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RESEARCH ARTICLE

Edge-Version of Fault-Tolerant Resolvability in Networks

MUHAMMAD FAHEEM[1], MUHAMMAD AHMAD[1], ZOHAIB ZAHID[1], MUHAMMAD JAVAID[1], AND MAMO ABEBE ASHEBO[2]

[1]Department of Mathematics, University of Management and Technology, Lahore 54770, Pakistan
[2]Department of Mathematics, Wollega University, Nekemte 395, Ethiopia

Corresponding author: Mamo Abebe Ashebo (mamoabebe37@gmail.com)

ABSTRACT Fault tolerance refers to a system's capacity to continue functioning as intended, even when one of its components fails. Such a system is known as a fault-tolerant, self-stable system. The idea of fault-tolerant resolving sets (FTRS) arises from the concept that removing any vertex from a resolving set (RS) still results in another RS, hence designated as an FTRS. The minimum size of this set is called the fault-tolerant metric dimension (FTMD). This paper extends the concept to edges by introducing the edge version of the fault-tolerant resolving set (EVFTRS) and its corresponding edge version of the fault-tolerant metric dimension (EVFTMD), which is based on edge distances in the network analysis. We calculate the EVFTMD values for n-sunlet and cycle with chord networks, demonstrating that these values remain constant. These findings illustrate the reliability of these network topologies in environments prone to edge failures, offering valuable insights for designing resilient communication systems such as optical networks and smart grids. By adopting an edge-based perspective, this study advances fault tolerance analysis in graph theory and practical network design.

INDEX TERMS n-sunlet graph, cycle with chord graph, edge computing, fault-tolerance, metric dimension.

I. INTRODUCTION

In real-world situations, networks frequently experience defects or failures in their edges or vertices for a variety of reasons, including purposeful attacks, network outages, and hardware problems. This encourages the research of network fault tolerance, where defects are present yet the network should nevertheless function and maintain its key characteristics. FTMD is a notion associated with the study of network fault tolerance and metric dimension (MD). The degree to which a subset of vertices known as "landmarks" may be employed to distinguish a network's vertices from one another is known as the network's MD.

Slater used the term "landmarks" for RSs in [1] and [2]. Moreover, another mathematician, Harary, independently described the concept of RSs and MD with the help of Melter [3]. It can be used in a number of fields, including facility location and network design. The concept of a resolving set (RS) finds diverse applications across computer networks, robot navigation, chemical structures, and electric circuits. It plays a crucial role in identifying intruders in indoor security systems, where a chain of sensors (nodes) acts as a means of detection. Later, Chartrand and Zhang [4] and [5] applied this notion in the fields of chemistry and robotics. The study in [6] examined the edge locating numbers and metric size of coffee chemical structures to understand coffee molecules. It features a network of their properties and highlights key compounds like caffeine, cafestol, kahweol, chlorogenic acid, caffeic acid, gallotannins, and ellagitannins. In [7], Koam examined the edge MD of some convex polytope structures and compares the results with previous resolvability parameter cardinalities. Manuel et al. [8] and Wang et al. [9] used the RSs as sensors (nodes) in computer networks.

However, in real-world situations, networks frequently experience defects or failures in their edges or vertices for a variety of reasons, including purposeful attacks, network outages, and hardware problems. This encourages the research of network fault tolerance, where defects are present yet the network should nevertheless function and

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maintain its key characteristics. This problem was initially solved by Hernando et al. [10], who also introduced the FTRS invariant. That is, if the removal of any element from a RS results in another RS, then such an RS is termed a FTRS, and the minimum cardinality of the FTRS is referred to as FTMD.

Researchers have investigated the relationship between fault tolerance and various network parameters, as well as algorithms and approaches to compute or estimate FTMD. Identifying how resilient a network is to maintaining its metric properties in the face of errors is the goal of the study of FTMD. It is used in fault-tolerant network design and analysis, where secure communication and routing are critical. To ensure the resilience and reliability of networks during failures or disruptions, researchers and engineers can construct fault-tolerant systems with a better understanding of FTMD [11].

In communication networks like the internet or wireless sensor networks, ensuring fault tolerance is crucial to maintaining uninterrupted connectivity. The FTMD can help in designing network topologies that can withstand node or link failures without causing significant disruptions in communication [12]. Similarly, in transportation systems like railway networks or airline routes, fault tolerance is essential to prevent widespread disruptions caused by accidents, infrastructure failures, or natural calamities. FTMD can assist in designing robust transportation networks that can adapt to unforeseen circumstances. Power distribution networks are critical to ensuring uninterrupted electricity supply to consumers. To maintain high levels of reliability, designers can use FTMD techniques to create robust power grid architectures. These architectures can isolate faults and reduce the impact of failures on the overall system performance, as stated in [13]. Apart from power distribution, FTMD can also be useful in other domains, such as social networking platforms, healthcare networks, and other areas where network reliability and resilience are crucial. Organizations can ensure the continued operation of their systems by incorporating fault tolerance into network design and optimization processes, even in the face of unforeseen events or failures.

FTMD has been studied for a variety of network classes, including general networks, trees, and grids. Hernando et al. [10] pioneered the application of the invariant FTMD by computing it for tree networks. They demonstrated that the values of the FTMD are bound. This analysis was then expanded to different network families by Prabhu et al. [14]. In another study, Zheng et al. [15] investigated the FTMD of wheel-related networks. In [16] and [17], certain interconnection networks have been discussed for the FTMD problem. Researchers in [18] and [19] investigated convex polytopes, determining their FTMD. Furthermore, the FTMD is thought to be important for studying other structures, such as FTRSs of cycle networks [20], and the FTMD of King's networks studied in [21]. Basak et al. [22] made significant contributions by computing FTMD for circulant networks. In [23], Hussain et al. established constant FTMD for specific families of gear networks.

The precise MD and FTMD values of linear phenylene, linear heptagon, and cyclic hexagonal square chain structures were determined by Nadeem et al. in [24]. Also, Ahmad et al. [25] found the FTRSs of certain chemical structures. Laxman [26] precisely determined the exact value of FTMD of the cube of paths. Raza et al. [27] computed some bounds for the FTMD of anti-prism networks. Guo et al. [28] contributed great effort in this invariant and found constant FTMD for some families of the line networks. Ahmad et al. [29] proved that the FTMD of the barycentric subdivision of the Cayley network is constant. Ali et al. [30] introduce the fault-tolerant mixed MD as a new parameter for resolvability, comparing a lab network issue to designing a circular lab for optimal device placement to ensure internet connectivity.

The FTMD of the some families of networks was computed by Faheem et al. [31]. In the study [32], Nadeem et al. identify the fault-tolerant beacon set for the hexagonal Mobius ladder network $H(\alpha, \beta)$ and demonstrate that all variations of α and β within $H(\alpha, \beta)$ maintain a constant FTMD. It was discovered by Sharma and Bhat [33] that the families of the double antiprism networks can be resolved with just 4 vertices. This invariant was found for the three types of ladder networks by Wang et al. [34]. Several (edge) metric-based theories for the construction of the hollow coronoid were investigated in [35]. In a different study, Ahmad et al. [36] examined the FTMD problem for the $P(n, 2) \odot K_1$, and Simic et al. [37] computed the FTMD for grid networks. For the FTMD applications in engineering, we refer [38] and [39].

The novelty and contribution of this paper lie in the introduction of the concepts of EVFTRS and EVFTMD, which utilize the edge distance within a network, for details: (see [40], [41]). The introduction of EVFTMD as an expansion of the FTMD establishes a novel framework for examining edge resilience, distinguishing it from earlier research that focused exclusively on vertices. Our findings regarding n-sunlet and cycle with chord networks provide new perspectives on the fault tolerance of networks subject to edge perturbations, laying the groundwork for future studies on different types of networks.

This research addresses a gap in the literature by focusing on edge-based tolerance, which is significant for network design, routing, communication in distributed networks, electric circuits, and fault recovery in large-scale infrastructure systems [42]. By formulating the problem for these specific network families, we enable the identification of distinct combinatorial properties that have not been explored previously, thereby expanding our understanding of both theoretical graph theory and its practical applications.

Our research on EVFTMD enhances availability and reliability by ensuring distinct edge identification despite faults. While we emphasize mathematical modeling over system implementation, EVFTMD is crucial for failure detection and recovery in network structures, facilitating accurate fault localization. We simplify the design process by leveraging effective combinatorial characteristics of structures, such as n-sunlet and cycle with chord networks. Although the study does not directly address cost or latency, it establishes a theoretical basis for creating resilient networks. Overall, our framework promotes structural robustness and supports resilient network design, while influencing data consistency and mitigating SPOF (single point of failure) issues.

The following sections provide a detailed discussion of the remaining part of the article: Section 2 covers notations and basic definitions that are useful in computing EVFTMD. Moving on to Section 3, we delve into the computation of EVFTMD for n-sunlet networks. In Section 4, we investigate EVFTMD for cycles with chord networks. Section 5 is dedicated to addressing the implications and applications of our findings. Finally, Section 6 concludes this article by offering an opinion.

II. PRELIMINARIES AND METHODOLOGY

Suppose $\mathbf{G}$ is a connected, undirected, and simple network consisting of the set of vertices $V(\mathbf{G})$ and $E(\mathbf{G})$ is the set of edges. Let $v_1, v_2 \in V(\mathbf{G})$, then the distance $d(v_1, v_2)$ is the smallest path between v_1 and v_2. For every subset $U \subseteq V(\mathbf{G})$ is termed a RS of $\mathbf{G}$ if there exists a vertex $u \in U$ such that it resolves every pair of vertices $v_1, v_2 \in V(\mathbf{G})$. The metric basis of a RS is the RS having minimum cardinality. The number of vertices in such a basis is referred to as its MD, or $\beta(\mathbf{G})$. The MD of the line network of a network is called EVMD, introduced by Nasir et al. in [43]. In this paper, we named the FTMD of the line network of a network the EVFTMD of the network. Now, in order to investigate the EVFTMD for certain networks, the following definitions are useful:

Definition 1: Let $\mathbf{G}(V(\mathbf{G}), E(\mathbf{G}))$ be an undirected and connected network, where $V(\mathbf{G})$ and $E(\mathbf{G})$ are the sets of nodes (vertices) and branches (edges), respectively. The degree "$d_{\mathbf{G}}(v)$" of "$v \in V(\mathbf{G})$" is the cardinality of edges that are incident to a vertex v.

Definition 2: Let $v_1 - v_2$ be a path for any $v_1, v_2 \in V(\mathbf{G})$, then the distance between $v_1, v_2 \in V(\mathbf{G})$ is the minimum cardinality of the edges in $v_1 - v_2$ path.

Definition 3: Let $K = \{k_1, k_2, \ldots, k_t\} \subseteq V(\mathbf{G})$, then the absolute difference code have t-vector $(|d_{\mathbf{G}}(l_1, k_1) - d_{\mathbf{G}}(l_2, k_1)|, \ldots, |d_{\mathbf{G}}(l_1, k_t) - d_{\mathbf{G}}(l_2, k_t)|)$ for any $l_1, l_2 \in V(\mathbf{G})$ with respect to K, denoted by $AD((l_1, l_2)|K)$.

Definition 4: Let $K = \{k_1, k_2, \ldots, k_t\} \subseteq V(\mathbf{G})$, then $\big(d_{\mathbf{G}}(l, k_1), d_{\mathbf{G}}(l, k_2), \ldots, d_{\mathbf{G}}(l, k_p)\big)$ be the distance code $r(l|K)$ of order t for a node $l \in V(\mathbf{G})$. If $r(l_1|K) \neq r(l_2|K)$ for every different $l_1, l_2 \in V(\mathbf{G})$, then K is known as RS of the network $\mathbf{G}$. Furthermore, K is called the RS if $AD((l_1, l_2)|K)$ for every different $l_1, l_2 \in V(\mathbf{G})$ have at least one non-zero in their t-vector. The MD, represented by $\beta(\mathbf{G})$, is the least cardinality of K.

Definition 5: A RS $K' = \{k'_1, k'_2, \ldots, k'_t\} \subseteq V(\mathbf{G})$ is known as a FTRS, if $K' \setminus \{k'\}$ is again a RS, for each $k' \in K'$. Furthermore, K' is called the FTRS if $AD((l_1, l_2)|K')$ for every different $l_1, l_2 \in V(\mathbf{G})$ have at least two non-zeros in their t-vector. The FTMD, represented by $\beta'(\mathbf{G})$, is the least cardinality of K'.

Definition 6: The line network $L(\mathbf{G})$ of a network $\mathbf{G}$, whose vertices are the edges of $\mathbf{G}$ and two edges e_1 and e_2 have a common end vertex in $\mathbf{G}$ if and only if they are connected in $L(\mathbf{G})$.

Definition 7: The minimum length of a path between any two nodes $e_1, e_2 \in L(\mathbf{G})$ is considered as the edge distance between edges $e_1, e_2 \in E(\mathbf{G})$ and it is denoted by $d_E(e_1, e_2)$.

Definition 8: The edge version of absolute difference code consists of a t-vector $(|d_E(e_1, g_1) - d_E(e_2, g_1)|, \ldots, |d_E(e_1, g_t) - d_E(e_2, g_t)|)$ for any $e_1, e_2 \in E(\mathbf{G})$ with respect to $K_E = \{g_1, g_2, \ldots, g_t\} \subseteq E(\mathbf{G})$ and it is denoted by $AD_E((e_1, e_2)|K_E)$.

Definition 9: Let $K_E = \{g_1, g_2, \ldots, g_t\} \subseteq E(\mathbf{G})$, and $e_1 \in E(\mathbf{G})$, then the t-tuple $\big(d_E(e_1, g_1), d_E(e_1, g_2), \ldots, d_E(e_1, g_t)\big)$ is the edge version of distance code of e_1 with respect to K_E and is denoted by $r_E(e_1|K_E)$. If the edge version of distance codes $r_E(e_1|K_E)$ are distinct for every edge $e_1 \in E(\mathbf{G})$, then K_E is known as the EVRS. On the other hand, if $AD_E((e_1, e_2)|K_E)$ has minimum one non-zero in its t-vector for every $e_1 \neq e_2 \in E(\mathbf{G})$, then K_E is known as the EVRS. The least cardinality of the EVRS is known as the EVMD, denoted by $\beta_E(\mathbf{G})$.

Definition 10: For the EVRS K'_E, if $K'_E \setminus \{g_1\}$ is also an EVRS for any $g_1 \in K'_E$, then K'_E is called the EVFTRS. Moreover, if $AD_E((e_1, e_2)|K'_E)$ has minimum two non-zeros in its t-vector for every $e_1 \neq e_2 \in E(\mathbf{G})$, then K'_E is known as the EVFTRS. The least cardinality of K'_E is known as the EVFTMD, denoted by $\beta'_E(\mathbf{G})$.

For every network $\mathbf{G}$, Estrado et al. determined some important bounds, which are given below:

Lemma 1 ([44]): Let $\mathbf{G}$ be any network, then $\beta(\mathbf{G}) < \beta'(\mathbf{G})$.

Lemma 2 ([44]): Let $\mathbf{G} \neq P_n$ be any network, then $\beta'(\mathbf{G}) \geq 3$.

Lemma 3: If $\beta'(\mathbf{G}) = 3$ for any network $\mathbf{G}$ and $\{e_1, e_2, e_3\} \subset V(\mathbf{G})$ is a FTRS in $\mathbf{G}$, then $d_{\mathbf{G}}(e_1) \leq 3$, $d_{\mathbf{G}}(e_2) \leq 3$ and $d_{\mathbf{G}}(e_3) \leq 3$.

It should be noted that Lemmas 1, 2, and 3, also holds for $\beta_E(\mathbf{G})$ and $\beta'_E(\mathbf{G})$.

A. METHODOLOGY

The process employed to determine the EVFTMD for networks structured as n-sunlets and cycles with chords involves several systematic steps. Suppose $\mathbf{G}$ is a connected, undirected, and simple network. The EVFTMD for the network $\mathbf{G}$ will be calculated using the steps of the algorithm outlined below:

- The adjacency matrix of the network **G**, represented by $A = [a_{ij}]$ is constructed as under:

$$a_{ij} = \begin{cases} 1, & \textit{if } y_i \text{ and } y_j \text{ are adjacent edges in } \mathbf{G}; \\ 0, & \textit{if } \text{ otherwise.} \end{cases}$$

- Compute the edge distances (shortest path lengths between each pair of edges in the network) of **G** from the adjacency matrix A using MATLAB programming.
- Iteratively identify the EVFTRSs. Step 2 is repeated if the EVFTRSs cannot be successfully identified, changing the strategy or parameters as needed to find the EVFTRSs. The cardinality of the smallest EVFTRS is then used to calculate the EVFTMD.
- Generalization of the EVFTRSs and EVFTMD for the class of network structures under consideration.
- Compute the distance codes for all the edges of the networks relative to the corresponding EVFTRSs.
- Generalize the distance codes for the class of network structures under consideration.
- Contradictions are used to confirm the minimality and uniqueness of the EVFTRS, ensuring that it accurately represents the EVFTMD.

III. THE EVFTMD FOR THE *N*−SUNLET NETWORKS

The $n-$sunlet network S_n, is constructed by adding n pendant branches to a cycle network C_n. The family of $n-$sunlet networks with edge set $E(S_n) = \{e_i, f_i : 1 \leq i \leq n\}$ is shown in Figure 1.

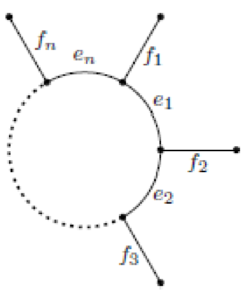

FIGURE 1. *n*−**Sunlet Network (*S_n*).**

Now the line network $L(S_n)$ is needed to compute $\beta'_E(S_n)$. The line networks of the family of $n-$sunlet networks $L(S_n)$ consists of an inner cycle of vertices $\{e_i : 1 \leq i \leq n\}$ and the outer vertices $\{f_i : 1 \leq i \leq n\}$ as shown in Figure 2.

The known result of the EVMD for the family of $n-$sunlet networks is presented below.

Theorem 1 ([43]): For any integer $n \geq 4$, we have

$$\beta_E(S_n) = \begin{cases} 2, & \textit{if } \text{ n is even;} \\ 3, & \textit{if } \text{ n is odd.} \end{cases}$$

The results about the $\beta'_E(S_n)$ are presented below.

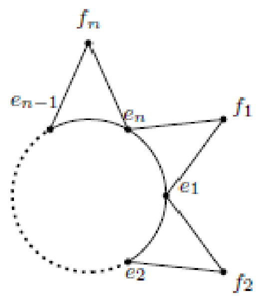

FIGURE 2. **Line network of *n*-Sunlet network (*L*(*S_n*)).**

Theorem 2: Let $n \geq 4$ be any integer, then

$$\beta'_E(S_n) = \begin{cases} 3, & \textit{if } n = 6; \\ 4, & \text{else.} \end{cases}$$

Proof: To prove the stated theorem, we will calculate the EVFTMD, denoted as $\beta'_E(\mathbf{S_n})$, specifically for the n-sunlet network S_n. To ensure a comprehensive understanding of the network's properties, we proceed by examining the following two cases based on the value of n:

Case 1: (when n is odd)

Take $K'_E = \{e_1, e_2, e_{\frac{n+3}{2}}, e_{\frac{n+5}{2}}\} \subseteq E(S_n)$ for odd integers $n \geq 5$. For the edges e_p, the distance codes are shown in Table 1.

TABLE 1. **Distance codes for the edges e_p, where $1 \leq p \leq n$.**

$r_E(e_p\|K'_E)$	p
$(-1+p, -p+2, \frac{n-1}{2}, \frac{n-5+2p}{2})$	$1 \leq p \leq 2$
$(p-1, -2+p, \frac{n+3-2p}{2}, \frac{n+5-2p}{2})$	$3 \leq p \leq \frac{n+1}{2}$
$(\frac{-1+n}{2}, \frac{-1+n}{2}, 0, 1)$	$p = \frac{n+3}{2}$
$(-p+n+1, -p+n+2, \frac{-n-3+2p}{2}, \frac{-n-5+2p}{2})$	$\frac{n+5}{2} \leq p \leq n$

For the edges f_p, where $1 \leq p \leq 5$, the distance codes are provided in Table 2.

TABLE 2. **Distance codes for the edges f_p, where $1 \leq p \leq 5$.**

$r_E(f_p\|K'_E)$	p
$(1, 2, 2, 1)$	$p = 1$
$(1, 1, 3, 2)$	$p = 2$
$(-1+p, -2+p, -p+5, -p+6)$	$3 \leq p \leq 4$
$(2, 3, 1, 1)$	$p = 5$

The distance codes of the edges f_p, where $n \geq 7$ and $1 \leq p \leq n$ are provided in Table 3.

We can verify that for every $e, f \in E(S_n)$, at least two elements in the 4-vector $AD_E((e,f)|K'_E)$ are non-zero.

TABLE 3. Distance codes for the edges f_p, where $n \geq 7$ and $1 \leq p \leq n$.

$r_E(f_p\|K_E^{'})$	p
$(1, 2, \frac{-1+n}{2}, \frac{-3+n}{2})$	$p = 1$
$(1, 1, \frac{1+n}{2}, \frac{-1+n}{2})$	$p = 2$
$(-1+p, -2+p, \frac{5+n-2p}{2}, \frac{7+n-2p}{2})$	$3 \leq p \leq \frac{3+n}{2}$
$(\frac{-1+n}{2}, \frac{1+n}{2}, 1, 1)$	$p = \frac{5+n}{2}$
$(-p+2+n, -p+3+n, \frac{-n+2p-3}{2}, \frac{-n+2p-5}{2})$	$\frac{7+n}{2} \leq p \leq n$

This shows that $\beta_E'(S_n) \leq 4$. Using the results of Lemma 1 and Theorem 1, we have $\beta_E'(S_n) \geq 4$. Hence $\beta_E'(S_n) = 4$.

Case 2: (when n is even)

It can be seen that $K_E' = \{e_1, e_2, e_3, e_4\} \subseteq E(S_4)$ and $K_E' = \{f_1, f_3, f_5\} \subseteq E(S_6)$ are EVFTRS for $n = 4$ and $n = 6$, respectively.

Take $K_E' = \{e_1, e_2, e_{\frac{n+2}{2}}, e_{\frac{n+4}{2}}\} \subseteq E(S_n)$ for even integers $n \geq 8$. For the edges e_p, where $1 \leq p \leq n$, the distance codes are provided in Table 4.

TABLE 4. Distance codes for the edges e_p, where $1 \leq p \leq n$.

$r_E(e_p\|K_E^{'})$	p
$(0, 1, \frac{n}{2}, \frac{-2+n}{2})$	$p = 1$
$(-1+p, -2+p, \frac{n+-2p+2}{2}, \frac{-2p+4+n}{2})$	$2 \leq p \leq \frac{2+n}{2}$
$(-p+1+n, -p+2+n, \frac{2p-n-2}{2}, \frac{-n+2p-4}{2})$	$\frac{4+n}{2} \leq p \leq n$

For the edges f_p, the distance codes are provided in Table 5.

TABLE 5. Distance codes for the edges e_p, where $1 \leq p \leq n$.

$r_E(f_p\|K_E^{'})$	p
$(1, 2, \frac{n}{2}, \frac{-2+n}{2})$	$p = 1$
$(1, 1, \frac{n}{2}, \frac{n}{2})$	$p = 2$
$(-1+p, -2+p, \frac{-2p+4+n}{2}, \frac{-2p+n+6}{2})$	$3 \leq p \leq \frac{2+n}{2}$
$(\frac{n}{2}, \frac{n}{2}, 1, 1)$	$p = \frac{4+n}{2}$
$(-p+2+n, -p+3+n, \frac{-n+2p-2}{2}, \frac{-n+2p-4}{2})$	$\frac{6+n}{2} \leq p \leq n$

We can verify that for every $e, f \in E(S_n)$, at least two elements in the 4-vector $AD_E((e,f)|K_E')$ are non-zero. So, $\beta_E'(S_n) \leq 4$. Now, in order to prove that $\beta_E'(S_n) \geq 4$ for $n \neq 6$, suppose contrarily that $\beta_E'(S_n) = 3$ and using Lemma 3, we present all the conditions:

Case A: (when $n = 4$)

There are four possibilities $K_{E_1}' = \{f_1, f_2, f_3\}$, $K_{E_2}' = \{f_1, f_3, f_4\}$, $K_{E_3}' = \{f_1, f_2, f_4\}$ and $K_{E_4}' = \{f_2, f_3, f_4\}$ are the subsets of $E(S_4)$. It can be easily verified that there is no EVFTRS of cardinality 3.

Case B: (when $n \geq 8$)

Let $K_E' = \{f_p, f_s, f_t\} \subseteq E(S_n)$ with $1 \leq p < s < t \leq \frac{n}{2}$, then

$$AD_E((f_n, e_{n-1})|K_E') = \begin{cases} (0, 0, 1), & if\ t = \dfrac{n}{2}; \\ (0, 0, 0), & if\ else. \end{cases}$$

So, K_E' is not the EVFTRS.

Now, let $K_E' = \{f_p, f_s, f_t\} \subseteq E(S_n)$ with $1 \leq p < s < \frac{n}{2} \leq t \leq n$, then $AD_E((f_n, e_{n-1})|W_E') = (0, 0, 1)$. So, K_E' is not the EVFTRS.

Above absolute difference codes represent that there is no EVFTRS with cardinality 3. So, $\beta_E'(S_n) \geq 4$. Hence, $\beta_E'(S_n) = 4$. □

IV. THE EVFTMD FOR THE CYCLES WITH CHORD NETWORKS

The family of cycles with chord network C_n^t is determined by joining two nodes of the cycle network C_n at t distance. The edge set of this family is $E(C_n^t) = \{f_i : 1 \leq i \leq n+1\}$, as shown in Figure 3.

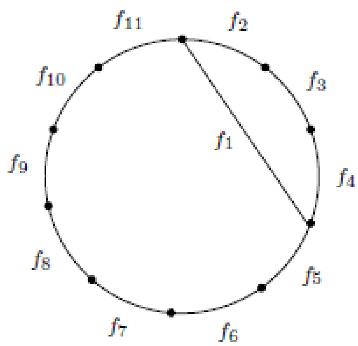

FIGURE 3. Cycles with chord network C_{10}^3.

Now, to compute $\beta_E'(C_n^t)$, we need to convert the family of cycles with chord networks into their line networks. The line networks of the family of cycles with chord networks $L(C_n^t)$ consists of vertices $\{f_i : 1 \leq i \leq n+1\}$ as shown in Figure 4.

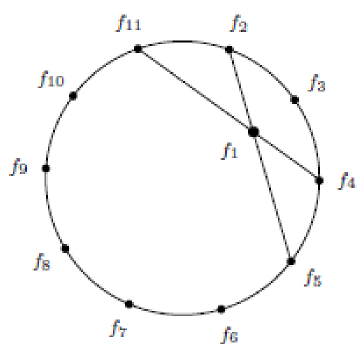

FIGURE 4. Line network of cycles with chord network ($L(C_{10}^3)$).

The known result of the EVMD of the network C_n^t is presented below.

Theorem 3: For any integers $2 \leq t \leq \lfloor \frac{n}{2} \rfloor$ and $n \geq 4$, we have $\beta_E(C_n^t) = 2$.

The results about the $\beta'_E(C_n^t)$ are presented below.

Theorem 4: For any integers $2 \leq t \leq \lfloor \frac{n}{2} \rfloor$ and $n \geq 4$, we have

$$\beta'_E(C_n^t) = \begin{cases} 4, & \text{if } t = 2 \text{ or } 6 \leq n \leq 7, t = 3 \\ & \text{or } n = 8, t = 4; \\ 3, & \text{else.} \end{cases}$$

Proof: To prove the stated theorem, we will calculate the EVFTMD, denoted as $\beta'_E(\mathbf{C_n^t})$, specifically for the cycle with chord network C_n^t. To ensure a comprehensive understanding of the network's properties, we proceed by examining the following two cases based on the values of n and t:

Case 1: (when both n and t are even)

Let $n = 4$ and $t = 2$, take $K'_E = \{f_2, f_3, f_4, f_5\} \subseteq E(C_4^2)$. It is quite simple to verify that K'_E is the EVFTRS.

Take $K'_E = \{f_2, f_3, f_4, f_{n+1}\} \subseteq E(C_n^2)$, where $n \geq 6$ and $t = 2$. The distance codes of the edges f_p, for $1 \leq p \leq n+1$, are provided in Table 6.

TABLE 6. Distance codes for the edges f_p, where $1 \leq p \leq n+1$.

$r_E(f_p \mid K'_E)$	p
$(-p+2, 1, p, 1)$	$1 \leq p \leq 2$
$(1, 0, 1, 2)$	$p = 3$
$(-2+p, -3+p, -4+p, -2+p)$	$4 \leq p \leq \frac{2+n}{2}$
$(\frac{n}{2}, \frac{-2+n}{2}, \frac{-4+n}{2}, \frac{n}{2})$	$p = \frac{4+n}{2}$
$(\frac{-2+n}{2}, \frac{n}{2}, \frac{-2+n}{2}, \frac{-4+n}{2})$	$p = \frac{6+n}{2}$
$(-p+2+n, -p+3+n, -p+3+n, -p+1+n)$	$\frac{8+n}{2} \leq p \leq n+1$

We can verify that for every $e, f \in E(C_n^2)$, at least two elements in the 4-vector $AD_E((e,f)|K'_E)$ are non-zero. So, $\beta'_E(C_n^2) \leq 4$.

Now to prove that $\beta'_E(C_n^2) \geq 4$ for $n \neq 6$, suppose contrarily that $\beta'_E(C_n^2) = 3$ and using Lemma 3, we present all the conditions:

Case A: (when $n = 4, 6$)

It is quite simple to demonstrate that there are no EVFTRs that exist which have a cardinality of 3.

Case B: (when $n \geq 8$)

(i) Let $K'_E = \{f_2, f_i, f_j\} \subseteq E(C_n^2)$ for $3 \leq i < j \leq \frac{n+2}{2}$, then

$$AD_E((f_1, f_3)|K'_E) = \begin{cases} (0, 1, 0), & \textit{if } i = 3; \\ (0, 0, 0), & \textit{if else}. \end{cases}$$

So, K'_E is not the EVFTRS.

(ii) Let $K'_E = \{f_2, f_i, f_j\} \subseteq E(C_n^2)$ for $3 \leq i \leq \frac{n}{2} < j \leq n+1$, then $AD_E((f_1, f_2)|K'_E) = (1, 0, 0)$ for $i = 3$, $\frac{n+6}{2} \leq j \leq n+1$ and

$$AD_E((f_1, f_3)|K'_E) = \begin{cases} (0, 1, 0), & \textit{if } i = 3, \frac{n+2}{2} \leq j \leq \frac{n+4}{2}; \\ (0, 0, 0), & \textit{if } 4 \leq i \leq \frac{n}{2}, \frac{n+2}{2} \leq j \leq \frac{n+4}{2}; \\ (0, 0, 1), & \textit{if else}. \end{cases}$$

So, K'_E is not the EVFTRS.

Above absolute difference codes represent that there is no EVFTRS with cardinality 3. So, $\beta'_E(C_n^2) \geq 4$. Hence, $\beta'_E(C_n^2) = 4$ for any $n \geq 4$.

Now for $n = 8$ and $t = 4$, take $K'_E = \{f_2, f_3, f_4, f_5\} \subseteq E(C_8^4)$. It is quite simple to verify that K'_E is the EVFTRS. Since there is no EVFTRS with a cardinality of 3, it follows that $\beta'_E(C_8^4) = 4$.

Take $K'_E = \{f_{\frac{t+2}{2}}, f_{\frac{t+6}{2}}, f_{n+1}\} \subseteq E(C_n^t)$, where $n \geq 10$ and $t \geq 4$. The distance codes of the edges f_p, for $1 \leq p \leq n+1$, are provided in Table 7.

TABLE 7. Distance codes for the edges f_p, where $1 \leq p \leq n+1$.

$r_E(f_p \mid K'_E)$	p
$(\frac{t}{2}, \frac{-2+t}{2}, 1)$	$p = 1$
$(\frac{-2+t}{2}, \frac{t}{2}, 1)$	$p = 2$
$(\frac{-2p+2+t}{2}, \lvert\frac{-2p+6+t}{2}\rvert, -1+p)$	$3 \leq p \leq \frac{2+t}{2}$
$(\frac{-t+2p-2}{2}, \lvert\frac{-2p+6+t}{2}\rvert, -p+3+t)$	$\frac{4+t}{2} \leq p \leq 1+t$
$(\frac{n}{2}, \frac{-4+n}{2}, \frac{-t+n}{2})$	$2+t \leq p \leq \frac{t+n}{2}$
$(-p+2+n, -p+3+n, -p+3+n, -p+1+n)$	$p = \frac{t+n+2}{2}$
$(\frac{-2p+t+2n+2}{2}, \frac{-2p+2n+t+2}{2}, n-p+1)$	$\frac{n+4+t}{2} \leq p \leq n+1$

We can verify that for every $e, f \in E(C_n^t)$, at least two elements in the 3-vector $AD_E((e,f)|K'_E)$ are non-zero. So, $\beta'_E(C_n^t) \leq 3$. Using Lemma 2, $\beta'_E(C_n^t) \geq 3$. Hence, $\beta'_E(C_n^t) = 3$.

Case 2: (when t is even and n is odd)

Take $K'_E = \{f_1, f_2, f_3, f_{n+1}\} \subseteq E(C_n^2)$, where $n \geq 5$ and $t = 2$. For the edges f_p, where $1 \leq p \leq n+1$, the distance codes are provided in Table 8.

TABLE 8. Distance codes for the edges f_p, where $1 \leq p \leq n+1$.

$r_E(e_p \mid K'_E)$	p
$(-1+p, -p+2, 1, 1)$	$1 \leq p \leq 2$
$(1, 1, 0, 2)$	$p = 3$
$(-3+p, -2+p, -3+p, -2+p)$	$4 \leq p \leq \frac{3+n}{2}$
$(\frac{-1+n}{2}, \frac{-1+n}{2}, \frac{-1+n}{2}, \frac{-3+n}{2})$	$p = \frac{5+n}{2}$
$(-p+2+n, -p+2+n, -p+3+n, -p+1+n)$	$\frac{7+n}{2} \leq p \leq n+1$

We can verify that for every $e, f \in E(C_n^2)$, at least two elements in the 4-vector $AD_E((e,f)|K'_E)$ are non-zero. So, $\beta'_E(C_n^2) \leq 4$.

Now to prove that $\beta'_E(C_n^2) \geq 4$, suppose contrarily that $\beta'_E(C_n^2) = 3$ and using Lemma 3, we present all the conditions:

Case A: (when $n = 5, 7$)

It is easy to verify that there are no EVFTRS of cardinality 3.

Case B: (when $n \geq 9$)

(i) Let $K'_E = \{f_2, f_i, f_j\} \subseteq E(C_n^2)$ for $3 \leq i < j \leq \frac{n+1}{2}$, then

$$AD_E((f_1, f_3)|K'_E) = \begin{cases} (0,1,0), & \text{if } i = 3; \\ (0,0,0), & \text{if else.} \end{cases}$$

So, K'_E is not the EVFTRS.

(ii) Let $K'_E = \{f_2, f_i, f_j\} \subseteq E(C_n^2)$ for $3 \leq i \leq \frac{n-1}{2} < j \leq n+1$, then $AD_E((f_1, f_2)|K'_E) = (1, 0, 0)$ for $i = 3$, $\frac{n+5}{2} \leq j \leq n+1$ and

$$AD_E((f_1, f_3)|K'_E) = \begin{cases} (0,1,0), & \text{if } i = 3, \frac{n+1}{2} \leq j \leq \frac{n+3}{2}; \\ (0,0,0), & \text{if } 4 \leq i \leq \frac{n-1}{2}, \frac{n+1}{2} \leq j \leq \frac{n+5}{2}; \\ (0,0,1), & \text{if else.} \end{cases}$$

So, K'_E is not the EVFTRS.

The above absolute difference codes represent that there is no EVFTRS with cardinality 3. So, $\beta'_E(C_n^2) \geq 4$. Hence, $\beta'_E(C_n^2) = 4$ for any $n \geq 5$.

Now take $K'_E = \{f_{\frac{t+2}{2}}, f_{\frac{t+6}{2}}, f_{n+1}\} \subseteq E(C_n^t)$, where $n \geq 9$ and $t \geq 4$. For the edges f_p, where $1 \leq p \leq n+1$, the distance codes are provided in Table 9.

TABLE 9. Distance codes for the edges f_p, where $1 \leq p \leq n+1$.

$r_E(f_p\|K'_E)$	p
$(\frac{t}{2}, \frac{-2+t}{2}, 1)$	$p = 1$
$(\frac{-2+t}{2}, \frac{t}{2}, 1)$	$p = 2$
$(\|\frac{-2p+t+2}{2}\|, \frac{-2p+t+6}{2}, -1+p)$	$3 \leq p \leq \frac{4+t}{2}$
$(\|\frac{-2p+t+2}{2}\|, \frac{-t+2p-6}{2}, -p+t+3)$	$\frac{6+t}{2} \leq p \leq 1+t$
$(\|\frac{-2p+2+t}{2}\|, \frac{-t+2p-6}{2}, -t+p)$	$2+t \leq p \leq \frac{t+n+1}{2}$
$(\frac{-1+n}{2}, \frac{-3+n}{2}, \frac{-t+n-1}{2})$	$p = \frac{t+n+3}{2}$
$(\frac{-2p+2n+t+2}{2}, \frac{-2p+t+2n+2}{2}, -p+1+n)$	$\frac{t+n+5}{2} \leq p \leq n+1$

We can verify that for every $e, f \in E(C_n^t)$, at least two elements in the 3-vector $AD_E((e,f)|K'_E)$ are non-zero. So, $\beta'_E(C_n^t) \leq 3$. Using Lemma 2, $\beta'_E(C_n^t) \geq 3$. Hence, $\beta'_E(C_n^t) = 3$.

Case 3: (when t is odd and n is even)

Take $K'_E = \{f_1, f_2, f_3, f_4\} \subseteq E(C_6^3)$ for $n = 6$ and $t = 3$. We can verify that for every $e, f \in E(C_6^3)$, at least two elements in the 4-vector $AD_E((e,f)|K'_E)$ are non-zero. So, the EVRS K'_E becomes a EVFTRS. Hence, $\beta'_E(C_6^3) \leq 4$. Now to prove that $\beta'_E(C_6^3) \geq 4$, suppose that $\beta'_E(C_6^3) = 3$. It is simple to prove that no EVFTRS of cardinality 3 exists. So, $\beta'_E(C_6^3) = 4$.

Now for $n = 8$ and $t = 3$, take $K'_E = \{f_3, f_6, f_8\} \subseteq E(C_8^3)$. It is quite simple to verify that K'_E is the EVFTRS.

Take $K'_E = \{f_{\frac{t+3}{2}}, f_{t+3}, f_{\frac{n+t+5}{2}}\} \subseteq E(C_n^t)$, where $n \geq 10$ and $t \geq 3$. The distance codes of the edges f_p, where $1 \leq p \leq n+1$, the distance codes are provided in Table 10.

TABLE 10. Distance codes for the edges f_p, where $1 \leq p \leq n+1$.

$r_E(f_p\|K'_E)$	p
$(\frac{1+t}{2}, 2, \frac{-t+n-1}{2})$	$p = 1$
$(\frac{-2p+t+3}{2}, 1+p, \frac{2p-t+n-5}{2})$	$2 \leq p \leq \frac{1+t}{2}$
$(0, \frac{3+t}{2}, \frac{-2+n}{2})$	$p = \frac{3+t}{2}$
$(\frac{-t+2p-3}{2}, \|-p+t+3\|, \frac{-2p+n+t+3}{2})$	$\frac{5+t}{2} \leq p \leq 1+t$
$(\frac{-t+2p-3}{2}, \|-p+t+3\|, \frac{-2p+t+n+5}{2})$	$2+t \leq p \leq \frac{t+n+3}{2}$
$(\frac{-2p+t+2n+3}{2}, \|-p+t+3\|, \frac{-n-t+2p-5}{2})$	$\frac{n+t+5}{2} \leq p \leq \frac{n+t+7}{2}$
$(\frac{-2p+t+2n+3}{2}, n-p+4, \frac{-n-t+2p-5}{2})$	$\frac{n+t+9}{2} \leq p \leq n+1$

We can verify that for every $e, f \in E(C_n^t)$, at least two elements in the 3-vector $AD_E((e,f)|K'_E)$ are non-zero. So, $\beta'_E(C_n^t) \leq 3$. Using Lemma 2, $\beta'_E(C_n^t) \geq 3$. Hence, $\beta'_E(C_n^t) = 3$ for $n \neq 6$ and $t \neq 3$.

Case 4: (when both n and t are odd)

Take $K'_E = \{f_1, f_2, f_3, f_4\} \subseteq E(C_7^3)$ for $n = 7$ and $t = 3$. We can verify that for every $e, f \in E(C_7^3)$, at least two elements in the 4-vector $AD_E((e,f)|K'_E)$ are non-zero. So, the EVRS K'_E is a EVFTRS. Hence, $\beta'_E(C_7^3) \leq 4$. Now to prove that $\beta'_E(C_7^3) \geq 4$, suppose that $\beta'_E(C_7^3) = 3$. It is simple to prove that no EVFTRS of cardinality 3 exists. So, $\beta'_E(C_7^3) = 4$

Take $K'_E = \{f_{\frac{t+3}{2}}, f_{t+2}, f_{n+1}\} \subseteq E(C_n^t)$, where $n \geq 9$ and $t \geq 3$. For the edges f_p, where $1 \leq p \leq n+1$, the distance codes are provided in the Table 11.

TABLE 11. Distance codes for the edges f_p, where $1 \leq p \leq n+1$.

$r_E(f_p\|K'_E)$	p
$(\frac{1+t}{2}, 1, 1)$	$p = 1$
$(\|\frac{-2p+t+3}{2}\|, p, -1+p)$	$2 \leq p \leq \frac{1+t}{2}$
$(0, \frac{1+t}{2}, \frac{1+t}{2})$	$p = \frac{3+t}{2}$
$(\|\frac{-2p+t+3}{2}\|, \|-p+t+2\|, -p+t+3)$	$\frac{5+t}{2} \leq p \leq 1+t$
$(\|\frac{-2p+t+3}{2}\|, \|-p+2+t\|, -t+p)$	$2+t \leq p \leq \frac{t+n}{2}$
$(\frac{-1+n}{2}, \frac{-t+n-2}{2}, \frac{-t+n}{2})$	$p = \frac{n+t+2}{2}$
$(\frac{-1+n}{2}, \frac{-t+n}{2}, \frac{-t+n-2}{2})$	$p = \frac{n+t+4}{2}$
$(\frac{-2p+t+2n+3}{2}, -p+3+n, -p+1+n)$	$\frac{n+t+6}{2} \leq p \leq n+1$

We can verify that for every $e, f \in E(C_n^t)$, at least two elements in the 3-vector $AD_E((e,f)|K'_E)$ are non-zero. So, $\beta'_E(C_n^t) \leq 3$. Using Lemma 2, $\beta'_E(C_n^t) \geq 3$. Hence, $\beta'_E(C_n^t) = 3$ for $n \geq 9$ and $t \geq 3$. □

V. APPLICATIONS, COMPARATIVE ANALYSIS AND LIMITATIONS

In the field of network design and routing, the idea of FTMD is a metric used to assess how resilient a network is to node (or edge) failures while preserving effective routing capabilities. One particular kind of network structure that is frequently researched in the context of network design and routing is the cycle with chord network, denoted as C_n^t, for more details: (see [42] and [45]). Consider the following example to better understand the idea of EVFTMD and its application in network design and routing: Assume we have a network, C_{10}^3, consisting of edges $\{f_i : 1 \leq i \leq 10+1\}$ arranged in a cycle. Additionally, there is an extra chord that connects the edges 2, 11 from one end and the edges 4, 5 from the other end. It is important to note that these edges are not adjacent to each other within the cycle. Here is a visual representation of the C_{10}^3 network as illustrated in Figure 5.

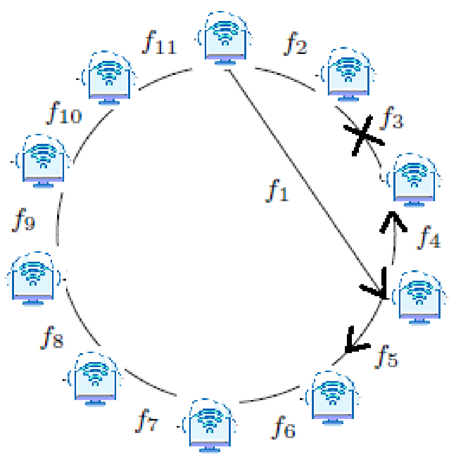

FIGURE 5. Network Design and Routing As C_{10}^3.

Let us take a closer look at how the EVFTMD is utilized in this context. The EVFTMD represents the minimum number of edges required to uniquely identify all other edges in a network, even in the event of edge failure. In essence, it measures the network's ability to maintain effective routing despite edge failures. A C_n^t network, which includes a cycle and chord, presents fault-tolerant and effective routing properties for network design and routing. Analyzing the EVFTMD of C_n^t can help determine the network's resilience to edge failures. Based on the EVFTMD, effective routing algorithms can be developed, ensuring that the network can continue to function and deliver messages successfully even if certain edges fail.

Suppose that edge 3 in our C_{10}^3 network fails. However, this network can still route messages between any pair of edges if it has the appropriate fault-tolerant routing algorithm based on EVFTMD. The redundancy provided by the chord and cycle topology allows the edges in the network to collaborate and find alternative paths around the failed edge. Therefore, to build robust and efficient networks that can withstand edge failures while maintaining connection and routing capabilities, the cycle with chord network C_n^t and the concept of EVFTMD are crucial.

A. APPLICATIONS IN NETWORK RESILIENCE

The EVFTMD in graph theory improves fault tolerance in communication networks by assessing their ability to maintain integrity during edge failures. An n-sunlet network consists of a cycle with n vertices connected to pendant nodes. The constant EVFTMD ensures that even with edge deletions, remaining nodes can be identified based on their distances to an RS. This is crucial for local area networks (LANs) to maintain device identifiability through alternative paths. Networks like cycles with chords enhance fault tolerance in metropolitan area networks (MANs) by ensuring connectivity despite primary link failures. EVFTMD promotes redundant pathways, resilient node identification, fault-tolerant backbones, and efficient resource distribution. In optical networks, fiber optic rings with connecting chords allow sustained data transmission through alternate routes during link failures, leading to more robust designs.

1) CASE STUDY

The EVFTMD concept, applied to network structures like n-sunlet and cycles with chord, significantly benefits real-world network applications by enhancing fault tolerance and resilience. Case studies from wireless mesh networks, smart grid communication systems, and optical networks illustrate the impact of these graph-theoretic structures on network design and maintenance.

2) WIRELESS MESH NETWORKING

Wireless mesh networks mostly use ring and spoke topologies where nodes (devices) connect in a cycle to them with additional links (chords) as redundant networks, closely resembling a cycle with a chords network. In a wireless mesh networks, each node serves as both a host and a router. Devices are connected through multiple redundant paths to ensure data can still be transmitted, even if there are link or node failures. The EVFTMD is important in this regard because it tells us that much of the network can withstand link failures to avoid faulty and accurate routing [46].

For example, in city-wide Wi-Fi systems such as San Francisco and Philadelphia, there are extra links between non-adjacent nodes that provide network continuity through localized failures via backup data paths to other access points or connections if more than one primary endpoint goes down. This model enables resilient link failures and rigorous scalability without consuming too many resources for stable service in large networks [47].

3) POWER GRID COMMUNICATION SYSTEMS

Smart grid systems are similar to n-sunlet networks, where central hubs (substations) connect to outer nodes (distribution points), and redundancy ensures that connectivity persists

TABLE 12. Comparative analysis of *EVFTMD* with other distance related parameters.

Aspects	MD	FTMD	EMD (Edge Metric Dimension)	FTEMD (Fault-Tolerant Edge Metric Dimension)	EVMD (Edge Version of Metric Dimension)	EVFTMD
Main Focus	Determining a unique identification of vertices by their distances from a RS	Guaranteeing unique identification of vertices even under vertex failures	Identifying edges uniquely by measuring distances from a RS	Preserving edge uniqueness despite edge failures	Evaluating distances among edges via RSs without accommodating fault tolerance	Edge-based fault tolerance, addressing failures in edges
Basic Metric Set	A collection of vertices that can uniquely identify all other vertices within the network	Subset of vertices that can uniquely identify all other vertices, even in the event of some failures	Subset of vertices that can uniquely identify all edges within the network	A collection of vertices that can uniquely identify all edges, even if some edges fail	A collection of edges that can uniquely identify all other edges without accounting for potential failures	A collection of edges that can uniquely identify edges while dynamically considering edge failures
Tolerance of Failure	Assumes there are no faults; necessitates that every vertex and edge function correctly	Accommodates vertex failures, making sure all vertices remain resolvable	Lacks fault tolerance; presumes all edges are operational	Accommodates edge failures, ensuring that all edges can still be resolved	Presumes that all edges are operational; emphasizes distances between edges	Accommodates edge failures while maintaining the capability to differentiate between edges
Structural Challenges	It depends on network topology when resolving all vertices without considering faults	It necessitates redundancy in RSs to ensure vertex resolution in the presence of faults	It depends on the network's structure for edge resolvability without considering faults	It requires redundancy in resolving sets to sustain edge resolution despite faults	There is no redundancy; it emphasizes the metric relationships between edges	It balances edge resolvability with resilience to edge perturbations
Novelty and Scope	A fundamental concept in graph theory that emphasizes the identification of vertices	Expands on MD by incorporating solutions for vertex failures and enhancing network resilience	Introduces resolving sets based on edges, broadening MD to encompass edge identification	Integrates EMD with fault tolerance, guaranteeing resilience in edge-based systems	Examines edge relationships with an emphasis on metric distances rather than fault tolerance	Combines principles from FTMD and EMD, centering on resilience in edge-based contexts and unique properties
Applications	Fundamental graph analysis, coding theories, navigation systems, and robotics	Sensor placement that can withstand faults, navigation, and communication networks	Cable networks, identification of edges in infrastructure, and communication links	Communication networks based on edges that are fault−resilient, including optical fiber systems	Analysis of edge−to−edge relationships in transportation and logistics	Network design that is resilient, ensuring edge stability in systems vulnerable to disruptions
Theoretical Contribution	Foundational concept with broad applications and theoretical relevance	Enhances comprehension of vertex resilience in networks	Broadens network metric research to incorporate edges, facilitating new applications	Initiates studies on fault-resilient edges, valuable in critical network scenarios	Extends graph theory to edge−to−edge metrics without considering faults	Connects edge-resilience to metric studies, creating innovative robust frameworks

even if a central link goes down. The fault tolerance of the n-sunlet design guarantees that essential functions in the grid remain connected, even during outages. For instance, Southern California Edison's smart grid configuration employs a structure such as n-sunlet, featuring central substations alongside peripheral distribution nodes. In the event of a central link failure, the network continues to support communication, allowing for ongoing real-time data flow and monitoring. This design enhances the grid's reliability, facilitating uninterrupted service delivery and robust communication for critical operations, even amid edge failures; for more details see [48] and [49].

4) OPTICAL NETWORKS

In the case of optical networks, especially in MANs (metropolitan area networks) and long-distance systems, cycle-based topologies with additional chords provide more resilience to fiber cuts or equipment blowouts in a single location. SONET/SDH rings exemplify this setup, where nodes are arranged in a circular structure and interconnected with extra fiber links (chords) to non-adjacent nodes. These chords ensure network connectivity remains intact during multiple fiber failures, such as natural disasters or accidents [50], [51]. For example, Verizon's optical ring networks utilize SONET/SDH rings with chordal links to ensure uninterrupted connectivity across cities, rerouting traffic through redundant fibers if a primary link fails. With chordal cycles, optical networks achieve high availability and cost-efficiency, as the fault tolerance boosts reliability without high costs or structural complexity. The implementation of chordal cycles guarantees that the network can withstand several link failures while maintaining connectivity between essential data centers or backbone nodes. The stable EVFTMD of cycles with chords allows the distance relationships among nodes to be preserved, enabling routing algorithms to adjust effectively to failures.

These case studies highlight the importance of graph-theoretic structures, such as n-sunlets and cycles with chords, in enhancing fault tolerance and reliability in various networks, including smart grids and wireless mesh communications. These structures ensure high availability even during edge or node failures.

B. COMPARATIVE ANALYSIS

A concise comparative study distinguishing EVFTMD from other related concepts in fault-tolerance and MDs as discussed in Table 12.

As a result, EVFTMD integrates fault tolerance, edge identification, and MDs in a unique way. It emphasizes edge-specific robustness in contrast to MD or FTMD. Because it places more emphasis on edge-to-edge fault tolerance than EMD, FTEMD, or EVMD, it is essential for fault-prone environments such as data infrastructure, transportation systems, and communication networks.

C. LIMITATIONS

There are some limitations for the current study discussed as under:

- The study is limited to n-sunlet and cycle networks with chords, which may not reflect larger, diverse network complexities.
- Theoretical results lack empirical validation and need experiments or simulations for real-world relevance.
- It focuses primarily on edge fault tolerance without considering interactions between vertex and edge failures.
- There are scalability issues regarding the computational complexity of applying EVFTMD to larger networks, affecting its practicality in extensive systems.

VI. CONCLUSION

This study presents a novel approach, called the EVFTMD, which focuses on understanding how networks can endure potential failures, particularly by examining the edges, and the connections between various network points. By analyzing specific types of networks, such as n-sunlet and cycle networks, the research identifies consistent values for EVFTMD, which helps us understand how resilient these edge-focused systems are. These findings are crucial for designing communication networks that remain functional and structurally sound even when some links fail.

The results presented in the study include comprehensive computations that ensure the research is thorough and precise. However, because it focuses on just a few kinds of network types, the study does have some limitations. Future studies could investigate different network types, such as circulant networks or hierarchical structures, to broaden the applicability of the EVFTMD concept. For instance, hierarchical networks may demonstrate how to secure multilayered systems, while circulant networks' distinctive characteristics may reveal novel approaches to managing defects.

Considering the aforementioned limitations, this study establishes the groundwork for designing robust networks. The consistent EVFTMD values can lead to more effective resource management, meticulous backup planning, and the development of networks that can effectively handle failures, particularly in rapidly evolving settings such as wireless and optical systems. However, putting these findings to the test through real-world examples or simulations remains a challenge. Addressing this issue will enhance both the theoretical insights and their practical use in network design.

Open Problem 1: Computing EVFTMD for the circulant networks $C_n(1, t)$ for all values of $t \geq 2$.

DATA AVAILABILITY STATEMENT

For additional details regarding the data, readers are encouraged to contact the corresponding author, as this article covers all of the material provided.

CONFLICT OF INTEREST

The authors declare no conflicts of interest.

AUTHOR'S CONTRIBUTIONS

Muhammad Faheem and Muhammad Ahmad were engaged participants in the problem discussion, ensuring the validation of results, and conducting the final review. Zohaib Zahid played a pivotal role in initiating the problem, gathering essential data, and providing continuous supervision. Muhammad Javaid played a crucial role in analyzing and computing the results and initiating the drafting process of the study. Mamo Abebe Ashebo contributed to the methodology, engaged in discussing the problem and meticulously proofread the final version.

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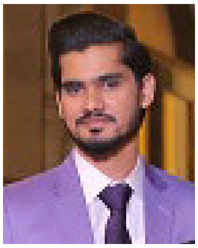
MUHAMMAD FAHEEM received the B.Sc. and M.Sc. degrees in mathematics from the University of Punjab, the M.Phil. degree in mathematics from Minhaj University Lahore, and the Ph.D. degree from the University of Management and Technology (UMT), Lahore, Pakistan, in 2022. His main research interest includes combinatorial optimization in graph theory.



MUHAMMAD AHMAD received the B.Sc. and M.Sc. degrees in mathematics from the University of Punjab, the M.Phil. degree in mathematics from Minhaj University Lahore, and the Ph.D. degree from the University of Management and Technology (UMT), Lahore, Pakistan, in 2022. He has published 12 research articles in reputed international journals. His main area of research interests include graph theory, fuzzy graph theory, network optimization, and combinatorics.

ZOHAIB ZAHID received the B.S. degree from the Department of Mathematics, University of the Punjab, Lahore, Pakistan, in 2008, and the Ph.D. degree from the Abdus Salam School of Mathematical Sciences, Lahore, in 2014. Since 2014, he has been with the University of Management and Technology (UMT), Lahore, where he is currently an Associate Professor. His research interests include computational algebra and graph theory.

MUHAMMAD JAVAID received the M.Sc. degree from the University of the Punjab, Lahore, Pakistan, in 2002, the M.Phil. degree from Government College University, Lahore, in 2008, and the Ph.D. degree in mathematics from the National University of Computer and Emerging Sciences, Lahore, in 2014. He was a Postdoctoral Researcher in mathematics with the School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui, China, in 2017. He is currently an Associate Professor of mathematics with the Department of Mathematics, School of Science, University of Management and Technology, Lahore. He has supervised eight Ph.D., 50 M.S., and M.Phil. students. He has published more than 150 research articles in different well-reputed international journals. His research interests include different areas of graph theory, such as spectral theory of graphs, computational theory of graphs and networks, and extremal theory of graphs. He is a Reviewer of *Mathematical Reviews* and IEEE Access. He is also included in the list of top 2 percent scientists of the world by Stanford University, USA, in October 2022 (based on document published on November 2022).

MAMO ABEBE ASHEBO received the B.Ed. degree in mathematics from Alemaya University, Alemaya, Ethiopia, in 2005, the M.Sc. degree in mathematics (numerical analysis) from Addis Ababa University, Addis Ababa, Ethiopia, in 2010, and the Ph.D. degree in graph theory from Wollega University, Nekemte, Ethiopia, in 2020. He is currently an Associate Professor in mathematics with the Department of Mathematics, College of Natural and Computational Sciences, Wollega University. He has co-supervised three Ph.D. students and supervised 24 M.Sc. students. He has published 24 research articles and one book chapter under his name in reputed international journals. His research interests include graph theory, fuzzy graph theory, networks, combinatorics, and numerical analysis.