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RESEARCH ARTICLE

Suboptimal Linear Distributed Control-Estimation Synthesis for Stochastic Multi-Agent System

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ABSTRACT This paper considers the distributed cooperative control problem for a linear stochastic multi-agent system (MAS). The optimal cooperative control design for each agent is challenging due to the limited neighboring information, contingent upon the MAS network topology. A synthesized distributed control-estimation framework is proposed to address the computationally tractable suboptimal solution. In particular, a distributed estimator extends MAS information beyond neighboring agents, allowing interactions with non-neighboring agents. The proposed control-estimation law is theoretically verified and further validated using numerical simulations.

INDEX TERMS Distributed control, optimal control, multi-agent systems, cooperative control.

I. INTRODUCTION

Distributed cooperative control has been vital in many networked multi-agent systems (MASs) with diverse applications over the past decades [1], [2], [3]. In addition to the theoretical foundations of the MAS's cardinal characteristics [4], [5], [6], a wide range of MAS applications have been investigated, including but not limited to formation maneuvers [1], sensor networks [7], distributed computing [8], etc. The key to enabling these applications is the inter-agent interaction specified by the embedded distributed control law [3]. Difficulties in distributed cooperative control have been investigated in the MAS operational context, such as controlling individual agents with limited local information [9], [10], [11], uncertain system and network dynamics [12], [13], [14], [15], [16], [17], [18], [19], [20], and stochastic characteristics [21].

Despite extensive research on distributed control for cooperative MAS, their optimality remains a challenging problem. Designing an optimal controller with network topology constraints is NP-hard [22], and the optimal controller may not be linear even for linear MAS dynamics under Gaussian noise and quadratic cost [23]. To solve this problem, some preliminary studies have attempted to derive sufficient conditions in terms of MAS network topology [24], [25], [26], [27] and/or the global cost [28], [29], [30], whereby the optimal distributed control problem is tractable. However, these conditions enforce a specific form of network topology and cost function, resulting in restrictive problem formulation. Optimal MAS control laws for more general network topological constraints and cost functions have been developed in [31] and [32]. However, they are limited to single and double integrator dynamics for each agent, which is not applicable to general MAS dynamics.

To circumvent the complications that arise from general MAS problem formulation (i.e., network topology, cost function, and agent dynamics), other approaches have proposed approximation techniques that address suboptimal distributed control laws [33], [34], [35], [36], [37]. In addition, other researchers have merely considered the local cost function from the perspective of individual agents; without accounting for the global cost function [38], [39], [40]. Consequently, the aforementioned studies are prone to degrade the overall performance of the entire MAS. Furthermore, most suboptimal results lack theoretical performance guarantees with reference to the global optima.

Within the scope of linear control structures, another line of work has focused on reformulating the original optimal

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distributed control problem into a convex optimization problem, in which the resulting optimal control law is linear. Different parameterization techniques, such as Youla [41], system-level [42], and input-output [43], have shown a tractable way to find the optimal solution under the notion of quadratic invariance (QI), which is a sufficient and necessary condition for exact convex reformulation [44]. However, as in the aforementioned works [26], [29], the QI condition poses a restriction on the class of problems, whereas non-QI cases are intractable. Various approximation techniques, such as convex relaxation [45] and restriction [46], have been introduced for those that are not under the notion of QI. Nevertheless, these methods entail additional difficulties in determining feasible solutions. Recently, a gradient descent-based method was proposed to achieve optimal control for non-QI cases [47]. However, this method is valid under a new notion, called uniquely stationary, which hardly resolves general structural input constraints. Therefore, computing the optimal linear distributed control law for general MAS dynamics with arbitrary network topology remains a significant challenge.

In this paper, the linear distributed cooperative control problem for stochastic linear MAS without specific constraints on network topology is solved from the perspective of joint distributed control and estimation design. The key idea is to augment the information for each agent using a distributed estimation algorithm. This enables individual agents to interact beyond their neighbors based on the estimated information, which in turn relaxes the network topological constraints for the distributed control law design. Compared to recent work in which the estimator complements decentralized control [48], our problem substantially differs in MAS dynamics and information-sharing patterns. More importantly, to the best of the authors' knowledge, our approach for the first time leverages the information of non-neighboring agents for control, offering a novel pathway in the development of an optimal distributed control law.

It is worth noting that this work is founded on a series of previous works [49], [50], [51]. The proposed distributed estimator was initially introduced, focusing on the estimation of non-neighboring agents [49]. Based on this distributed estimator, [50] attempted to augment distributed control by exploiting non-neighboring estimation information. The resulting distributed estimation-based control showed impressive performance, but its optimal design was out of focus. Finally, [51] tackled the distributed estimation-control synthesis problem, which optimally integrates and designs a distributed estimator and controller. In this paper, we take a step further by synthesizing a new and advanced control-estimation framework along with the rigorous proof of the theoretical performance guarantee that was unattainable in the previous method [51]. Furthermore, the computational complexity of the proposed framework is significantly reduced. In contrast to [51], which needs to solve the optimization problem for each agent, the method we propose here forms a single optimization problem regardless of the number of agents. This makes our framework especially well-suited for large-scale MAS applications, where scalability is a crucial factor. The contributions of this paper are as follows.

1) Reformulating the primal optimal distributed control problem into a joint control-estimation problem by synthesizing a distributed estimator and controller for the individual agents.
2) Establishing an iterative optimization framework to compute the suboptimal distributed control-estimation law for a general stochastic linear MAS.
3) Verifying the theoretical performance guarantee relative to the global optima and numerically validating its effectiveness through comparative simulations with existing work.

The structure of this paper is organized as follows: Section II introduces the problem setup including the MAS dynamics and cost function targeted for optimal distributed control. Section III presents the design of a suboptimal distributed control-estimation law, and Section IV presents theoretical performance verification. Numerical simulations are presented in Section V. Section VI discusses the conclusions and directions for future research.

Notation: $\mathbb{R}$ and $\mathbb{N}$ are the symbols for the set of all real numbers and natural numbers, respectively. $\mathbb{E}[x]$ is the expectation of random variable x, and $\mathbb{E}[x|y]$ stands for the conditional expectation of x given y. $\otimes$ is the Kronecker product of the two matrices. I_p is the identity matrix, and 0_p is the zero matrix, each with dimensions of $p \times p$. $1_p \in \mathbb{R}^p$ is a vector with all entries equal to 1. $\|\cdot\|_2$ and $\|\cdot\|_F$ represent Euclidean and Frobenius norms, respectively. The expression $D = blkdg(D_1, \cdots, D_n)$ signifies a block-diagonal matrix composed of the individual matrices $D_1, \cdots, D_n$. $diag(c_1, \cdots, c_n)$ denotes a diagonal matrix where the entries are $c_1, \cdots, c_n$. The transpose of a matrix D is denoted as D^{T}, and its trace is represented by $\mathrm{Tr}(D)$. The notation $x = [x_1^{\mathrm{T}} \cdots x_N^{\mathrm{T}}]^{\mathrm{T}}$ represents a vector x formed by concatenating the transposes of individual vectors $x_i, \forall i \in 1, \cdots, N$, followed by taking the transpose of the resulting row vector to form a column vector. For a symmetric matrix $X = X^{\mathrm{T}}, X \succ 0$ and $X \succeq 0$ represent positive definite and positive semidefinite, respectively.

II. PROBLEM FORMULATION

A. DYNAMICAL STOCHASTIC MULTI-AGENT SYSTEM MODEL

Let us consider a MAS that includes N homogeneous stochastic linear time-invariant agent dynamics described by

$$x_i(k+1) = Ax_i(k) + Bu_i(k) + w_i(k), \forall i \in \{1, .., N\} \quad (1)$$

where $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^p$ are the state and control inputs of the i^{th} agent, respectively. w_i is the process noise of a white Gaussian nature, whose covariance is a positive definite covariance matrix $\Theta_i \succ 0$, and $k \in \mathbb{N}$ represents the discrete-time index. Matrix pair (A,B) represents the dynamics of each agent and satisfies the controllability condition. Correspondingly, the dynamics of the entire MAS are as follows.

$$\begin{aligned} x(k+1) &= \tilde{A}x(k) + \tilde{B}u(k) + \tilde{w}(k) \\ \tilde{A} &= (I_N \otimes A)\,,\ \tilde{B} = (I_N \otimes B) \\ x(k) &= \left[x_1^{\mathrm{T}}(k)\cdots x_N^{\mathrm{T}}(k)\right]^{\mathrm{T}},\ u(k) = \left[u_1^{\mathrm{T}}(k)\cdots u_N^{\mathrm{T}}(k)\right]^{\mathrm{T}} \\ \tilde{w}(k) &= \left[w_1^{\mathrm{T}}(k)\cdots w_N^{\mathrm{T}}(k)\right]^{\mathrm{T}} \end{aligned} \tag{2}$$

Over a finite time horizon, the stacked vectors of the states and inputs of the entire MAS are denoted as follows.

$$\begin{aligned} \mathbf{x} &= [x(0)^{\mathrm{T}}\dots x(T)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nn(T+1)} \\ \mathbf{u} &= \sum_{i}^{N}(I_T\otimes M_i^{\mathrm{T}})\mathbf{u}_i \in \mathbb{R}^{NpT} \\ \mathbf{u}_i &= [u_i(0)^{\mathrm{T}}\dots u_i(T-1)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{pT} \end{aligned} \tag{3}$$

where $M_i = [0_p \cdots I_p \cdots 0_p] \in \mathbb{R}^{p\times Np}$ is the block matrix filled i^{th} block entry with I_p, and 0_p in the other block entries. In addition to the dynamics of the provided agents, the inter-agent network topology can be represented by a graph model. The individual agents and network connections between them are defined as a set of nodes $\mathcal{V} = \{1,\ 2\,.\,.\ N\}$ and edges $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$, respectively. Subsequently, $(i,j)\in\mathcal{E}$ indicates that the i^{th} agent has network connectivity with the j^{th} agent. $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is an adjacency matrix that expresses the network connectivity of the entire graph model, where its element $a_{ij} = 1$ if $(i,j)\in\mathcal{E}$, and $a_{ij} = 0$ otherwise. $\mathcal{D} = diag(\sum_j a_{1j}\cdots\sum_j a_{Nj})$, and $\mathcal{L} = \mathcal{D}-\mathcal{A}$ are the degree and Laplacian matrices, respectively. The network graph considered in this paper is a directed graph. This assumption is based on the authors' previous works [49], [50], [51], and can apply to both communication and sensing-based networks. In these networks, agents may have different capabilities in observing each other, leading to directed network connections. The set of i^{th} agent's neighboring agents and its cardinality are defined as Ω_i and $|\Omega_i|$, respectively. Given $\mathcal{L}$, the i^{th} agent acquires noisy measurements of its neighboring as follows [49].

$$z_{ij}(k) = a_{ij}\left(x_j(k)+v_{ij}(k)\right),\qquad \forall j\in\mathcal{V} \tag{4}$$

where v_{ij} is the measurement noise of white Gaussian nature with covariance $\xi_{ij} \succ 0$. Furthermore, $Z_i(k) = \left[z_{i1}^{\mathrm{T}}(k)\cdots z_{iN}^{\mathrm{T}}(k)\right]^{\mathrm{T}}$ and $v_i(k) = \left[v_{i1}^{\mathrm{T}}(k)\cdots v_{iN}^{\mathrm{T}}(k)\right]^{\mathrm{T}}$ are the measurements and noise sets of the i^{th} agent, respectively. One can rewrite the MAS dynamics (2) over the finite time horizon in a static form using (3) as follows [47].

$$\begin{aligned} \mathbf{x} &= P_{11}\mathbf{w} + P_{12}\mathbf{u} \\ P_{11} &= (I-\mathrm{D}\bar{A})^{-1},\ P_{12} = (I-\mathrm{D}\bar{A})^{-1}\mathrm{D}\bar{B} \\ \bar{A} &= I_{T+1}\otimes\tilde{A},\qquad \bar{B} = \begin{bmatrix} I_T\otimes\tilde{B} \\ 0_{Nn\times NpT}\end{bmatrix} \\ \mathrm{D} &= \begin{bmatrix} 0_{Nn\times NnT} & 0_{Nn\times Nn} \\ I_{NnT} & 0_{NnT\times Nn}\end{bmatrix} \end{aligned} \tag{5}$$

where $\mathbf{w} = [x(0)^{\mathrm{T}}\ \tilde{w}(0)^{\mathrm{T}}\ \dots\tilde{w}(T-1)^{\mathrm{T}}] \in \mathbb{R}^{Nn(T+1)}$. Moreover, individual agents interact with their neighborhoods according to embedded control laws. Without loss of generality, a control input for each agent is designed using the following linear output feedback control law [47].

$$\mathbf{u}_i = (I_T\otimes M_i)\mathcal{F}Z_{i,(0:T-1)} = (I_T\otimes M_i)\mathcal{F}C(\mathbf{x}+\mathbf{v}_i),\ \forall i\in\mathcal{V} \tag{6}$$

where $\mathbf{v}_i = [v_i(0)^{\mathrm{T}}\dots v_i(T)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nn(T+1)}$, $Z_{i,(0:T-1)} = [Z_i(0)^{\mathrm{T}}\cdots Z_i(T-1)^{\mathrm{T}}] \in \mathbb{R}^{NnT}, \forall i \in \mathcal{V}$, and $C = [I_{NnT}\ 0_{NnT\times Nn}]$. Here, $\mathcal{F}\in\mathbb{F}$ denotes the control law matrix which is our design parameter, and $\mathbb{F} \subset \mathbb{R}^{NpT\times NnT}$ is a subspace that ensures $\mathcal{F}$ to be a causal feedback policy by forcing zero to the entries corresponding to the future outputs, it also imposes the structural constraint induced by the network topology of the MAS.

B. SYNTHESIZED DISTRIBUTED CONTROL-ESTIMATION PROBLEM

Given the dynamical MAS model with the embedded linear control law (6), the objective of the distributed cooperative control of the MAS can be expressed as the following problem.

Problem 1: Optimal linear distributed control law subject to network topological constraints [47]

$$\min_{\mathcal{F}\in\mathbb{F}} \quad \mathbb{E}\left[\mathbf{x}^{\mathrm{T}}\mathcal{Q}\mathbf{x}+\mathbf{u}^{\mathrm{T}}\mathcal{R}\mathbf{u}\right]$$
$$\text{subject to (5), (6)},\ \forall i\in\mathcal{V}$$

where $\mathcal{Q}\in\mathbb{R}^{Nn(T+1)\times Nn(T+1)} \succeq 0$, and $\mathcal{R}\in\mathbb{R}^{NpT\times NpT} \succ 0$ denote the associated weight matrices. The formulated quadratic cost function is widely used to express different cooperative behaviors of MAS, such as consensus [52], formation [5], rendezvous [53], etc. Notably, quadratic state cost plays a vital role in regulating disparities among agent states, enabling the coordination of relative states to align with specific task objectives.

Problem 1 is highly non-convex because of the structural constraints encoded in the subspace of the control law matrix $\mathbb{F}$ [33]. To address this challenging problem, we utilized the notion of *virtual interaction*. This notion involves incorporating estimation-based feedback control using the previously proposed distributed estimator in [49] to emulate interactions between non-neighboring agents as if they had access to each other's information. Correspondingly, a virtual network topology is defined as the network connections associated with these virtual interactions regardless of the actual network connections of the MAS. Notably, the capability of our proposed distributed estimator to estimate the entire MAS provides significant flexibility in designing the virtual network topology. Among arbitrary design choices for virtual networks, we adopt a fully connected virtual network topology to simplify the analysis and, remove any constraints on the virtual interactions.

Definition 1: Given the measurement set of the i^{th} agent up to time k, the state estimate of the MAS from the i^{th}

agent's perspective and the corresponding estimation error covariance are denoted by ${}^{i}\hat{x}(k) := \mathbb{E}\left[x(k)|Z_{i,(0:k)}\right]$, and ${}^{ii}\Sigma(k) := \mathbb{E}\left[\left(x(k) - {}^{i}\hat{x}(k)\right)\left(x(k) - {}^{i}\hat{x}(k)\right)^{\mathrm{T}}|Z_{i,(0:k)}\right], \forall i \in \mathcal{V}$, respectively.

The state estimate of the MAS from the perspective of the i^{th} agent is given by the following Kalman-like filter.[1]

$$ {}^{i}\hat{x}(k) = {}^{i}\hat{x}^{-}(k) + L_i(k)\left(H_i Z_i(k) - H_i{}^{i}\hat{x}^{-}(k)\right) \tag{7} $$

where ${}^{i}\hat{x}^{-}(k) := \mathbb{E}\left[x(k)|Z_{i,(0:k-1)}\right]$ is the predicted state estimate from the i^{th} agent's perspective. $L_i \in \mathbb{R}^{nN\times n|\Omega_i|}$ denotes the estimator gain, whereas $H_i \in \mathbb{R}^{n|\Omega_i|\times nN}$ is used to exclude the state entries of the non-neighboring agents of the i^{th} agent from MAS such that $H_i = [h_1\ h_2\ \cdots h_{|\Omega_i|}]^{\mathrm{T}} \otimes I_n$ where $h_m \in \mathbb{R}^N$, $m = 1, 2, \ldots, |\Omega_i|$ are the non-zero column vectors of matrix $diag(a_{i1}, a_{i2} \ldots, a_{iN})$. Note that the innovation term $\left(H_i Z_i(k) - H_i{}^{i}\hat{x}^{-}(k)\right)$, only measures the neighboring agents' state information due to the network topological constraint.

By virtue of the virtual interaction through the distributed estimator (7), we amend the distributed control law (6) to accommodate the MAS state estimate information, that is the estimation-based feedback control law.

$$ \mathbf{u}_i = (I_T \otimes M_i)\mathcal{F}C\ {}^{i}\hat{\mathbf{x}},\ \forall i \in \mathcal{V} \tag{8} $$

where ${}^{i}\hat{\mathbf{x}} = \mathbb{E}\left[\mathbf{x}|Z_{i,(0:k)}\right]$ is the stacked vector of the state estimates over the time horizon up to k. The resulting control input space is no longer constrained by the network topology. Alternately, the control performance relies upon the accuracy of the estimator whose design parameter is the estimator gain, $L_i,\ \forall i \in \mathcal{V}$.

Optimizing the cost of Problem 1 subject to a general network topology is intractable [22], [54], [55]. Accordingly, we focus on reformulating the original linear distributed control problem into a joint control-estimation problem. We then simultaneously optimize the distributed control and estimation law according to the linear structures (8) and (7), which are widely adopted in MAS problems [34], [35].

Problem 2: Virtual interaction based optimal linear distributed control-estimation law

$$ \min_{\mathcal{F}\in\tilde{\mathbb{F}}, \Upsilon_i, \forall i\in\mathcal{V}} \quad J(\mathcal{F}, \Upsilon_1, \cdots, \Upsilon_N) $$
$$ \text{subject to (5), (7), (8)} $$

where $J(\mathcal{F}, \Upsilon_1, \cdots, \Upsilon_N) := \mathbb{E}\left[\mathbf{x}^{\mathrm{T}}\mathcal{Q}\mathbf{x} + \mathbf{u}^{\mathrm{T}}\mathcal{R}\mathbf{u}\right]$ and $\Upsilon_i := \{L_i(k)|k = 0, \cdots, T\}$ is the set of estimator gains over the time horizon T for the i^{th} agent.

Remark 1: Unlike $\mathbb{F}$, the subspace $\tilde{\mathbb{F}} \subset \mathbb{R}^{NpT\times NnT}$ is associated only with the causality of the control policy without any structural constraints [47].

Despite the absence of the structural constraint, Problem 2 is still non-trivial to solve, as the controller and estimator mutually affect each other [49]. To resolve this problem, we sequentially design distributed control laws and estimator laws using an iterative optimization framework. The overall optimization procedure is shown in Figure. 1.

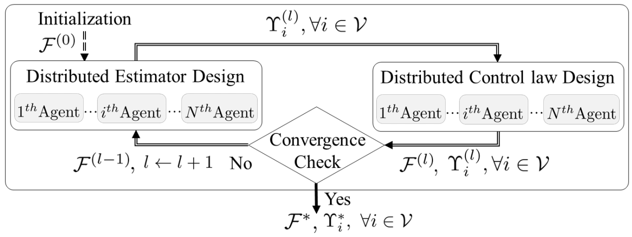

FIGURE 1. Synthesized suboptimal linear distributed control-estimation design procedure.

Specifically, at the l^{th} iteration, two design steps are sequentially executed such that i) *distributed estimator design* computes a set of estimation laws $\Upsilon_i^{(l)}, \forall i \in \mathcal{V}$ based on the distributed control law computed from the previous iteration; and ii) *distributed control law design* optimizes the control law $\mathcal{F}^{(l)}$, with the given state estimation errors induced by the estimation laws from the foregoing distributed estimator design step. Finally, the control and estimation laws at the current iteration are checked for convergence and are applied to the next iteration. Once the optimization process converges under the pre-set stopping conditions, we get the suboptimal linear distributed control-estimation law represented by $\mathcal{F}^*$ and $\Upsilon_i^*, \forall i \in \mathcal{V}$. Throughout the optimization process, we make use of information from the entire MAS to improve overall costs. This includes considering interactions beyond proximity, particularly those influenced by non-neighboring agents. It's worth noting that while our design approach leverages information from the entire MAS to enhance cooperative behavior and optimize overall performance, the deployment of the MAS performs in a distributed manner, as detailed in forthcoming assumptions.

Assumption 1: In the deployment of the distributed MAS, the i^{th} agent is initialized with the information about the dynamics of MAS and the distributed estimation law, i.e., $A, B, \mathcal{F}^*$, and Υ_i^*. Subsequently, this information is utilized as prior knowledge for state estimation and incorporated during the initialization of the algorithm, remaining unchanged throughout the distributed operation.

These assumptions are not critical from both theoretical and practical standpoints, and can be readily applied in real-world scenarios [1], [2], [3]. Moreover, leveraging prior MAS information, as commonly applied in existing works [50], [51], has been shown to enhance performance and achieve sophisticated functionalities. In contrast, scenarios that do not utilize prior information focus more on robustness than optimality [12], [13], [14].

III. ALGORITHM DEVELOPMENT

This section describes the detailed procedure of the proposed control-estimation synthesis for the finite time horizon case.

[1]Kalman-like filter is referred to as recursive linear estimation which minimizes the estimation error covariances.

A. DISTRIBUTED ESTIMATOR DESIGN

The objective of the distributed estimator design step is to optimize the estimator gains for individual agents, $\Upsilon_i^{(l)}, \forall i \in \mathcal{V}$. Note that the whole design procedure is done offline wherein the agent dynamics (A, B), neighboring agents (Ω_i), and the control law designed in the previous iteration $\mathcal{F}^{(l-1)}$ information can be exploited for the estimator design. For brevity, $\mathcal{F}^{(l-1)}$ is denoted as $\mathcal{F}$ in this subsection. Based on the Bayesian estimation framework in (7), individual agents exploit the estimation error when updating their state estimate of the entire MAS, ${}^i\hat{x}(k), \forall k \in \{0, \cdots, T\}$.

Definition 2: The MAS state estimation error and its covariance from the i^{th} agent's perspective are denoted by ${}^ie(k) := x(k) - {}^i\hat{x}(k)$, and ${}^{ii}\Sigma(k)$, respectively. The concatenated estimation errors from different agents' perspectives and their covariance are defined as $e(k) = [{}^1e(k)^{\mathrm{T}} \cdots {}^Ne(k)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{NnN}$, and $\Sigma(k) := \mathbb{E}[e(k)e(k)^{\mathrm{T}}] \in \mathbb{R}^{NnN \times NnN}$. The counterparts of the predicted state estimate ${}^i\hat{x}^-(k)$, such as ${}^{ii}e^-(k)$, ${}^{ii}\Sigma^-(k)$, $e^-(k)$, $\Sigma^-(k)$ are defined in a similar manner [49].

With the given information, the i^{th} agent can construct an estimation-based control input, as follows.

$$u_i(k) = M_i \sum_{s=0}^{k} \mathcal{F}_{sk}\, {}^i\hat{x}(s) \tag{9}$$

where block matrix $\mathcal{F}_{sk} \in \mathbb{R}^{pN \times nN}$ represents the feedback gain matrix at time step k applied to ${}^i\hat{x}(s)$. By applying (9) to Definition 2, we can rewrite the dynamics of MAS (2) as follows.

$$x(k+1) = \tilde{A}x(k) + \tilde{B}\mathcal{F}_{kk}x(k) + \sum_{s=0}^{k-1} \tilde{B}\mathcal{F}_{sk}x(s) - \sum_{s=0}^{k} \tilde{\tilde{B}}\tilde{M}\tilde{\mathcal{F}}_{sk}e(s) + \tilde{w}(k) \tag{10}$$

where $\tilde{M} = blkdg(\bar{M}_1, \ldots, \bar{M}_N) \in \mathbb{R}^{NpN \times NpN}$ is a block diagonal-matrix with $\bar{M}_i = M_i^{\mathrm{T}} M_i \in \mathbb{R}^{pN \times pN}$. $\tilde{\mathcal{F}}_{sk} = I_N \otimes \mathcal{F}_{sk}$, $\tilde{\tilde{B}} = 1_N^{\mathrm{T}} \otimes \tilde{B}$. The predicted MAS state estimate can be written as

$${}^i\hat{x}^-(k+1) = \tilde{A}\, {}^i\hat{x}(k) + \tilde{B}\mathcal{F}_{kk}{}^i\hat{x}(k) + \sum_{s=0}^{k-1} \tilde{B}\mathcal{F}_{sk}{}^i\hat{x}(s) \tag{11}$$

Then, one can obtain the predicted estimation errors, $e^-(k+1)$, by concatenating ${}^ie^-(k+1), \forall i \in \mathcal{V}$ after calculating ${}^ie^-(k+1)$ by subtracting (11) from (10). Additionally, the predicted estimation covariance, $\Sigma^-(k+1)$, can be determined based on $e^-(k+1)$. Note that $\Sigma^-(k+1)$ consists of block-matrices where the off-diagonal block-matrices, ${}^{ij}\Sigma^-(k+1) = \mathbb{E}[{}^ie^-(k+1){}^je^{-\mathrm{T}}(k+1)], \forall i \neq j \in \mathcal{V}$, are the cross-covariances between two different agents' predicted state estimates, and the diagonal block-matrices, ${}^{ii}\Sigma^-(k+1), \forall i \in \mathcal{V}$, are the predicted estimation error covariance from the i^{th} agent's perspective. Updating the predicted state estimate through (7), the estimator gain L_i is designed in a spirit similar to Kalman filtering such that it minimizes the trace of the updated estimation error covariance, which yields

$$L_i(k+1) = {}^{ii}\Sigma^-(k+1)H_i^{\mathrm{T}}(S_i(k+1))^{-1} \tag{12}$$

where $S_i(k+1) = H_i({}^{ii}\Sigma^-(k+1) + {}^i\xi)H_i^{\mathrm{T}}$, and ${}^i\xi = blkdg(\xi_{i1}, \xi_{i2}, \ldots \xi_{iN})$. Subsequently, the updated estimation error covariance $\Sigma(k+1)$, is computed. Furthermore, the cross-covariance between the predicted and updated state estimates over different time steps, i.e., $\mathbb{E}[e^-(k+1)e(s)^{\mathrm{T}}]$ and $\mathbb{E}[e(k+1)e(s)^{\mathrm{T}}], \forall s < k$, can be computed. Starting with the initial error covariance conditions $\Sigma(0)$, one can recursively update ${}^i\hat{x}(k)$, $\Sigma^-(k)$, and $\Sigma(k)$ given the optimized $L_i(k)$ over the finite time horizon T. This completes the design of the linear distributed estimators for individual agents parameterized by the estimator gains $\Upsilon_i, \forall i \in \mathcal{V}$. And the resulting estimation error covariance information will be used for the distributed controller design, presented in the next subsection. The key result regarding the stability of the proposed distributed estimator is presented by the following lemma and theorem.

Lemma 1: The distributed estimation error covariances ${}^{ii}\Sigma(k), \forall i \in \mathcal{V}$ for all k are positive definite and bounded if the following system is observable [50].

$$\begin{aligned} \bar{x}(k+1) &= \mathcal{L}\bar{x}(k) \in \mathbb{R}^N \\ \bar{Z}_i(k) &= \bar{H}_i\bar{x}(k) \in \mathbb{R}^{|\Omega_i|}, \ \forall i \in \mathcal{V} \end{aligned} \tag{13}$$

where $\bar{H}_i = [h_1\ h_2\ \cdots h_{|\Omega_i|}]^{\mathrm{T}} \in \mathbb{R}^{|\Omega_i| \times N}$ is an observer matrix that gathers the measurements from the i^{th} agent's perspective, i.e., those that are neighboring agents of the i^{th} agent. The row vectors of $\bar{H}_i$ consist of $h_q \in \mathbb{R}^N, q = 1, 2, \ldots, |\Omega_i|$, which are non-zero column vectors of the matrix $diag(a_{i1}, a_{i2} \ldots, a_{iN})$.

Theorem 1: Given the MAS dynamics (2) and the distributed estimator (7), the proposed distributed estimator is stable in the sense of Lyapunov if (13) is observable.

Proof. The proof is referred to in [50]. ∎

Remark 2: When operating the MAS with a network topology satisfying the network observability condition in Theorem 1, the designed distributed estimator processes only the neighboring measurements available to each agent. Nevertheless, each agent can estimate MAS states, including non-neighboring agents.

Remark 3: The estimation error covariance is subject to the network topology $\mathcal{L}$. However, directly representing the estimation error covariance as the function of the network condition is non-trivial, as it also depends on $\mathcal{F}$ and $\Upsilon_i, \forall i \in \mathcal{V}$.

B. DISTRIBUTED CONTROL LAW DESIGN

This subsection describes the design procedure of the distributed control law for solving Problem 2, with the aid of a distributed estimator embedded in individual agents. Note that $\mathcal{F}$ is now the design parameter to be optimized, eventually be $\mathcal{F}^{(l)}$ at the l^{th} iteration, whereas the estimator design is based on $\mathcal{F}^{(l-1)}$.

Definition 3: The concatenated estimation errors over time from the i^{th} agent's viewpoint are denoted as ${}^{i}\mathbf{e} := \mathbf{x} - {}^{i}\hat{\mathbf{x}}$. The estimation error covariances from individual agents' perspectives over time horizon T are defined as $\Sigma_{ii} := \mathbb{E}[{}^{i}\mathbf{e}{}^{i}\mathbf{e}^{\mathrm{T}}], \forall i \in \mathcal{V}$, and the covariance between different agents' perspectives $\Sigma_{ij} := \mathbb{E}[{}^{i}\mathbf{e}{}^{j}\mathbf{e}^{\mathrm{T}}], \forall i \neq j \in \mathcal{V}$. These covariances can be constructed using the results of the previous estimator design procedure.

Recalling (3) and (8), the estimation-based control input of the entire MAS over the time horizon T can be written in terms of the estimation errors as follows.

$$\mathbf{u} = \sum_{i}^{N} \mathcal{M}_i \mathcal{F} C\, {}^{i}\hat{\mathbf{x}} = \mathcal{F} C \mathbf{x} - \sum_{i}^{N} \mathcal{M}_i \mathcal{F} C\, {}^{i}\mathbf{e} \tag{14}$$

where $\mathcal{M}_i = I_T \otimes (M_i^{\mathrm{T}} M_i), \forall i \in \mathcal{V}$. By substituting (14) into (5), the state and the input vector are rewritten in terms of the disturbance and the estimation errors as follows

$$\begin{aligned}\mathbf{x} &= (I - P_{12}\mathcal{F}C)^{-1} P_{11}\mathbf{w} \\ &\quad - (I - P_{12}\mathcal{F}C)^{-1} P_{12} \sum_{i}^{N} \mathcal{M}_i \mathcal{F} C\, {}^{i}\mathbf{e} \\ \mathbf{u} &= \mathcal{F}C(I - P_{12}\mathcal{F}C)^{-1} P_{11}\mathbf{w} \\ &\quad - (I - \mathcal{F}CP_{12})^{-1} \sum_{i}^{N} \mathcal{M}_i \mathcal{F} C\, {}^{i}\mathbf{e}. \end{aligned} \tag{15}$$

Substituting (15) into the quadratic cost function, and using the following fact for any matrix P,

$$\begin{aligned}\mathbb{E}[\mathbf{w}^{\mathrm{T}} P \mathbf{w}] &= \mathrm{Tr}(P\Sigma_w) + \mu_w^{\mathrm{T}} P \mu_w, \\ \mathbb{E}[{}^{i}\mathbf{e}^{\mathrm{T}} P\, {}^{j}\mathbf{e}] &= \mathrm{Tr}(P\Sigma_{ij}), \\ \mathrm{Tr}(P^{\mathrm{T}} P) &= \|P\|_F^2, \end{aligned}$$

the global cost of MAS in Problem 2 can be expressed by

$$\begin{aligned} &J(\mathcal{F}, \Upsilon_1, \cdots, \Upsilon_N) \\ &= \|\mathcal{Q}^{\frac{1}{2}}(I - P_{12}\mathcal{F}C)^{-1} P_{11}\Sigma_w^{\frac{1}{2}}\|_F^2 \\ &\quad + \|\mathcal{R}^{\frac{1}{2}}(I - \mathcal{F}CP_{12})^{-1}\mathcal{F}CP_{11}\Sigma_w^{\frac{1}{2}}\|_F^2 + \sum_{i,j}\|\mathcal{Q}^{\frac{1}{2}}P_{12} \\ &\quad \times (I - \mathcal{F}CP_{12})^{-1}(\mathcal{M}_i\mathcal{F}C\Sigma_{ij}C^{\mathrm{T}}\mathcal{F}^{\mathrm{T}}\mathcal{M}_j^{\mathrm{T}})^{\frac{1}{2}}\|_F^2 \\ &\quad + \sum_{i,j}\|\mathcal{R}^{\frac{1}{2}}(I - \mathcal{F}CP_{12})^{-1}(\mathcal{M}_i\mathcal{F}C\Sigma_{ij}C^{\mathrm{T}}\mathcal{F}^{\mathrm{T}}\mathcal{M}_j^{\mathrm{T}})^{\frac{1}{2}}\|_F^2 \\ &\quad + \|\mathcal{Q}^{\frac{1}{2}}(I - P_{12}\mathcal{F}C)^{-1}P_{11}\mu_w\|_2^2 \\ &\quad + \|\mathcal{R}^{\frac{1}{2}}(I - \mathcal{F}CP_{12})^{-1}\mathcal{F}CP_{11}\mu_w\|_2^2 \end{aligned} \tag{16}$$

where $\Sigma_w := \mathbb{E}[(\mathbf{w} - \mu_w)(\mathbf{w} - \mu_w)^{\mathrm{T}}] \in \mathbb{R}^{Nn(T+1)\times Nn(T+1)}$, $\mu_w := \mathbb{E}[\mathbf{w}] \in \mathbb{R}^{Nn(T+1)}$. A comparable derivation is detailed in [47].

As seen in (16), the optimization variables, i.e., $\mathcal{F}$ and $\Upsilon_i, \forall i \in \mathcal{V}$, are the latent variables of $\Sigma_{ij}, \forall i, j \in \mathcal{V}$. This implies that the cost of the joint control-estimation problem is difficult for a single optimization, motivating the sequential iterative optimization framework. Inspired by the Alternating Direction Method of Multipliers (ADMM) [56], we alternate the target optimization variables and sequentially optimize the distributed control and estimation laws. It is worth noting that we optimize the estimator gain instead of the estimation error covariance because the latter depends on the distributed control and estimation laws. This ensures a straightforward and interpretable optimization process. The overall steps of this iterative optimization start by optimizing the distributed estimator gains $\Upsilon_i, \forall i \in \mathcal{V}$ with an arbitrarily initialized distributed control law $\mathcal{F}$ as mentioned in the distributed estimator design step. This results in optimized estimator gains and the corresponding estimation error covariance, $\Sigma_{ij}, \forall i, j \in \mathcal{V}$. Subsequently, given that the estimation error covariance in (16) is fixed as constant, we optimize only $\mathcal{F}$ without adjusting $\Upsilon_i, \forall i \in \mathcal{V}$. Once the new distributed control law is obtained, we repeat the process. This sequential iterative optimization continues until the predefined convergence rule is satisfied.

The cost function with constant $\Sigma_{ij}, \forall i, j \in \mathcal{V}$, is problematic owing to its non-convex combination. To this end, we approximate different estimation error covariances using their upper bounds.

Definition 4: The estimation error covariances of all agents are bounded above by Σ_{max} defined as follows.

$$\Sigma_{max} = \begin{bmatrix} \Sigma_{max}(0,0) & \cdots & \Sigma_{max}(0,T) \\ \vdots & \ddots & \vdots \\ \Sigma_{max}(T,0) & \cdots & \Sigma_{max}(T,T) \end{bmatrix}$$

where

$$\begin{aligned} \Sigma_{max}(k,s) &:= \mathbb{E}[{}^{\tilde{i}}e(k)\, {}^{\tilde{j}}e^{\mathrm{T}}(s)] \\ \tilde{i}, \tilde{j} &= \arg\max_{i,j\in\mathcal{V}} \|\mathbb{E}[{}^{i}e(k)\, {}^{j}e^{\mathrm{T}}(s)]\|_F \end{aligned}$$

such that $\|\Sigma_{max}(k,s)\|_F \geq \|\mathbb{E}[{}^{i}e(k)\, {}^{j}e^{\mathrm{T}}(s)]\|_F, \forall, i, j \in \mathcal{V}$ [57].

Substituting all $\Sigma_{ij}, \forall i, j \in \mathcal{V}$ with Σ_{max}, the original cost (16) can be approximated in a form compatible with convexification, although the result would be a conservative solution. Amidst the iterative optimization framework, the control law optimization for the approximated cost at the l^{th} iteration can be expressed as follows.

Problem 3: Optimal distributed control with maximum estimation error covariance between agents.

$$\min_{\mathcal{F}^{(l)} \in \tilde{\mathbb{F}}} J_{apx}^{(l)}(\mathcal{F}^{(l)})$$

where

$$\begin{aligned} J_{apx}^{(l)}(\mathcal{F}^{(l)}) &= \|\mathcal{Q}^{\frac{1}{2}}(I - P_{12}\mathcal{F}^{(l)}C)^{-1}P_{11}\Sigma_w^{\frac{1}{2}}\|_F^2 \\ &\quad + \|\mathcal{R}^{\frac{1}{2}}(I - \mathcal{F}^{(l)}CP_{12})^{-1}\mathcal{F}^{(l)}CP_{11}\Sigma_w^{\frac{1}{2}}\|_F^2 \\ &\quad + \|\mathcal{Q}^{\frac{1}{2}}(I - P_{12}\mathcal{F}^{(l)}C)^{-1}P_{12}\mathcal{F}^{(l)}C\Sigma_{max}^{(l)\frac{1}{2}}\|_F^2 \\ &\quad + \|\mathcal{R}^{\frac{1}{2}}(I - \mathcal{F}^{(l)}CP_{12})^{-1}\mathcal{F}^{(l)}C\Sigma_{max}^{(l)\frac{1}{2}}\|_F^2 \end{aligned}$$

$$+ \|\mathcal{Q}^{\frac{1}{2}}(I - P_{12}\mathcal{F}^{(l)}C)^{-1}P_{11}\mu_w\|_2^2 + \|\mathcal{R}^{\frac{1}{2}}(I - \mathcal{F}^{(l)}CP_{12})^{-1}\mathcal{F}^{(l)}CP_{11}\mu_w\|_2^2 \quad (17)$$

where $\Sigma_{max}^{(l)}$ is the upper bound error covariance at the l^{th} iteration.

Now, the cost function (17), which is still non-convex with respect to $\mathcal{F}^{(l)}$, can be reformulated into an equivalent convex formulation in view of QI [47].

Definition 5: A subspace $\tilde{\mathbb{K}}$ is called QI with respect to $\mathbb{W}$ if and only if $\tilde{K}\mathbb{W}\tilde{K} \in \tilde{\mathbb{K}}, \forall \tilde{K} \in \tilde{\mathbb{K}}$ [44].

Remark 4: By Definition 5, it is trivial to show that the subspace $\tilde{\mathbb{F}}$ is QI with respect to CP_{12}, because $\mathcal{F}^{(l)}CP_{12}\mathcal{F}^{(l)} \in \tilde{\mathbb{F}}$, $\forall \mathcal{F}^{(l)} \in \tilde{\mathbb{F}}$. Note that this is generally not the case if the control input is subject to the structural constraint. However, our problem can achieve QI owing to the distributed estimator embedded in each agent.

Definition 6: Let us introduce a new optimization variable $\Phi \in \mathbb{R}^{NpT \times NnT}$, and formulate the cost function $\tilde{J}^{(l)} : \mathbb{R}^{NpT \times nNT} \mapsto \mathbb{R}, \forall i \in \mathcal{V}$ with respect to $\Phi^{(l)}$ as

$$\tilde{J}^{(l)}(\Phi^{(l)}) = \|\mathcal{Q}^{\frac{1}{2}}(I + P_{12}\Phi^{(l)}C)P_{11}\Sigma_w^{\frac{1}{2}}\|_F^2 + \|\mathcal{R}^{\frac{1}{2}}\Phi^{(l)}CP_{11}\Sigma_w^{\frac{1}{2}}\|_F^2 + \|\mathcal{Q}^{\frac{1}{2}}P_{12}\Phi^{(l)}C\Sigma_{max}^{(l)\frac{1}{2}}\|_F^2 + \|\mathcal{R}^{\frac{1}{2}}\Phi^{(l)}C\Sigma_{max}^{(l)\frac{1}{2}}\|_F^2 + \|\mathcal{R}^{\frac{1}{2}}\Phi^{(l)}CP_{11}\mu_w\|_2^2 + \|\mathcal{Q}^{\frac{1}{2}}(I + P_{12}\Phi^{(l)}C)P_{11}\mu_w\|_2^2 \quad (18)$$

and the non-linear mapping $h : \mathbb{R}^{NpT \times NnT} \mapsto \mathbb{R}^{NpT \times NnT}$ as follows.

$$h(\Phi) = (I + \Phi CP_{12})^{-1}\Phi \quad (19)$$

Lemma 2: Given the non-linear mapping in (19), which establishes a bijective correspondence between $\mathcal{F}$ and Φ, we have that $\mathcal{F} = h(h^{-1}(\mathcal{F}))$. Thus, the cost function $\tilde{J}^{(l)}$ in (18) is equivalent to $J_{apx}^{(l)}$ in (17).

Proof. The proof follows from the definition of h and utilization of the matrix inversion lemma, as demonstrated in Lemma 1 of [47]. ■

Then, from Theorem 1 in [47], the equivalent convex problem of Problem 3 can be formulated as follows.

Problem 4: Equivalent convex problem for optimal distributed control with maximum estimation error covariance between agents.

$$\min_{\Phi^{(l)} \in h^{-1}(\tilde{\mathbb{F}})} \tilde{J}^{(l)}(\Phi^{(l)}) \quad (20)$$

Finally, Problem 4 can be solved through convex programming, and $\mathcal{F}^{(l)}$ is obtained from the optimized $\Phi^{(l)}$ by taking the inverse mapping h^{-1} of (19).

C. CONVERGENCE CHECK

Once the control law and estimator are optimized at the current iteration, i.e., $\mathcal{F}^{(l)}$, and $\Upsilon_i^{(l)}$, $\forall i \in \mathcal{V}$, we evaluate them to decide whether to move on to the next iteration or terminate the iteration. First, a set of optimized variables is retained over the iterations as follows.

$$S := \left\{ s^{(l)} \middle| s^{(l)} = \left(\mathcal{F}^{(l)}, \Upsilon_1^{(l)}, \cdots, \Upsilon_N^{(l)}\right), \ l \in \mathbb{N} \right\}$$

The iterative optimization process ends if i) the number of iterations exceeds the threshold number N_{max}, or ii) the following stopping condition is met, which implies the convergence of the consecutive iteration.

$$\triangle J(l, l-1) \leq \epsilon_{stop} \quad (21)$$

where $\triangle J(l, l-1) := |J(\mathcal{F}^{(l)}, \Upsilon_1^{(l)}, \cdots, \Upsilon_N^{(l)}) - J(\mathcal{F}^{(l-1)}, \Upsilon_1^{(l-1)}, \cdots, \Upsilon_N^{(l-1)})|$. The threshold ϵ_{stop} is set sufficiently small. By substituting $\mathcal{F}^{(l)}$, and $\Upsilon_i^{(l)}, \forall i \in \mathcal{V}$ into (16), the cost at the l^{th} iteration is computed. Once the iteration terminates, the resulting optimized distributed control-estimation laws are given by

$$\mathcal{F}^* = \mathcal{F}^{(\kappa)}, \ \Upsilon_i^* = \Upsilon_i^{(\kappa)}, \ \forall i \in \mathcal{V}$$
$$\text{where } \kappa = \arg\min_{\forall l \in |S|} J(\mathcal{F}^{(l)}, \Upsilon_1^{(l)}, \cdots, \Upsilon_N^{(l)}) \quad (22)$$

A comprehensive overview of the proposed iterative suboptimal linear distributed control-estimation law design procedure is presented in Algorithm 1. In the runtime phase, the designed $\mathcal{F}^*$ and Υ_i^* are embedded into each agent to execute the distributed control and estimation. The runtime operation of the individual agents is summarized in Algorithm 2.

Remark 5: The proposed distributed control-estimation synthesis is performed in the offline design phase before the deployment of the MAS. The main computational complexity of the design (Algorithm 1) is from the convex optimization in solving Problem 4. Except for the computational burden in convex optimization, the calculation of the inverse matrices, as shown in (12) and (19), has a computational complexity of $\mathcal{O}((Nn)^3)$. As for the runtime operation (Algorithm 2), it is viable for the limited onboard computing resources of individual agents, because it only involves computing the state estimate and control input based on the designed gains, as shown in (7) and (9).

Remark 6: The central versus distributed paradigms for MAS can be distinctly applied across the design and runtime phases. In the design phase, Algorithm 1 leverages all MAS information to optimize complex MAS tasks in a centralized fashion, similar to [33], [34], and [35]. This contrasts with distributed design, which relies solely on information from neighboring agents, allowing each agent to independently design its protocol in a decentralized manner. During the runtime phase through Algorithm 2, our method adopts a distributed control strategy that depends only on local measurements from neighboring agents. This is in contrast to centralized control, which requires full MAS measurements and is impractical for online operations with network topological constraints.

Remark 7: The proposed algorithm can be extended to heterogeneous agents with more complex system dynamics.

The structure of the algorithm remains the same, except for the parts that involve agent dynamics, e.g., (5) and (10). Although it is feasible to derive an algorithm for a heterogeneous MAS, we opted to focus on a homogeneous MAS in this paper to clearly convey our core idea.

Remark 8: The proposed iterative optimization process does not guarantee monotonic performance improvement with each iteration. Therefore, to avoid an infinite run of the iterative loop, the maximum number of allowed iterations is set as the initial condition, N_{max}. It is worth noting that the focus of this paper is not on identifying the conditions where the optimal solution can be tractable, but on developing an optimization framework to effectively approximate the solution for a wider scope of intractable problems, specifically those that do not satisfy well-known network conditions, e.g., the QI condition [44].

Algorithm 1 (Design phase)
Virtual Network Based Suboptimal linear Distributed Control-Estimation Synthesi

Initialization
- Set the MAS dynamics information A, B, $\mathcal{L}$, $\Sigma(0)$, ϵ_{stop}, N_{max}, $\mathcal{F}^{(0)}$, and the cost metrics $\mathcal{Q}$, $\mathcal{R}$.

$l = 0$
While $l \leq N_{max}$
a) **Distributed estimator design**
  While $k \leq T$
  1) Given $\mathcal{F}^{(l)}$, solve for the estimator gains for each agent, $L_i(k+1)$, $\forall i \in \mathcal{V}$ using (12)
  $k \leftarrow k+1$
  end while
  Output $\Longrightarrow$ $\Upsilon_i^{(l)}$, $\Sigma_{ij}^{(l)}, \forall i \in \mathcal{V}$
b) **Distributed control law design**
  4) Given $\Sigma_{ij}^{(l)}, \forall i, j \in \mathcal{V}$, compute $\Sigma_{max}^{(l)}$ by Definition 4
  5) Solve the convex problem in (20), and compute $\mathcal{F}^{(l)}$ by applying the inverse of the non-linear mapping in (19)
  Output $\Longrightarrow$ $\mathcal{F}^{(l)}$
c) **Convergence check**
  6) Store $\mathcal{F}^{(l)}$ and $\Upsilon_i^{(l)}, \forall i \in \mathcal{V}$ in the set S
  7) **If** (21) is satisfied $\longrightarrow$ **Break**
  else $l \leftarrow l+1$
end while
Return $\Longrightarrow$ $\mathcal{F}^*$, and Υ_i^*, $\forall i \in \mathcal{V}$ using (22)

IV. THEORETIC PERFORMANCE ANALYSIS

This section presents a verification of the theoretical performance of the proposed solution. Beforehand, we consider an auxiliary problem for an ideal MAS, having fully connected network topology ($\Omega_i = \mathcal{V}$, $\forall i \in \mathcal{V}$) and no measurement noise ($v_i = 0$, $\forall i \in V$) while retaining the same dynamics as in (2). The optimal distributed control for such a MAS can be implemented using full-state feedback without requiring a distributed estimator. The cost associated with this auxiliary problem serves as a benchmark for comparing it with other scenarios, such as those involving network topology constraints or measurement noise.

Algorithm 2 (Runtime phase)
Virtual Interaction Based Suboptimal Linear Distributed Control for the i^{th} Agent

Initialization
- Set the MAS dynamics information A, B, H_i, the initial condition ${}^i\hat{x}(0)$, the optimized control and estimator gains $(\mathcal{F}^*, \Upsilon_i^*)$.

$k = 0$
While $k \leq T$ do
1) Execute the control input $u_i(k)$ by (9)
2) The MAS state $x(k)$ is evolved to $x(k+1)$ by (2)
3) Measure the state of neighboring agents, $Z_i(k+1)$ by (4)
4) Update ${}^i\hat{x}(k+1)$ using (11), and (7)

$k \leftarrow k+1$
end while

Problem 5: Full-state feedback-based optimal distributed control of a noise-free MAS with a fully connected network topology.

$$\min_{\mathcal{F}\in\bar{\mathbb{F}}} \quad \bar{J}(\mathcal{F})$$
$$\text{subject to (5), and } \mathbf{u}_i = (I_T \otimes M_i)\mathcal{F}C\mathbf{x}$$

where the quadratic cost $\bar{J} = \mathbb{E}\left[\mathbf{x}^{\mathrm{T}}\mathcal{Q}\mathbf{x} + \mathbf{u}^{\mathrm{T}}\mathcal{R}\mathbf{u}\right]$ can be written by

$$\begin{aligned}\bar{J}(\mathcal{F}) = &\|\mathcal{Q}^{\frac{1}{2}}(I - P_{12}\mathcal{F}C)^{-1}P_{11}\Sigma_w^{\frac{1}{2}}\|_F^2\\ &+ \|\mathcal{R}^{\frac{1}{2}}(I - \mathcal{F}CP_{12})^{-1}\mathcal{F}CP_{11}\Sigma_w^{\frac{1}{2}}\|_F^2\\ &+ \|\mathcal{Q}^{\frac{1}{2}}(I - P_{12}\mathcal{F}C)^{-1}P_{11}\mu_w\|_2^2\\ &+ \|\mathcal{R}^{\frac{1}{2}}(I - \mathcal{F}CP_{12})^{-1}\mathcal{F}CP_{11}\mu_w\|_2^2. \end{aligned} \tag{23}$$

Using the non-linear mapping technique (19), Problem 5 can be solved through the following equivalent convex problem.

Problem 6: Equivalent convex problem for Problem 5.

$$\min_{\Phi\in h^{-1}(\bar{\mathbb{F}})} \bar{\bar{J}}(\Phi)$$

where

$$\begin{aligned}\bar{\bar{J}}(\Phi) = &\|\mathcal{Q}^{\frac{1}{2}}(I + P_{12}\Phi C)P_{11}\Sigma_w^{\frac{1}{2}}\|_F^2 + \|\mathcal{R}^{\frac{1}{2}}\Phi CP_{11}\Sigma_w^{\frac{1}{2}}\|_F^2\\ &+ \|\mathcal{R}^{\frac{1}{2}}\Phi CP_{11}\mu_w\|_2^2 + \|\mathcal{Q}^{\frac{1}{2}}(I + P_{12}\Phi C)P_{11}\mu_w\|_2^2.\end{aligned}$$

Along with Problem 5, whose solution is denoted by $\mathcal{F}^{free}$, the preceding optimization problems are characterized by the following definition.

Definition 7: Let us define a set of cost formulas as follows.

$$\begin{aligned} J^{opt} &:= J(\mathcal{F}^{opt}, \Upsilon_1^{opt}, \cdots, \Upsilon_N^{opt}) \\ J^{apx} &:= J_{apx}^{(\kappa)}(\mathcal{F}^*) \\ J^{free} &:= \bar{J}(\mathcal{F}^{free}) \\ J^* &:= J(\mathcal{F}^*, \Upsilon_1^*, \cdots, \Upsilon_N^*) \end{aligned}$$

which respectively refer to the global optima by the exact optimal solution (denoted by $\mathcal{F}^{opt}$, $\Upsilon_i^{opt}, \forall i \in \mathcal{V}$) to Problem 2; the optimal solution to the approximated cost for Problem 3; the optimal solution to the noise-free cost for Problem 5; and the cost achieved by the proposed distributed control-estimation synthesis.

Note that J^{apx} and J^{free} are readily computed from equivalent convex problems, i.e., Problems 4 and 6, respectively. However, the global optimum J^{opt}, is not a convex problem and is thus intractable to solve. Then, we are interested in evaluating the performance of $\mathcal{F}^*$, $\Upsilon_i^*, \forall i \in \mathcal{V}$ as to $\mathcal{F}^{opt}$, $\Upsilon_i^{opt}, \forall i \in \mathcal{V}$ in terms of the ratio between J^* and J^{opt}.

Definition 8: Let $\nabla := \frac{J^*}{J^{opt}}$ be the performance ratio of the proposed solution relative to global optima.

Without knowing the exact J^{opt}, we provide a finite bound on the ratio ∇ by using the computable measures J^{apx} and J^{free}. Remark that ∇ gets closer to 1 when J^* achieves the comparable performance to J^{opt} [58].

Theorem 2: Given the primal distributed control-estimation problem in the form of Problem 2, this ratio is bounded above by the following inequality.

$$\nabla \leq \frac{J^{apx}}{J^{free}} \tag{24}$$

where J^{apx} and J^{free} are solved from Problem 4 and 6, respectively.

Proof. First, from (23), (16), and (17), $\bar{J}$ is the common part of J and $J_{apx}^{(l)}$ such that

$$\begin{aligned} J(\mathcal{F}, \Upsilon_1, \cdots, \Upsilon_N) &= \bar{J}(\mathcal{F}) + \Pi(\mathcal{F}, \Upsilon_1, \cdots, \Upsilon_N) \\ J_{apx}^{(l)}(\mathcal{F}) &= \bar{J}(\mathcal{F}) + \Pi_{apx}^{(l)}(\mathcal{F}) \end{aligned} \tag{25}$$

where Π and $\Pi_{apx}^{(l)}$ are the cost terms induced by the estimation error covariances as follows.

$$\begin{aligned} &\Pi(\mathcal{F}, \Upsilon_1, \cdots, \Upsilon_N) \\ &= \sum_{i,j} \|\mathcal{R}^{\frac{1}{2}}(I - \mathcal{F}CP_{12})^{-1}(\mathcal{M}_i\mathcal{F}C\Sigma_{ij}C^{\mathrm{T}}\mathcal{F}^{\mathrm{T}}\mathcal{M}_j^{\mathrm{T}})^{\frac{1}{2}}\|_F^2 \\ &\quad + \sum_{i,j} \|\mathcal{Q}^{\frac{1}{2}}P_{12}(I - \mathcal{F}CP_{12})^{-1}(\mathcal{M}_i\mathcal{F}C\Sigma_{ij}C^{\mathrm{T}}\mathcal{F}^{\mathrm{T}}\mathcal{M}_j^{\mathrm{T}})^{\frac{1}{2}}\|_F^2 \\ &\Pi_{apx}^{(l)}(\mathcal{F}) = \|\mathcal{Q}^{\frac{1}{2}}(I - P_{12}\mathcal{F}C)^{-1}P_{12}\mathcal{F}C\Sigma_{max}^{(l)\frac{1}{2}}\|_F^2 \\ &\quad + \|\mathcal{R}^{\frac{1}{2}}(I - \mathcal{F}CP_{12})^{-1}\mathcal{F}C\Sigma_{max}^{(l)\frac{1}{2}}\|_F^2 \end{aligned} \tag{26}$$

By Definition 4, $\Pi(\mathcal{F}^{(l)}, \Upsilon_1^{(l)}, \cdots, \Upsilon_N^{(l)}) \leq \Pi_{apx}^{(l)}(\mathcal{F}^{(l)})$ holds for all $l \in |S|$ including the case $l = \kappa$, and thus $J^* \leq J^{apx}$. Dividing this inequality by J^{opt} yields

$$\frac{J^*}{J^{opt}} =: \nabla \leq \frac{J^{apx}}{J^{opt}}.$$

Further, from the fact that $\bar{J}$ is included in J,

$$\bar{J}(\mathcal{F}^{free}) \leq \bar{J}(\mathcal{F}^{opt}) \leq J(\mathcal{F}^{opt}, \Upsilon_1^{opt}, \cdots, \Upsilon_N^{opt}) = J^{opt}.$$

Hence, $J^{free} \leq J^{opt}$ holds. By this inequality,

$$\nabla \leq \frac{J^{apx}}{J^{opt}} \leq \frac{J^{apx}}{J^{free}}$$

which yields (24). This ends the proof. ∎

Remark 9: Given a MAS network topology satisfying the network observability condition in Theorem 1, ∇ varies with the extent of disturbance and measurement noise. As an ideal case, for deterministic MAS dynamics, i.e., $\Sigma_w \approx 0$ and $\Sigma_{ij} \approx 0, \forall i,j \in \mathcal{V}$, the approximated cost induced by the estimation error covariances becomes negligible, i.e., $\Pi_{apx}^{(\kappa)} \approx 0$, by (26). Thus, $J^{apx} \approx J^{free}$ from (25). Furthermore, observing the cost comparison $0 \leq J^{free} \leq J^{opt} \leq J^* \leq J^{apx}$, it follows that $1 \leq \frac{J^*}{J^{opt}}$. Consequently, $1 \leq \nabla \leq \frac{J^{apx}}{J^{free}} \approx 1$ implying that the proposed solution approaches the global optima of Problem 2, $J^* \approx J^{opt}$. Moreover, when the initial state $x(0)$ has a zero mean, we have $J^{free} \equiv J^{apx} \equiv \bar{J}(\mathcal{F}^*)$. In this case, the bounds can be further refined, resulting in $1 \leq \frac{J^*}{J^{opt}} \equiv \frac{J^{apx}}{J^{opt}} \equiv \frac{J^{apx}}{J^{free}} \equiv 1$. This clearly indicates that the upper bound of ∇ is tightened to one.

V. NUMERICAL SIMULATION

This section validates the proposed suboptimal linear distributed control-estimation synthesis via numerical simulations using the following parameter sets $A = 1$, $B = 1$, $\Theta_i = 1$, ${}^i\xi = I_5, \forall i \in \mathcal{V}$, $\mathcal{Q} = I_6 \otimes (5I_5 - 1_5 1_5^{\mathrm{T}})$, $\mathcal{R} = I_{25}$, and $T = 5$. The considered network topology is an undirected string graph with five agents, represented by the following Laplacian matrix.

$$\mathcal{L} = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 \\ -1 & 0 & 2 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix} \tag{27}$$

The primary goal of the defined cost metrics is to establish a consensus protocol, where agents aim to converge to the same state, thereby reducing the relative distance between them to zero. For the iterative optimization process in the design phase, the stopping criterion is set as $\epsilon_{stop} = 0.05$ with the maximum number of iterations as $N_{max} = 10$. In Figure. 2, the convergence of the proposed iterative optimization process is demonstrated with the cost difference of consecutive iterations. In the aforementioned problem set-up, iteration terminates at the 5^{th} iterations.

The resulting distributed control law, $\mathcal{F}^*$ is color-coded in Figure. 3 to provide a clearer insight into the proposed virtual interaction concept. Displaying different colors with respect to the gain magnitude, each entry of the computed

TABLE 1. Comparative analysis of computational complexity and measurement loads.

Algorithms	Optimal solution with fully connected network [47]	Proposed solution with partially connected network	Suboptimal solution with partially connected network [33]
Computational complexity (Design phase)	$\mathcal{O}(n^3N^3)$	$\mathcal{O}(n^3N^3)$	$\mathcal{O}(\sum_i^N n^3\vert\Omega_i\vert^3)$
Computational complexity per agent (Runtime phase)	$\mathcal{O}(n^2N^2)$	$\mathcal{O}(n^2N^2)$	$\mathcal{O}(n^2\vert\Omega_i\vert^2)$
Load of measurements per sampling time	$\mathcal{O}(N^2)$	$\mathcal{O}(\sum_{i,j}^N a_{ij})$	$\mathcal{O}(\sum_{i,j}^N a_{ij})$

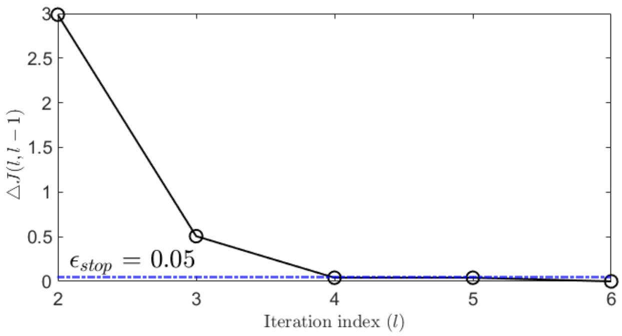

FIGURE 2. **Cost differences over the consecutive iteration of the proposed algorithm.**

control gain matrix $\mathcal{F}^*$ indicates the weights of the (possibly virtual) interactions between agents (e.g., the top row and second column of $\mathcal{F}^*$ represent the virtual interaction weights between the 1^{st} agent and the 2^{nd} agent, from the 1^{st} agent's perspective at time step $k = 0$). Specifically, at time step $k = 2$, $\mathcal{F}^*_{22}$ exemplifies the optimal control law for each time step, in which a large gain value is assigned to those that correspond to non-zero entries of $\mathcal{L}$. This implies that agents interact more intensively with their neighbors by virtue of directly available measurements. More importantly, even for zero entries in $\mathcal{L}$, the corresponding control gain value is not zero. This signifies virtual interactions between non-neighboring agents through the fully connected virtual network topology established by the distributed estimator.

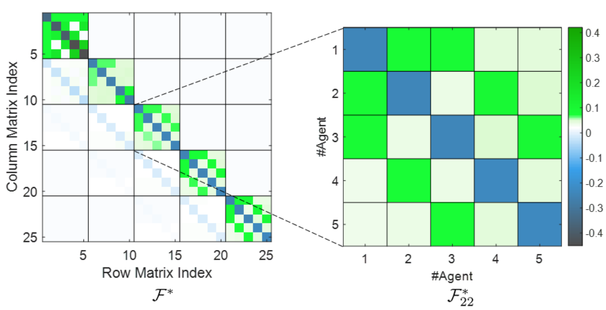

FIGURE 3. **The entry of the distributed control law, under $\theta_i = 1$ and ${}^i\xi = I_5, \forall i \in \mathcal{V}$.**

To further demonstrate the underlying mechanism of the proposed distributed control-estimation synthesis, we have performed additional simulations under different measurement noise scenarios when the noise covariance of measuring the 4^{th} agent is ten times bigger than all the other agents, i.e.,

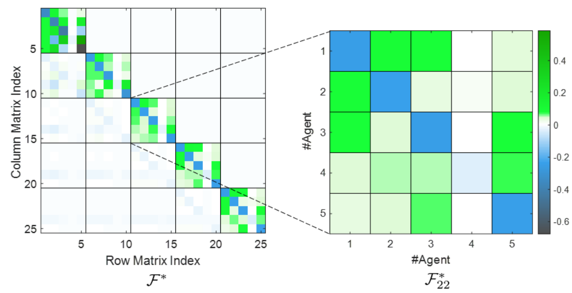

FIGURE 4. **The entry of the distributed control law, under $\theta_i = 1$ and ${}^i\xi = diag(1, 1, 1, 10, 1), \forall i \in \mathcal{V}$.**

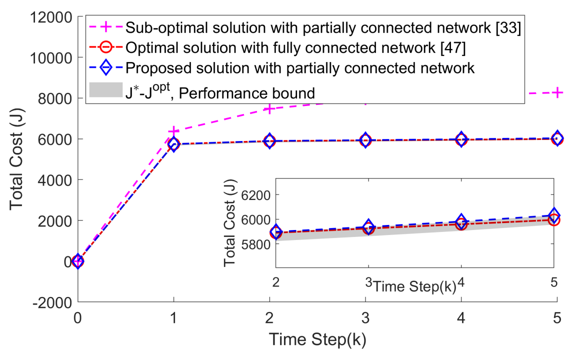

FIGURE 5. **Total costs over finite time horizon (Monte Carlo simulations with 100 runs).**

${}^i\xi = diag(1, 1, 1, 10, 1), \forall i \in \mathcal{V}$. The resulting control gain is color-mapped in Figure. 4. It is highlighted that increasing the noise of the 4^{th} agent reduces the interaction weights between the 4^{th} agent and all other agents. When optimizing the control law, the cost function encompasses the estimation errors of all individual agents. Therefore, the control law is designed to diminish the effect of the estimation error from measurement noise. This trend is well understood by intuition because less interaction is preferable if the measured information is inaccurate.

Two different distributed control schemes are carried out for the comparative analysis. The first scheme employs the suboptimal distributed control law developed in [33] for MAS subject to the same partially connected network topology, $\mathcal{L}$. The second scheme considers the distributed output feedback control under a fully connected network, i.e., each agent can directly interact with every other agent. Hence, a globally optimal distributed control solution can be easily obtained [47]. Together with our distributed control-estimation law scheme, three schemes are showcased through Monte Carlo simulations. The average cumulative total costs for different schemes are shown in Figure. 5. The proposed control-estimation law, which facilitates virtual interactions among agents that are not direct neighbors, surpasses the suboptimal solution for the same partially connected network [33], and its performance nearly matches the optimal solution with a fully connected network [47]. By Theorem 2, we can theoretically infer the solution to Problem 2, which is not tractable but guaranteed to be within the gray-shaded area. Additionally, it is worth noting that the optimal solution with a fully connected network serves as the lower bound for the solution to the original Problem 1, since the problem of finding an optimal solution under the fully connected network is an unconstrained optimization of Problem 1. Therefore, by comparing the proposed solution with the optimal solution with a fully connected network, we can indirectly gauge the performance of the proposed algorithm relative to the solution to Problem 1.

Table 1 provides a comprehensive comparison between existing methods and our proposed method in terms of computational complexity and measurement acquisition load. These criteria are assessed using the big $\mathcal{O}$ notation, which is desired to be small in practical applications. The suboptimal solution stands out because it exhibits a minimal measurement load and computational complexity during both the design and runtime phases, albeit with poor performance. Conversely, the optimal solution with a fully connected network ranks least favorably in terms of the measurement load, because it requires establishing network links between every possible pair of agents in the entire MAS. Our proposed method takes advantage of network connection efficiency by relying only on neighboring measurements. This in turn reduces the network overhead, while the performance nearly matches that of a fully connected network MAS.

VI. CONCLUSION

In this paper, a synthesized suboptimal linear distributed control-estimation framework has been proposed to address the non-convex complexity imposed by the MAS network topological constraint. The key innovation of the proposed framework is that the distributed estimator expands the information available to individual agents beyond their neighbors, empowering the distributed control law to realize interactions between non-neighboring agents. The theoretical performance guarantees relative to the global optima of the joint distributed control and estimation problems have been discussed. Furthermore, numerical simulations have demonstrated that the performance of distributed control-estimation synthesis is superior to existing suboptimal solutions within the same partially connected network topology and is comparable to the globally optimal solution within a fully connected network. The proposed work has taken the first step toward a new paradigm on optimal distributed control, opening up many future works including but not limited to:

1) extending the proposed distributed control-estimation framework to an infinite time horizon case for practical use; and
2) applying the proposed distributed control-estimation framework to facilitate complex real-world applications, subject to time-varying network topology, non-linear stochastic dynamics, heterogeneous agents, physical constraints, and process delays.

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