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Selection of Cloud Security by Employing MABAC Technique in the Environment of Hesitant Bipolar Complex Fuzzy Information

HAFIZ MUHAMMAD WAQAS[1], WALID EMAM[2], TAHIR MAHMOOD[1], UBAID UR REHMAN[1] AND SHI YIN[3]

[1]Department of Mathematics and Statistics, International Islamic University Islamabad 44000, Pakistan.
[2]Department of Statistics and Operations Research, Faculty of Science, King Saud University, Riyadh 11451, Saudi Arabia.
[3]College of Economics and Management, Hebei Agricultural University, Baoding 071001, China.
hafizmwaqas009@gmail.com; wemam.c@ksu.edu.sa; tahirbakhat@iiu.edu.pk;ubaid5@outlook.com; acadch@hrbeu.edu.cn;

Corresponding author: hafizmwaqas009@gmail.com

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ABSTRACT The term "cloud security (CS)" describes the collection of procedures and tools intended to defend networks, data, apps, and systems used in cloud computing from possible security risks and unauthorized access. Data breaches, identity and access management, network security, adherence to industry and governmental standards, and the security of third-party services and apps are a few of the major issues with CS. Selecting the best CS becomes critical for resolving all these problems. Within the context of hesitant bipolar complex fuzzy sets (HBCFSs) theory, we address in this study optimal selection utilizing various conceptions of aggregation operators (AOs). The notion of HBCFSs gives us a valuable framework by providing the hesitancy nature of any object along with its positive and negative aspects. Moreover, HBCFSs are a valuable tool to eliminate the vagueness and uncertainty of any given information. In this manuscript, by utilizing the framework of HBCFSs we developed some new AOs which are obliging to convert the set of information into a singleton value. Then by utilizing these AOs we calculate and aggregate all the numerical significance of CS. To handle our supposed problem of CS the mainly developed AOs are hesitant bipolar complex fuzzy (HBCF) weighted averaging (HBCFWA), HBCF ordered weighted averaging (HBCFOWA), HBCF weighted geometric (HBCFWG), HBCF ordered weighted geometric (HBCFOWG), generalized HBCF weighted averaging (GHBCFWA), generalized HBCF weighted geometric (GHBCFWG) operators. Furthermore, we develop the multi-attributive border approximation area comparison (MABAC) method to address our multi-attribute group decision-making (MAGDM) problem of CS. Moreover, in this manuscript, we propose and analyze a CS-related numerical case study to identify the optimal CS. Lastly; to demonstrate the advantages and superiority of the interpretive work, we compared our suggested methodology with other extant ideas.

INDEX TERMS Cloud security; hesitant bipolar complex fuzzy set; Aggregation Operators; MABAC technique; MAGDM.

I. INTRODUCTION

Cloud security is the practice of protecting cloud-based data, applications, and infrastructure from unauthorized access, theft, damage, or other cybersecurity threats. It is essential because cloud computing has become increasingly popular as businesses move their operations to the cloud to reduce costs, improve scalability, and enhance their ability to collaborate and access data from anywhere. Here are some reasons why cloud security is so important:

- Protection of data: Data is the basis on which any organization sits, and security in the cloud guarantees information safety from identity theft or data breaches. CS processes like encryption, access controls and data backup & recovery are the ways to save data from thieves and online criminals.

- Compliance: Some businesses must cope with industry regulations, e.g. HIPAA or General Data Protection Regulation, and they must ensure that they have data protection measures in place. Cloud security is meant to facilitate even the most demanding compliance level by providing suitable security measures and controls.
- Business continuity: Cloud security can aid in keeping business tasks critical and not to be interrupted due to cases of cyberattacks or other issues. The companies can respond to the incidents expeditiously through the security systems in place so that they can minimize downtime.
- Reputation: A data breach or cyber-attack will destroy the brand's reputation and can attract customers and revenue loss. Digital security might reduce the risk of such cases and guarantee the company's name.

Cloud computing security is of high order to safeguard an organization's classified data, guaranteeing compliance with regulations, continuity of business, and maintaining their brand value. Thus, Duncan and Whittington [1] discussed the importance of cloud security. Mather et al. [2] studied cloud security from an enterprise perspective. Kandukuri and Rakshit [3] described the issues related to cloud security. Al-Issa et al. [4] studied eHealth cloud security. The threats and solutions related to cloud security were investigated by Coppolino et al. [5]. Nassif et al. [6] interpreted machine learning in terms of cloud security. The could security technologies were discussed by Muttik and Barton [7]. Decision-makers (DMs) may not always be able to express their choices in terms of crisp information. Taking into consideration the constraints of crisp information the idea of the fuzzy set (FS) was developed by Zadeh [7]. It contains the membership degree (MD), which is restricted to the unit interval $[0,1]$. FS offers decision-makers a variety of options for resolving problems in daily life. They have been used by several researchers in various domains. Tariq et al. [8] investigated cloud computing with the assistance of fuzzy logic. Alruwavthi and Nygard [9] employed the technique of fuzzy logic for cloud security. Pandeeswari and Kumar [10] utilized fuzzy clustering in the setting of cloud security. Thakare et al. [11] employ fuzzy theory for the evaluation model of cloud security. No doubt FS is the backbone of FS theory, although FS has several restrictions. By extending the concept of FS, Torra [13] introduced the theory of hesitant fuzzy set (HFS) in which the MD of an element is the set of some values in $[0,1]$. HFS provides uncertain information more precisely than FS. Many researchers have utilized them in separate fields of life. Qian et al. [14] studied Generalized HFS and their use in decision support systems. Dual HFS (DHFS) was developed by Zhu et al. [15]. Beg and Rashid [16] introduce group decision-making (GDM) using intuitionistic HFS (IHFS). Mahmood et al. [17] studied Some generalized AOs for cubic HFS and their applications to multi-criteria decision-making (MCDM). Abbas et al. [18] discuss the concept of partitioned hammy mean AOs for MCGDM in the MAIRCA framework with q-rung ortho-pair fuzzy 2-tuple linguistic information. Abbas et al. [19] introduce the idea of an integrated GDM method under q-rung ortho-pair fuzzy 2-tuple linguistic context with partial weight information. Farhadinia [20] extended the knowledge of HFS and announced the concept of correlation for DHFS and dual interval-valued HFS. The hesitation on MD can be managed using numerous types of techniques. For example, Khan et al. [21] introduce the novel dual-partitioned Maclaurin symmetric mean operators for the selection of computer network security systems with complex intuitionistic FSs. Moreover, Khan et al. [22] discuss the concept of extension of the GRA method for MGDM problem under linguistic Pythagorean fuzzy setting with incomplete weight information. Wang and Li [23] initiate the picture hesitant fuzzy set (PHFS) and its application to MCDM. However, many DM problems are important to solve with the bipolar fuzzy (BF) information. BF information provides positive and negative facets of any item. Keeping these requirements in mind the idea of the bipolar fuzzy set (BFS) given by Zhang [24]. BFS is a couple of degrees namely positive membership degree (PMD) and negative membership degree (NMD). The PMD belongs to $[0,1]$ and NMD belongs to $[-1,0]$. The invention of BFS gives a large motivation to many researchers. Mandal and Ranadive [25] extended BFS into Hesitant BFS and bipolar-valued HFS and their applications in multi-attribute group decision-making (MAGDM). Wang et al. [26] invent the idea of HBF soft sets and their application in DM. Moreover, Riaz and Tehrim [27] introduce MAGDM based on cubic BF information using averaging AOs. Jana et al. [28] set up BF Dombi AOs and their application in the multiattribute attribute decision-making (MADM) process. Wei et al. [29] deduced Hamacher AOs for BFS. Gao et al. [30] utilized dual hesitant BF Hamacher prioritized AOs in MADM. The idea of novel bipolar soft rough set approximations and their application in DM was given by Gul et al. [31]. Over time, various researchers expressed their ideas and gave more work on BFS. Lu et al. [32] proposed Bipolar 2-tuple linguistic AOs in MADM. No doubt FS, IFS, PFS, HFS, and BFS can solve many DM problems, but all these sets cannot able to solve the information in complex form, to face that kind of problems in FS theory, Ramot et al. [33] modified the FS to invent complex fuzzy set (CFS). The geometric AOs for CFS were discussed by Bi et al. [34]. After the invention of CFS, researchers got into a huge field. Considering the limitations of CFS many researchers give their ideas in this field and Greenfield et al. [35] planned the idea of Interval-valued CF logic. Rani and Garg [36] set up distance measures among the complex intuitionistic FS and their applications to DM. Jan et al. [37] proposed the idea of a robust hybrid DM model for human-computer interaction in the environment of BCF picture FSs. Mahmood and Ur Rehman [38] developed the well-known notion of BCFS.

Mahmood and Ur Rehman [39] also utilized analysis and application of Aczel-Alsina AOs based on BCF information and application in MADM. Gwak et al. [40] introduce the hybrid integrated DM algorithm for clustering analysis based on BCF soft sets. Mahmood et al. [41] developed BCF soft sets (SSs) and their applications in DM. Mahmood and Ur Rehman [42] investigated Dombi AOs for BCF information and Ur Rehman and Mahmood [43] deduced dice similarity measures for BCF information. The MADM approach is one of the most expressive and reliable techniques that draw the attention of many researchers in diverse areas of science and technology. MADM is a genuine life method that can be stated as the advantages of mental and reasoning procedures for the cataloging and verification of appropriate alternatives established on defined attributes. In the preceding familiar works, the researchers have investigated a variety of MADM techniques, including the VIKOR approach, TOPOSIS approach, AHP approach, and many other methods using different fuzzy environments. The MABAC method was first developed by Pamucar and Cirovic [44] to calculate the distance between the border approximation area (BAA) and the alternatives. It has several characteristics, including (1) stable computing results based on the MABAC method, (2) straightforward equations for this purpose, and (3) ease of combination with other methods. Hence, the MABAC model is a useful tool for producing good DM results. Verma [45] expands the MABAC technique to FS. Jana [46] studied the MABAC approach under the BFS environment and Liu and Zhang [47] studied the hesitant fuzzy MABAC technique. Wang et al. [48] introduce the MABAC method for MAGDM under a q-rung ortho-pair fuzzy environment. Wei et al. [49] extend the MABAC method for MAGDM with probabilistic uncertain linguistic information. Verma [50] introduces the idea of intuitionistic fuzzy order-α divergence and entropy measures with the MABAC method for MAGDM. Moreover, Jiang et al. [51] discuss the concept of the picture fuzzy MABAC method based on prospect theory for MAGDM and its application to suppliers' selection. Jia et al. [52] introduce the MABAC method for MCGDM based on IF rough numbers. Peng and Dai [53] gave the idea about algorithms for interval neutrosophic MADM based on MABAC, similarity measure, and EDAS.

A. RESEARCH GAPS:

We observe that the application domains of HFSs, DHFSs, IHFSs, PHFSs, BHFSs, and BCFSs are related to some conditions and restrictions such that HFS is capable of dealing with hesitancy only on MD and cannot deal with any other aspect of any object. DHFSs, IHFSs, and PHFSs are considered to have two kinds of hesitation membership and non-membership with several restrictions. The DHFSs and IHFSs fulfill the requirement that the sum of their maximum membership and non-membership values is one or less. Moreover, DHFSs and IHFSs cannot manage the information that contains positive and negative aspects of any objects. PHFSs assure the condition that the sum of its maximum membership, non-membership, and natural grade with square is one or less. BHFSs provide hesitation on positive and negative aspects of an object and it is near to our theory but it also cannot solve our theory-related problems. BHFSs are similar to our theory because hesitancy and positive and negative aspects are the same as our theory aspects but due to the complex information like the imaginary part it is different from our work and this shows that the generalization of our work. FSs, HFSs, DHFSs, IHFSs, PHFSs, BHFSs, and CFSs have a wide range of abilities in many fields of life, but their environments and coordination mechanisms are constrained. Such structures handle the data that is sorted according to the term with just one dimension of data at a time, which results in a clear loss of different types of information. BCFS deals the complex environments or frameworks where it becomes essential to add another term for positive and negative grades. BCFS provides PMD and NMD in the model of the complex environment but the hesitation on PMD and NMD of BCFS cannot be developed yet. Therefore, the application of BCFS is restricted to solving DM problems in which DMs can't freely discuss their hesitations. So, this is the main motivation.

B. MOTIVATION OF THE PROPOSED WORK AND FOCUS OF THE STUDY:

To minimize the above-discussed problems in research gapes we need a new theory or framework that can manage all the aspects of any object like hesitancy nature aspect, positive and negative aspects at the same time. So, keeping in mind all those problems Aslam et al. [43] introduce the idea of HBCFSs and their Dombi AOs. Therefore, the motivation of this paper is to develop some new algebraic AOs for best CS. Moreover, our proposed theory of AOs provides a peaceful environment for DMs to their preferences and decisions in the form of hesitation. We also introduce the MABAC technique to handle HBCF information in MAGDM problems. If we apply our generalized HBCFS to handle GDM problems, there will be no restriction and no hesitation, and the result will be different as compared to other existing techniques or simple DM. Further, in this manuscript, we study and select CS with the assistance of the developed MABAC technique under HBCFSs. The main contributions of this manuscript are discussed below.

- Development of some new AOs such as HBCFWA, HBCFOWA, HBCFWG, and HBCFOWG. Further, we discuss some generalized AOs such as the GHBCFWA and GHBCFWG operators.
- Development of MAGDM technique using HBCF environment.
- Development of the MABAC model for making the best decision using the HBCF environment.

- Development of best CS case study and numerical examples.

- Comparative analysis of the proposed work.

C. ORGANIZATION OF THE PAPER:

This article is arraigned as follows in section 2 we discussed an overview of some elementary notions of HFSs, BCFSs, and HBCFSs with their basic operations and properties. In section 3 we construct some new AOs and their related theorems and properties. The MABAC model of the DM technique under the framework of HBCFSs and numerical examples of CS are discussed in section 4. In section 5, we compare our work with other prevailing notions in literature. Section 6 gives the concluding remarks of the overall work.

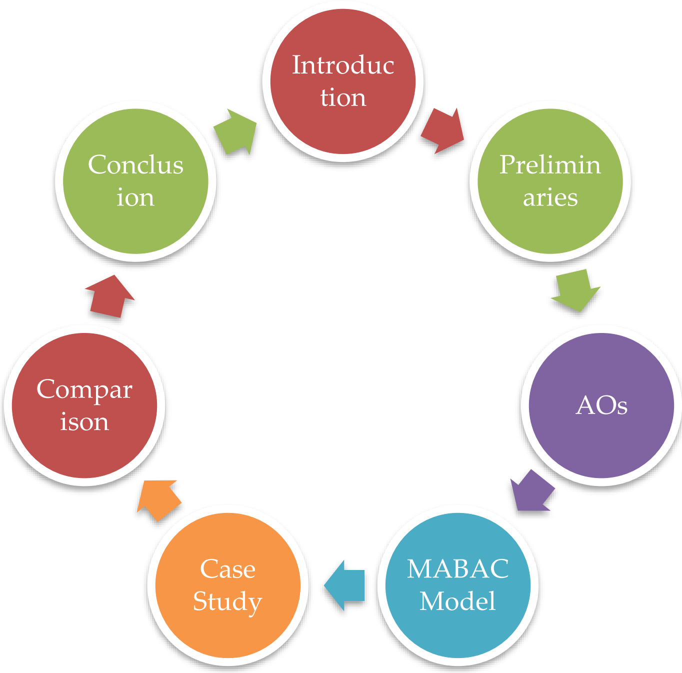

Figure 1: Graphical overview of the present manuscript.

II. PRELIMINARIES

Some elementary notions with their essential properties and operations which are related to our proposed work are reviewed in this section. We discussed here only HFSs, BCFSs and HBCFSs.

Definition 1: [13] Let $\hat{X}$ be a reference set. The HFS $\hat{\mathfrak{h}}$ on X is shown by

$$\hat{\mathfrak{h}} = \{< \breve{\epsilon}, \breve{I}_{\hat{\mathfrak{h}}}(\breve{\epsilon}) > \mid \breve{\epsilon} \in \hat{\mathfrak{h}}\} \quad (1)$$

Where $\breve{I}_{\hat{\mathfrak{h}}}(\breve{\epsilon})$ is the set of some finite values in $[0,1]$, shows the MD of each $\breve{\epsilon} \in \hat{\mathfrak{h}}$. For easiness $\breve{I} = \breve{I}_{\hat{\mathfrak{h}}}(\breve{\epsilon})$ Would be employed for hesitant fuzzy elements (HFE). $\Delta\hat{I}$

Definition 2: [13] For three HFEs $\breve{I}_1, \breve{I}_2$ and $\breve{I}_3$

a) $\breve{I}^c = \cup_{\ddot{3} \in \breve{I}}\{1 - \ddot{3}\}$,

b) $\breve{I}_1 \cup \breve{I}_2 = \cup_{\ddot{3}_1 \in \breve{I}_1, \ddot{3}_2 \in \breve{I}_2} \max\{\ddot{3}_1, \ddot{3}_2\}$,

c) $\breve{I}_1 \cap \breve{I}_2 = \cup_{\ddot{3}_1 \in \breve{I}_1, \ddot{3}_2 \in \breve{I}_2} \min\{\ddot{3}_1, \ddot{3}_2\}$

Definition 3: [13] Let $\breve{I}_1, \breve{I}_2$ and $\breve{I}_3$ be three HFEs and $\lambda > 0$, then.

a) $\breve{I}^{\lambda} = \cup_{\ddot{3} \in \breve{I}}\{\ddot{3}^{\lambda}\}$

b) $\lambda\breve{I} = \cup_{\ddot{3} \in \breve{I}}\{1 - (1 - \ddot{3})^{\lambda}\}$

c) $\breve{I}_1 \oplus \breve{I}_2 = \cup_{\ddot{3}_1 \in \breve{I}_1, \ddot{3}_2 \in \breve{I}_2}\{\ddot{3}_1 + \ddot{3}_2 - \ddot{3}_1\ddot{3}_2\}$

d) $\breve{I}_1 \otimes \breve{I}_2 = \cup_{\ddot{3}_1 \in \breve{I}_1, \ddot{3}_2 \in \breve{I}_2}\{\ddot{3}_1\ddot{3}_2\}$

Definition 4: [38] Let $\hat{X}$ be a reference set. The BCFS $\hat{\mathfrak{h}}$ on $\hat{X}$ is shown by

$$\hat{\mathfrak{h}} = \left\{< \breve{\epsilon}, \left(\breve{I}^{+}_{\hat{\mathfrak{h}}}(\breve{\epsilon}), \breve{I}^{-}_{\hat{\mathfrak{h}}}(\breve{\epsilon})\right) > \mid \breve{\epsilon} \in \hat{X}\right\} \quad (2)$$

Where $\breve{I}^{+}_{\hat{\mathfrak{h}}}(\breve{\epsilon})$ and $\breve{I}^{-}_{\hat{\mathfrak{h}}}(\breve{\epsilon})$ are PMD and NMD respectively for each $\breve{\epsilon} \in \hat{X}$. All values of $\breve{I}^{+}_{\hat{\mathfrak{h}}}(\breve{\epsilon})$ and $\breve{I}^{-}_{\hat{\mathfrak{h}}}(\breve{\epsilon})$ lies in the unit square of a complex plane. Note that the PMD and NMD are in the form of $\breve{I}^{+}_{\hat{\mathfrak{h}}}(\breve{\epsilon}) = \ddot{3}^{+\widehat{Re}}_{\hat{\mathfrak{h}}}(\breve{\epsilon}) + \iota\ddot{3}^{+\widehat{IM}}_{\hat{\mathfrak{h}}}(\breve{\epsilon})$, and $\breve{I}^{-}_{\hat{\mathfrak{h}}}(\breve{\epsilon}) = \ddot{3}^{-\widehat{Re}}_{\hat{\mathfrak{h}}}(\breve{\epsilon}) + \iota\ddot{3}^{-\widehat{IM}}_{\hat{\mathfrak{h}}}(\breve{\epsilon})$ respectively. Where $\ddot{3}^{+\widehat{Re}}_{\hat{\mathfrak{h}}}(\breve{\epsilon}), \ddot{3}^{+\widehat{IM}}_{\hat{\mathfrak{h}}}(\breve{\epsilon}) \in [0,1]$ and $\ddot{3}^{-\widehat{Re}}_{\hat{\mathfrak{h}}}(\breve{\epsilon}), \ddot{3}^{-\widehat{IM}}_{\hat{\mathfrak{h}}}(\breve{\epsilon}) \in [-1,0]$.

Definition 5: [38] Let $\breve{\dot{I}} = (\breve{\dot{I}}^{+}, \breve{\dot{I}}^{-}) = (ӟ^{+\widehat{Re}} + \iota ӟ^{+\widehat{IM}}, ӟ^{-\widehat{Re}} + \iota ӟ^{-\widehat{IM}})$, $\breve{\dot{I}}_1 = (\breve{\dot{I}}_1^{+}, \breve{\dot{I}}_1^{-}) = (ӟ_1^{+\widehat{Re}} + \iota ӟ_1^{+\widehat{IM}}, ӟ_1^{-\widehat{Re}} + \iota ӟ_1^{-\widehat{IM}})$ and $\breve{\dot{I}}_2 = (\breve{\dot{I}}_2^{+}, \breve{\dot{I}}_2^{-}) = (ӟ_2^{+\widehat{Re}} + \iota ӟ_2^{+\widehat{IM}}, ӟ_2^{-\widehat{Re}} + \iota ӟ_2^{-\widehat{IM}})$ are three BCFEs, then

a) $\breve{\dot{I}}^{c} = \left\{ \begin{matrix} \left((1 - ӟ^{+\widehat{Re}}) + \iota(1 - ӟ^{+\widehat{IM}}) \right), \\ \left((-1 - ӟ^{-\widehat{Re}}) + \iota(-1 - ӟ^{-\widehat{IM}}) \right) \end{matrix} \right\}$

b) $\breve{\dot{I}}_1 \cup \breve{\dot{I}}_2 = \left\{ \begin{pmatrix} (\max(ӟ_1^{+\widehat{Re}}, ӟ_2^{+\widehat{Re}}) + \iota \max(ӟ_1^{+\widehat{IM}}, ӟ_2^{+\widehat{IM}})), \\ (\min(ӟ_1^{-\widehat{Re}}, ӟ_2^{-\widehat{Re}}) + \iota \min(ӟ_1^{-\widehat{IM}}, ӟ_2^{-\widehat{IM}})) \end{pmatrix} \right\}$

c) $\breve{\dot{I}}_1 \cap \breve{\dot{I}}_2 = \left\{ \begin{pmatrix} (\min(ӟ_1^{+\widehat{Re}}, ӟ_2^{+\widehat{Re}}) + \iota \min(ӟ_1^{+\widehat{IM}}, ӟ_2^{+\widehat{IM}})), \\ (\max(ӟ_1^{-\widehat{Re}}, ӟ_2^{-\widehat{Re}}) + \iota \max(ӟ_1^{-\widehat{IM}}, ӟ_2^{-\widehat{IM}})) \end{pmatrix} \right\}$

Definition 6: [38] Let $\breve{\dot{I}} = (\breve{\dot{I}}^{+}, \breve{\dot{I}}^{-}) = (ӟ^{+\widehat{Re}} + \iota ӟ^{+\widehat{IM}}, ӟ^{-\widehat{Re}} + \iota ӟ^{-\widehat{IM}})$, $\breve{\dot{I}}_1 = (\breve{\dot{I}}_1^{+}, \breve{\dot{I}}_1^{-}) = (ӟ_1^{+\widehat{Re}} + \iota ӟ_1^{+\widehat{IM}}, ӟ_1^{-\widehat{Re}} + \iota ӟ_1^{-\widehat{IM}})$ and $\breve{\dot{I}}_2 = (\breve{\dot{I}}_2^{+}, \breve{\dot{I}}_2^{-}) = (ӟ_2^{+\widehat{Re}} + \iota ӟ_2^{+\widehat{IM}}, ӟ_2^{-\widehat{Re}} + \iota ӟ_2^{-\widehat{IM}})$ be three BCFEs, and $\lambda > 0$, Then

a) $\breve{\dot{I}}_1 \oplus \breve{\dot{I}}_2 = \begin{pmatrix} \begin{pmatrix} (ӟ_1^{+\widehat{Re}} + ӟ_2^{+\widehat{Re}} - ӟ_1^{+\widehat{Re}} ӟ_2^{+\widehat{Re}}) + \\ \iota(ӟ_1^{+\widehat{IM}} + ӟ_2^{+\widehat{IM}} - ӟ_1^{+\widehat{IM}} ӟ_2^{+\widehat{IM}}) \end{pmatrix}, \\ \left((ӟ_1^{+\widehat{Re}} ӟ_2^{+\widehat{Re}}) + \iota(ӟ_1^{+\widehat{IM}} ӟ_2^{+\widehat{IM}}) \right) \end{pmatrix}$

b) $\breve{\dot{I}}_1 \otimes \breve{\dot{I}}_2 = \begin{pmatrix} \left((ӟ_1^{+\widehat{Re}} ӟ_2^{+\widehat{Re}}) + \iota(ӟ_1^{+\widehat{IM}} ӟ_2^{+\widehat{IM}}) \right), \\ \begin{pmatrix} (ӟ_1^{-\widehat{Re}} + ӟ_2^{-\widehat{Re}} + ӟ_1^{-\widehat{Re}} ӟ_2^{-\widehat{Re}}) + \\ \iota(ӟ_1^{-\widehat{IM}} + ӟ_2^{-\widehat{IM}} + ӟ_1^{-\widehat{IM}} ӟ_2^{-\widehat{IM}}) \end{pmatrix} \end{pmatrix}$

c) $\breve{\dot{I}}^{\lambda} = \begin{pmatrix} \left((ӟ^{+\widehat{Re}})^{\lambda} + \iota (ӟ^{+\widehat{IM}})^{\lambda} \right), \\ \begin{pmatrix} \left(-1 + (1 + ӟ^{-\widehat{Re}})^{\lambda}\right) + \\ \iota\left(-1 + (1 + ӟ^{-\widehat{IM}})^{\lambda}\right) \end{pmatrix} \end{pmatrix}$

d) $\lambda \breve{\dot{I}} = \begin{pmatrix} \begin{pmatrix} \left(1 - (1 - ӟ^{+\widehat{Re}})^{\lambda}\right) + \\ \iota\left(1 - (1 - ӟ^{+\widehat{IM}})^{\lambda}\right) \end{pmatrix}, \\ \left(\left(-|ӟ^{-\widehat{Re}}|^{\lambda}\right) + \iota\left(-|ӟ^{-\widehat{IM}}|^{\lambda}\right) \right) \end{pmatrix}$

Definition 7: [38] Let $\breve{\dot{I}} = (\breve{\dot{I}}^{+}, \breve{\dot{I}}^{-}) = (ӟ^{+\widehat{Re}} + \iota ӟ^{+\widehat{IM}}, ӟ^{-\widehat{Re}} + \iota ӟ^{-\widehat{IM}})$ be a BCFE then the score and accuracy function are initiated by:

$$\breve{s}(\breve{\dot{I}}) = \frac{1}{4}(2 + ӟ^{+\widehat{Re}} + ӟ^{+\widehat{IM}} + ӟ^{-\widehat{Re}} + ӟ^{-\widehat{IM}}), \breve{s}(\breve{\dot{I}}) \in [0, 1]$$

$$\breve{\alpha}(\breve{\dot{I}}) = \frac{1}{4}(ӟ^{+\widehat{Re}} + ӟ^{+\widehat{IM}} - ӟ^{-\widehat{Re}} - ӟ^{-\widehat{IM}}), \quad \breve{\alpha}(\breve{\dot{I}}) \in [0, 1]$$

Definition 8: [54] Let $\hat{X}$ be a reference set. A HBCFS $\hat{ϟ}$ is shown as:

$$\hat{ϟ} = \{< \breve{\epsilon}, \breve{\dot{I}}_{\hat{ϟ}}(\breve{\epsilon}) > | \breve{\epsilon} \in \hat{X}\} = \{< \breve{\epsilon}, (\breve{\dot{I}}_{\hat{ϟ}}^{+}(\breve{\epsilon}), \breve{\dot{I}}_{\hat{ϟ}}^{-}(\breve{\epsilon})) > | \breve{\epsilon} \in \hat{X}\} \quad (3)$$

Where, $\breve{\dot{I}}_{\hat{ϟ}}^{+}(\breve{\epsilon}) = \{ӟ_{\hat{ϟ}_{j}}^{+\widehat{Re}}(\breve{\epsilon}) + \iota ӟ_{\hat{ϟ}_{j}}^{+\widehat{IM}}(\breve{\epsilon}), \quad j = 1,2, \ldots ŋ\}$ is a positive part of membership degree and $\breve{\dot{I}}_{\hat{ϟ}}^{-}(\breve{\epsilon}) = \{ӟ_{\hat{ϟ}_{ҝ}}^{-\widehat{Re}}(\breve{\epsilon}) + \iota ӟ_{\hat{ϟ}_{ҝ}}^{-\widehat{IM}}(\breve{\epsilon}), \quad ҝ = 1,2, \ldots ɱ\}$ is a negative part of the membership degree and both sets lie in the unit square of complex plane and $\forall j, ӟ_{\hat{ϟ}_{j}}^{+\widehat{Re}}(\breve{\epsilon}), ӟ_{\hat{ϟ}_{j}}^{+\widehat{IM}}(\breve{\epsilon}) \in [0, 1]$ and $\forall ҝ, ӟ_{\hat{ϟ}_{ҝ}}^{-\widehat{Re}}(\breve{\epsilon}), ӟ_{\hat{ϟ}_{ҝ}}^{-\widehat{IM}}(\breve{\epsilon}) \in [-1, 0]$. For simplicity, we shall use the symbol $\breve{\dot{I}} = (\breve{\dot{I}}^{+}, \breve{\dot{I}}^{-}) = (ӟ^{+\widehat{Re}} + \iota ӟ^{+\widehat{IM}}, ӟ^{-\widehat{Re}} + \iota ӟ^{-\widehat{IM}})$ For hesitant bipolar complex fuzzy element (HBCFE).

Definition 9: [54] Let $\breve{\dot{I}} = (\breve{\dot{I}}^{+}, \breve{\dot{I}}^{-}) = (ӟ^{+\widehat{Re}} + \iota ӟ^{+\widehat{IM}}, ӟ^{-\widehat{Re}} + \iota ӟ^{-\widehat{IM}})$, $\breve{\dot{I}}_1 = (\breve{\dot{I}}_1^{+}, \breve{\dot{I}}_1^{-}) = (ӟ_1^{+\widehat{Re}} + \iota ӟ_1^{+\widehat{IM}}, ӟ_1^{-\widehat{Re}} + \iota ӟ_1^{-\widehat{IM}})$ and $\breve{\dot{I}}_2 = (\breve{\dot{I}}_2^{+}, \breve{\dot{I}}_2^{-}) = (ӟ_2^{+\widehat{Re}} + \iota ӟ_2^{+\widehat{IM}}, ӟ_2^{-\widehat{Re}} + \iota ӟ_2^{-\widehat{IM}})$ be three HBCFEs, then

a) $\breve{\dot{I}}^{c} = \begin{pmatrix} \bigcup_{ӟ^{+} \in \breve{\dot{I}}^{+}} \left\{ \begin{matrix} (1 - ӟ^{+\widehat{Re}}) \\ +\iota(1 - ӟ^{+\widehat{IM}}) \end{matrix} \right\}, \\ \bigcup_{ӟ^{-} \in \breve{\dot{I}}^{-}} \left\{ \begin{matrix} (-1 - ӟ^{-\widehat{Re}}) \\ +\iota(-1 - ӟ^{-\widehat{IM}}) \end{matrix} \right\} \end{pmatrix}$

b) $\breve{\dot{I}}_1 \cup \breve{\dot{I}}_2 = \begin{pmatrix} \bigcup_{ӟ_1^{+} \in \breve{\dot{I}}_1^{+}, ӟ_2^{+} \in \breve{\dot{I}}_2^{+}} \left\{ \begin{matrix} \max(ӟ_1^{+\widehat{Re}}, ӟ_2^{+\widehat{Re}}) + \\ \iota \max(ӟ_1^{+\widehat{IM}}, ӟ_2^{+\widehat{IM}}) \end{matrix} \right\}, \\ \bigcup_{ӟ_1^{-} \in \breve{\dot{I}}_1^{-}, ӟ_2^{-} \in \breve{\dot{I}}_2^{-}} \left\{ \begin{matrix} \min(ӟ_1^{-\widehat{Re}}, ӟ_2^{-\widehat{Re}}) + \\ \iota \min(ӟ_1^{-\widehat{IM}}, ӟ_2^{-\widehat{IM}}) \end{matrix} \right\} \end{pmatrix}$

c) $\breve{\dot{I}}_1 \cap \breve{\dot{I}}_2 = \begin{pmatrix} \bigcup_{ӟ_1^{+} \in \breve{\dot{I}}_1^{+}, ӟ_2^{+} \in \breve{\dot{I}}_2^{+}} \left\{ \begin{matrix} \min(ӟ_1^{+\widehat{Re}}, ӟ_2^{+\widehat{Re}}) + \\ \iota \min(ӟ_1^{+\widehat{IM}}, ӟ_2^{+\widehat{IM}}) \end{matrix} \right\}, \\ \bigcup_{ӟ_1^{-} \in \breve{\dot{I}}_1^{-}, ӟ_2^{-} \in \breve{\dot{I}}_2^{-}} \left\{ \begin{matrix} max[ӟ_1^{-\widehat{Re}}, ӟ_2^{-\widehat{Re}}] + \\ \iota \max(ӟ_1^{-\widehat{IM}}, ӟ_2^{-\widehat{IM}}) \end{matrix} \right\} \end{pmatrix}$

Definition 10: [54] Let $\breve{\dot{I}}_1 = (\breve{\dot{I}}_1^{+}, \breve{\dot{I}}_1^{-}) = (ӟ_1^{+\widehat{Re}} + \iota ӟ_1^{+\widehat{IM}}, ӟ_1^{-\widehat{Re}} + \iota ӟ_1^{-\widehat{IM}})$ and $\breve{\dot{I}}_2 = (\breve{\dot{I}}_2^{+}, \breve{\dot{I}}_2^{-}) = (ӟ_2^{+\widehat{Re}} + \iota ӟ_2^{+\widehat{IM}}, ӟ_2^{-\widehat{Re}} + \iota ӟ_2^{-\widehat{IM}})$ be two HBCFEs and $\lambda > 0$, then

a) $\breve{\dot{I}}_1 \oplus \breve{\dot{I}}_2 =$

$$\begin{pmatrix} \bigcup_{ӟ_1^{+} \in \breve{\dot{I}}_1^{+}, ӟ_2^{+} \in \breve{\dot{I}}_2^{+}} \left\{ \begin{matrix} \begin{pmatrix} ӟ_1^{+\widehat{Re}} + ӟ_2^{+\widehat{Re}} \\ -ӟ_1^{+\widehat{Re}} ӟ_2^{+\widehat{Re}} \end{pmatrix} + \\ \iota \begin{pmatrix} ӟ_1^{+\widehat{IM}} + ӟ_2^{+\widehat{IM}} \\ -ӟ_1^{+\widehat{IM}} ӟ_2^{+\widehat{IM}} \end{pmatrix} \end{matrix} \right\}, \\ \bigcup_{ӟ_1^{-} \in \breve{\dot{I}}_1^{-}, ӟ_2^{-} \in \breve{\dot{I}}_2^{-}} \left\{ \begin{matrix} (-ӟ_1^{-\widehat{Re}} ӟ_2^{-\widehat{Re}}) + \\ \iota(-ӟ_1^{-\widehat{IM}} ӟ_2^{-\widehat{IM}}) \end{matrix} \right\} \end{pmatrix}$$

b) $\breve{\dot{I}}_1 \otimes \breve{\dot{I}}_2 =$

$$\begin{pmatrix} \bigcup_{ӟ_1^{+} \in \breve{\dot{I}}_1^{+}, ӟ_2^{+} \in \breve{\dot{I}}_2^{+}} \left\{ \begin{matrix} (ӟ_1^{+\widehat{Re}} ӟ_2^{+\widehat{Re}}) + \\ \iota(ӟ_1^{+\widehat{IM}} ӟ_2^{+\widehat{IM}}) \end{matrix} \right\}, \\ \bigcup_{ӟ_1^{-} \in \breve{\dot{I}}_1^{-}, ӟ_2^{-} \in \breve{\dot{I}}_2^{-}} \left\{ \begin{matrix} \begin{pmatrix} ӟ_1^{-\widehat{Re}} + ӟ_2^{-\widehat{Re}} + \\ ӟ_1^{-\widehat{Re}} ӟ_2^{-\widehat{Re}} \end{pmatrix} + \\ \iota \begin{pmatrix} ӟ_1^{-\widehat{IM}} + ӟ_2^{-\widehat{IM}} + \\ ӟ_1^{-\widehat{IM}} ӟ_2^{-\widehat{IM}} \end{pmatrix} \end{matrix} \right\} \end{pmatrix}$$

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c) $$\breve{I}^{\lambda} = \left(\begin{array}{c} \bigcup_{\ddot{z}^{+}\in\breve{I}^{+}} \left\{ \left(\left(\ddot{z}^{+\widehat{Re}}\right)^{\lambda}\right) + \iota\left(\left(\ddot{z}^{+\widehat{IM}}\right)^{\lambda}\right)\right\}, \\ \bigcup_{\ddot{z}^{-}\in\breve{I}^{-}} \left\{ \begin{array}{c} \left(-1+\left(1+\ddot{z}^{-\widehat{Re}}\right)^{\lambda}\right)+ \\ \iota\left(-1+\left(1+\ddot{z}^{-\widehat{IM}}\right)^{\lambda}\right)\end{array}\right\}\end{array}\right)$$

d) $$\lambda\breve{I} = \left(\begin{array}{c} \bigcup_{\ddot{z}^{+}\in\breve{I}^{+}} \left\{ \begin{array}{c} \left(1-\left(1-\ddot{z}^{+\widehat{Re}}\right)^{\lambda}\right)+ \\ \iota\left(1-\left(1-\ddot{z}^{+\widehat{IM}}\right)^{\lambda}\right)\end{array}\right\}, \\ \bigcup_{\ddot{z}^{-}\in\breve{I}^{-}} \left\{ \left(-\left|\ddot{z}^{-\widehat{Re}}\right|^{\lambda}\right) + \iota\left(-\left|\ddot{z}^{-\widehat{IM}}\right|^{\lambda}\right)\right\}\end{array}\right)$$

Definition 11: [54] Let $\breve{I} = \left(\breve{I}^{+},\breve{I}^{-}\right) = \left(\ddot{z}^{+\widehat{Re}} + \iota\ddot{z}^{+\widehat{IM}}, \ddot{z}^{-\widehat{Re}} + \iota\ddot{z}^{-\widehat{IM}}\right)$ be an HBCFE then the score and accuracy function are initiated by:

$$\breve{S}\left(\breve{I}\right) = \frac{1}{4}\left(\begin{array}{c} 2 + \frac{1}{l_{\ddot{z}^{+\widehat{Re}}}}\sum_{\ddot{z}^{+}\in\breve{I}^{+}} \ddot{z}^{+\widehat{Re}} + \frac{1}{l_{\ddot{z}^{+\widehat{IM}}}}\sum_{\ddot{z}^{+}\in\breve{I}^{+}} \ddot{z}^{+\widehat{IM}} + \\ \frac{1}{l_{\ddot{z}^{-\widehat{Re}}}}\sum_{\ddot{z}^{-}\in\breve{I}^{-}} \ddot{z}^{-\widehat{Re}} + \frac{1}{l_{\ddot{z}^{-IM}}}\sum_{\ddot{z}^{-}\in\breve{I}^{-}} \ddot{z}^{-\widehat{IM}} \end{array}\right),$$
$$\breve{S}\left(\breve{I}\right) \in [0,1] \quad (4)$$

$$\breve{A}\left(\breve{I}\right) = \frac{1}{4}\left(\begin{array}{c} \frac{1}{l_{\ddot{z}^{+\widehat{Re}}}}\sum_{\ddot{z}^{+}\in\breve{I}^{+}} \ddot{z}^{+\widehat{Re}} + \frac{1}{l_{\ddot{z}^{+\widehat{IM}}}}\sum_{\ddot{z}^{+}\in\breve{I}^{+}} \ddot{z}^{+\widehat{IM}} \\ -\frac{1}{l_{\ddot{z}^{-\widehat{Re}}}}\sum_{\ddot{z}^{-}\in\breve{I}^{-}} \ddot{z}^{-\widehat{Re}} - \frac{1}{l_{\ddot{z}^{-IM}}}\sum_{\ddot{z}^{-}\in\breve{I}^{-}} \ddot{z}^{-\widehat{IM}} \end{array}\right),$$
$$\breve{A}\left(\breve{I}\right) \in [0,1] \quad (5)$$

Where l is the length of numbers, using the above equations (4) and (5) for two HBCFEs $\breve{I}_1$ and $\breve{I}_2$we have the following criteria for comparison if $\breve{S}\left(\breve{I}_1\right) < \breve{S}\left(\breve{I}_2\right)$, then $\breve{I}_1 < \breve{I}_2$, if $\breve{S}\left(\breve{I}_1\right) > \breve{S}\left(\breve{I}_2\right)$, then $\breve{I}_1 > \breve{I}_2$ if $\breve{S}\left(\breve{I}_1\right) = \breve{S}\left(\breve{I}_2\right)$, then $\breve{A}\left(\breve{I}_1\right) < \breve{A}\left(\breve{I}_2\right)$, then $\breve{I}_1 < \breve{I}_2$ if $\breve{A}\left(\breve{I}_1\right) > \breve{A}\left(\breve{I}_2\right)$, then $\breve{I}_1 > \breve{I}_2$ if $\breve{A}\left(\breve{I}_1\right) = \breve{A}\left(\breve{I}_2\right)$ then $\breve{I}_1 = \breve{I}_2$.

Definition 12: Let $\breve{I}_1 = \left(\breve{I}_1^{+},\breve{I}_1^{-}\right) = \left(\ddot{z}_1^{+\widehat{Re}} + \iota\ddot{z}_1^{+\widehat{IM}}, \ddot{z}_1^{-\widehat{Re}} + \iota\ddot{z}_1^{-\widehat{IM}}\right)$ and $\breve{I}_2 = \left(\breve{I}_2^{+},\breve{I}_2^{-}\right) = \left(\ddot{z}_2^{+\widehat{Re}} + \iota\ddot{z}_2^{+\widehat{IM}}, \ddot{z}_2^{-\widehat{Re}} + \iota\ddot{z}_2^{-\widehat{IM}}\right)$ be two HBCFEs, then the distance between $\breve{I}_1$ and $\breve{I}_2$ is defined as:

$$d\left(\breve{I}_1,\breve{I}_2\right) = \frac{1}{4}\left(\begin{array}{c} \frac{1}{l}\sum_{\ddot{z}_1^{+\widehat{Re}}\in\breve{I}_1^{+},\ddot{z}_2^{+\widehat{Re}}\in\breve{I}_2^{+}} \left|\ddot{z}_1^{+\widehat{Re}} - \ddot{z}_2^{+\widehat{Re}}\right| + \\ \frac{1}{l}\sum_{\ddot{z}_1^{+\widehat{IM}}\in\breve{I}_1^{+},\ddot{z}_2^{+\widehat{IM}}\in\breve{I}_2^{+}} \left|\ddot{z}_1^{+\widehat{IM}} - \ddot{z}_2^{+\widehat{IM}}\right| \\ +\frac{1}{l}\sum_{\ddot{z}_1^{-\widehat{Re}}\in\breve{I}_1^{-},\ddot{z}_2^{-\widehat{Re}}\in\breve{I}_2^{-}} \left|\ddot{z}_1^{-\widehat{Re}} - \ddot{z}_2^{-\widehat{Re}}\right| + \\ \frac{1}{l}\sum_{\ddot{z}_1^{-\widehat{IM}}\in\breve{I}_1^{-},\ddot{z}_2^{-\widehat{IM}}\in\breve{I}_2^{-}} \left|\ddot{z}_1^{-\widehat{IM}} - \ddot{z}_2^{-\widehat{IM}}\right| \end{array}\right) \quad (6)$$

III. AGGREGATION OPERATORS FOR HBCFEs

In this section, we develop several new AOs such as HBCFWA, HBCFOWA, HBCFWG, HBCFOWG, GHBCFWA, and GHBCFWG operators based on HBCFEs and discussed their properties.

Definition 13: Let $\breve{I}_t = \left(\breve{I}_t^{+},\breve{I}_t^{-}\right)(t = 1,2,\ldots,n)$ be a collection of HBCFEs and $\dot{W} = \left(\dot{W}_1,\dot{W}_2,\ldots,\dot{W}_n\right)^T$ be the weight vector (WV) of $\breve{I}_t = \left(\breve{I}_t^{+},\breve{I}_t^{-}\right)(t = 1,2,\ldots,n)$ with $\dot{W}_t \in [0,1], \sum_{t=1}^{n}\dot{W}_t = 1$, then the hesitant bipolar complex fuzzy weighted averaging (HBCFWA) operator is a mapping HBCFWA: $E^n \longrightarrow E$, where

$$\mathrm{HBCFWA}\left(\breve{I}_1,\breve{I}_2,\ldots,\breve{I}_n\right) = \oplus_{t=1}^{n}\dot{W}_t\breve{I}_t = \dot{W}_1\breve{I}_1 \oplus \dot{W}_2\breve{I}_2 \oplus \ldots\ldots \oplus \dot{W}_n\breve{I}_n \quad (7)$$

Theorem 1: By employing the above eq. (14), we get the HBCFEs and

$$\mathrm{HBCFWA}\left(\breve{I}_1,\breve{I}_2,\ldots,\breve{I}_n\right) = \left(\begin{array}{c} \bigcup_{\ddot{z}_1^{+}\in\breve{I}_1^{+},\ldots,\ddot{z}_n^{+}\in\breve{I}_n^{+}} \left\{ \begin{array}{c} \left(1-\prod_{t=1}^{n}\left(1-\ddot{z}_t^{+\widehat{Re}}\right)^{\dot{W}_t}\right)+ \\ \iota\left(1-\prod_{t=1}^{n}\left(1-\ddot{z}_t^{+\widehat{IM}}\right)^{\dot{W}_t}\right)\end{array}\right\}, \\ \bigcup_{\ddot{z}_1^{-}\in\breve{I}_1^{-},\ldots,\ddot{z}_n^{-}\in\breve{I}_n^{-}} \left\{ \begin{array}{c} \left(-\prod_{t=1}^{n}\left(\left|\ddot{z}_t^{-\widehat{Re}}\right|\right)^{\dot{W}_t}\right)+ \\ \iota\left(-\prod_{t=1}^{n}\left(\left|\ddot{z}_t^{-\widehat{IM}}\right|\right)^{\dot{W}_t}\right)\end{array}\right\} \end{array}\right) \quad (8)$$

Proof: We prove eq. (8) by using a well-known method of mathematical induction (MI), assume for $n = 2$, we have

$$\dot{W}_1\breve{I}_1 = \left(\begin{array}{c} \bigcup_{\ddot{z}_1^{+}\in\breve{I}_1^{+}} \left\{ \left(1-\left(1-\ddot{z}_1^{+\widehat{Re}}\right)^{\dot{W}_1}\right) + \iota\left(1-\left(1-\ddot{z}_1^{+\widehat{IM}}\right)^{\dot{W}_1}\right)\right\}, \\ \bigcup_{\ddot{z}_1^{-}\in\breve{I}_1^{-}} \left\{ \left(-\left|\ddot{z}_1^{-\widehat{Re}}\right|^{\dot{W}_1}\right) + \iota\left(-\left|\ddot{z}_1^{-\widehat{IM}}\right|^{\dot{W}_1}\right)\right\} \end{array}\right)$$

$$\dot{W}_2\breve{I}_2 = \left(\begin{array}{c} \bigcup_{\ddot{z}_2^{+}\in\breve{I}_2^{+}} \left\{ \left(1-\left(1-\ddot{z}_2^{+\widehat{Re}}\right)^{\dot{W}_2}\right) + \iota\left(1-\left(1-\ddot{z}_2^{+\widehat{IM}}\right)^{\dot{W}_2}\right)\right\}, \\ \bigcup_{\ddot{z}_2^{-}\in\breve{I}_2^{-}} \left\{ \left(-\left|\ddot{z}_2^{-\widehat{Re}}\right|^{\dot{W}_2}\right) + \iota\left(-\left|\ddot{z}_2^{-\widehat{IM}}\right|^{\dot{W}_2}\right)\right\} \end{array}\right)$$

$$\dot{W}_1\breve{I}_1 \oplus \dot{W}_2\breve{I}_2 = \left(\begin{array}{c} \bigcup_{\ddot{z}_1^{+}\in\breve{I}_1^{+}} \left\{ \begin{array}{c} \left(1-\left(1-\ddot{z}_1^{+\widehat{Re}}\right)^{\dot{W}_1}\right) \\ +\iota\left(1-\left(1-\ddot{z}_1^{+\widehat{IM}}\right)^{\dot{W}_1}\right)\end{array}\right\}, \\ \bigcup_{\ddot{z}_1^{-}\in\breve{I}_1^{-}} \left\{ \left(-\left|\ddot{z}_1^{-\widehat{Re}}\right|^{\dot{W}_1}\right) + \iota\left(-\left|\ddot{z}_1^{-\widehat{IM}}\right|^{\dot{W}_1}\right)\right\} \end{array}\right)$$

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$$\oplus\left(\begin{array}{c}\bigcup_{\ddot{z}_2^{+}\in\breve{I}_2^{+}}\left\{\begin{array}{c}\left(1-\left(1-\ddot{z}_2^{+\widehat{Re}}\right)^{\dot{W}_2}\right)\\+\iota\left(1-\left(1-\ddot{z}_2^{+\widehat{IM}}\right)^{\dot{W}_2}\right)\end{array}\right\},\\\bigcup_{\ddot{z}_2^{-}\in\breve{I}_2^{-}}\left\{\left(-\left|\ddot{z}_2^{-\widehat{Re}}\right|^{\dot{W}_2}\right)+\iota\left(-\left|\ddot{z}_2^{-\widehat{IM}}\right|^{\dot{W}_2}\right)\right\}\end{array}\right)$$

$$=\left(\begin{array}{c}\bigcup_{\ddot{z}_1^{+}\in\breve{I}_1^{+},\ddot{z}_2^{+}\in\breve{I}_2^{+}}\left\{\begin{array}{c}1-\left(1-\ddot{z}_1^{+\widehat{Re}}\right)^{\dot{W}_1}\left(1-\ddot{z}_2^{+\widehat{Re}}\right)^{\dot{W}_2}+\\\iota\left(1-\left(1-\ddot{z}_1^{+\widehat{IM}}\right)^{\dot{W}_1}\left(1-\ddot{z}_2^{+\widehat{IM}}\right)^{\dot{W}_2}\right)\end{array}\right\},\\\bigcup_{\ddot{z}_1^{-}\in\breve{I}_1^{-},\ddot{z}_2^{-}\in\breve{I}_2^{-}}\left\{\begin{array}{c}-\left(\left|\ddot{z}_1^{-\widehat{Re}}\right|^{\dot{W}_1}\left|\ddot{z}_2^{-\widehat{Re}}\right|^{\dot{W}_2}\right)+\\\iota\left(-\left(\left|\ddot{z}_1^{-\widehat{IM}}\right|^{\dot{W}_1}\left|\ddot{z}_2^{-\widehat{IM}}\right|^{\dot{W}_2}\right)\right)\end{array}\right\}\end{array}\right)$$

$$=\left(\begin{array}{c}\bigcup_{\ddot{z}_1^{+}\in\breve{I}_1^{+},\ddot{z}_2^{+}\in\breve{I}_2^{+}}\left\{\begin{array}{c}\left(1-\prod_{ŧ=1}^{2}\left(1-\ddot{z}_ŧ^{+\widehat{Re}}\right)^{\dot{W}_ŧ}\right)+\\\iota\left(1-\prod_{ŧ=1}^{2}\left(1-\ddot{z}_ŧ^{+\widehat{IM}}\right)^{\dot{W}_ŧ}\right)\end{array}\right\},\\\bigcup_{\ddot{z}_1^{-}\in\breve{I}_1^{-},\ddot{z}_2^{-}\in\breve{I}_2^{-}}\left\{\begin{array}{c}\left(-\prod_{ŧ=1}^{2}\left(\left|\ddot{z}_ŧ^{-\widehat{Re}}\right|\right)^{\dot{W}_ŧ}\right)+\\\iota\left(-\prod_{ŧ=1}^{2}\left(\left|\ddot{z}_ŧ^{-\widehat{IM}}\right|\right)^{\dot{W}_ŧ}\right)\end{array}\right\}\end{array}\right)$$

Next, we suppose that Eq. (8) is true for $\underline{n}=Ķ$ so,

$$HBCFWA\left(\breve{I}_1,\breve{I}_2,\ldots,\breve{I}_Ķ\right)=\left(\begin{array}{c}\bigcup_{\ddot{z}_1^{+}\in\breve{I}_1^{+},\ldots,\ddot{z}_Ķ^{+}\in\breve{I}_Ķ^{+}}\left\{\begin{array}{c}\left(1-\prod_{ŧ=1}^{Ķ}\left(1-\ddot{z}_ŧ^{+\widehat{Re}}\right)^{\dot{W}_ŧ}\right)+\\\iota\left(1-\prod_{ŧ=1}^{Ķ}\left(1-\ddot{z}_ŧ^{+\widehat{IM}}\right)^{\dot{W}_ŧ}\right)\end{array}\right\},\\\bigcup_{\ddot{z}_1^{-}\in\breve{I}_1^{-},\ldots,\ddot{z}_Ķ^{-}\in\breve{I}_Ķ^{-}}\left\{\begin{array}{c}\left(-\prod_{ŧ=1}^{Ķ}\left(\left|\ddot{z}_ŧ^{-\widehat{Re}}\right|\right)^{\dot{W}_ŧ}\right)+\\\iota\left(-\prod_{ŧ=1}^{Ķ}\left(\left|\ddot{z}_ŧ^{-\widehat{IM}}\right|\right)^{\dot{W}_ŧ}\right)\end{array}\right\}\end{array}\right)$$

For $\underline{n}=Ķ+1$, we have

$$HBCFWA\left(\breve{I}_1,\breve{I}_2,\ldots,\breve{I}_Ķ,\breve{I}_{Ķ+1}\right)$$

$$=\left(\begin{array}{c}\bigcup_{\ddot{z}_1^{+}\in\breve{I}_1^{+},\ldots,\ddot{z}_Ķ^{+}\in\breve{I}_Ķ^{+}}\left\{\begin{array}{c}\left(1-\prod_{ŧ=1}^{Ķ}\left(1-\ddot{z}_ŧ^{+\widehat{Re}}\right)^{\dot{W}_ŧ}\right)+\\\iota\left(1-\prod_{ŧ=1}^{Ķ}\left(1-\ddot{z}_ŧ^{+\widehat{IM}}\right)^{\dot{W}_ŧ}\right)\end{array}\right\},\\\bigcup_{\ddot{z}_1^{-}\in\breve{I}_1^{-},\ldots,\ddot{z}_Ķ^{-}\in\breve{I}_Ķ^{-}}\left\{\begin{array}{c}\left(-\prod_{ŧ=1}^{Ķ}\left(\left|\ddot{z}_ŧ^{-\widehat{Re}}\right|\right)^{\dot{W}_ŧ}\right)+\\\iota\left(-\prod_{ŧ=1}^{Ķ}\left(\left|\ddot{z}_ŧ^{-\widehat{IM}}\right|\right)^{\dot{W}_ŧ}\right)\end{array}\right\}\end{array}\right)$$

$$\oplus\left(\begin{array}{c}\bigcup_{\ddot{z}_{Ķ+1}^{+}\in\breve{I}_{Ķ+1}^{+}}\left\{\begin{array}{c}\left(1-\left(1-\ddot{z}_{Ķ+1}^{+\widehat{Re}}\right)^{\dot{W}_{Ķ+1}}\right)+\\\iota\left(1-\left(1-\ddot{z}_{Ķ+1}^{+\widehat{IM}}\right)^{\dot{W}_{Ķ+1}}\right)\end{array}\right\},\\\bigcup_{\ddot{z}_{Ķ+1}^{-}\in\breve{I}_{Ķ+1}^{-}}\left\{\left(-\left|\ddot{z}_{Ķ+1}^{-\widehat{Re}}\right|^{\dot{W}_{Ķ+1}}\right)+\iota\left(-\left|\ddot{z}_{Ķ+1}^{-\widehat{IM}}\right|^{\dot{W}_{Ķ+1}}\right)\right\}\end{array}\right)$$

$$=\left(\begin{array}{c}\bigcup_{\ddot{z}_1^{+}\in\breve{I}_1^{+},\ldots,\ddot{z}_Ķ^{+}\in\breve{I}_Ķ^{+}}\left\{\begin{array}{c}\left(1-\prod_{ŧ=1}^{Ķ}\left(1-\ddot{z}_ŧ^{+\widehat{Re}}\right)^{\dot{W}_ŧ}\right)+\\\left(1-\left(1-\ddot{z}_{Ķ+1}^{+\widehat{Re}}\right)^{\dot{W}_{Ķ+1}}\right)-\\\left(1-\prod_{ŧ=1}^{Ķ}\left(1-\ddot{z}_ŧ^{+\widehat{Re}}\right)^{\dot{W}_ŧ}\right)\\\left(1-\left(1-\ddot{z}_{Ķ+1}^{+\widehat{Re}}\right)^{\dot{W}_{Ķ+1}}\right)\\+\iota\left(\begin{array}{c}\left(1-\prod_{ŧ=1}^{Ķ}\left(1-\ddot{z}_ŧ^{+\widehat{IM}}\right)^{\dot{W}_ŧ}\right)+\\\left(1-\left(1-\ddot{z}_{Ķ+1}^{+\widehat{IM}}\right)^{\dot{W}_{Ķ+1}}\right)-\\\left(1-\prod_{ŧ=1}^{Ķ}\left(1-\ddot{z}_ŧ^{+\widehat{IM}}\right)^{\dot{W}_ŧ}\right)\\\left(1-\left(1-\ddot{z}_{Ķ+1}^{+\widehat{IM}}\right)^{\dot{W}_{Ķ+1}}\right)\end{array}\right)\end{array}\right\},\\\bigcup_{\ddot{z}_1^{-}\in\breve{I}_1^{-},\ldots,\ddot{z}_Ķ^{-}\in\breve{I}_Ķ^{-}}\left\{\begin{array}{c}-\left(\begin{array}{c}\left(-\prod_{ŧ=1}^{Ķ}\left(\left|\ddot{z}_ŧ^{-\widehat{Re}}\right|\right)^{\dot{W}_ŧ}\right)\\\left(-\left|\ddot{z}_{Ķ+1}^{-\widehat{Re}}\right|^{\dot{W}_{Ķ+1}}\right)\end{array}\right)\\+\iota\left(\begin{array}{c}-\left(-\prod_{ŧ=1}^{Ķ}\left(\left|\ddot{z}_ŧ^{-\widehat{IM}}\right|\right)^{\dot{W}_ŧ}\right)\\\left(-\left|\ddot{z}_{Ķ+1}^{-\widehat{IM}}\right|^{\dot{W}_{Ķ+1}}\right)\end{array}\right)\end{array}\right\}\end{array}\right)$$

$$= \left(\begin{array}{c} \bigcup_{\ddot{z}_1^+\in\breve{I}_1^+,\dots,\ddot{z}_K^+\in\breve{I}_K^+} \left\{ \begin{array}{c} \left(\frac{1-\prod_{t=1}^{K}\left(1-\ddot{z}_t^{+\widehat{Re}}\right)^{\dot{W}_t}}{\left(1-\ddot{z}_{K+1}^{+\widehat{Re}}\right)^{\dot{W}_{K+1}}} \right) \\ +\iota\left(\frac{1-\prod_{t=1}^{K}\left(1-\ddot{z}_t^{+\widehat{IM}}\right)^{\dot{W}_t}}{\left(1-\ddot{z}_{K+1}^{+\widehat{IM}}\right)^{\dot{W}_{K+1}}} \right) \end{array} \right\}, \\ \bigcup_{\ddot{z}_1^-\in\breve{I}_1^-,\dots,\ddot{z}_K^-\in\breve{I}_K^-} \left\{ \begin{array}{c} \left(\frac{-\left(\prod_{t=1}^{K}\left(\left|\ddot{z}_t^{-\widehat{Re}}\right|\right)^{\dot{W}_t}\right)}{\left(\left|\ddot{z}_{K+1}^{-\widehat{Re}}\right|\right)^{\dot{W}_{K+1}}} \right) + \\ \iota\left(\frac{-\left(\prod_{t=1}^{K}\left(\left|\ddot{z}_t^{-\widehat{IM}}\right|\right)^{\dot{W}_t}\right)}{\left(\left|\ddot{z}_{K+1}^{-\widehat{IM}}\right|\right)^{\dot{W}_{K+1}}} \right) \end{array} \right\} \end{array} \right)$$

$$= \left(\begin{array}{c} \bigcup_{\ddot{z}_1^+\in\breve{I}_1^+,\dots,\ddot{z}_K^+\in\breve{I}_K^+,\ddot{z}_{K+1}^+\in\breve{I}_{K+1}^+} \left\{ \begin{array}{c} \left(1-\prod_{t=1}^{K+1}\left(1-\ddot{z}_t^{+\widehat{Re}}\right)^{\dot{W}_t}\right)+ \\ \iota\left(1-\prod_{t=1}^{K+1}\left(1-\ddot{z}_t^{+\widehat{IM}}\right)^{\dot{W}_t}\right) \end{array} \right\}, \\ \bigcup_{\ddot{z}_1^+\in\breve{I}_1^+,\dots,\ddot{z}_K^+\in\breve{I}_K^+,\ddot{z}_{K+1}^+\in\breve{I}_{K+1}^+} \left\{ \begin{array}{c} -\prod_{t=1}^{K+1}\left(\left|\ddot{z}_t^{-\widehat{Re}}\right|\right)^{\dot{W}_t}+ \\ \iota\left(-\prod_{t=1}^{K+1}\left(\left|\ddot{z}_t^{-\widehat{IM}}\right|\right)^{\dot{W}_t}\right) \end{array} \right\} \end{array} \right)$$

This shows that Eq. (8) holds for $\underline{n} \geq 0$.

The following properties are held for HBCFWA operators.

Theorem 2: (Idempotency property) Let us assume $\breve{I}_t = (\breve{I}_t^+, \breve{I}_t^-) = (\ddot{z}_t^{+\widehat{Re}} + \iota\ddot{z}_t^{+\widehat{IM}}, \ddot{z}_t^{-\widehat{Re}} + \iota\ddot{z}_t^{-\widehat{IM}})(t = 1,2,\dots,\underline{n})$ be the set of HBCFEs, and $\dot{W} = (\dot{W}_1, \dot{W}_2, \dots, \dot{W}_{\underline{n}})^T$ be the well-known weights of the HBCFEs $\breve{I}_t = (\breve{I}_t^+, \breve{I}_t^-)(t = 1,2,\dots,\underline{n})$. Moreover, note that $\dot{W}_t \in [0,1], \sum_{t=1}^{\underline{n}} \dot{W}_t = 1$. If $\breve{I}_t = \breve{I}\ \forall\, t$ then to show the

$$HBCFWA(\breve{I}_1, \breve{I}_2, \dots, \breve{I}_{\underline{n}}) = \breve{I} \tag{9}$$

Theorem 3: (Monotonicity property) Let $\breve{I}_t = (\breve{I}_t^+, \breve{I}_t^-) = (\ddot{z}_t^{+\widehat{Re}} + \iota\ddot{z}_t^{+\widehat{IM}}, \ddot{z}_t^{-\widehat{Re}} + \iota\ddot{z}_t^{-\widehat{IM}})$ and $\breve{I}'_t = (\breve{I}'^+_t, \breve{I}'^-_t) = (\ddot{z}'^{+\widehat{Re}}_t + \iota\ddot{z}'^{+\widehat{IM}}_t, \ddot{z}'^{-\widehat{Re}}_t + \iota\ddot{z}'^{-\widehat{IM}}_t)\ (t = 1,2,\dots,\underline{n})$ be a collection of two HBCFEs and $\dot{W} = (\dot{W}_1, \dot{W}_2, \dots, \dot{W}_{\underline{n}})^T$ be the WV with $\dot{W}_t \in [0,1], \sum_{t=1}^{\underline{n}} \dot{W}_t = 1$. If $\ddot{z}_t^{+\widehat{Re}} \leq \ddot{z}'^{+\widehat{Re}}_t, \ddot{z}_t^{+\widehat{IM}} \leq \ddot{z}'^{+\widehat{IM}}_t, \ddot{z}_t^{-\widehat{Re}} \leq \ddot{z}'^{-\widehat{Re}}_t, \ddot{z}_t^{-\widehat{IM}} \leq \ddot{z}'^{-\widehat{IM}}_t\ \forall\, t$, then $HBCFWA(\breve{I}_1, \breve{I}_2, \dots, \breve{I}_{\underline{n}})$

$$\leq HBCFWA(\breve{I}'_1, \breve{I}'_2, \dots, \breve{I}'_{\underline{n}}) \tag{10}$$

Theorem 4: (Boundedness property) Let $\breve{I}_t = (\breve{I}_t^+, \breve{I}_t^-) = (\ddot{z}_t^{+\widehat{Re}} + \iota\ddot{z}_t^{+\widehat{IM}}, \ddot{z}_t^{-\widehat{Re}} + \iota\ddot{z}_t^{-\widehat{IM}})(t = 1,2,\dots,\underline{n})$ be a collection of HBCFEs, let

$$\breve{I}^- = \begin{pmatrix} \min_t\{\ddot{z}_t^{+\widehat{Re}}\} + \iota\min_t\{\ddot{z}_t^{+\widehat{IM}}\}, \\ \max_t\{\ddot{z}_t^{-\widehat{Re}}\} + \iota\max_t\{\ddot{z}_t^{-\widehat{IM}}\} \end{pmatrix}, \text{ and}$$

$$\breve{I}^+ = \begin{pmatrix} \max_t\{\ddot{z}_t^{+\widehat{Re}}\} + \iota\max_t\{\ddot{z}_t^{+\widehat{IM}}\}, \\ \min_t\{\ddot{z}_t^{-\widehat{Re}}\} + \iota\min_t\{\ddot{z}_t^{-\widehat{IM}}\} \end{pmatrix}, \text{ then}$$

$$\breve{I}^- \leq HBCFWA(\breve{I}_1, \breve{I}_2, \dots, \breve{I}_{\underline{n}}) \leq \breve{I}^+ \tag{11}$$

Definition 14: Let $\breve{I}_t = (\breve{I}_t^+, \breve{I}_t^-)(t = 1,2,\dots,\underline{n})$ be a collection of HBCFEs and $\dot{W} = (\dot{W}_1, \dot{W}_2, \dots, \dot{W}_{\underline{n}})^T$ be the WV of $\breve{I}_t = (\breve{I}_t^+, \breve{I}_t^-)(t = 1,2,\dots,\underline{n})$ with $\dot{W}_t \in [0,1], \sum_{t=1}^{\underline{n}} \dot{W}_t = 1$, then A hesitant bipolar complex fuzzy ordered weighted averaging (HBCFOWA) operator is a mapping HBCFOWA: $E^n \longrightarrow E$, where

$$HBCFOWA(\breve{I}_1, \breve{I}_2, \dots, \breve{I}_{\underline{n}}) = \oplus_{t=1}^{\underline{n}} \dot{W}_t \breve{I}_{o(t)}$$
$$= \dot{W}_1\breve{I}_{o(1)} \oplus \dot{W}_2\breve{I}_{o(2)} \oplus \dots \oplus \dot{W}_{\underline{n}}\breve{I}_{o(\underline{n})} \tag{12}$$

Where $(o(1), o(2), o(3), \dots, o(\underline{n}))$ is a permutation of $(1,2,\dots,\underline{n})$ such that $\breve{I}_{o(t-1)} \geq \breve{I}_{o(t)}\ \forall\, t$.

Theorem 5: By employing the above eq. (12), we get the HBCFEs and

$HBCFOWA(\breve{I}_1, \breve{I}_2, \dots, \breve{I}_{\underline{n}})$

$$= \left(\begin{array}{c} \bigcup_{\ddot{z}_1^+\in\breve{I}_1^+,\dots,\ddot{z}_{\underline{n}}^+\in\breve{I}_{\underline{n}}^+} \left\{ \begin{array}{c} \left(1-\prod_{t=1}^{\underline{n}}\left(1-\ddot{z}_{o(t)}^{+\widehat{Re}}\right)^{\dot{W}_t}\right)+ \\ \iota\left(1-\prod_{t=1}^{\underline{n}}\left(1-\ddot{z}_{o(t)}^{+\widehat{IM}}\right)^{\dot{W}_t}\right) \end{array} \right\}, \\ \bigcup_{\ddot{z}_1^-\in\breve{I}_1^-,\dots,\ddot{z}_{\underline{n}}^-\in\breve{I}_{\underline{n}}^-} \left\{ \begin{array}{c} \left(-\prod_{t=1}^{\underline{n}}\left(\left|\ddot{z}_{o(t)}^{-\widehat{Re}}\right|\right)^{\dot{W}_t}\right)+ \\ \iota\left(-\prod_{t=1}^{\underline{n}}\left(\left|\ddot{z}_{o(t)}^{-\widehat{IM}}\right|\right)^{\dot{W}_t}\right) \end{array} \right\} \end{array} \right) \tag{13}$$

The following properties are held for HBCFOWA operators.

Theorem 6: (Idempotency property) Let us assume that the collection of HBCFEs is $\breve{I}_t = (\breve{I}_t^+, \breve{I}_t^-) = (\ddot{z}_t^{+\widehat{Re}} + \iota\ddot{z}_t^{+\widehat{IM}}, \ddot{z}_t^{-\widehat{Re}} + \iota\ddot{z}_t^{-\widehat{IM}})(t = 1,2,\dots,\underline{n})$ and the weights of HBCFEs is given as $\dot{W} = (\dot{W}_1, \dot{W}_2, \dots, \dot{W}_{\underline{n}})^T, \breve{I}_t = (\breve{I}_t^+, \breve{I}_t^-)(t = 1,2,\dots,\underline{n})$ where note that the $\dot{W}_t \in [0,1], \sum_{t=1}^{\underline{n}} \dot{W}_t = 1$ and if $\breve{I}_t = \breve{I}\ \forall\, t$ then to show

$$HBCFOWA(\breve{I}_1, \breve{I}_2, \dots, \breve{I}_{\underline{n}}) = \breve{I} \tag{14}$$

Theorem 7: (Boundedness property) Let $\breve{I}_t = (\breve{I}_t^+, \breve{I}_t^-) = (\ddot{z}_t^{+\widehat{Re}} + \iota\ddot{z}_t^{+\widehat{IM}}, \ddot{z}_t^{-\widehat{Re}} + \iota\ddot{z}_t^{-\widehat{IM}})(t = 1,2,\dots,\underline{n})$ be a collection of HBCFNs, let

$$\breve{I}^- = \begin{pmatrix} \min_t\{\ddot{z}_t^{+\widehat{Re}}\} + \iota\min_t\{\ddot{z}_t^{+\widehat{IM}}\}, \\ \max_t\{\ddot{z}_t^{-\widehat{Re}}\} + \iota\max_t\{\ddot{z}_t^{-\widehat{IM}}\} \end{pmatrix}, \text{ and}$$

$$\breve{I}^{+}=\begin{pmatrix}\max_{t}\{\ddot{z}_{t}^{+\widehat{Re}}\}+\iota\max_{t}\{\ddot{z}_{t}^{+\widehat{IM}}\},\\ \min_{t}\{\ddot{z}_{t}^{-\widehat{Re}}\}+\iota\min_{t}\{\ddot{z}_{t}^{-\widehat{IM}}\}\end{pmatrix}, \text{then}$$

$$\breve{I}^{-}\leq \mathrm{HBCFOWA}(\breve{I}_{1},\breve{I}_{2},\ldots,\breve{I}_{\underline{n}})\leq \breve{I}^{+} \quad (15)$$

Theorem 8: (Monotonicity property) Let $\breve{I}_{t}=(\breve{I}_{t}^{+},\breve{I}_{t}^{-})=(\ddot{z}_{t}^{+\widehat{Re}}+\iota\ddot{z}_{t}^{+\widehat{IM}},\ddot{z}_{t}^{-\widehat{Re}}+\iota\ddot{z}_{t}^{-\widehat{IM}})$ and $\breve{I}'_{t}=(\breve{I}'^{+}_{t},\breve{I}'^{-}_{t})=(\ddot{z}'^{+\widehat{Re}}_{t}+\iota\ddot{z}'^{+\widehat{IM}}_{t},\ddot{z}'^{-\widehat{Re}}_{t}+\iota\ddot{z}'^{-\widehat{IM}}_{t})$ $(t=1,2,\ldots,\underline{n})$ be a collection of two HBCFNs and $\dot{W}=(\dot{W}_{1},\dot{W}_{2},\ldots,\dot{W}_{\underline{n}})^{T}$ be the WV with $\dot{W}_{t}\in[0,1],\sum_{t=1}^{\underline{n}}\dot{W}_{t}=1$. If $\ddot{z}_{t}^{+\widehat{Re}}\leq \ddot{z}'^{+\widehat{Re}}_{t},\ddot{z}_{t}^{+\widehat{IM}}\leq \ddot{z}'^{+\widehat{IM}}_{t},\ddot{z}_{t}^{-\widehat{Re}}\leq \ddot{z}'^{-\widehat{Re}}_{t},\ddot{z}_{t}^{-\widehat{IM}}\leq \ddot{z}'^{-\widehat{IM}}_{t}$ $\forall\, t$ then

$$\mathrm{HBCFOWA}(\breve{I}_{1},\breve{I}_{2},\ldots,\breve{I}_{\underline{n}})\leq \mathrm{HBCFOWA}(\breve{I}'_{1},\breve{I}'_{2},\ldots,\breve{I}'_{\underline{n}}) \quad (16)$$

Definition 15: Let $\breve{I}_{t}=(\breve{I}_{t}^{+},\breve{I}_{t}^{-})(t=1,2,\ldots,\underline{n})$ be a collection of HBCFEs and $\dot{W}=(\dot{W}_{1},\dot{W}_{2},\ldots,\dot{W}_{\underline{n}})^{T}$ be the WV of $\breve{I}_{t}=(\breve{I}_{t}^{+},\breve{I}_{t}^{-})(t=1,2,\ldots,\underline{n})$ with $\dot{W}_{t}\in[0,1],\sum_{t=1}^{\underline{n}}\dot{W}_{t}=1$, then A hesitant bipolar complex fuzzy weighted geometric (HBCFWG) operator is a mapping $\mathrm{HBCFWG}:E^{n}\longrightarrow E$, where

$$\mathrm{HBCFWG}(\breve{I}_{1},\breve{I}_{2},\ldots,\breve{I}_{\underline{n}})=\otimes_{t=1}^{\underline{n}}\breve{I}_{t}^{\dot{W}_{t}}=\breve{I}_{1}^{\dot{W}_{1}}\otimes\breve{I}_{2}^{\dot{W}_{2}}\otimes\ldots\otimes\breve{I}_{\underline{n}}^{\dot{W}_{\underline{n}}} \quad (17)$$

Theorem 9: By employing the above eq. (17), we get the HBCFEs and

$$\mathrm{HBCFWG}(\breve{I}_{1},\breve{I}_{2},\ldots,\breve{I}_{\underline{n}})=\begin{pmatrix}\bigcup_{\ddot{z}_{1}^{+}\in\breve{I}_{1}^{+},\ldots,\ddot{z}_{\underline{n}}^{+}\in\breve{I}_{\underline{n}}^{+}}\left\{\begin{matrix}\left(\prod_{t=1}^{\underline{n}}(\ddot{z}_{t}^{+\widehat{Re}})^{\dot{W}_{t}}\right)+\\ \iota\left(\prod_{t=1}^{\underline{n}}(\ddot{z}_{t}^{+\widehat{IM}})^{\dot{W}_{t}}\right)\end{matrix}\right\},\\ \bigcup_{\ddot{z}_{1}^{-}\in\breve{I}_{1}^{-},\ldots,\ddot{z}_{\underline{n}}^{-}\in\breve{I}_{\underline{n}}^{-}}\left\{\begin{matrix}\left(-1+\prod_{t=1}^{\underline{n}}(1+\ddot{z}_{t}^{-\widehat{Re}})^{\dot{W}_{t}}\right)+\\ \iota\left(-1+\prod_{t=1}^{\underline{n}}(1+\ddot{z}_{t}^{-\widehat{IM}})^{\dot{W}_{t}}\right)\end{matrix}\right\}\end{pmatrix} \quad (18)$$

Proof: We prove eq. (18) By using a well-known method of mathematical induction (MI), assume for $\underline{n}=2$, we have

$$\breve{I}_{1}^{\dot{W}_{1}}=\begin{pmatrix}\bigcup_{\ddot{z}_{1}^{+}\in\breve{I}_{1}^{+}}\left\{(\ddot{z}_{1}^{+\widehat{Re}})^{\dot{W}_{1}}+\iota(\ddot{z}_{1}^{+\widehat{IM}})^{\dot{W}_{1}}\right\},\\ \bigcup_{\ddot{z}_{1}^{-}\in\breve{I}_{1}^{-}}\left\{\begin{matrix}\left(-1+(1+\ddot{z}_{1}^{-\widehat{Re}})^{\dot{W}_{1}}\right)+\\ \iota\left(-1+(1+\ddot{z}_{1}^{-\widehat{IM}})^{\dot{W}_{2}}\right)\end{matrix}\right\}\end{pmatrix}$$

$$\breve{I}_{2}^{\dot{W}_{2}}=\begin{pmatrix}\bigcup_{\ddot{z}_{2}^{+}\in\breve{I}_{2}^{+}}\left\{(\ddot{z}_{2}^{+\widehat{Re}})^{\dot{W}_{2}}+\iota(\ddot{z}_{2}^{+\widehat{IM}})^{\dot{W}_{2}}\right\},\\ \bigcup_{\ddot{z}_{2}^{-}\in\breve{I}_{2}^{-}}\left\{\begin{matrix}\left(-1+(1+\ddot{z}_{2}^{-\widehat{Re}})^{\dot{W}_{2}}\right)+\\ \iota\left(-1+(1+\ddot{z}_{2}^{-\widehat{IM}})^{\dot{W}_{2}}\right)\end{matrix}\right\}\end{pmatrix}$$

$$\breve{I}_{1}^{\dot{W}_{1}}\otimes\breve{I}_{2}^{\dot{W}_{2}}=\begin{pmatrix}\bigcup_{\ddot{z}_{1}^{+}\in\breve{I}_{1}^{+}}\left\{(\ddot{z}_{1}^{+\widehat{Re}})^{\dot{W}_{1}}+\iota(\ddot{z}_{1}^{+\widehat{IM}})^{\dot{W}_{1}}\right\},\\ \bigcup_{\ddot{z}_{1}^{-}\in\breve{I}_{1}^{-}}\left\{\begin{matrix}\left(-1+(1+\ddot{z}_{1}^{-\widehat{Re}})^{\dot{W}_{1}}\right)+\\ \iota\left(-1+(1+\ddot{z}_{1}^{-\widehat{IM}})^{\dot{W}_{1}}\right)\end{matrix}\right\}\end{pmatrix}\otimes\begin{pmatrix}\bigcup_{\ddot{z}_{2}^{+}\in\breve{I}_{2}^{+}}\left\{(\ddot{z}_{2}^{+\widehat{Re}})^{\dot{W}_{2}}+\iota(\ddot{z}_{2}^{+\widehat{IM}})^{\dot{W}_{2}}\right\},\\ \bigcup_{\ddot{z}_{2}^{-}\in\breve{I}_{2}^{-}}\left\{\begin{matrix}\left(-1+(1+\ddot{z}_{2}^{-\widehat{Re}})^{\dot{W}_{2}}\right)+\\ \iota\left(-1+(1+\ddot{z}_{2}^{-\widehat{IM}})^{\dot{W}_{2}}\right)\end{matrix}\right\}\end{pmatrix}$$

$$=\begin{pmatrix}\bigcup_{\ddot{z}_{1}^{+}\in\breve{I}_{1}^{+},\ddot{z}_{2}^{+}\in\breve{I}_{2}^{+}}\left\{\begin{matrix}(\ddot{z}_{1}^{+\widehat{Re}})^{\dot{W}_{1}}(\ddot{z}_{2}^{+\widehat{Re}})^{\dot{W}_{2}}+\\ \iota\left((\ddot{z}_{1}^{+\widehat{IM}})^{\dot{W}_{1}}(\ddot{z}_{2}^{+\widehat{IM}})^{\dot{W}_{2}}\right)\end{matrix}\right\},\\ \bigcup_{\ddot{z}_{1}^{-}\in\breve{I}_{1}^{-},\ddot{z}_{2}^{-}\in\breve{I}_{2}^{-}}\left\{\begin{matrix}\left(-1+(1+\ddot{z}_{1}^{-\widehat{Re}})^{\dot{W}_{1}}(1+\ddot{z}_{2}^{-\widehat{Re}})^{\dot{W}_{2}}\right)+\\ \iota\left(\left(-1+(1+\ddot{z}_{1}^{-\widehat{IM}})^{\dot{W}_{1}}(1+\ddot{z}_{2}^{-\widehat{IM}})^{\dot{W}_{2}}\right)\right)\end{matrix}\right\}\end{pmatrix}$$

$$=\begin{pmatrix}\bigcup_{\ddot{z}_{1}^{+}\in\breve{I}_{1}^{+},\ddot{z}_{2}^{+}\in\breve{I}_{2}^{+}}\left\{\begin{matrix}\left(\prod_{t=1}^{2}(\ddot{z}_{t}^{+\widehat{Re}})^{\dot{W}_{t}}\right)+\\ \iota\left(\prod_{t=1}^{2}(\ddot{z}_{t}^{+\widehat{IM}})^{\dot{W}_{t}}\right)\end{matrix}\right\},\\ \bigcup_{\ddot{z}_{1}^{-}\in\breve{I}_{1}^{-},\ddot{z}_{2}^{-}\in\breve{I}_{2}^{-}}\left\{\begin{matrix}\left(-1+\prod_{t=1}^{2}(1+\ddot{z}_{t}^{-\widehat{Re}})^{\dot{W}_{t}}\right)+\\ \iota\left(-1+\prod_{t=1}^{2}(1+\ddot{z}_{t}^{-\widehat{IM}})^{\dot{W}_{t}}\right)\end{matrix}\right\}\end{pmatrix}$$

Next, we suppose that Eq. (18) is true for $\underline{n}=K$ so,

$$\mathrm{HBCFWG}(\breve{I}_{1},\breve{I}_{2},\ldots,\breve{I}_{K})=\begin{pmatrix}\bigcup_{\ddot{z}_{1}^{+}\in\breve{I}_{1}^{+},\ldots,\ddot{z}_{K}^{+}\in\breve{I}_{K}^{+}}\left\{\begin{matrix}\left(\prod_{t=1}^{K}(\ddot{z}_{t}^{+\widehat{Re}})^{\dot{W}_{t}}\right)+\\ \iota\left(\prod_{t=1}^{K}(\ddot{z}_{t}^{+\widehat{IM}})^{\dot{W}_{t}}\right)\end{matrix}\right\},\\ \bigcup_{\ddot{z}_{1}^{-}\in\breve{I}_{1}^{-},\ldots,\ddot{z}_{K}^{-}\in\breve{I}_{K}^{-}}\left\{\begin{matrix}\left(-1+\prod_{t=1}^{K}(1+\ddot{z}_{t}^{-\widehat{Re}})^{\dot{W}_{t}}\right)+\\ \iota\left(-1+\prod_{t=1}^{K}(1+\ddot{z}_{t}^{-\widehat{IM}})^{\dot{W}_{t}}\right)\end{matrix}\right\}\end{pmatrix}$$

For $\underline{n}=K+1$, we have

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$$\mathrm{HBCFWG}\left(\breve{I}_1,\breve{I}_2,\ldots,\breve{I}_{K+1}\right)$$

$$=\left(\begin{array}{c}\bigcup_{\ddot{z}_1^{+}\in\breve{I}_1^{+},\ldots,\ddot{z}_K^{+}\in\breve{I}_K^{+}}\left\{\begin{array}{c}\left(\prod_{t=1}^{K}\left(\ddot{z}_t^{+\widehat{Re}}\right)^{\dot{W}_t}\right)+\\ \iota\left(\prod_{t=1}^{K}\left(\ddot{z}_t^{+\widehat{IM}}\right)^{\dot{W}_t}\right)\end{array}\right\},\\ \bigcup_{\ddot{z}_1^{-}\in\breve{I}_1^{-},\ldots,\ddot{z}_K^{-}\in\breve{I}_K^{-}}\left\{\begin{array}{c}\left(-1+\prod_{t=1}^{K}\left(1+\ddot{z}_t^{-\widehat{Re}}\right)^{\dot{W}_t}\right)+\\ \iota\left(-1+\prod_{t=1}^{K}\left(1+\ddot{z}_t^{-\widehat{IM}}\right)^{\dot{W}_t}\right)\end{array}\right\}\end{array}\right)$$

$$\otimes\left(\begin{array}{c}\bigcup_{\ddot{z}_{K+1}^{+}\in\breve{I}_{K+1}^{+}}\left\{\left(\ddot{z}_{K+1}^{+\widehat{Re}}\right)^{\dot{W}_{K+1}}+\iota\left(\ddot{z}_{K+1}^{+\widehat{IM}}\right)^{\dot{W}_{K+1}}\right\},\\ \bigcup_{\ddot{z}_{K+1}^{-}\in\breve{I}_{K+1}^{-}}\left\{\begin{array}{c}\left(-1+\left(1+\ddot{z}_{K+1}^{-\widehat{Re}}\right)^{\dot{W}_{K+1}}\right)+\\ \iota\left(-1+\left(1+\ddot{z}_{K+1}^{-\widehat{IM}}\right)^{\dot{W}_{K+1}}\right)\end{array}\right\}\end{array}\right)$$

$$=\left(\begin{array}{c}\bigcup_{\ddot{z}_1^{+}\in\breve{I}_1^{+},\ldots,\ddot{z}_K^{+}\in\breve{I}_K^{+},\ddot{z}_{K+1}^{+}\in\breve{I}_{K+1}^{+}}\left\{\begin{array}{c}\left(\prod_{t=1}^{K+1}\left(\ddot{z}_t^{+\widehat{Re}}\right)^{\dot{W}_t}\right)+\\ \iota\left(\prod_{t=1}^{K+1}\left(\ddot{z}_t^{+\widehat{IM}}\right)^{\dot{W}_t}\right)\end{array}\right\},\\ \bigcup_{\ddot{z}_1^{-}\in\breve{I}_1^{-},\ldots,\ddot{z}_K^{-}\in\breve{I}_K^{-},\ddot{z}_{K+1}^{-}\in\breve{I}_{K+1}^{-}}\left\{\begin{array}{c}\left(-1+\prod_{t=1}^{K+1}\left(1+\ddot{z}_t^{-\widehat{Re}}\right)^{\dot{W}_t}\right)+\\ \iota\left(-1+\prod_{t=1}^{K+1}\left(1+\ddot{z}_t^{-\widehat{IM}}\right)^{\dot{W}_t}\right)\end{array}\right\}\end{array}\right)$$

This shows that Eq. (18) holds for $\underline{n}\geq 0$.

Underneath properties are holds for HBCFWG operators.

Theorem 10: (Idempotency property) Let $\breve{I}_t=\left(\breve{I}_t^{+},\breve{I}_t^{-}\right)=\left(\ddot{z}_t^{+\widehat{Re}}+\iota\ddot{z}_t^{+\widehat{IM}},\ddot{z}_t^{-\widehat{Re}}+\iota\ddot{z}_t^{-\widehat{IM}}\right)(t=1,2,\ldots,\underline{n})$ be a collection of HBCFEs, $\dot{W}=\left(\dot{W}_1,\dot{W}_2,\ldots,\dot{W}_{\underline{n}}\right)^T$ be the WV of $\breve{I}_t=\left(\breve{I}_t^{+},\breve{I}_t^{-}\right)(t=1,2,\ldots,\underline{n})$ with $\dot{W}_t\in[0,1],\sum_{t=1}^{\underline{n}}\dot{W}_t=1$. If $\breve{I}_t=\breve{I}\ \forall\ t$ then $\mathrm{HBCFWG}\left(\breve{I}_1,\breve{I}_2,\ldots,\breve{I}_{\underline{n}}\right)=\breve{I}$ (26)

Theorem 11: (Boundedness property) Let $\breve{I}_t=\left(\breve{I}_t^{+},\breve{I}_t^{-}\right)=\left(\ddot{z}_t^{+\widehat{Re}}+\iota\ddot{z}_t^{+\widehat{IM}},\ddot{z}_t^{-\widehat{Re}}+\iota\ddot{z}_t^{-\widehat{IM}}\right)(t=1,2,\ldots,\underline{n})$ be a collection of HBCFEs, let

$$\breve{I}^{-}=\left(\begin{array}{c}\min_t\left\{\ddot{z}_t^{+\widehat{Re}}\right\}+\iota\min_t\left\{\ddot{z}_t^{+\widehat{IM}}\right\},\\ \max_t\left\{\ddot{z}_t^{-\widehat{Re}}\right\}+\iota\max_t\left\{\ddot{z}_t^{-\widehat{IM}}\right\}\end{array}\right),\text{ and}$$

$$\breve{I}^{+}=\left(\begin{array}{c}\max_t\left\{\ddot{z}_t^{+\widehat{Re}}\right\}+\iota\max_t\left\{\ddot{z}_t^{+\widehat{IM}}\right\},\\ \min_t\left\{\ddot{z}_t^{-\widehat{Re}}\right\}+\iota\min_t\left\{\ddot{z}_t^{-\widehat{IM}}\right\}\end{array}\right),\text{ then}$$

$$\breve{I}^{-}\leq\mathrm{HBCFWG}\left(\breve{I}_1,\breve{I}_2,\ldots,\breve{I}_{\underline{n}}\right)\leq\breve{I}^{+}\qquad(19)$$

Theorem 12: (Monotonicity property). Let $\breve{I}_t=\left(\breve{I}_t^{+},\breve{I}_t^{-}\right)=\left(\ddot{z}_t^{+\widehat{Re}}+\iota\ddot{z}_t^{+\widehat{IM}},\ddot{z}_t^{-\widehat{Re}}+\iota\ddot{z}_t^{-\widehat{IM}}\right)$ and $\breve{I}'_t=\left(\breve{I}'^{+}_t,\breve{I}'^{-}_t\right)=\left(\ddot{z}'^{+\widehat{Re}}_t+\iota\ddot{z}'^{+\widehat{IM}}_t,\ddot{z}'^{-\widehat{Re}}_t+\iota\ddot{z}'^{-\widehat{IM}}_t\right)(t=1,2,\ldots,\underline{n})$ be a collection of two HBCFEs and $\dot{W}=\left(\dot{W}_1,\dot{W}_2,\ldots,\dot{W}_{\underline{n}}\right)^T$ be the WV with $\dot{W}_t\in[0,1],\sum_{t=1}^{\underline{n}}\dot{W}_t=1$. If $\ddot{z}_t^{+\widehat{Re}}\leq\ddot{z}'^{+\widehat{Re}}_t,\ddot{z}_t^{+\widehat{IM}}\leq\ddot{z}'^{+\widehat{IM}}_t,\ddot{z}_t^{-\widehat{Re}}\leq\ddot{z}'^{-\widehat{Re}}_t,\ddot{z}_t^{-\widehat{IM}}\leq\ddot{z}'^{-\widehat{IM}}_t\ \forall\ t$, then

$$\mathrm{HBCFWG}\left(\breve{I}_1,\breve{I}_2,\ldots,\breve{I}_{\underline{n}}\right)\leq\mathrm{HBCFWG}\left(\breve{I}'_1,\breve{I}'_2,\ldots,\breve{I}'_{\underline{n}}\right)\qquad(20)$$

Definition 16: Let $\breve{I}_t=\left(\breve{I}_t^{+},\breve{I}_t^{-}\right)(t=1,2,\ldots,\underline{n})$ be a collection of HBCFEs and $\dot{W}=\left(\dot{W}_1,\dot{W}_2,\ldots,\dot{W}_{\underline{n}}\right)^T$ be the WV of $\breve{I}_t=\left(\breve{I}_t^{+},\breve{I}_t^{-}\right)(t=1,2,\ldots,\underline{n})$ with $\dot{W}_t\in[0,1],\sum_{t=1}^{\underline{n}}\dot{W}_t=1$, then A hesitant bipolar complex fuzzy ordered weighted geometric (HBCFOWG) operator is a mapping HBCFOWG: $E^n\longrightarrow E$, where

$$\mathrm{HBCFOWG}\left(\breve{I}_1,\breve{I}_2,\ldots,\breve{I}_{\underline{n}}\right)=\otimes_{t=1}^{\underline{n}}\breve{I}_{o(t)}^{\dot{W}_t}=\breve{I}_{o(1)}^{\dot{W}_1}\otimes\breve{I}_{o(2)}^{\dot{W}_2}\otimes\ldots\otimes\breve{I}_{o(\underline{n})}^{\dot{W}_{\underline{n}}}\qquad(21)$$

Where $\left(o(1),o(2),o(3),\ldots,o(\underline{n})\right)$ is a permutation of $(1,2,\ldots,\underline{n})$ such that $\breve{I}_{o(t-1)}\geq\breve{I}_{o(t)}\ \forall\ t$.

Theorem 13: By employing the above Eq. (21), we get the HBCFNs and

$$\mathrm{HBCFOWG}\left(\breve{I}_1,\breve{I}_2,\ldots,\breve{I}_{\underline{n}}\right)$$

$$=\left(\begin{array}{c}\bigcup_{\ddot{z}_1^{+}\in\breve{I}_1^{+},\ldots,\ddot{z}_{\underline{n}}^{+}\in\breve{I}_{\underline{n}}^{+}}\left\{\begin{array}{c}\left(\prod_{t=1}^{\underline{n}}\left(\ddot{z}_{o(t)}^{+\widehat{Re}}\right)^{\dot{W}_t}\right)+\\ \iota\left(\prod_{t=1}^{\underline{n}}\left(\ddot{z}_{o(t)}^{+\widehat{IM}}\right)^{\dot{W}_t}\right)\end{array}\right\},\\ \bigcup_{\ddot{z}_1^{-}\in\breve{I}_1^{-},\ldots,\ddot{z}_{\underline{n}}^{-}\in\breve{I}_{\underline{n}}^{-}}\left\{\begin{array}{c}\left(-1+\prod_{t=1}^{\underline{n}}\left(1+\ddot{z}_{o(t)}^{-\widehat{Re}}\right)^{\dot{W}_t}\right)+\\ \iota\left(-1+\prod_{t=1}^{\underline{n}}\left(1+\ddot{z}_{o(t)}^{-\widehat{IM}}\right)^{\dot{W}_t}\right)\end{array}\right\}\end{array}\right)\qquad(22)$$

Following properties are holds for HBCFOWG.

Theorem 14: (Idempotency property) Let us assume that the collection of HBCEs is $\breve{I}_t=\left(\breve{I}_t^{+},\breve{I}_t^{-}\right)=\left(\ddot{z}_t^{+\widehat{Re}}+\iota\ddot{z}_t^{+\widehat{IM}},\ddot{z}_t^{-\widehat{Re}}+\iota\ddot{z}_t^{-\widehat{IM}}\right)(t=1,2,\ldots,\underline{n})$ and the well-known weights vectors are given as $\dot{W}=\left(\dot{W}_1,\dot{W}_2,\ldots,\dot{W}_{\underline{n}}\right)^T$

For the collection of HBCF numbers $\breve{I}_t=\left(\breve{I}_t^{+},\breve{I}_t^{-}\right)(t=1,2,\ldots,\underline{n})$ where note that the $\dot{W}_t\in[0,1],\sum_{t=1}^{\underline{n}}\dot{W}_t=1$. Now if $\breve{I}_t=\breve{I}\ \forall\ t$ then to show

$$\mathrm{HBCFOWG}\left(\breve{I}_1,\breve{I}_2,\ldots,\breve{I}_{\underline{n}}\right)=\breve{I}\qquad(23)$$

Theorem 15: (Boundedness property) Let $\breve{I}_t=\left(\breve{I}_t^{+},\breve{I}_t^{-}\right)=\left(\ddot{z}_t^{+\widehat{Re}}+\iota\ddot{z}_t^{+\widehat{IM}},\ddot{z}_t^{-\widehat{Re}}+\iota\ddot{z}_t^{-\widehat{IM}}\right)(t=1,2,\ldots,\underline{n})$ be a collection of HBCFEs, let

$$\breve{I}^{-}=\left(\begin{array}{c}\min_t\left\{\ddot{z}_t^{+\widehat{Re}}\right\}+\iota\min_t\left\{\ddot{z}_t^{+\widehat{IM}}\right\},\\ \max_t\left\{\ddot{z}_t^{-\widehat{Re}}\right\}+\iota\max_t\left\{\ddot{z}_t^{-\widehat{IM}}\right\}\end{array}\right),\text{ and}$$

$$\breve{I}^{+}=\begin{pmatrix}\max_t\{\ddot{z}_t^{+\widehat{Re}}\}+\iota\max_t\{\ddot{z}_t^{+\widehat{IM}}\},\\ \min_t\{\ddot{z}_t^{-\widehat{Re}}\}+\iota\min_t\{\ddot{z}_t^{-\widehat{IM}}\}\end{pmatrix}, \text{ then}$$

$$\breve{I}^{-}\le HBCFOWG(\breve{I}_1,\breve{I}_2,\dots,\breve{I}_{\underline{n}})\le\breve{I}^{+} \quad (24)$$

Theorem 16: (Monotonicity property) Let $\breve{I}_t=(\breve{I}_t^{+},\breve{I}_t^{-})=(\ddot{z}_t^{+\widehat{Re}}+\iota\ddot{z}_t^{+\widehat{IM}},\ddot{z}_t^{-\widehat{Re}}+\iota\ddot{z}_t^{-\widehat{IM}})$ and $\breve{I}'_t=(\breve{I}'^{+}_t,\breve{I}'^{-}_t)=(\ddot{z}'^{+\widehat{Re}}_t+\iota\ddot{z}'^{+\widehat{IM}}_t,\ddot{z}'^{-\widehat{Re}}_t+\iota\ddot{z}'^{-\widehat{IM}}_t)$ $(t=1,2,\dots,\underline{n})$ be a collection of two HBCFEs and $\dot{W}=(\dot{W}_1,\dot{W}_2,\dots,\dot{W}_{\underline{n}})^T$ be the WV with $\dot{W}_t\in[0,1],\sum_{t=1}^{\underline{n}}\dot{W}_t=1$. If $\ddot{z}_t^{+\widehat{Re}}\le\ddot{z}'^{+\widehat{Re}}_t,\ddot{z}_t^{+\widehat{IM}}\le\ddot{z}'^{+\widehat{IM}}_t,\ddot{z}_t^{-\widehat{Re}}\le\ddot{z}'^{-\widehat{Re}}_t,\ddot{z}_t^{-\widehat{IM}}\le\ddot{z}'^{-\widehat{IM}}_t\ \forall\, t$ then

$$HBCFOWG(\breve{I}_1,\breve{I}_2,\dots,\breve{I}_{\underline{n}})\le HBCFOWG(\breve{I}'_1,\breve{I}'_2,\dots,\breve{I}'_{\underline{n}}) \quad (25)$$

Definition 17: Let $\breve{I}_t=(\breve{I}_t^{+},\breve{I}_t^{-})(t=1,2,\dots,\underline{n})$ be a collection of HBCFEs and $\dot{W}=(\dot{W}_1,\dot{W}_2,\dots,\dot{W}_{\underline{n}})^T$ be the WV of $\breve{I}_t=(\breve{I}_t^{+},\breve{I}_t^{-})(t=1,2,\dots,\underline{n})$ with $\dot{W}_t\in[0,1]$, $\sum_{t=1}^{\underline{n}}\dot{W}_t=1$, and $\lambda>0$ then A generalized hesitant bipolar complex fuzzy weighted averaging (GHBCFWA) operator is a mapping GHBCFWA: $E^n\longrightarrow E$, where

$$GHBCFWA_\lambda(\breve{I}_1,\breve{I}_2,\dots,\breve{I}_{\underline{n}})=\oplus_{t=1}^{\underline{n}}(\dot{W}_t\breve{I}_t^{\lambda})^{\frac{1}{\lambda}} \quad (26)$$

Theorem 17: By employing the above Eq. (26), we get the HBCFEs and

$$GHBCFWA_\lambda(\breve{I}_1,\breve{I}_2,\dots,\breve{I}_{\underline{n}})=\begin{pmatrix}\bigcup_{\ddot{z}_1^{+}\in\breve{I}_1^{+},\dots,\ddot{z}_{\underline{n}}^{+}\in\breve{I}_{\underline{n}}^{+}}\left\{\begin{array}{l}\left(1-\prod_{t=1}^{\underline{n}}\left(1-\left(\ddot{z}_t^{+\widehat{Re}}\right)^{\lambda}\right)^{\dot{W}_t}\right)^{\frac{1}{\lambda}}\\+\iota\left(1-\prod_{t=1}^{\underline{n}}\left(1-\left(\ddot{z}_t^{+\widehat{IM}}\right)^{\lambda}\right)^{\dot{W}_t}\right)^{\frac{1}{\lambda}}\end{array}\right\},\\ \bigcup_{\ddot{z}_1^{-}\in\breve{I}_1^{-},\dots,\ddot{z}_{\underline{n}}^{-}\in\breve{I}_{\underline{n}}^{-}}\left\{\begin{array}{l}\left(-1+\left(1-\prod_{t=1}^{\underline{n}}\left|-1+\left(1+\ddot{z}_t^{-\widehat{Re}}\right)^{\lambda}\right|^{\dot{W}_t}\right)\right)^{\frac{1}{\lambda}}\\+\iota\left(-1+\left(1-\prod_{t=1}^{\underline{n}}\left|-1+\left(1+\ddot{z}_t^{-\widehat{IM}}\right)^{\lambda}\right|^{\dot{W}_t}\right)\right)^{\frac{1}{\lambda}}\end{array}\right\}\end{pmatrix} \quad (27)$$

Definition 18: Let $\breve{I}_t=(\breve{I}_t^{+},\breve{I}_t^{-})(t=1,2,\dots,\underline{n})$ be a collection of HBCENs and $\dot{W}=(\dot{W}_1,\dot{W}_2,\dots,\dot{W}_{\underline{n}})^T$ be the WV of $\breve{I}_t=(\breve{I}_t^{+},\breve{I}_t^{-})(t=1,2,\dots,\underline{n})$ with $\dot{W}_t\in[0,1],\sum_{t=1}^{\underline{n}}\dot{W}_t=1$, then A generalized hesitant bipolar complex fuzzy weighted geometric (GHBCFWG) operator is a mapping GHBCFWG: $E^n\longrightarrow E$, where

$$GHBCFWG_\lambda(\breve{I}_1,\breve{I}_2,\dots,\breve{I}_{\underline{n}})=\frac{1}{\lambda}\left(\otimes_{t=1}^{\underline{n}}(\lambda\breve{I}_t)^{\dot{W}_t}\right) \quad (28)$$

Theorem 18: By employing the above Eq. (28), we get the HBCFNs and

$$GHBCFWG_\lambda(\breve{I}_1,\breve{I}_2,\dots,\breve{I}_{\underline{n}})=\begin{pmatrix}\bigcup_{\ddot{z}_1^{+}\in\breve{I}_1^{+},\dots,\ddot{z}_{\underline{n}}^{+}\in\breve{I}_{\underline{n}}^{+}}\left\{\begin{array}{l}\left(1-\left(1-\prod_{t=1}^{\underline{n}}\left(1-\left(1-\ddot{z}_t^{+\widehat{Re}}\right)^{\lambda}\right)^{\dot{W}_t}\right)\right)^{\frac{1}{\lambda}}\\+\iota\left(1-\left(1-\prod_{t=1}^{\underline{n}}\left(1-\left(1-\ddot{z}_t^{+\widehat{Re}}\right)^{\lambda}\right)^{\dot{W}_t}\right)\right)^{\frac{1}{\lambda}}\end{array}\right\},\\ \bigcup_{\ddot{z}_1^{-}\in\breve{I}_1^{-},\dots,\ddot{z}_{\underline{n}}^{-}\in\breve{I}_{\underline{n}}^{-}}\left\{\begin{array}{l}\left(-\left|-1+\prod_{t=1}^{\underline{n}}\left(1-\left(\left|\ddot{z}_t^{-\widehat{Re}}\right|\right)^{\lambda}\right)^{\dot{W}_t}\right|^{\frac{1}{\lambda}}\right)\\+\iota\left(-\left|-1+\prod_{t=1}^{\underline{n}}\left(1-\left(\left|\ddot{z}_t^{-\widehat{Re}}\right|\right)^{\lambda}\right)^{\dot{W}_t}\right|^{\frac{1}{\lambda}}\right)\end{array}\right\}\end{pmatrix} \quad (29)$$

Relationships among the developed operators are discussed as.

Theorem 19: Let $\breve{I}_t=(\breve{I}_t^{+},\breve{I}_t^{-})(t=1,2,\dots,\underline{n})$ be a collection of HBCFEs and $\dot{W}=(\dot{W}_1,\dot{W}_2,\dots,\dot{W}_{\underline{n}})^T$ be the WV of $\breve{I}_t=(\breve{I}_t^{+},\breve{I}_t^{-})(t=1,2,\dots,\underline{n})$ with $\dot{W}_t\in[0,1],\sum_{t=1}^{\underline{n}}\dot{W}_t=1$ and $\lambda>0$, then we have

a) $\oplus_{t=1}^{\underline{n}}\dot{W}_t\breve{I}_t^{c}=\left(\otimes_{t=1}^{\underline{n}}\breve{I}_t^{\dot{W}_t}\right)^{c}$

b) $\otimes_{t=1}^{\underline{n}}(\breve{I}_t^{c})^{\dot{W}_t}=\left(\oplus_{t=1}^{\underline{n}}\dot{W}_t\breve{I}_t\right)^{c}$

c) $\left(\oplus_{t=1}^{\underline{n}}\dot{W}_t\left(\breve{I}_t^{c}\right)^{\lambda}\right)^{\frac{1}{\lambda}}=\left(\frac{1}{\lambda}\left(\otimes_{t=1}^{\underline{n}}(\lambda\breve{I}_t)^{\dot{W}_t}\right)\right)^{c}$

d) $\frac{1}{\lambda}\left(\otimes_{t=1}^{\underline{n}}(\lambda\breve{I}_t^{c})^{\dot{W}_t}\right)=\left(\left(\oplus_{t=1}^{\underline{n}}(\dot{W}_t\breve{I}_t^{\lambda})\right)^{\frac{1}{\lambda}}\right)^{c}$

IV. MABAC MODEL IN THE ENVIRONMENT OF HBCFSS

In this section, we develop the MABAC model which is one of the best techniques for solving DM problems. We set up here MABAC for the environment of HBCFNs and used it next to solve the MAGDM issue. Let us assume that there be a set of $\mathfrak{T}$ alternatives $\{C_1,C_2,\dots,C_{\mathfrak{T}}\}$, and ς attributes $\{B_1,B_2,\dots,B_\varsigma\}$ with a correlated set of WV $\{\dot{W}_1,\dot{W}_2,\dots,\dot{W}_\varsigma\}$ and μ experts $\{R_1,R_2,\dots,R_\mu\}$ with the weighting vector $\{\phi_1,\phi_2,\dots,\phi_\mu\}$, then HBCF evaluation matrix $M=[C_{\rho\Psi}^{\mu}]_{\mathfrak{T}\times\varsigma}=\left((\breve{I}_{\rho\Psi}^{+})^{\mu},(\breve{I}_{\rho\Psi}^{-})^{\mu}\right)_{\mathfrak{T}\times\varsigma}=\begin{pmatrix}(\ddot{z}_{\rho\Psi}^{+\widehat{Re}})^{\mu}+\iota(\ddot{z}_{\rho\Psi}^{+\widehat{IM}})^{\mu},\\(\ddot{z}_{\rho\Psi}^{-\widehat{Re}})^{\mu}+\iota(\ddot{z}_{\rho\Psi}^{-\widehat{IM}})^{\mu}\end{pmatrix}_{\mathfrak{T}\times\varsigma},\rho=1,2,\dots,\mathfrak{T},\Psi=1,2,\dots,\varsigma.$

Where $(\breve{I}_{\rho\Psi}^{+})^{\mu}$ is PMD which is the set of finite values in the unit square of a complex plane and $(\breve{I}_{\rho\Psi}^{-})^{\mu}$ is NMD which is also the set of finite values in the unit square of a complex

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plane, where $\left(\breve{I}^{+}_{\rho\Psi}\right)^{\mu} \in [0,1]$ and $\left(\breve{I}^{-}_{\rho\Psi}\right)^{\mu} \in [-1,0]$ then HBCF MABAC approach follows the following steps.

Step 1 Evaluation of HBCF matrix formulation

$$M = \left[C^{\mu}_{\rho\Psi}\right]_{\mathfrak{T}\times\varsigma} = \begin{pmatrix} \left(\breve{I}^{+}_{\rho\Psi}\right)^{\mu}, \\ \left(\breve{I}^{-}_{\rho\Psi}\right)^{\mu} \end{pmatrix}_{\mathfrak{T}\times\varsigma} = \begin{pmatrix} \left(\ddot{z}^{+\widehat{Re}}_{\rho\Psi}\right)^{\mu} + \iota\left(\ddot{z}^{+\widehat{IM}}_{\rho\Psi}\right)^{\mu}, \\ \left(\ddot{z}^{-\widehat{Re}}_{\rho\Psi}\right)^{\mu} + \iota\left(\ddot{z}^{-\widehat{IM}}_{\rho\Psi}\right)^{\mu} \end{pmatrix}_{\mathfrak{T}\times\varsigma}$$

$\rho = 1,2,\ldots,\mathfrak{T}, I = 1,2,\ldots,\varsigma$ is given as below:

$$M = \left[C^{\mu}_{\rho\Psi}\right]_{\mathfrak{T}\times\varsigma} = \begin{bmatrix} \begin{pmatrix} \left(\ddot{z}^{+\widehat{Re}}_{11}\right)^{\mu} + \iota\left(\ddot{z}^{+\widehat{IM}}_{11}\right)^{\mu}, \\ \left(\ddot{z}^{-\widehat{Re}}_{11}\right)^{\mu} + \iota\left(\ddot{z}^{-\widehat{IM}}_{11}\right)^{\mu} \end{pmatrix} & \begin{pmatrix} \left(\ddot{z}^{+\widehat{Re}}_{12}\right)^{\mu} + \iota\left(\ddot{z}^{+\widehat{IM}}_{12}\right)^{\mu}, \\ \left(\ddot{z}^{-\widehat{Re}}_{12}\right)^{\mu} + \iota\left(\ddot{z}^{-\widehat{IM}}_{12}\right)^{\mu} \end{pmatrix} & \cdots & \begin{pmatrix} \left(\ddot{z}^{+\widehat{Re}}_{1\varsigma}\right)^{\mu} + \iota\left(\ddot{z}^{+\widehat{IM}}_{1\varsigma}\right)^{\mu}, \\ \left(\ddot{z}^{-\widehat{Re}}_{1\varsigma}\right)^{\mu} + \iota\left(\ddot{z}^{-\widehat{IM}}_{1\varsigma}\right)^{\mu} \end{pmatrix} \\ \begin{pmatrix} \left(\ddot{z}^{+\widehat{Re}}_{21}\right)^{\mu} + \iota\left(\ddot{z}^{+\widehat{IM}}_{21}\right)^{\mu}, \\ \left(\ddot{z}^{-\widehat{Re}}_{21}\right)^{\mu} + \iota\left(\ddot{z}^{-\widehat{IM}}_{21}\right)^{\mu} \end{pmatrix} & \begin{pmatrix} \left(\ddot{z}^{+\widehat{Re}}_{22}\right)^{\mu} + \iota\left(\ddot{z}^{+\widehat{IM}}_{22}\right)^{\mu}, \\ \left(\ddot{z}^{-\widehat{Re}}_{22}\right)^{\mu} + \iota\left(\ddot{z}^{-\widehat{IM}}_{22}\right)^{\mu} \end{pmatrix} & \cdots & \begin{pmatrix} \left(\ddot{z}^{+\widehat{Re}}_{2\varsigma}\right)^{\mu} + \iota\left(\ddot{z}^{+\widehat{IM}}_{2\varsigma}\right)^{\mu}, \\ \left(\ddot{z}^{-\widehat{Re}}_{2\varsigma}\right)^{\mu} + \iota\left(\ddot{z}^{-\widehat{IM}}_{2\varsigma}\right)^{\mu} \end{pmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{pmatrix} \left(\ddot{z}^{+\widehat{Re}}_{\mathfrak{T}1}\right)^{\mu} + \iota\left(\ddot{z}^{+\widehat{IM}}_{\mathfrak{T}1}\right)^{\mu}, \\ \left(\ddot{z}^{-\widehat{Re}}_{\mathfrak{T}1}\right)^{\mu} + \iota\left(\ddot{z}^{-\widehat{IM}}_{\mathfrak{T}1}\right)^{\mu} \end{pmatrix} & \begin{pmatrix} \left(\ddot{z}^{+\widehat{Re}}_{\mathfrak{T}2}\right)^{\mu} + \iota\left(\ddot{z}^{+\widehat{IM}}_{\mathfrak{T}2}\right)^{\mu}, \\ \left(\ddot{z}^{-\widehat{Re}}_{\mathfrak{T}2}\right)^{\mu} + \iota\left(\ddot{z}^{-\widehat{IM}}_{\mathfrak{T}2}\right)^{\mu} \end{pmatrix} & \cdots & \begin{pmatrix} \left(\ddot{z}^{+\widehat{Re}}_{\mathfrak{T}\varsigma}\right)^{\mu} + \iota\left(\ddot{z}^{+\widehat{IM}}_{\mathfrak{T}\varsigma}\right)^{\mu}, \\ \left(\ddot{z}^{-\widehat{Re}}_{\mathfrak{T}\varsigma}\right)^{\mu} + \iota\left(\ddot{z}^{-\widehat{IM}}_{\mathfrak{T}\varsigma}\right)^{\mu} \end{pmatrix} \end{bmatrix} \quad (30)$$

Where the $C^{\mu}_{\rho\Psi} = \begin{pmatrix} \left(\breve{I}^{+}_{\rho\Psi}\right)^{\mu}, \\ \left(\breve{I}^{-}_{\rho\Psi}\right)^{\mu} \end{pmatrix} = \begin{pmatrix} \left(\ddot{z}^{+\widehat{Re}}_{\rho\Psi}\right)^{\mu} + \iota\left(\ddot{z}^{+\widehat{IM}}_{\rho\Psi}\right)^{\mu}, \\ \left(\ddot{z}^{-\widehat{Re}}_{\rho\Psi}\right)^{\mu} + \iota\left(\ddot{z}^{-\widehat{IM}}_{\rho\Psi}\right)^{\mu} \end{pmatrix}$ $(\rho = 1,2,\ldots,\mathfrak{T};\ \Psi = 1,2,\ldots,\varsigma)$ denotes the formula of BCHF information of alternatives C_{ρ} based on the $B_{\Psi}(\Psi = 1,2,\ldots,\varsigma)$ attributes by R^{μ} experts.

Step 2 Using the above-defined aggregation operators HBCFWA or HBCFWG, we aggregate $C^{\mu}_{\rho\Psi}$ to $C_{\rho\Psi}$ then the fused HBCFNs matrix is given below.

$$M = \left[C_{\rho\Psi}\right]_{\mathfrak{T}\times\varsigma} = \begin{bmatrix} \begin{pmatrix} \ddot{z}^{+\widehat{Re}}_{11} + \\ \iota\ddot{z}^{+\widehat{IM}}_{11}, \\ \ddot{z}^{-\widehat{Re}}_{11} + \\ \iota\ddot{z}^{-\widehat{IM}}_{11} \end{pmatrix} & \begin{pmatrix} \ddot{z}^{+\widehat{Re}}_{12} + \\ \iota\ddot{z}^{+\widehat{IM}}_{12}, \\ \ddot{z}^{-\widehat{Re}}_{12} + \\ \iota\ddot{z}^{-\widehat{IM}}_{12} \end{pmatrix} & \cdots & \begin{pmatrix} \ddot{z}^{+\widehat{Re}}_{1\varsigma} + \\ \iota\ddot{z}^{+\widehat{IM}}_{1\varsigma}, \\ \ddot{z}^{-\widehat{Re}}_{1\varsigma} + \\ \iota\ddot{z}^{-\widehat{IM}}_{1\varsigma} \end{pmatrix} \\ \begin{pmatrix} \ddot{z}^{+\widehat{Re}}_{21} + \\ \iota\ddot{z}^{+\widehat{IM}}_{21}, \\ \ddot{z}^{-\widehat{Re}}_{21} + \\ \iota\ddot{z}^{-\widehat{IM}}_{21} \end{pmatrix} & \begin{pmatrix} \ddot{z}^{+\widehat{Re}}_{22} + \\ \iota\ddot{z}^{+\widehat{IM}}_{22}, \\ \ddot{z}^{-\widehat{Re}}_{22} + \\ \iota\ddot{z}^{-\widehat{IM}}_{22} \end{pmatrix} & \cdots & \begin{pmatrix} \ddot{z}^{+\widehat{Re}}_{2\varsigma} + \\ \iota\ddot{z}^{+\widehat{IM}}_{2\varsigma}, \\ \ddot{z}^{-\widehat{Re}}_{2\varsigma} + \\ \iota\ddot{z}^{-\widehat{IM}}_{2\varsigma} \end{pmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{pmatrix} \ddot{z}^{+\widehat{Re}}_{\mathfrak{T}1} + \\ \iota\ddot{z}^{+\widehat{IM}}_{\mathfrak{T}1}, \\ \ddot{z}^{-\widehat{Re}}_{\mathfrak{T}1} + \\ \iota\ddot{z}^{-\widehat{IM}}_{\mathfrak{T}1} \end{pmatrix} & \begin{pmatrix} \ddot{z}^{+\widehat{Re}}_{\mathfrak{T}2} + \\ \iota\ddot{z}^{+\widehat{IM}}_{\mathfrak{T}2}, \\ \ddot{z}^{-\widehat{Re}}_{\mathfrak{T}2} + \\ \iota\ddot{z}^{-\widehat{IM}}_{\mathfrak{T}2} \end{pmatrix} & \cdots & \begin{pmatrix} \ddot{z}^{+\widehat{Re}}_{\mathfrak{T}\varsigma} + \\ \iota\ddot{z}^{+\widehat{IM}}_{\mathfrak{T}\varsigma}, \\ \ddot{z}^{-\widehat{Re}}_{\mathfrak{T}\varsigma} + \\ \iota\ddot{z}^{-\widehat{IM}}_{\mathfrak{T}\varsigma} \end{pmatrix} \end{bmatrix} \quad (31)$$

Where the $C_{\rho\Psi} = \left(\breve{I}^{+}_{\rho\Psi}, \breve{I}^{-}_{\rho\Psi}\right) = \left(\ddot{z}^{+\widehat{Re}}_{\rho\Psi} + \iota\ddot{z}^{+\widehat{IM}}_{\rho\Psi}, \ddot{z}^{-\widehat{Re}}_{\rho\Psi} + \iota\ddot{z}^{-\widehat{IM}}_{\rho\Psi}\right)(\rho = 1,2,\ldots,\mathfrak{T};\ \Psi = 1,2,\ldots,\varsigma)$ symbolizes the formula of HBCF information of alternatives C_{ρ} based on the $B_{\Psi}(\Psi = 1,2,\ldots,\varsigma)$ attributes by R^{μ} experts.

Step 3: Normalize the fuse matrix $M = \left[C_{\rho\Psi}\right]_{\mathfrak{T}\times\varsigma}(\rho = 1,2,\ldots,\mathfrak{T};\ \Psi = 1,2,\ldots,\varsigma)$ based on the nature of each attribute by the given formula:

For benefit attributes:

$$M_{\rho\Psi} = C_{\rho\Psi} = \left(\breve{I}^{+}_{\rho\Psi}, \breve{I}^{-}_{\rho\Psi}\right) = \begin{pmatrix} \ddot{z}^{+\widehat{Re}}_{\rho\Psi} + \iota\ddot{z}^{+\widehat{IM}}_{\rho\Psi}, \\ \ddot{z}^{-\widehat{Re}}_{\rho\Psi} + \iota\ddot{z}^{-\widehat{IM}}_{\rho\Psi} \end{pmatrix} \quad (32)$$

Where $(\rho = 1,2,\ldots,\mathfrak{T};\ \Psi = 1,2,\ldots,\varsigma)$

For cost attributes:

$$M_{\rho\Psi} = \left(C_{\rho\Psi}\right)^{c} = \left(\breve{I}^{-}_{\rho\Psi}, \breve{I}^{+}_{\rho\Psi}\right) = \left(\ddot{z}^{-\widehat{Re}}_{\rho\Psi} + \iota\ddot{z}^{-\widehat{IM}}_{\rho\Psi}, \ddot{z}^{+\widehat{Re}}_{\rho\Psi} + \iota\ddot{z}^{+\widehat{IM}}_{\rho\Psi}\right)$$
$$(\rho = 1,2,\ldots,\mathfrak{T};\ \Psi = 1,2,\ldots,\varsigma) \quad (41)$$

Step 4: For normalized matrix $M = C_{\rho\Psi} = \left(\breve{I}^{+}_{\rho\Psi}, \breve{I}^{-}_{\rho\Psi}\right) = \left(\ddot{z}^{+\widehat{Re}}_{\rho\Psi} + \iota\ddot{z}^{+\widehat{IM}}_{\rho\Psi}, \ddot{z}^{-\widehat{Re}}_{\rho\Psi} + \iota\ddot{z}^{-\widehat{IM}}_{\rho\Psi}\right)(\rho = 1,2,\ldots,\mathfrak{T};\ \Psi = 1,2,\ldots,\varsigma)$ and by using the attribute's weights $\dot{W}_{\Psi}$ $(\Psi = 1,2,\ldots,\varsigma)$, then we established the normalized HBCF weighted matrix $\bar{W}M_{\rho\Psi} = \left(\breve{I}^{+\ *}_{\rho\Psi}, \breve{I}^{-\ *}_{\rho\Psi}\right), (\rho = 1,2,\ldots,\mathfrak{T};\ \Psi = 1,2,\ldots,\varsigma)$ by the following method:

$$\bar{W}M_{\rho\Psi} = \dot{W}_{\Psi} \oplus M_{\rho\Psi}, (\rho = 1,2,\ldots,\mathfrak{T};\ \Psi = 1,2,\ldots,\varsigma)$$

$$= \begin{pmatrix} \bigcup\limits_{\ddot{z}^{+}_{\rho\Psi}\in\breve{I}^{+}_{1},\ldots,\ddot{z}^{+}_{\mathfrak{T}\varsigma}\in\breve{I}^{+}_{\mathfrak{T}\varsigma}} \left\{ \begin{matrix} \left(1 - \prod\limits_{\Psi=1}^{\varsigma}\left(1 - \ddot{z}^{+\widehat{Re}}_{\rho\Psi}\right)^{\dot{W}_{\Psi}}\right) + \\ \iota\left(1 - \prod\limits_{\Psi=1}^{\varsigma}\left(1 - \ddot{z}^{+\widehat{IM}}_{\rho\Psi}\right)^{\dot{W}_{\Psi}}\right) \end{matrix} \right\}, \\ \bigcup\limits_{\ddot{z}^{-}_{\rho\Psi}\in\breve{I}^{-}_{\rho\Psi},\ldots,\ddot{z}^{-}_{\mathfrak{T}\varsigma}\in\breve{I}^{-}_{\mathfrak{T}\varsigma}} \left\{ \begin{matrix} \left(-\prod\limits_{\Psi=1}^{\varsigma}\left(\left|\ddot{z}^{-\widehat{Re}}_{\rho\Psi}\right|\right)^{\dot{W}_{\Psi}}\right) + \\ \iota\left(-\prod\limits_{\Psi=1}^{\varsigma}\left(\left|\ddot{z}^{-\widehat{IM}}_{\rho\Psi}\right|\right)^{\dot{W}_{\Psi}}\right) \end{matrix} \right\} \end{pmatrix} \quad (33)$$

Step 5: By evaluating the values of border approximation areas (BAA) and for BAA matrix $\mathfrak{I} = [\varkappa_{\Psi}]_{1\times\varsigma}$ can be computed as:

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$$\varkappa_{\Psi} = \left(\prod_{\rho=1}^{\mathfrak{T}} M_{\rho\Psi}\right)^{\frac{1}{\mathfrak{T}}}, (\rho = 1,2,\ldots,\mathfrak{T};\ \Psi = 1,2,\ldots,\varsigma)$$
$$= \left(\begin{array}{c} \bigcup_{\ddot{\mathfrak{z}}^{+}_{\rho\Psi}\in \check{I}^{+}_{1},\ldots,\ddot{\mathfrak{z}}^{+}_{\mathfrak{T}\varsigma}\in \check{I}^{+}_{\mathfrak{T}\varsigma}} \left\{ \begin{array}{c} \left(\prod_{\rho=1}^{\mathfrak{T}} \left(\ddot{\mathfrak{z}}^{+\widehat{Re}}_{\rho\Psi}\right)^{\frac{1}{\mathfrak{T}}}\right) + \\ \iota\left(\prod_{\rho=1}^{\mathfrak{T}} \left(\ddot{\mathfrak{z}}^{+\widehat{IM}}_{\rho\Psi}\right)^{\frac{1}{\mathfrak{T}}}\right) \end{array} \right\}, \\ \bigcup_{\ddot{\mathfrak{z}}^{-}_{\rho\Psi}\in \check{I}^{-}_{\rho\Psi},\ldots,\ddot{\mathfrak{z}}^{-}_{\mathfrak{T}\varsigma}\in \check{I}^{-}_{\mathfrak{T}\varsigma}} \left\{ \begin{array}{c} \left(-1 + \prod_{\rho=1}^{\mathfrak{T}} \left(1 + \ddot{\mathfrak{z}}^{-\widehat{Re}}_{\rho\Psi}\right)^{\frac{1}{\mathfrak{T}}}\right) + \\ \iota\left(-1 + \prod_{\rho=1}^{\mathfrak{T}} \left(1 + \ddot{\mathfrak{z}}^{-\widehat{IM}}_{\rho\Psi}\right)^{\frac{1}{\mathfrak{T}}}\right) \end{array} \right\} \end{array} \right) \quad (34)$$

Step 6: Compute the distance $\mathfrak{D} = \left[d_{\rho\Psi}\right]_{\mathfrak{T}\times\varsigma}$ Between each alternative and BAA matrix by the following criteria.

$$d_{\rho\Psi} = \begin{cases} d\left(\bar{\ddot{W}}M_{\rho\Psi}, \varkappa_{\Psi}\right), if \quad \bar{\ddot{W}}M_{\rho\Psi} > \varkappa_{\Psi} \\ 0 \qquad\qquad if \quad \bar{\ddot{W}}M_{\rho\Psi} = \varkappa_{\Psi} \\ -d\left(\bar{\ddot{W}}M_{\rho\Psi}, \varkappa_{\Psi}\right), if \quad \bar{\ddot{W}}M_{\rho\Psi} < \varkappa_{\Psi} \end{cases} \quad (35)$$

Where $d\left(\bar{\ddot{W}}M_{\rho\Psi}, \varkappa_{\Psi}\right)$ is the mean distance from $\bar{\ddot{W}}M_{\rho\Psi}$ to $\varkappa_{\Psi}$.

Step 7: Using the given criteria, sum the values of each alternative $d_{\rho\Psi}$

$$\hat{S}_{\rho} = \sum_{!=1}^{\varsigma} d_{\rho\Psi} \quad (36)$$

A. CASE STUDY AND NUMERICAL EXAMPLE

CS uses access restrictions and encryption to prevent unwanted access, guaranteeing the security and privacy of data stored online. It assists companies in complying with rules, protects against data breaches, and guarantees business continuity by promptly identifying and mitigating risks. To keep cloud services reliable and intact, regular security audits and upgrades are necessary. Suppose an organization requires the best CS for protecting the data and applications in cloud computing. The organization considered the following four CSs.

$\acute{C}_1$ **(Access control):** The process of controlling and limiting access to data or resources using the least privilege principle is known as access control. This entails selecting which resources or information may be shared, whether they can be accessed by anybody, and what can be done once access is granted. Since it prevents unauthorized parties from accessing, altering, or manipulating private data, access control is the core component of information security. Access control may be implemented through several ways, such as physical access control, network access control, and application access control. Ensuring access to physical sites, shops, and data centers is made possible by physical access control. Servers, routers, and switches are just a few of the devices that may be used to implement network access control, often known as access to network resources. Application access control is concerned with restricting access to digital resources such as databases, software programmers, and other resources. Technical approaches, policies, and procedures must all be combined for access control to be effective. User authentication, encryption, firewalls, intrusion detection systems, and network monitoring are a few technological approaches to cyber security. For correct implementation and improvement, it is also essential that rules and procedures for access restrictions be put in place.

$\acute{C}_2$ **(Application security):** Application security refers to methods for shielding programmers against misuse, manipulation, and unauthorized access. It encompasses a range of methods and strategies for addressing holes and flaws in software security, including online and mobile apps. Ensuring that sensitive information, such as financial and personal data as well as intellectual property, cannot be accessed by unauthorized parties is one of the main objectives of application security. Several methods may be used to do this, including safe coding principles, encryption, authentication and authorization systems, and access restrictions. Using some of the more well-liked methods, including threat modeling, penetration testing, vulnerability assessments, and code reviews, may greatly improve application security. The methods address app vulnerabilities by ensuring that they are addressed before thieves have a chance to take advantage of them. The other important goal area of application security, where application security is maintained even after deployment, is protecting the application from security breaches. This necessitates ongoing application maintenance and monitoring on the airtime, which includes frequent software upgrades and patches, security assessments, and user training.

$\acute{C}_3$ **(Conformity with regulations and their observance):** Following the guidelines and regulations that have previously been established by a company or regulatory body is known as compliance. A variety of expressions exist for compliance, such as financial, legal, and regulatory implementations. Adhering to the laws and guidelines that governments set out to ensure that companies and organizations operate in a risk-free way, morally and legally compliant is known as compliance with regulations, to give it a more precise definition. Many topics, including data security, labor legislation, environmental rules, and financial reporting, are included in the term "compliance with regulations." To continue operating within the law and protect their reputation, businesses and organizations must adhere to compliance requirements. Regulation noncompliance may result in fines, legal action, and reputational damage to the organization. This frequently results in businesses hiring compliance experts or creating specialized compliance departments. Maintaining the firms' compliance with all relevant rules and regulations is the major focus of their efforts.

$Ȼ_4$ **(Identity and access management):** The term identity and access management (IAM) refers to the set of protocols and tools used to control and authorize access to data, applications, and resources. Throughout an individual's tenure in an organization, IAM comprises all the user identification, authentication, authorization, and management capabilities that allow them to access certain resources. IAM's primary goal is to restrict access to organizational resources and data to just those who have been granted permission while preventing access for unauthorized users. Using privileged access management, identity and access administration, identity and access intelligence, and identity and access governance are all part of identity and access management or IAM. IAM is critical for all sizes of organizations because it improves operational efficiency, security, and compliance. Following IAM best practices may help organizations greatly reduce their risk of identity theft, data breaches, and other security incidents. The workers will have access to the tools required to carry out their duties, as well as those of the company's partners and clients. For the assessment of this 4-cloud security, the organization hired a team of 3 experts $Ɽ_\mu, \mu = 1,2,3$ and provide them the weight $(0.5, 0.3, 0.2)$. The team of experts will assess this cloud security by considering 4 attributes that are the $₿_1 =$ Centralized $₿_2 =$ Multi-layered Security, $₿_3 =$ Scalability, and $₿_4 =$ Flexibility along with their weight $(0.4, 0.3, 0.2, 0.1)$. The assessment values would be in the form of HBCFNs and underneath are the steps to solve this information.

Step 1: The assessment values interpreted by the experts are discussed in Tables 1 to 3 respectively.

Table 1. HBCF decision matrix M_1 interpreted by expert 1.

	$₿_1$	$₿_2$	$₿_3$	$₿_4$
$Ȼ_1$	$\left(\begin{Bmatrix}(0.2+\iota 0.3),\\(0.7+\iota 0.8)\end{Bmatrix}, \begin{Bmatrix}(-0.2-\iota 0.3),\\(-0.3-\iota 0.4),\\(-0.2-\iota 0.4)\end{Bmatrix}\right)$	$\left(\begin{Bmatrix}(0.3+\iota 0.1),\\(0.2+\iota 0.6),\\(0.11+\iota 0.23)\end{Bmatrix}, \begin{Bmatrix}(-0.2-\iota 0.26),\\(-0.2-\iota 0.3)\end{Bmatrix}\right)$	$\left(\begin{Bmatrix}(0.44+\iota 0.3),\\(0.2+\iota 0.21)\end{Bmatrix}, \begin{Bmatrix}(-0.11-\iota 0.4),\\(-0.12-\iota 0.34)\end{Bmatrix}\right)$	$\left(\begin{Bmatrix}(0.3+\iota 0.4),\\(0.31+\iota 0.2)\end{Bmatrix}, \begin{Bmatrix}(-0.77-\iota 0.13),\\(-0.88-\iota 0.24),\\(-0.11-0.23)\end{Bmatrix}\right)$
$Ȼ_2$	$\left(\begin{Bmatrix}(0.1+\iota 0.35),\\(0.2+\iota 0.9),\\(0.11+\iota 0.67)\end{Bmatrix}, \begin{Bmatrix}(-0.11-\iota 0.2),\\(-0.9-\iota 0.77)\end{Bmatrix}\right)$	$\left(\begin{Bmatrix}(0.66+\iota 0.3),\\(0.22+\iota 0.1)\end{Bmatrix}, \begin{Bmatrix}(-0.2-\iota 0.34),\\(-0.45-\iota 0.23)\end{Bmatrix}\right)$	$\left(\begin{Bmatrix}(0.3+\iota 0.5),\\(0.2+\iota 0.5),\\(0.2+\iota 0.33)\end{Bmatrix}, \begin{Bmatrix}(-0.1-\iota 0.23),\\(-0.4-\iota 0.4)\end{Bmatrix}\right)$	$\left(\begin{Bmatrix}(0.4+\iota 0.5),\\(0.09+\iota 0.77),\\(0.67+\iota 0.34)\end{Bmatrix}, \begin{Bmatrix}(-0.45-\iota 0.98),\\(-0.98-\iota 0.32)\end{Bmatrix}\right)$
$Ȼ_3$	$\left(\begin{Bmatrix}(0.9+\iota 0.1),\\(0.1+\iota 0.9)\end{Bmatrix}, \begin{Bmatrix}(-0.2-\iota 0.7),\\(-0.11-\iota 0.9)\end{Bmatrix}\right)$	$\left(\begin{Bmatrix}(0.9+\iota 0.8),\\(0.7+\iota 0.8)\end{Bmatrix}, \begin{Bmatrix}(-0.2-\iota 0.9),\\(-0.8-\iota 0.99),\\(-0.89-\iota 0.6)\end{Bmatrix}\right)$	$\left(\begin{Bmatrix}(0.8+\iota 0.5),\\(0.9+\iota 0.56)\end{Bmatrix}, \begin{Bmatrix}(-0.11-\iota 0.7),\\(-0.23-\iota 0.2)\end{Bmatrix}\right)$	$\left(\begin{Bmatrix}(0.2+\iota 0.1),\\(0.1+\iota 0.6),\\(0.11+\iota 0.9)\end{Bmatrix}, \begin{Bmatrix}(-0.6-\iota 0.1),\\(-0.9-\iota 0.2),\\(-0.33-\iota 0.9)\end{Bmatrix}\right)$
$Ȼ_4$	$\left(\begin{Bmatrix}(0.6+\iota 0.7),\\(0.81+\iota 0.1)\end{Bmatrix}, \begin{Bmatrix}(-0.9-\iota 0.1),\\(-0.1-\iota 0.9)\end{Bmatrix}\right)$	$\left(\begin{Bmatrix}(0.6+\iota 0.5),\\(0.6+\iota 0.3),\\(0.9+\iota 0.1)\end{Bmatrix}, \begin{Bmatrix}(-0.11-\iota 0.9),\\(-0.9-\iota 0.11),\\(-0.55-\iota 0.6)\end{Bmatrix}\right)$	$\left(\begin{Bmatrix}(0.8+\iota 0.1),\\(0.9+\iota 0.3)\end{Bmatrix}, \begin{Bmatrix}(-0.81-\iota 0.23),\\(-0.12-\iota 0.99)\end{Bmatrix}\right)$	$\left(\begin{Bmatrix}(0.9+\iota 0.2),\\(0.34+\iota 0.8)\end{Bmatrix}, \begin{Bmatrix}(-0.2-\iota 0.24),\\(-0.9-\iota 0.2)\end{Bmatrix}\right)$

Table 2. HBCF decision matrix M_2 interpreted by expert 2.

$₿_1$	$₿_2$	$₿_3$	$₿_4$

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$\acute{C}_1$	$(\{(0.10+\iota 0.20),(0.39+\iota 0.32)\},\{(-0.17-\iota 0.72),(-0.09-\iota 0.22)\})$	$(\{(0.19+\iota 0.8),(0.67+\iota 0.12),(0.8+\iota 0.34)\},\{(-0.06-\iota 0.02),(-0.90-\iota 0.12)\})$	$(\{(0.11+\iota 0.45),(0.9+\iota 0.2)\},\{(-0.7-\iota 0.3),(-0.08-\iota 0.07)\})$	$(\{(0.21+\iota 0.9),(0.2+\iota 0.1)\},\{(-0.1-\iota 0.1),(-0.61-\iota 0.8),(-0.9-\iota 0.2)\})$
$\acute{C}_2$	$(\{(0.2+\iota 0.2),(0.6+\iota 0.1),(0.12+\iota 0.3)\},\{(-0.3-\iota 0.1),(-0.3-\iota 0.8),(-0.8-\iota 0.1)\})$	$(\{(0.9+\iota 0.2),(0.17+\iota 0.8)\},\{(-0.19-\iota 0.9),(-0.01-\iota 0.2),(-0.98-\iota 0.1)\})$	$(\{(0.09+\iota 0.0),(0.9+\iota 0.9)\},\{(-0.10-\iota 0.8),(-0.2-\iota 0.1)\})$	$(\{(0.81+\iota 0.21),(0.9+\iota 0.28)\},\{(-0.3-\iota 0.1),(-0.6-\iota 0.4)\})$
$\acute{C}_3$	$(\{(0.19+\iota 0.22),(0.76+\iota 0.78)\},\{(-0.39-\iota 0.12),(-0.94-\iota 0.20)\})$	$(\{(0.59+\iota 0.9),(0.8+\iota 0.1)\},\{(-0.2-\iota 0.6),(-0.6-\iota 0.8)\})$	$(\{(0.5+\iota 0.6),(0.19+\iota 0.1),(0.78+\iota 0.7)\},\{(-0.4-\iota 0.3),(-0.7-\iota 0.98),(-0.19-\iota 0.02)\})$	$(\{(0.23+\iota 0.21),(0.9+\iota 0.55)\},\{(-0.23-\iota 0.01),(-0.08-\iota 0.12)\})$
$\acute{C}_4$	$(\{(0.81+\iota 0.21),(0.89+\iota 0.42)\},\{(-0.71-\iota 0.61),(-0.9-\iota 0.42),(-0.8-\iota 0.89)\})$	$(\{(0.29+\iota 0.72),(0.49+\iota 0.92)\},\{(-0.88-\iota 0.76),(-0.91-\iota 0.77)\})$	$(\{(0.19+\iota 0.22),(0.49+\iota 0.62)\},\{(-0.39-\iota 0.98),(-0.43-\iota 0.65),(-0.26-\iota 0.46)\})$	$(\{(0.41+\iota 0.2),(0.9+\iota 0.9),(0.23+\iota 0.31)\},\{(-0.2-\iota 0.7),(-0.71-\iota 0.02),(-0.92-\iota 0.76)\})$

Table 3. HBCF decision matrix M_3 interpreted by expert 3.

	Ƀ_1	Ƀ_2	Ƀ_3	Ƀ_4
$\acute{C}_1$	$(\{(0.11+\iota 0.23),(0.71+\iota 0.30)\},\{(-0.23-\iota 0.11),(-0.15-\iota 0.11),(-0.12-\iota 0.19)\})$	$(\{(0.23+\iota 0.56),(0.67+\iota 0.67),(0.11+\iota 0.10)\},\{(-0.57-\iota 0.19),(-0.17-\iota 0.89)\})$	$(\{(0.15+\iota 0.41),(0.13+\iota 0.65)\},\{(-0.21-\iota 0.19),(-0.77-\iota 0.76)\})$	$(\{(0.18+\iota 0.60),(0.13+\iota 0.25)\},\{(-0.78-\iota 0.5),(-0.8-\iota 0.89),(-0.18-\iota 0.23)\})$
$\acute{C}_2$	$(\{(0.13+\iota 0.23),(0.76+\iota 0.68),(0.1+\iota 0.2)\},\{(-0.8-\iota 0.69),(-0.6-\iota 0.54)\})$	$(\{(0.45+\iota 0.24),(0.10+\iota 0.20)\},\{(-0.6-\iota 0.49),(-0.7-\iota 0.19),(-0.5-\iota 0.39)\})$	$(\{(0.14+\iota 0.76),(0.89+\iota 0.60),(0.11+\iota 0.10)\},\{(-0.7-\iota 0.09),(-0.5-\iota 0.32)\})$	$(\{(0.11+\iota 0.56),(0.19+\iota 0.01),(0.11+\iota 0.16)\},\{(-0.6-\iota 0.09),(-0.77-\iota 0.69)\})$
$\acute{C}_3$	$(\{(0.11+\iota 0.10),(0.56+\iota 0.96)\},\{(-0.54-\iota 0.19),(-0.81-\iota 0.99)\})$	$(\{(0.18+\iota 0.20),(0.11+\iota 0.2)\},\{(-0.18-\iota 0.88),(-0.4-\iota 0.99),(-0.2-\iota 0.19)\})$	$(\{(0.11+\iota 0.90),(0.80+\iota 0.60),(0.23+\iota 0.90)\},\{(-0.76-\iota 0.8),(-0.4-\iota 0.7)\})$	$(\{(0.19+\iota 0.90),(0.13+\iota 0.20),(0.11+\iota 0.05)\},\{(-0.5-\iota 0.23),(-0.90-\iota 0.87)\})$
$\acute{C}_4$	$(\{(0.11+\iota 0.70),(0.90+\iota 0.77)\},\{(-0.7-\iota 0.70),(-0.5-\iota 0.80),(-0.6-\iota 0.88)\})$	$(\{(0.10+\iota 0.20),(0.1+\iota 0.2),(0.1+\iota 0.2)\},\{(-0.8-\iota 0.9),(-0.68-\iota 0.8)\})$	$(\{(0.11+\iota 0.21),(0.17+\iota 0.20)\},\{(-0.89-\iota 0.19),(-0.76-\iota 0.29)\})$	$(\{(0.11+\iota 0.70),(0.90+\iota 0.3)\},\{(-0.7-\iota 0.19),(-0.81-\iota 0.19)\})$

Step 2: Using the HBCFWA operator we aggregate the above decision matrices M_1, M_2, M_3 to M using experts WV, which is described in Table 4.

Table 4. Aggregated matrix of M_1, M_2, and M_3.

	$Ƀ_1$	$Ƀ_2$	$Ƀ_3$	$Ƀ_4$
$Ć_1$	$\left(\{(0.153366+\iota 0.257367), (0.631329+\iota 0.629074), (0.631329+\iota 0.629074)\}, \{(-0.19588-\iota 0.31918), (-0.18199-\iota 0.25825), (-0.14211-\iota 0.28808)\}\right)$	$\left(\{(0.254594+\iota 0.50327), (0.486191+\iota 0.512384), (0.431302+\iota 0.241497)\}, \{(-0.17184-\iota 0.11312), (-0.304-\iota 0.28327), (-0.304-\iota 0.28327)\}\right)$	$\left(\{(0.300489+\iota 0.370743), (0.564038+\iota 0.215211), (0.564038+\iota 0.215211)\}, \{(-0.21811-\iota 0.31617), (-0.1541-\iota 0.24856), (-0.1541-\iota 0.24856)\}\right)$	$\left(\{(0.250796+\iota 0.676788), (0.244464+\iota 0.181856), (0.244464+\iota 0.181856)\}, \{(-0.41847-\iota 0.15731), (-0.7735-\iota 0.44769), (-0.22805-\iota 0.22056)\}\right)$
$Ć_2$	$\left(\{(0.137117+\iota 0.284381), (0.489254+\iota 0.756049), (0.111028+\iota 0.506367)\}, \{(-0.22103-\iota 0.20811), (-0.59688-\iota 0.72552), (-0.80108-\iota 0.3888)\}\right)$	$\left(\{(0.740694+\iota 0.259306), (0.182252+\iota 0.44018), (0.182252+\iota 0.44018)\}, \{(-0.24534-\iota 0.48984), (-0.1569-\iota 0.21229), (-0.58045-\iota 0.1991)\}\right)$	$\left(\{(0.210847+\iota 0.468471), (0.711714+\iota 0.704949), (0.562052+\iota 0.598315)\}, \{(-0.14758-\iota 0.2771), (-0.33973-\iota 0.25238), (-0.33973-\iota 0.25238)\}\right)$	$\left(\{(0.540189+\iota 0.440933), (0.541628+\iota 0.566299), (0.718723+\iota 0.289069)\}, \{(-0.42206-\iota 0.30652), (-0.80604-\iota 0.39899), (-0.80604-\iota 0.39899)\}\right)$
$Ć_3$	$\left(\{(0.709983+\iota 0.13782), (0.475343+\iota 0.894528), (0.475343+\iota 0.894528)\}, \{(-0.29807-\iota 0.31773), (-0.31213-\iota 0.18474), (-0.31213-\iota 0.20424)\}\right)$	$\left(\{(0.767406+\iota 0.785645), (0.669822+\iota 0.585613), (0.669822+\iota 0.585613)\}, \{(-0.19583-\iota 0.79335), (-0.63886-\iota 0.92869), (-0.58661-\iota 0.5197)\}\right)$	$\left(\{(0.645118+\iota 0.661075), (0.784845+\iota 0.464939), (0.809443+\iota 0.70835)\}, \{(-0.23849-\iota 0.55758), (-0.35876-\iota 0.41391), (-0.24261-\iota 0.12878)\}\right)$	$\left(\{(0.207153+\iota 0.442289), (0.537592+\iota 0.523995), (0.538073+\iota 0.753675)\}, \{(-0.4339-\iota 0.0592), (-0.43541-\iota 0.23024), (-0.26365-\iota 0.4884)\}\right)$
$Ć_4$	$\left(\{(0.624566+\iota 0.598881), (0.858161+\iota 0.399525), (0.858161+\iota 0.399525)\}, \{(-0.79711-\iota 0.25387), (-0.13368-\iota 0.69938), (-0.13383-\iota 0.89297)\}\right)$	$\left(\{(0.1441201+\iota 0.5384416), (0.494001+\iota 0.624946), (0.747001+\iota 0.574729)\}, \{(-0.30525-\iota 0.85549), (-0.85376-\iota 0.29327), (-0.66741-\iota 0.68492)\}\right)$	$\left(\{(0.589854+\iota 0.160008), (0.751065+\iota 0.401448), (0.751065+\iota 0.401448)\}, \{(-0.66289-\iota 0.34197), (-0.25456-\iota 0.68261), (-0.2189-\iota 0.61535)\}\right)$	$\left(\{(0.1736286+\iota 0.342499), (0.743095+\iota 0.791294), (0.526065+\iota 0.627446)\}, \{(-0.25695-\iota 0.31578), (-0.82072-\iota 0.09921), (-0.88706-\iota 0.29547)\}\right)$

Step 3: The information is benefits type so the normalized matrix is the same as interpreted in Table 4.
Step 4: The normalized weighted matrix is interpreted in Table 5.

Table 5. Normalized weighted matrix

	$Ƀ_1$	$Ƀ_2$	$Ƀ_3$	$Ƀ_4$

$\acute{C}_1$	$\left(\left\{(0.064426+\iota 0.112211),(0.329103+\iota 0.327465),(0.329103+\iota 0.327465)\right\},\left\{(-0.52095-\iota 0.63331),(-0.50585-\iota 0.58185),(-0.45819-\iota 0.60786)\right\}\right)$	$\left(\left\{(0.084375+\iota 0.189345),(0.181082+\iota 0.193836),(0.155763+\iota 0.079578)\right\},\left\{(-0.58958-\iota 0.52007),(-0.69962-\iota 0.68495),(-0.69962-\iota 0.68495)\right\}\right)$	$\left(\left\{(0.06898+\iota 0.088481),(0.152988+\iota 0.047312),(0.152988+\iota 0.047312)\right\},\left\{(-0.73745-\iota 0.7943),(-0.68796-\iota 0.75698),(-0.68796-\iota 0.75698)\right\}\right)$	$\left(\left\{(0.028462+\iota 0.1068),(0.027643+\iota 0.019872),(0.027643+\iota 0.019872)\right\},\left\{(-0.91653-\iota 0.83114),(-0.97464-\iota 0.92276),(-0.86259-\iota 0.85971)\right\}\right)$
$\acute{C}_2$	$\left(\left\{(0.057284+\iota 0.125273),(0.235668+\iota 0.43125),(0.045985+\iota 0.246017)\right\},\left\{(-0.54674-\iota 0.53372),(-0.8135-\iota 0.87955),(-0.91511-\iota 0.68531)\right\}\right)$	$\left(\left\{(0.332973+\iota 0.086115),(0.058575+\iota 0.159738),(0.058575+\iota 0.159738)\right\},\left\{(-0.65604-\iota 0.80726),(-0.5737-\iota 0.62817),(-0.84944-\iota 0.6162)\right\}\right)$	$\left(\left\{(0.046255+\iota 0.118737),(0.220233+\iota 0.216608),(0.152217+\iota 0.166746)\right\},\left\{(-0.68203-\iota 0.77362),(-0.8058-\iota 0.7593),(-0.8058-\iota 0.7593)\right\}\right)$	$\left(\left\{(0.074752+\iota 0.05649),(0.075043+\iota 0.080146),(0.119127+\iota 0.033543)\right\},\left\{(-0.91735-\iota 0.88848),(-0.97867-\iota 0.91221),(-0.97867-\iota 0.91221)\right\}\right)$
$\acute{C}_3$	$\left(\left\{(0.390506+\iota 0.057591),(0.227408+\iota 0.593317),(0.227408+\iota 0.593317)\right\},\left\{(-0.61621-\iota 0.63215),(-0.62767-\iota 0.50889),(-0.62767-\iota 0.52973)\right\}\right)$	$\left(\left\{(0.354376+\iota 0.370001),(0.282826+\iota 0.232247),(0.282826+\iota 0.232247)\right\},\left\{(-0.61315-\iota 0.93291),(-0.87422-\iota 0.97805),(-0.85213-\iota 0.82172)\right\}\right)$	$\left(\left\{(0.187138+\iota 0.194583),(0.264555+\iota 0.117566),(0.282197+\iota 0.218422)\right\},\left\{(-0.75075-\iota 0.88974),(-0.81463-\iota 0.83826),(-0.75332-\iota 0.6637)\right\}\right)$	$\left(\left\{(0.022945+\iota 0.056719),(0.74231+\iota 0.071544),(0.074328+\iota 0.130738)\right\},\left\{(-0.9199-\iota 0.75376),(-0.92022-\iota 0.86341),(-0.87519-\iota 0.93085)\right\}\right)$
$\acute{C}_4$	$\left(\left\{(0.324207+\iota 0.30608),(0.542156+\iota 0.184549),(0.542156+\iota 0.184549)\right\},\left\{(-0.91329-\iota 0.57789),(-0.44712-\iota 0.86673),(-0.44732-\iota 0.95573)\right\}\right)$	$\left(\left\{(0.160198+\iota 0.206996),(0.184836+\iota 0.254877),(0.337881+\iota 0.226251)\right\},\left\{(-0.70048-\iota 0.95425),(-0.95368-\iota 0.69212),(-0.88577-\iota 0.89267)\right\}\right)$	$\left(\left\{(0.163265+\iota 0.034272),(0.242788+\iota 0.097556),(0.242788+\iota 0.097556)\right\},\left\{(-0.92106-\iota 0.80686),(-0.7606-\iota 0.92648),(-0.73799-\iota 0.90745)\right\}\right)$	$\left(\left\{(0.124788+\iota 0.041064),(0.127074+\iota 0.145025),(0.071949+\iota 0.094019)\right\},\left\{(-0.87294-\iota 0.89112),(-0.98044-\iota 0.7937),(-0.98809-\iota 0.88522)\right\}\right)$

Step 5: The values of border approximation areas (BAA) as follows

$$\varkappa_1=\left(\left\{(0.000117+\iota 0.0000619477),(0.002391+\iota 0.003866),(0.000466+\iota 0.002205)\right\},\left\{(-0.97158-\iota 0.9748),(-0.97487-\iota 0.96729),(-0.9743-\iota 0.96833)\right\}\right),$$

$$\varkappa_2=\left(\left\{(0.000399+\iota 0.000312),(0.000139+\iota 0.000458),(0.000218+\iota 0.000167)\right\},\left\{(-0.97179-\iota 0.95946),(-0.96174-\iota 0.96413),(-0.95711-\iota 0.96332)\right\}\right),$$

$$\varkappa_3=\left(\left\{(0.0000244+\iota 0.0000175),(0.000541+\iota 0.0000294),(0.000399+\iota 0.000042)\right\},\left\{(-0.96161-\iota 0.95755),(-0.96196-\iota 0.95724),(-0.96372-\iota 0.96168)\right\}\right),$$

$$\varkappa_4 = \left(\begin{Bmatrix} \begin{pmatrix} 0.00000152 + \\ \iota 0.00000351 \end{pmatrix}, \\ \begin{pmatrix} 0.00000489 + \\ \iota 0.00000413 \end{pmatrix}, \\ \begin{pmatrix} 0.0000044 + \\ \iota 0.00000205 \end{pmatrix} \end{Bmatrix}, \begin{Bmatrix} \begin{pmatrix} -0.94838 - \\ \iota 0.9552 \end{pmatrix}, \\ \begin{pmatrix} -0.94196 - \\ \iota 0.952 \end{pmatrix}, \\ \begin{pmatrix} -0.94633 - \\ \iota 0.94943 \end{pmatrix} \end{Bmatrix} \right)$$

Step 6: The distance $\mathfrak{D} = [d_{\rho\Psi}]_{\mathfrak{T}\times\varsigma}$ between each alternative and BAA matrix is displayed in Table 6.

Table 6. Distance between alternative and BAA.

	$Ƀ_1$	$Ƀ_2$	$Ƀ_3$	$Ƀ_4$
$Ƈ_1$	0.333652	0.231755	0.158261	0.051793
$Ƈ_2$	0.216	0.208396	0.174805	0.056907
$Ƈ_3$	0.364	0.207171	0.193	0.0717
$Ƈ_4$	0.308	0.172328	0.131708	0.08718

Step 7: Sum the values of each alternative $d_{\rho\Psi}$ using (45), so we have

$$\hat{S}_1 = (0.333652) + (0.231755) + (0.158261) + (0.051793) = 0.775461858$$

$$\hat{S}_2 = (0.216) + (0.208396) + (0.174805) + (0.056907) = 0.656$$

$$\hat{S}_3 = (0.364) + (0.207171) + (0.193) + (0.0717) = 0.836$$

$$\hat{S}_4 = (0.308) + (0.172328) + (0.131708) + (0.08718) = 0.69935$$

To find the better result or good choice we have the order list according to the above values.

$$Ƈ_3 > Ƈ_1 > Ƈ_4 > Ƈ_2$$

As the value of $Ƈ_3$ is greater than all other alternatives so, $Ƈ_3$ is the best cloud security.

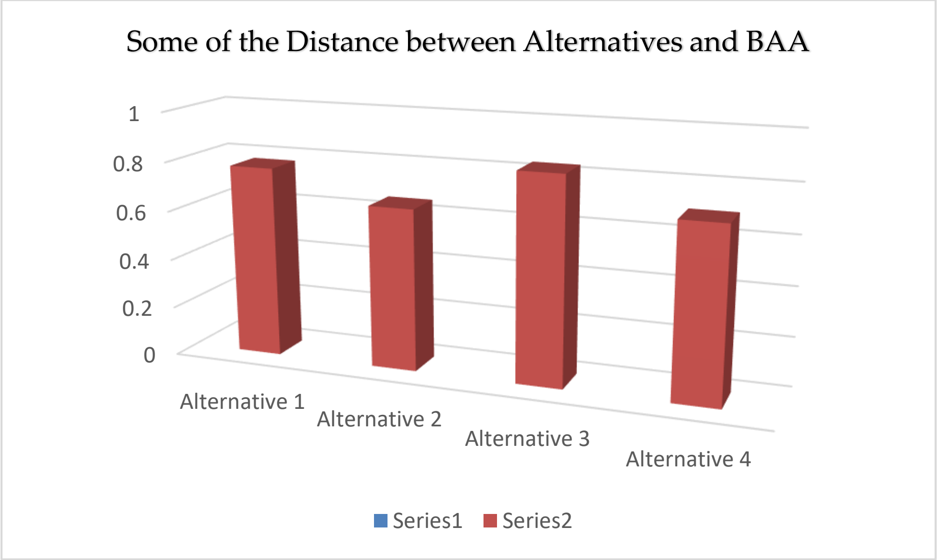

Figure 2: Sum of the distance between alternatives and BAA.

V. COMPARATIVE ANALYSIS

To demonstrate the worth and values of the diagnosed operator, we make a comparison between diagnosed work and existing works. This is because comparison plays a very important role in viewing the importance and effectiveness of any newly developed work. We cannot tell the difference

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between good and bad unless we compare. So, this study aims to compare the investigated theory with some prevailing theories. We select some aggregation theories

related to HFSs, CFSs, BFSs, and BCFSs and try to make a comparison with our proposed work to show the usefulness of our work. Hesitant fuzzy (HF) geometric Bonferroni means (HFGBM) was invented by Zhu et al. [55]. Complex fuzzy (CF) geometric AO by Bi et al. [34]. Bipolar fuzzy (BF) Hamacher AOs by Wei et al. [29]. BF Dombi AOs by Jana et al. [28]. Dombi AOs under BCF information by Mahmood and Ur Rehman [37]. BCF Hamacher AOs by Mahmood et al. [40]. A detailed comparison is discussed in Table 7.

Table 2. Comparison between the proposed and existing theories

Source	Methods	$\breve{S}(\acute{C}_1)$	$\breve{S}(\acute{C}_2)$	$\breve{S}(\acute{C}_3)$	$\breve{S}(\acute{C}_4)$	Ranking
Zhu et al. [55]	**HFGBM**	×××××	×××××	×××××	×××××	×××××
Bi et al. [34]	**CFWG**	×××××	×××××	×××××	×××××	×××××
Wei et al. [29]	**BFHG**	×××××	×××××	×××××	×××××	×××××
Jana et al. [28]	**BFDWGA**	×××××	×××××	×××××	×××××	×××××
Mahmood and Ur Rehman [42]	**BCFDWG**	×××××	×××××	×××××	×××××	×××××
Mahmood et al. [40]	**BCFHGA**	×××××	×××××	×××××	×××××	×××××
Proposed work	**HBCFWA**	0.563125	0.442083	0.578125	0.515208	$\acute{C}_3 > \acute{C}_1 > \acute{C}_4 > \acute{C}_2$

Based on the above data given in table 8, we have observed that all the supposed theories cannot be able to solve our data. Zhu et al. [55] theory of HFGBM cannot solve our information because HFS and aggregation on this set only deal with MDs and cannot be able to solve two-dimensional information. Similarly, CF AO operators given by Bi et al. [34] are capable of dealing with complex information but cannot be able to solve the negative aspects and hesitation qualities of any object. Theories of BF Hamacher AOs by Wei et al. [29] and BF Dombi AOs by Jana et al. [28] are bipolar-based theories. These theories can only solve the positive and negative aspects of any object but cannot handle our HBCF data. Dombi AOs under BCF information by Mahmood and Ur Rehman [42] and BCF Hamacher aggregation operators by Mahmood et al. [29] are near to our approach but aggregation on BCF information cannot be able to solve our data because the notion of BCFS can't cope with hesitation. So, all the above-supposed theories cannot solve our two-dimensional opinion in the form of hesitation and cannot decide the best alternative. Our proposed work and aggregation can solve two-dimensional data with negative and positive aspects and hesitation. In this case, firstly, we aggregated the information presented in Table 4 and then find the score values of the aggregated values. So, it shows the importance and effectiveness of our new work.

We reconsider the data given of numerical examples and make another comparison of our proposed MABAC approach under the environment of HBCFS with some other existing MABAC approaches. For this, we take the Fuzzy MABAC method and bipolar fuzzy MABAC method proposed by Varma [45] and Jana [46] respectively. The comparison of diagnosed MABAC work and existing MABAC work is given in Table 8.

Table 3. Comparison of proposed MABAC theory with existing MABAC theories

Source	Method	Sum the distance of alternative and BAA	Ranking
Varma [45]	Fuzzy-MABAC	×××××	×××××
Jana [46]	BF-MABAC	×××××	×××××
Proposed work	HBCF-MABAC	$\hat{S}_1 = 0.776, \hat{S}_2 = 0.656,$ $\hat{S}_3 = 0.836, \hat{S}_4 = 0.6994$	$\acute{C}_3 > \acute{C}_1 > \acute{C}_4 > \acute{C}_2$

Table 8 shows that the prevailing theories such as Varma [45] and Jana [46] are unable to handle the data in the structure of HBCFS. The MABAC approach deduced by Varma [45] deals only with MDs and cannot be able to solve negative grades, 2nd dimensional, and hesitation. Moreover, Jana [46] invented the bipolar fuzzy MABAC approach that deals with positive and negative aspects of any object or the negative and positive opinions of a human, but this theory can only handle one-dimensional information and fails to solve two-dimensional data along with hesitation. The proposed MABAC method using HBCFNs is more advanced more generalized and capable of solving any kind of data in the form of two dimensions with positive and negative grades and hesitation. Furthermore, the proposed MABAC technique is also able to solve the information described in the structure of FS, BFS, HFS, BHFS, CFS, complex HFS, and BCFS. Therefore, the proposed work is more effective and dominant in solving many real-life challenges and provides a void range to decision-makers.

VI. CONCLUSION

The term "CS" refers to a variety of computer applications and tools that are sent over the internet by servers belonging to unaffiliated third parties. Any organization's total security posture must include cloud security. It is essential to make sure that the right security measures are in place when utilizing cloud services since businesses are entrusting a third-party provider with their sensitive data and applications. Choosing the finest cloud security is therefore a very important and crucial task. In this script, we selected the best CS by taking artificial data in the environment of HBCFSs and utilizing the MABAC technique of DM using an HBCF environment. For this first in this script, we introduced the novel notions of different AOs under the environment of HBCFS which provide us a valuable framework to assess vague information that consists of negative and positive aspects along with 2^{nd} dimension and hesitation. Moreover, we deduced some AOs such as HBCFWA, HBCFOWA, HBCFWG, HBCFOWG, GHBCFWA, and GHBCFWG operators. We also deduced the related properties of these investigated AOs. After that, we developed an MABAC approach in the environment of HBCFSs and established a case study of CS with numerical demonstration. At the end of this manuscript, we compare our investigated work with various other theories to display the superiority and practicality of the proposed work.

In the future, we are hoping to expand this concept to other work such as hesitant bipolar complex fuzzy soft set [46], dual hesitant FSs [30], and bipolar complex spherical fuzzy information [57] and we will extend this idea into some other concepts. Furthermore, we extend this idea in fuzzy uncertain linguistic AOs By Mahmood et al. [59] VIKOR method is also popular for making decisions so in the future we must extend our work in this field whose basic idea was given by Riaz et al. [60]. Furthermore, we extend the AOs using logarithmic t-norm and t-conorm given by Song et al. [61].

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Hafiz Muhammad Waqas, received the MSC degrees in Mathematics from International Islamic University Islamabad, Pakistan, in 2018. He received his M.S. degrees in Mathematics from International Islamic University Islamabad, Pakistan, in 2021. Currently, He is a Ph.D. Student in mathematics from International Islamic

University Islamabad, Pakistan. His areas of interest are algebraic structures, aggregation operators, similarity measures, soft set, hesitant bipolar fuzzy set, complex fuzzy set, fuzzy logic, fuzzy decision making, and their applications. He has published 02 articles in reputed journals.

Walid Emam (M'84) received the B.S. degree in special mathematics in May 2007, the M.S. degree in mathematical statistics in July. 2015, and the PH. D. degree in Mathematical Statistics in January 2018 from Al Azhar University Faculty of Science, EGYPT. His research interests include econometrics, multivariate analysis, data mining, regression analysis, survival analysis, public health, biostatistics, probability distributions, statistical inference, environmental statistics, and economic statistics.

Tahir Mahmood is Associate Professor of Mathematics at Department of Mathematics and Statistics, International Islamic University Islamabad, Pakistan. He received his Ph.D. degree in Mathematics from Quaid-i-Azam University Islamabad, Pakistan in 2012. His areas of interest are Algebraic structures, Fuzzy Algebraic structures, soft sets and their generalizations. He has published more than 300 international publications and he has also produced more than 50 MS students and 9 Ph.D. students.

Ubaid ur Rehman, received the MSC degrees in Mathematics from International Islamic University Islamabad, Pakistan, in 2018. He received has M.S. degrees in Mathematics from International Islamic University Islamabad, Pakistan, in 2020. Currently, He is a Student of Ph.D in mathematics from International Islamic University Islamabad, Pakistan. His areas of interest are algebraic structures, aggregation operators, similarity measures, soft set, bipolar fuzzy set, complex fuzzy set, fuzzy logic, fuzzy decision making, and their applications. He has published more than 50 articles in reputed journals.

SHI YIN received master's and Ph.D. degrees in management science and engineering from Harbin Engineering University, in 2019. In 2020, he was a Teacher at Hebei Agricultural University. His current research interests include fuzzy mathematics and management science. He is a member of the editorial board of Humanities and Social Sciences Communications (SSCI/AHCI) and PLOS One (SCI).