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Synchronization Analysis for State-dependent Spiking Switched Neural Networks

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ABSTRACT This paper investigates the exponential synchronization problem for the proposed state-dependent spiking switched neural network. Under certain conditions, it is proven that state-dependent spiking switched systems can be transformed into fixed-time spiking switched systems, and exponential synchronization of homologous comparison systems implies the exact synchronization of the considered systems. Then, an exponential synchronization criterion is obtained for the proposed systems. Finally, three numerical examples are given to illustrate the validity of our results.

INDEX TERMS B-equivalence, exponential synchronization, neural networks, state-dependent spikes, switch.

I. INTRODUCTION

In recent decades, synchronization, an essential collective behavior of dynamical networks, has received more and more attention in many fields [1]-[3]. Network synchronization has potential application prospects in information science, secure communication, parallel image processing, mechanical engineering, etc. [4]-[11]. From the perspective of computer science, a neural network can be considered a mathematical model containing many parameters, and this model is generated via the substitution of several functions (nested) [12]-[13]. The equilibrium points of a neural network can be found by using machine learning algorithms or mathematical methods. Since the synchronous networks have almost identical dynamics, appropriate neural networks can be selected to study the relevant problems. So, it is essential to investigate the synchronization problem of dynamic systems.

Spiking phenomena exist in many research areas, like epidemic prevention, economics, etc. [14]-[23]. Spiking systems can be mainly divided into two categories: *fixed-time spiking systems* (FTSSs) and *state-dependent spiking systems* (SDSSs). SDSSs usually depend on the state, and thus different solutions of SDSSs have other moments of spikes. Up to the present, many books and papers have concentrated on the FTSSs [22]-[29], and there are only a few reports on SDSSs [30]-[34]. However, in reality, the spikes of many systems such as ecological systems, physiological systems, population control systems, and some circuit control systems do not arise at fixed moments [34]. SDSSs are fundamentally more important in modeling and control than FTSSs.

Switching systems could be utilized to model natural systems whose dynamics are selected from a series of options in the light of a switching signal [35]. Particularly, many practical systems are naturally multimodal in that several dynamical subsystems are required to depict their behaviors that may depend on various environmental factors [36]-[37]. Generally, spiking systems [38] and switched systems [35] are two widely studied types of hybrid systems.

Based on the previous statement, it is vital to investigate state-dependent spikes and switched systems. In [39]-[41], the researchers only studied state-dependent spiking systems without switching. Especially, the consensus of multi-agent

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systems was studied using state-dependent impulsive control [41]. However, many complex nonlinear systems that are unstable by a single controller can be stabilized by switching between finitely many controllers [42]-[43]. Meanwhile, a hybrid system, which is precisely a spiking and switched system [44], has been extensively studied in recent years [45]-[47]. In [45], the global stability of spiking-switched Hopfield neural networks was investigated, but only fixed-time spikes were considered, which happen at switched instants. That is, both the spikes and the switch occur at the same fixed moments, as introduced in [46] and [47]. In this paper, the global exponential synchronization of *state-dependent spiking switched neural networks* (SDSSNNs) is investigated, where the spikes are state-dependent and do not arise at switched instants. It is more consistent with the situation in practice and thus has more practical values.

Based on the above discussions, the main contributions of this paper are listed below:

1. State-dependent spiking switched systems are studied, where the spikes are state-dependent and do not arise at switched instants.

2. To the best of our knowledge, this paper is the first to investigate the global exponential synchronization of S-DSSNNs.

3. It is proven that state-dependent spiking switched systems can be transformed into fixed-time spiking switched systems under certain conditions, and the linear relation between original and homologous new jump operators can be obtained.

The rest of this paper is organized as follows: in Section 2, the proposed system is introduced, and some preliminaries are provided; in Section 3, the assumptions for the absence of beating are given, and a corresponding B-equivalent system is formulated; in Section 4, a criterion is established for global exponential synchronization of SDSSNNs; in Section 5, three numerical examples are given to demonstrate the validity of our results; finally, Section 6 summarizes this paper and presents the following research direction.

II. MODEL DESCRIPTION

In this paper, Z_+ and R_+ represent the set of positive integers and the set of positive real numbers, respectively; R^n denotes the n-dimensional Euclidean space, and $\Gamma_i = \{(x, y(x)) \in R_+ \times G : x = \theta_i + \tau_i(y(x)), x \in R_+, i \in Z_+, y \in G, G \subset R^n\}$ denotes the ith surface of discontinuity. P^T and I represent the transpose of matrix P and the identity matrix, respectively; $diag\{\cdots\}$ denotes the block-diagonal matrix. For $y \in R^n$, $\|y\|$ represents the Euclidean norm of y. For matrix $P \in R^{n\times n}$, $\|P\| = \sqrt{\max\{|\lambda(P^TP)|\}}$, where $\lambda(\cdot)$ denotes the eigenvalue value.

Now, SDSSNNs are proposed as follows:

$$\begin{cases} \dot{y}(x) = -A_{\phi(i+1)}(x)y(x) + B_{\phi(i+1)}(x)f_{\phi(i+1)}(y(x)) \\ \qquad + I_{\phi(i+1)}(x), \quad x \in (\theta_i, \theta_{i+1}], \\ \qquad \text{and } x \neq \theta_i + \tau_i(y(x)), \\ \Delta y(x) = J_i(y(x)),\ x = \theta_i + \tau_i(y(x)), \end{cases} \tag{1}$$

with the switched neural networks as continuous subsystems:

$$\begin{aligned} \dot{y}(x) = &-A_{\phi(i+1)}(x)y(x) + B_{\phi(i+1)}(x)f_{\phi(i+1)}(y(x)) \\ &+ I_{\phi(i+1)}(x), \quad x \in (\theta_i, \theta_{i+1}], \\ &\text{and } x \neq \theta_i + \tau_i(y(x)), \end{aligned} \tag{2}$$

and discrete subsystem:

$$\Delta y(x) = J_i(y(x)),\ x = \theta_i + \tau_i(y(x)), \tag{3}$$

where y is the state variable, $y = (y_1, y_2, \cdots, y_n)^T \in G \subset R^n$, $\phi : Z_+ \to U = \{1, 2, \cdots, m\}$, $m \in Z_+$, i.e., $\{\phi(1), \phi(2), \cdots, \phi(i), \cdots\} = \{1, 2, \cdots, m\}$. The time sequence $\{\theta_i\}$ satisfies $\theta_0 = 0 < \theta_1 < \theta_2 < \cdots < \theta_i < \theta_{i+1} < \cdots$, and $\theta_i \to \infty$ as $i \to \infty$. For $k \in U, A_k(x) = diag(a_1^{(k)}(x), a_2^{(k)}(x), \cdots, a_n^{(k)}(x))$ has positive entries, $B_k(x) = (b_{ul}^{(k)}(x)) \in R^{n\times n}$, and $f_k(y) = (f_1^{(k)}(y_1), f_2^{(k)}(y_2), \cdots, f_n^{(k)}(y_n))^T \in R^n$ represents activation functions. $\Delta y \mid_{x=\xi_i} = y(\xi_i+) - y(\xi_i)$, where $y(\xi_i+) = \lim_{x\to\xi_i+0} y(x)$ represents the state jump at ξ_i that satisfies $\xi_i = \theta_i + \tau_i(y(\xi_i))$. Generally, it is assumed that $y(\xi_i-) = \lim_{x\to\xi_i-0} y(x) = y(\xi_i)$, i.e., the solution $y(x)$ is left continuous at ξ_i.

Now, system (1) can be rewritten in the following form:

$$\begin{cases} \dot{y}(x) = -A_{\phi(i+1)}(x)y(x) + H_{\phi(i+1)}(x, y(x)), \\ \qquad x \in (\theta_i, \theta_{i+1}], \text{ and } x \neq \theta_i + \tau_i(y(x)), \\ \Delta y(x) = J_i(y(x)),\ x = \theta_i + \tau_i(y(x)), \end{cases} \tag{4}$$

where $H_{\phi(i+1)}(x, y(x)) = B_{\phi(i+1)}(x)f_{\phi(i+1)}(y(x)) + I_{\phi(i+1)}(x)$. System (1) and system (4) are equivalent. System (4) will be utilized to perform analysis in the next section. The following assumptions are made in this paper:

(H1). $\|A_j(x)\| \leq \zeta$, for any $j \in U$, where ζ is a positive constant. $H_{\phi(i+1)}(x, y(x))$ is continuous and satisfy the Lipschitz condition with respect to y, i.e., for all $x \in R_+$, there is a positive constant l_k such that $\|H_k(x, u) - H_k(x, v)\| \leq l_k\|u - v\|$, for any $u, v \in R^n$.

(H2). For each $y \in G, J_i(y) : G \to G, \tau_i(y) : G \to R$ are continuous and satisfy $J_i(0) = 0, \tau_i(0) = 0$, and there are positive constants l_J and l_τ such that $\|J_i(m) - J_i(n)\| \leq l_J\|m - n\|$ and $\|\tau_i(m) - \tau_i(n)\| \leq l_\tau\|m - n\|$, for any $m, n \in R^n$.

From Assumption (H1), one may obtain that $-A_{\phi(i+1)}(x)y(x)+H_{\phi(i+1)}(x, y(x))$ satisfies the local Lipschitz condition. By Theorem 5.2.1 in [31] (the local existence theorem), a solution to system (4) can be obtained, where the initial values $y(\theta_i)$ fall within $(\theta_i, \theta_{i+1}]$.

Finally, several definitions are given.

Definition 1 ([45]). *A piecewise continuous function $y(x) = y(x;\theta_0, y_0)$ is a solution to system (4) if*
(i) for $x \in [\theta_0, \theta_1]$, the solution coincides with the solution to

$$\begin{cases} \dot{y}(x) = -A_{\phi(1)}(x)y(x) + H_{\phi(1)}(x, y(x)), \\ y(\theta_0) = y_0, \end{cases}$$

(ii) suppose that the solution has been determined in the interval $[\theta_0, \theta_{i-1}]$. Then, for $(\theta_{i-1}, \theta_i]$, the solution coincides the solution to

$$\begin{cases} \dot{y}(x) = -A_{\phi(i)}(x)y(x) + H_{\phi(i)}(x, y(x)), \\ \qquad x \neq \theta_{i-1} + \tau_{i-1}(y(x)), \\ \Delta y(x) = J_{i-1}(y(x)),\ x = \theta_{i-1} + \tau_{i-1}(y(x)). \end{cases}$$

Based on Definition 1 and the statements mentioned above, one can obtain that a solution to system (4) exists.

Definition 2. *System (4) is said to achieve globally exponential synchronization, if there are some constants $\gamma > 0$ and $M > 0$ such that $\|y_i(x) - y_j(x)\| \leq Mexp(-\gamma(x - x_0))$, for any $i, j \in \{1, 2, \cdots, n\}$ and $x \geq x_0$.*

III. ABSENCE OF BEATING AND B-EQUIVALENCE

The following presents two assumptions that ensure the absence of beating, and then an FTSS is proposed as a comparison system of (4).

(H3). There exist three constants ν, $\underline{\vartheta}$ and $\bar{\vartheta}$ such that $0 < \tau_i(y) \leq \nu$, $\underline{\vartheta} < \theta_{i+1} - \theta_i < \bar{\vartheta}$, where $\underline{\vartheta} > \nu$, for each $i \in Z_+$.

(H4). Fix any $j \in Z_+$, and let $y(x) : (\theta_j, \theta_j + \nu] \to G$ be a solution to (4) in $(\theta_j, \theta_j + \nu]$. One of the following two conditions holds:

$$\text{(i)} \begin{cases} \dfrac{d\tau_j(y)}{dy}(-A_{\phi(j+1)}(x)y(x) + H_{\phi(j+1)}(x, y(x))) > 1, \\ \tau_j(y(\xi_j) + J_j(y(\xi_j))) \geq \tau_j(y(\xi_j)), \quad x = \xi_j, \end{cases}$$

$$\text{(ii)} \begin{cases} \dfrac{d\tau_j(y)}{dy}(-A_{\phi(j+1)}(x)y(x) + H_{\phi(j+1)}(x, y(x))) < 1, \\ \tau_j(y(\xi_j) + J_j(y(\xi_j))) \leq \tau_j(y(\xi_j)), \quad x = \xi_j, \end{cases}$$

where $x = \xi_j$ is the spiking point of system (4), that is, $\xi_j = \theta_j + \tau_j(y(\xi_j))$.

Lemma 1. *Suppose that the condition (H3) is satisfied, and $y(x) : R_+ \to G$ is a solution to system (4). Then, $y(x)$ traverses every surface Γ_i.*

The proof is omitted here.

Lemma 2. *Assume (H4) holds. Then, each solution to system (4) crosses over the surface Γ_i at most once.*

Proof. Assume there exists a solution $y(x)$ that traverses the surface Γ_j at $(s_1, y(s_1))$ and $(s_2, y(s_2))$. Generally, $s_1 < s_2$, and there is no spiking point of $y(x)$ between s_1 and s_2. Then, $s_1 = \theta_j + \tau_j(y(s_1))$ and $s_2 = \theta_j + \tau_j(y(s_2))$. For the situation (i) of (H4), we have

$$\begin{aligned} & s_2 - s_1 \\ =& \tau_j(y(s_2)) - \tau_j(y(s_1)) \\ \geq& \tau_j(y(s_2)) - \tau_j(y(s_1) + J_j(y(s_1))) \\ =& \tau_j(y(s_2)) - \tau_j(y(s_1+)) \\ =& \bigg(\frac{d\tau_j(y)}{dy}(-A_{\phi(j+1)}(x)y(x) \\ & + H_{\phi(j+1)}(x, y(x))) \bigg)_{x=\kappa\in(s_1,s_2]} (s_2 - s_1), \\ >& (s_2 - s_1). \end{aligned}$$

This is a contradiction. The situation (ii) of (H4) is similar. Therefore, the proof is completed. □

By Lemma 1 and Lemma 2, we have

Theorem 1. *Suppose that (H3) and (H4) hold. Then, each solution $y(x) : R_+ \to G$ to system (4) traverses every surface $\Gamma_i, i \in Z^+$ just once.*

Remark 1. *Without the absence of beating, the dynamics of system (4) are too complex to study. The conclusion of Theorem 1 ensures the absence of beating in system (4), and this is the basis for the main theorem in this paper.*

Now, a comparison system is constructed for system (4) by the B-equivalent method. Let $y^0(x) = y(x, \theta_i, y^0(\theta_i))$ be a solution to system (4) in $[\theta_i, \theta_{i+1}]$. Let ξ_i be the spiking moment when the solution encounters the discontinuity surface Γ_i, $\xi_i = \theta_i + \tau_i(y^0(\xi_i))$. Let $y^1(x)$ be a solution to system (4) in $[\theta_i, \theta_{i+1}]$ such that $y^1(\xi_i) = y^0(\xi_i^+) = y^0(\xi_i) + J_i(y^0(\xi_i))$.

Define the following map:

$$\begin{aligned} & W_i(y^0(\theta_i)) \\ =\ & y^1(\theta_i) - y^0(\theta_i) \\ =\ & y^1(\xi_i) - y^0(\theta_i) \\ & + \int_{\xi_i}^{\theta_i} (-A_{\phi(i+1)}(t)(y^1(t)) + H_{\phi(i+1)}(t, y^1(t)))dt \\ =\ & y^0(\xi_i) + J_i(y^0(\xi_i)) - y^0(\theta_i) \\ & + \int_{\xi_i}^{\theta_i} (-A_{\phi(i+1)}(t)(y^1(t)) + H_{\phi(i+1)}(t, y^1(t)))dt \\ =\ & \int_{\theta_i}^{\xi_i} (-A_{\phi(i+1)}(t)(y^0(t)) + H_{\phi(i+1)}(t, y^0(t)))dt \\ & + J_i\bigg(y^0(\theta_i) + \int_{\theta_i}^{\xi_i} (-A_{\phi(i+1)}(t)(y^0(t)) \\ & + H_{\phi(i+1)}(t, y^0(t)))dt\bigg) \\ & + \int_{\xi_i}^{\theta_i} (-A_{\phi(i+1)}(t)(y^1(t)) + H_{\phi(i+1)}(t, y^1(t)))dt. \end{aligned} \tag{5}$$

Remark 2. *$(\theta_i, y^0(\theta_i))$ is the common point of $[\theta_{i-1}, \theta_i]$ and $[\theta_i, \theta_{i+1}]$, and it meets the solution to*

$$\begin{cases} \dot{y}(x) = -A_{\phi(k)}(x)y(x) + H_{\phi(k)}(x, y(x)), \\ \qquad x \neq \theta_{k-1} + \tau_{k-1}(y(x)), \\ \Delta y(x) \mid_{x=\theta_{k-1}+\tau_{k-1}(y(x))} = J_{k-1}(y(x)). \end{cases}$$

for both $k = i$ and $k = i + 1$.

$y^0(x) = y(x, \theta_i, y^0(\theta_i))$ could be extended as the solution to system (4) in R_+ by Definition 1 and Remark 1. Furthermore, the following fixed-time spiking switched neural network in R_+ is considered.

$$\begin{cases} \dot{y}(x) = -A_{\phi(i+1)}(x)y(x) + H_{\phi(i+1)}(x, y(x)), \\ \qquad x \in (\theta_i, \theta_{i+1}], \\ \Delta y = W_i(y^0(\theta_i)), \ x = \theta_i. \end{cases} \tag{6}$$

Definition 3. *System (6) is the B-equivalent system of system (4) if $y^1(x) = y(x, \xi_i, y^0(\xi_i^+))$ could be extended as the solution to system (6) in R_+ by the definition of $W_i(y^0(\theta_i))$ and*

$$y^0(x) = y^1(x), x \in (\xi_i, \theta_{i+1}], \tag{7}$$

$$\begin{cases} y^1(\theta_i+) = y^0(\theta_i) + W_i(y^0(\theta_i)), \\ y^1(\xi_i) = y^0(\xi_i+) = y^0(\xi_i) + J_i(y^0(\xi_i)). \end{cases} \tag{8}$$

For a more explicit explanation, readers can refer to the book [31].

In addition, on $(\theta_i, \xi_i]$, let $h = \phi(i + 1)$, and we have

$$\begin{aligned} & y^1(x) - y^0(x) \\ =& y^0(\theta_i) + W_i(y^0(\theta_i)) \\ & + \int_{\theta_i}^{x} (-A_h(t)(y^1(t)) + H_h(t, y^1(t)))dt \\ & - y^0(\theta_i) - \int_{\theta_i}^{x} (-A_h(t)(y^0(t)) + H_h(t, y^0(t)))dt \\ =& W_i(y^0(\theta_i)) + \int_{\theta_i}^{x} [-A_h(t)(y^1(t) - y^0(t)) \\ & + H_h(t, y^1(t)) - H_h(t, y^0(t))]dt. \end{aligned}$$

From Gronwall-Bellman lemma, there is

$$\begin{aligned} & \|y^1(x) - y^0(x)\| \\ \leq & \|W_i(y^0(\theta_i))\| + (\zeta + l_h) \int_{\theta_i}^{x} \|y^1(t) - y^0(t)\|dt \\ \leq & \|W_i(y^0(\theta_i))\| \exp[(\zeta + l_h)\nu]. \end{aligned} \tag{9}$$

IV. A SYNCHRONIZATION CRITERION FOR THE SDSSNNS

Next, the global synchronization of spiking switched systems (6) and (4) is discussed, and the synchronization criteria for systems (6) and (4) are proposed respectively.

Theorem 2. *Under Theorem 1, assume a switching function $V_h(y(x))$ and some positive constants $\mu_h, \lambda_h, p, \alpha_h$ satisfy*

$$\mu_h \|y(x)\|^p \leq V_h(y(x)) \leq \lambda_h \|y(x)\|^p, \tag{10}$$

and

$$D^+ V_h(y(x)) \leq -\alpha_h V_h(y(x)), \ \ x \in (\theta_i, \theta_{i+1}], x \neq \xi_i, \tag{11}$$

where $y(x)$ is a solution to system (2), $h = \phi(i + 1)$. Then

(i) $\|y^0(\theta_i) + W_i(y^0(\theta_i))\| \leq \beta_h \|y^0(\theta_i)\|$,

(ii) $\|y^1(x) - y^0(x)\| \leq \delta_h \|y^0(\theta_i)\|$, for any $x \in (\theta_i, \xi_i]$,

where

$$\begin{aligned} \beta_h =& \left(1 - \frac{p}{\alpha_h}(1 - \exp(-\frac{\alpha_h \nu}{p}))(\zeta + l_h) \sqrt[p]{\mu_h^{-1}\lambda_h}\right)^{-1} \\ & (1 + l_J) \sqrt[p]{\mu_h^{-1}\lambda_h} > 0, \end{aligned}$$

$\delta_h = (1 + \beta_h)\exp[\nu(\zeta + l_h)]$, and $y^0(x) = y(x, \theta_i, y^0(\theta_i))$ is a solution to system (4), which traverses the surface Γ_i of the spike at ξ_i, i.e., $\xi_i = \theta_i + \tau_i(y^0(\xi_i))$. $y^1(x)$ is a solution to system (6) such that $y^1(\theta_i+) = y^0(\theta_i) + W_i(y^0(\theta_i))$ and $y^1(\xi_i) = y^0(\xi_i+) = y^0(\xi_i) + J_i(y^0(\xi_i))$, where $W_i(y^0(\theta_i))$ is defined by (5).

Proof. From conditions (10) and (11), we have

$$\sqrt[p]{\lambda_h^{-1} V_h(y(x))} \leq \|y(x)\| \leq \sqrt[p]{\mu_h^{-1} V_h(y(x))},$$

and

$$V_h(y(x)) \leq V_h(y(\theta_i^+)) \exp(-\alpha_h(x - \theta_i)), \ \ x \in (\theta_i, \xi_i].$$

By the last two inequalities, it can be found that, for $x \in (\theta_i, \xi_i]$,

$$\begin{aligned} \|y(x)\| \leq & \sqrt[p]{\mu_h^{-1} V_h(y(\theta_i+)) \exp(\alpha_h(x - \theta_i))} \\ \leq & \sqrt[p]{\mu_h^{-1}\lambda_h \exp(\alpha_h(x - \theta_i))} \|y(\theta_i+)\|. \end{aligned}$$

Furthermore,

$$\|y^0(x)\| \leq \sqrt[p]{\mu_h^{-1}\lambda_h \exp(-\alpha_h(x - \theta_i))} \|y^0(\theta_i)\|,$$

and

$$\|y^1(x)\| \leq \sqrt[p]{\mu_h^{-1}\lambda_h \exp(\alpha_h(x - \theta_i))} \|y^0(\theta_i) + W_i(y^0(\theta_i))\|.$$

Next, it is proven that the claim (i) holds. By (5), one can get

$$\begin{aligned} & \|y^0(\theta_i) + W_i(y^0(\theta_i))\| = \|y^1(\theta_i+)\| \\ = & \ \|y^1(\xi_i) - \int_{\theta_i}^{\xi_i} (-A_{\phi(i+1)}(t)(y^1(t)) + H_{\phi(i+1)}(t, y^1(t)))dt\| \\ \leq & \ \|y^0(\xi_i) + J_i(y^0(\xi_i))\| + (\zeta + l_h) \int_{\theta_i}^{\xi_i} \|y^1(t)\|dt \\ \leq & \ (1 + l_J)\|y^0(\xi_i)\| + (\zeta + l_h) \sqrt[p]{\mu_h^{-1}\lambda_h} \|y^0(\theta_i) \\ & + W_i(y^0(\theta_i))\| \int_{\theta_i}^{\xi_i} \exp(-\alpha_h(t - \theta_i)/p)dt \\ \leq & \ (1 + l_J) \sqrt[p]{\mu_h^{-1}\lambda_h} \|y^0(\theta_i)\| + \frac{p}{\alpha_h}(1 - \exp(-\alpha_h \nu/p)) \\ & (\zeta + l_h) \sqrt[p]{\mu_h^{-1}\lambda_h} \|y^0(\theta_i) + W_i(y^0(\theta_i))\|, \end{aligned}$$

which implies that

$$
\begin{aligned}
&\|y^0(\theta_i)+W_i(y^0(\theta_i))\| \\
\leq\ & \left(1-\frac{p}{\alpha_h}(1-\exp(-\frac{\alpha_h\nu}{p}))(\zeta+l_h)\sqrt[p]{\mu_h^{-1}\lambda_h}\right)^{-1} \\
&(1+l_J)\sqrt[p]{\mu_h^{-1}\lambda_h}\|y^0(\theta_i)\| \\
=\ & \beta_h\|y^0(\theta_i)\|.
\end{aligned}
$$

Finally, it is shown that the claim (ii) holds. From (9), we have

$$
\begin{aligned}
\|y^1(x)-y^0(x)\| \leq\ & \|W_i(y^0(\theta_i))\|\exp(\nu(\zeta+l_h)) \\
\leq\ & (1+\beta_h)\exp(\nu(\zeta+l_h))\|y^0(\theta_i)\| \\
=\ & \delta_h\|y^0(\theta_i)\|.
\end{aligned}
$$

The proof is completed. □

Remark 3. *Suppose all assumptions of Theorem 2 hold. Then, we have*

$$\{\delta_{\phi(1)},\delta_{\phi(2)},\cdots,\delta_{\phi(i)},\cdots\}=\{\delta_1,\delta_2,\cdots,\delta_m\}$$

by the above proof and

$$\{\phi(1),\phi(2),\cdots,\phi(i),\cdots\}=\{1,2,\cdots,m\}.$$

Let $\delta=\max\{\delta_1,\delta_2,\cdots,\delta_m\}$*, and then*

$$\|y^1(x)-y^0(x)\| \leq \delta\|y^0(\theta_i)\|,\quad x\in(\theta_i,\xi_i],\ \text{for any } i\in Z_+. \tag{12}$$

Remark 4. *Based on the aforementioned results, we have that for each solution* $y^0(x)$ *of system (4), there is a solution* $y^1(x)$ *to system (6) such that* $\|y^1(x)-y^0(x)\|\leq\delta\|y^0(\theta_i)\|$ *for* $x\in(\theta_i,\xi_i]$ *and* $y^1(x)=y^0(x)$ *for* $x\in[\theta_0,\theta_1]\cup(\xi_i,\theta_{i+1}]$*, and vice versa.*

Theorem 3. *Assume that the conditions of Theorem 2 are satisfied,* $h=\phi(i+1)$*, and let*

$$\psi(x)=\sum_{k=2}^{i+1}\ln(\rho\beta_{\phi(k)}^p)-\sum_{k=2}^{i}\alpha_{\phi(k)}(\theta_k-\theta_{k-1})-\alpha_h(x-\theta_i), \tag{13}$$

where $x\in(\theta_i,\theta_{i+1}]$*,* $\rho=\max(\lambda_u/\mu_l)_{u,l\in U}$*, and* $\psi(x)$ *is continuous on* R_+*.*

Then $\psi(x)\leq\varrho-\alpha(x-\theta_1)$*,* $x\geq\theta_1$*, with* $\varrho>0$ *and* $\alpha>0$ *being constants, implies that system (6) is globally exponentially synchronized.*

Proof. Similarly, by the conditions of Theorem 2, one can obtain

$$D^+V_h(x)\leq-\alpha_hV_h(x),\quad x\in(\theta_i,\theta_{i+1}].$$

So

$$V_h(x)\leq V_h(\theta_i+)\exp(-\alpha_h(x-\theta_i)),\quad x\in(\theta_i,\theta_{i+1}]. \tag{14}$$

It can be found that

$$
\begin{aligned}
V_h(\theta_i+) &\leq \lambda_h\|y(\theta_i+)\|^p \\
&\leq \beta_h^p\lambda_h\|y(\theta_i)\|^p \\
&\leq \rho\beta_h^pV_{\phi(i)}(\theta_i).
\end{aligned} \tag{15}
$$

Substituting (15) into (14), we obtain

$$V_h(x)\leq\rho\beta_h^p\exp(-\alpha_h(x-\theta_i))V_{\phi(i)}(\theta_i). \tag{16}$$

Repeating (16) on each interval, for $x\in(\theta_i,\theta_{i+1}]$, we have

$$
\begin{aligned}
V_h(x) \leq\ & \rho\beta_h^pV_{\phi(i)}(\theta_i)\exp(-\alpha_h(x-\theta_i)) \\
\leq\ & V_{\phi(2)}(\theta_1)\prod_{k=2}^{i+1}(\rho\beta_{\phi(k)}^p) \\
& \exp\left(\sum_{k=2}^{i}-\alpha_{\phi(k)}(\theta_k-\theta_{k-1})-\alpha_h(x-\theta_i)\right) \\
\leq\ & V_{\phi(2)}(\theta_1)\exp(\psi(x)).
\end{aligned}
$$

That is,

$$V_h(x)\leq V_{\phi(2)}(\theta_1)\exp(\psi(x)). \tag{17}$$

Substituting (10) into (17), we can obtain that

$$
\begin{aligned}
\|y(x)\| \leq\ & \sqrt[p]{\rho}\|y(\theta_1)\|\exp(\frac{\psi(x)}{p}), \\
\leq\ & \sqrt[p]{\rho}\exp(\frac{\varrho}{p})\|y(\theta_1)\|\exp(\frac{-\alpha(x-\theta_1)}{p}), x\geq\theta_1.
\end{aligned}
$$

Let $y_i^1(x)$ be any solution to system (6), and so is $y_j^1(x)$, then

$$\|y_i^1(x)-y_j^1(x)\|\leq 2\sqrt[p]{\rho}\exp(\frac{\varrho}{p})\|y(\theta_1)\|\exp(\frac{-\alpha(x-\theta_1)}{p}),$$

for any $x\geq\theta_1$. Therefore, system (6) is globally exponentially synchronized. This completes the proof. □

Remark 5. *Let* P_h *be a symmetric and positive definite matrix,* $V_h(x)=y(x)^TP_hy(x)$*,* $h=\phi(i+1)$*,* $x\in(\theta_i,\theta_{i+1}]$*, and* $V_h(x)$ *is a decreasing function. Then* $V_h(x)$ *can satisfy (10) and (11). That is, the assumptions of Theorem 2 can hold.*

Remark 6. *If*

$$\lim_{i\to\infty}\sum_{k=2}^{i+1}\ln(\rho\beta_{\phi(k)}^p)<\infty,$$

then let

$$\lim_{i\to\infty}\sum_{k=2}^{i+1}\ln(\rho\beta_{\phi(k)}^p)<\varrho,$$

and $\alpha=\min\{\alpha_1,\alpha_2,\cdots,\alpha_m\}$*,* $\psi(x)\leq\varrho-\alpha(x-\theta_1)$ *can be satisfied. In other words, the assumption of Theorem 3 is meaningful.*

Theorem 4. *Assume that the conditions of Theorem 3 hold. Then system (4) is globally exponentially synchronized.*

Proof. Let $y^0(x)=y(x,\theta_i,y^0(\theta_i))$ be a solution to system (4), for the homologous solution $y^1(x)$ to system (6), we may suppose that there are $M_1>0,\gamma_1>0$ such that $y^1(x)$ satisfies that $\|y^1(x)\|\leq M_1\exp(-\gamma_1(x-\theta_1))$ by the proof of Theorem 2, where $x\geq\theta_1$.

When $x\in(\xi_i,\theta_{i+1}]$, we have

$$\|y^0(x)\|=\|y^1(x)\|\leq M_1\exp(-\gamma_1(x-\theta_1)),$$

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and when $x \in (\theta_i, \xi_i]$, by (12), we obtain

$$\begin{aligned}\|y^0(x)\| \le& \|y^1(x) - y^0(x)\| + \|y^1(x)\| \\ \le& \delta\|y^0(\theta_i)\| + M_1\exp(-\gamma_1(x-\theta_1)) \\ \le& \delta M_1\exp(-\gamma_1(\theta_i-\theta_1)) + M_1\exp(-\gamma_1(x-\theta_1)) \\ =& M_1(1+\delta\exp(\gamma_1(x-\theta_i)))\exp(-\gamma_1(x-\theta_1)) \\ \le& M_1(1+\delta\exp(\gamma_1\nu))\exp(-\gamma_1(x-\theta_1)).\end{aligned}$$

Hence, there are $M_2 = M_1(1+\delta\exp(\gamma_1\nu)), \gamma_2 = \gamma_1$ such that $\|y^0(x)\| \le M_2\exp(-\gamma_2(x-\theta_1))$, which implies the results of Theorem 4. □

Remark 7. *By Theorem 2 and Theorem 4, we have*

$$\begin{aligned}\|y^1(x) - y^0(x)\| \le& \delta\|y^0(\theta_i)\| \\ \le& \delta M_1(1+\delta\exp(\gamma_1\nu))\exp(-\gamma_1(\theta_i-\theta_1)),\end{aligned}$$

where $y^1(x)$ and $y^0(x)$ are a solution to system (6) and a solution to system (4), respectively, $x \in (\theta_i, \theta_{i+1}]$. Evidently, $\theta_i \to \infty$ as $x \to \infty$. So system (4) and system (6) are exponentially synchronized.

V. THREE NUMERICAL EXAMPLES

Finally, three numerical examples are given to illustrate the validity of the aforementioned results. For simplicity, a two-dimensional SDSSNNs model is analyzed, where the switching systems have two or three subsystems and the corresponding switching sequence is $1 \to 2 \to 1 \to 2 \to \cdots$ or $1 \to 2 \to 3 \to 1 \to 2 \to 3 \to \cdots$.

Example 1. Consider the SDSSNNs with $I_1(x) = I_2(x) = 0$ for simplicity, and there is no spike in $[\theta_0, \theta_1]$:

$$\begin{cases} \dot{y}(x) = -A_1(x)y(x) + B_1(x)f_1(y(x)), x \in (KT, \\ \qquad KT+\sigma T], K = 0,1,2,\cdots, \\ \qquad \text{and } x \ne \theta_i + \tau_i(y(x)), \\ \Delta y(x) = J_1(y(x)),\ x = \theta_i + \tau_i(y(x)),\ i = 2K, \\ \qquad K = 1,2,3,\cdots, \\ \dot{y}(x) = -A_2(x)y(x) + B_2(x)f_2(y(x)), x \in (KT+\sigma T, \\ \qquad (K+1)T], K = 0,1,2,\cdots, \\ \qquad \text{and } x \ne \theta_{i+1} + \tau_{i+1}(y(x)), \\ \Delta y(x) = J_2(y(x)), x = \theta_{i+1} + \tau_{i+1}(y(x)), i = 2K, \\ \qquad K = 0,1,2,\cdots, \end{cases} \tag{18}$$

with $T = 2, \sigma = 0.5$, $\theta_i = i$, $\theta_{i+1} = i+1$, $\tau_i(y) = \tau_{i+1}(y) = 0.2\,\mathrm{arccot}(y_1^2)$ and $\nu = 0.1\pi$, $J_1(y) = 1.3y, J_2(y) = y$,

$$f_1(y(x)) = f_2(y(x)) = \begin{pmatrix} \sin y_1(x) \\ \tan y_2(x) \end{pmatrix},$$

and

$$A_1 = \begin{pmatrix} 2.2+\cos x & 0 \\ 0 & 3.2+\sin x \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0.8+\sin x & 0 \\ -0.7 & 1.2 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 2.4+\sin x & 0 \\ 0 & 3.1+\cos x \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0.6+\cos x & 0 \\ -0.8 & 1.5 \end{pmatrix}.$$

Note that

$$\begin{aligned} & \frac{d\tau_i(y)}{dy}\Big(-A_1(x)y(x) + B_1(x)f_1(y(x))\Big) \\ =\ & 0.4y_1\left(\frac{1}{1+y_1^4}, 0\right)\left(\begin{pmatrix} 2.2y_1 + y_1\cos x \\ 3.2y_2 + y_2\sin x \end{pmatrix} - \begin{pmatrix} 0.8\sin y_1 + \sin x\sin y_1 \\ -0.7\sin y_1 + 1.2\tan y_2 \end{pmatrix}\right) \\ =\ & 0.4y_1\frac{2.2y_1 + y_1\cos x - 0.8\sin y_1 - \sin x\sin y_1}{1+y_1^4} \\ <\ & \frac{2y_1^2}{1+y_1^4} \\ \le\ & 1. \end{aligned}$$

Moreover, $\tau_i(y+J_i(y)) - \tau_i(y) = 0.2(\mathrm{arccot}((2.3y_1)^2) - \mathrm{arccot}(y_1^2)) \le 0$, i.e., $\tau_i(y+J_i(y)) \le \tau_i(y)$.

Similarly,

$$\frac{d\tau_{i+1}(y)}{dy}(-A_2(x)y(x) + B_2(x)f_2(y(x))) < 1,$$

$\tau_{i+1}(y+J_i(y)) \le \tau_{i+1}(y)$. Therefore, assumption (H4) holds. It can be found that all conditions in Theorem 4 can be satisfied. So, system (18) is globally exponentially synchronized, as shown in Figure 1.

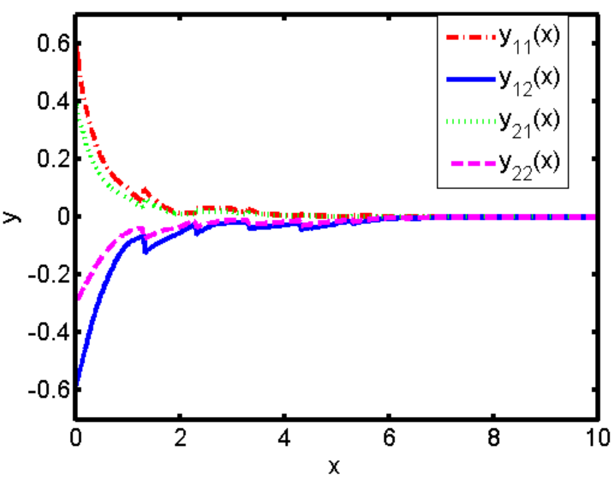

FIGURE 1. The curves of (18) in Example 1, where $y(0) = (0.6 \quad -0.6)^T$ and $y(0) = (0.4 \quad -0.3)^T$, respectively.

Remark 8. *For simplicity, the intended equilibrium can be shifted to the origin by $I_1(x) = I_2(x) = 0$. As a result, $y_{11}(x), y_{12}(x), y_{21}(x)$ and $y_{22}(x)$ converge to zero in Figure 1.*

Remark 9. *It can be seen from Figure 1 that the conditions are a little conservative, and we expect to improve these conditions in future research.*

Example 2. Consider again the SDSSNNs:

$$\begin{cases} \dot{y}(x) = -A_1(x)y(x) + B_1(x)f_1(y(x)), x \in (KT, \\ \qquad KT+\sigma T], K=0,1,2,\cdots, \\ \qquad \text{and } x \neq \theta_i + \tau_i(y(x)), \\ \Delta y(x) = J_1(y(x)),\ x = \theta_i + \tau_i(y(x)),\ i=2K, \\ \qquad K=1,2,3,\cdots, \\ \dot{y}(x) = -A_2(x)y(x) + B_2(x)f_2(y(x)), x \in (KT+\sigma T, \\ \qquad (K+1)T], K=0,1,2,\cdots, \\ \qquad \text{and } x \neq \theta_{i+1} + \tau_{i+1}(y(x)), \\ \Delta y(x) = J_2(y(x)), x = \theta_{i+1} + \tau_{i+1}(y(x)), i=2K, \\ \qquad K=0,1,2,\cdots, \end{cases} \tag{19}$$

with $T=2, \sigma=0.5,\ \theta_i=i,\ \theta_{i+1}=i+1, \tau_i(y)=\tau_{i+1}(y)=(\arctan(y_1))^2/(2\pi)$ and $\nu=0.125\pi,\ J_1(y)=-1.2y,\ J_2(y)=-1.1y,$

$$f_1(y(x)) = f_2(y(x)) = \begin{pmatrix} \sin y_1(x) \\ \sin y_2(x) \end{pmatrix},$$

and

$$A_1 = \begin{pmatrix} 0.6+\cos x & 0 \\ 0 & 0.8+0.6\sin x \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0.8+\cos x & 0 \\ -0.8 & 1.2 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0.8+\sin x & 0 \\ 0 & 0.6+0.5\cos x \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1.2+\sin x & 0 \\ -1.2 & 1 \end{pmatrix}.$$

Note that

$$\begin{aligned} &\frac{d\tau_i(y)}{dy}\Big(-A_1(x)y(x) + B_1(x)f_1(y(x))\Big) \\ =&\left(\frac{\arctan(y_1)}{\pi(1+y_1^2)}, 0\right)\left(-\begin{pmatrix} 0.6y_1 + y_1\cos x \\ 0.8y_2 + 0.6y_2\sin x \end{pmatrix} + \begin{pmatrix} 0.8\sin y_1 + \cos x \sin y_1 \\ -0.8\sin y_1 + 1.2\sin y_2 \end{pmatrix}\right) \\ =&\frac{\arctan(y_1)(-0.6y_1 - y_1\cos x + 0.8\sin y_1 + \cos x\sin y_1)}{\pi(1+y_1^2)} \\ \le&\frac{1.7|y_1|}{1+y_1^2} \\ <&1. \end{aligned}$$

Furthermore,

$$\begin{aligned} &\tau_i(y+J_i(y)) - \tau_i(y) \\ =&\frac{1}{2\pi}((\arctan((1+(-1.2))y_1))^2 - (\arctan(y_1))^2) \\ =&\frac{1}{2\pi}((\arctan(|-0.2y_1|))^2 - (\arctan(|y_1|))^2) \\ =&\frac{1}{2\pi}(\arctan(|0.2y_1|) + \arctan(|y_1|))(\arctan(|0.2y_1|) \\ &- \arctan(|y_1|)) \\ \le&0, \end{aligned}$$

that is, $\tau_i(y+J_i(y)) \le \tau_i(y)$.

Similarly,

$$\frac{d\tau_{i+1}(y)}{dy}(-A_2(x)y(x) + B_2(x)f_2(y(x))) < 1,$$

$\tau_{i+1}(y+J_i(y)) \le \tau_{i+1}(y)$. Thus, assumption (H4) holds. One can obtain that the assumptions in Theorem 4 can all hold for system (19). Therefore, system (19) is globally exponentially synchronized, as shown in Figure 2.

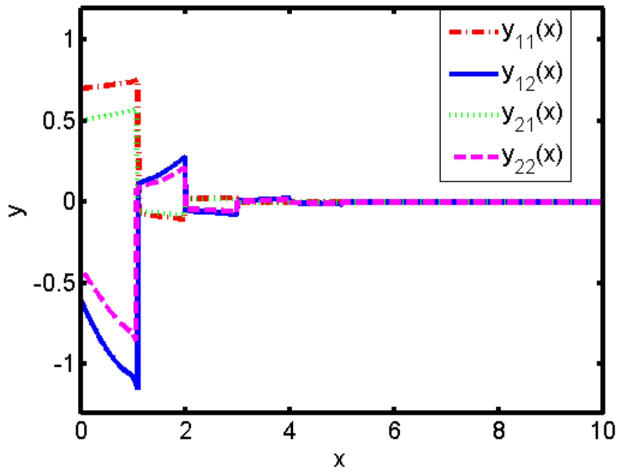

FIGURE 2. The curves of (19) in Example 2, where $y(0) = (0.7 \quad -0.6)^T$ and $y(0) = (0.5 \quad -0.4)^T$, respectively.

Remark 10. *It can be seen from Figure 2 that stabilizing impulses may stabilize the unstable continuous subsystem at its equilibrium point, which is consistent with the theoretical results.*

Example 3. Consider the SDSSNNs, where the switching systems have three subsystems, and the switching sequence is $1 \to 2 \to 3 \to 1 \to 2 \to 3 \to \cdots$.

$$\begin{cases} \dot{y}(x) = -A_1(x)y(x) + B_1(x)f_1(y(x)), x \in (KT, \\ \qquad KT+\sigma T], K=0,1,2,\cdots, \\ \qquad \text{and } x \neq \theta_i + \tau_i(y(x)), \\ \Delta y(x) = J_1(y(x)),\ x = \theta_i + \tau_i(y(x)),\ i=2K, \\ \qquad K=1,2,3,\cdots, \\ \dot{y}(x) = -A_2(x)y(x) + B_2(x)f_2(y(x)), x \in (KT+\sigma T, \\ \qquad KT+2\sigma T], K=0,1,2,\cdots, \\ \qquad \text{and } x \neq \theta_i + \sigma T + \tau_i(y(x)), \\ \Delta y(x) = J_2(y(x)), x = \theta_i + \sigma T + \tau_i(y(x)), i=2K, \\ \qquad K=0,1,2,\cdots, \\ \dot{y}(x) = -A_3(x)y(x) + B_3(x)f_3(y(x)), x \in (KT+2\sigma T, \\ \qquad (K+1)T], K=0,1,2,\cdots, \\ \qquad \text{and } x \neq \theta_i + 2\sigma T + \tau_i(y(x)), \\ \Delta y(x) = J_3(y(x)),\ x = \theta_i + 2\sigma T + \tau_i(y(x)),\ i=2K, \\ \qquad K=1,2,3,\cdots, \end{cases} \tag{20}$$

with $T = 2, \sigma = 1/3,\ \theta_i = i,\ \tau_i(y) = 0.2\,\mathrm{arccot}(y_1^2)$ and $\nu = 0.1\pi$, $J_1(y) = 1.3y$, $J_2(y) = y$, $J_3(y) = 1.1y$,

$$f_1(y(x)) = f_2(y(x)) = f_3(y(x)) = \begin{pmatrix} \sin y_1(x) \\ \tan y_2(x) \end{pmatrix},$$

and

$$A_1 = \begin{pmatrix} 2.2+\cos x & 0 \\ 0 & 3.2+\sin x \end{pmatrix},$$
$$B_1 = \begin{pmatrix} 0.8+\sin x & 0 \\ -0.7 & 1.2 \end{pmatrix},$$
$$A_2 = \begin{pmatrix} 2.4+\sin x & 0 \\ 0 & 3.1+\cos x \end{pmatrix},$$
$$B_2 = \begin{pmatrix} 0.6+\cos x & 0 \\ -0.8 & 1.5 \end{pmatrix}.$$
$$A_3 = \begin{pmatrix} 2.3+\cos x & 0 \\ 0 & 2.8+\cos x \end{pmatrix},$$
$$B_3 = \begin{pmatrix} 0.7+\sin x & 0 \\ -0.6 & 1.1 \end{pmatrix}.$$

Similar to Example 1, we have

$$\frac{d\tau_i(y)}{dy}\Big(-A_1(x)y(x) + B_1(x)f_1(y(x))\Big) < 1,$$
$$\frac{d\tau_i(y)}{dy}\Big(-A_2(x)y(x) + B_2(x)f_2(y(x))\Big) < 1,$$
$$\frac{d\tau_i(y)}{dy}\Big(-A_3(x)y(x) + B_3(x)f_3(y(x))\Big) < 1,$$

and $\tau_i(y + J_i(y)) \le \tau_i(y)$. Therefore, assumption (H4) holds. It can be found that all conditions in Theorem 4 can be satisfied. So, system (20) is globally exponentially synchronized, as shown in Figure 3.

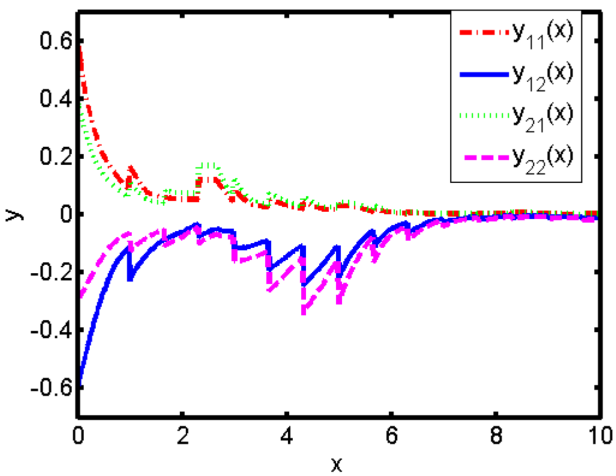

FIGURE 3. The curves of (20) in Example 3, where $y(0) = (0.6 \quad -0.6)^T$ and $y(0) = (0.4 \quad -0.3)^T$, respectively.

Remark 11. *It can be easily seen from Figure 3 that compared to Example 1, the time for the system to reach synchronization is prolonged because the unstable spike frequency increases.*

VI. CONCLUSIONS

This paper discusses the exponential synchronization of S-DSSNNs using B-equivalence. Under certain conditions, the state-dependent spiking switched systems can be transformed into fixed-time spiking switched systems. Based on this, an exponential synchronization criterion is formulated for the proposed neural network.

The dynamics of two synchronized neural networks are almost identical, and the dynamics of a neural network can be obtained by investigating its synchronous networks. Generally, the dynamics of a neural network can be investigated by using mathematical methods or machine learning algorithms. In fact, the conditions presented in this paper are a little conservative due to the complexity of the state-dependent spiking switched systems, such as the conditions of absence of beating and the estimation of the norm of the transformation map $W_i(x)$. We will relax these conservative conditions for certain neural networks and extend the presented method to delayed systems or more general spiking switched systems in future work.

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