cases. We then used these to reduce the Kgf equation into nonlinear ordinary differential equations. In certain cases, exact invariant solutions including the travelling wave solutions were found by solving these reduced equations. Some examples of soliton wave solutions and the graphical representation of these solutions were also represented in Section 5. In order to deduce further insight, we derived the local conservation laws using the partial Lagrangian approach via the partial Noether theorem. We found several cases for nonlinear damping and nonhomogeneous functions for the Kgf equation.

Regarding future work, further exploration will involve solving the set of reduced ordinary differential equations to gain further insight into the behaviour of the invariant solutions. This approach, known for its conciseness and efficacy, can also be applied to address other nonlinear partial differential equations.

The discovered conservation laws offer valuable tools for studying the dynamics and stability of physical systems, making this work relevant to a range of scientific endeavors that can be performed.

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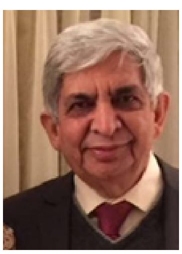

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