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RESEARCH ARTICLE

Classification of Cipher Text by Clustering of S-Topological Rough Group

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ABSTRACT Rough set theory provides valuable tools for handling and analyzing ciphertext, making it a prominent asset in cryptographic applications. Its ability to manage uncertainty and reduce complexity can enhance various aspects of ciphertext management, from pattern recognition, classification to cryptanalysis and security checks. By imposing the principles of rough sets, cryptographic systems can become more robust, efficient, and secure. The fundamental nature of the symmetric group within the context of rough topological groups makes it a powerful tool in both theoretical and applied mathematics. Some cryptographic protocols and coding theories depend on the properties of topological rough symmetric groups for security and error detection or correction. This paper aims to generalize topological rough group structures and investigate their properties. Additionally, an algorithm is established to classify the symmetric group S_n, and experimental result is provided to explore the effectiveness of the algorithm. It provides practical tools for analyzing imprecise or incomplete data, benefiting fields such as medical diagnostics, economic forecasting, and geographical information systems.

INDEX TERMS Rough set thoery, symmetric group, topological group, topological rough group.

I. INTRODUCTION

Topological rough groups generalize classical group theory, rough set theory, and topology. They incorporate group theory's algebraic operations and rough set theory's lower and upper approximations to handle uncertainty and imprecision in group elements. The topological aspect introduces continuity extending rough groups to topological spaces where the multiplication map and inverse map are continuous. Additionally, mathematical logic and set theory provide the foundational principles, such as equivalence relations and partitions, necessary for rigorously defining these structures. This integration creates a new structure for analyzing systems with inherent uncertainty, enhancing theoretical and practical applications.

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Rough set theory relies on equivalence relations to partition the universe of discourse into sets of objects that are indistinguishable with respect to certain attributes. Rough set theory analyses the uncertainty and vagueness through approximations on sets. The approximations are classified as upper and lower approximations where lower deals with certain objects and upper deals with both certain and uncertain objects [20]. By deriving decision rules, rough set theory captures attribute relationships and object classification. Consequently, it stands as an alternative approach to fuzzy set theory, finding utility across domains like machine learning, data mining, pattern recognition, and expert systems.

Rough topological groups blend topological group theory with rough set theory analysing group structures among incomplete or ambiguous data [2]. This structure introduces rough identity elements, a non-existence from the precise identities found in traditional topological groups. Unlike their standard counterparts, rough topological groups allow for elements that approximate identity with varying degrees of certainty. This novel feature yields unique insights into the group's structure, as elements may only be approximately neutral with respect to group operations [15]. Consequently, the presence of rough identity elements generates fresh outcomes and perspectives, enriching the study of rough topological groups beyond what is observed in conventional topological group theory.

In the study of a non-empty set, often conceptualized as a class of objects, various characteristics or attributes can be defined for these objects. The correlation between these characteristics, based on their belongingness within the set, can be examined. This exploration naturally leads to the definition of a correlation coefficient, quantifying the degree of association between different characteristics [8]. Motivated by this exploration, we aim to extend this concept to define a measure between the class of open sets on a symmetric group. This measure would allow us to analyze the relationships and interactions among open sets of symmetric group. A new type of topological structure on graphs is introduced and analyzed, as described in [19]. The concept of complementary soft neighborhoods is introduced, which is employed to create a model for covering soft rough sets [1].

The proposed structure advances the study of topological rough groups by integrating rough set approximations with classical group and topological group theories. The primary goal of this paper is to generalize the topological rough group structure by incorporating the influence of semi-continuity. Various theoretical results for these generalized topological rough group structures are investigated and illustrated. Additionally, an algorithm for the classification of S_n based on S-topological rough group structure is established, along with a practical application. This enhancement offers a deeper theoretical understanding of uncertain systems and practical applications in fields like medical diagnostics and economic forecasting. For instance, the algorithm aids in identifying hidden patterns in economic data and managing uncertainty in geographical information systems, leading to more accurate predictions and improved urban planning and environmental monitoring.

Federated learning is symmetric and is utilized for skin cancer detection and classification using privacy-aware algorithms [27]. In the realm of Internet traffic classification, the FLIC framework dynamically categorizes packets into applications, achieving 88% accuracy in traffic distribution and scaling to 92% accuracy with increasing client numbers [18]. In healthcare, federated learning enables the effective use of distributed medical data for detecting gastric cancer, ensuring privacy and security [11]. Similarly, in IoT environments, federated learning integrates decentralized data processing, privacy preservation, and scalability, enhancing the intelligence and security of IoT applications [14], [23]. In network systems, federated learning adopts a decentralized approach to optimize operational efficiency while safeguarding data privacy, advancing the development of smarter and more secure network infrastructures [24].

TABLE 1. List of acronyms and symbols used in the paper.

Abbreviations	Definitions
$\mathscr{U}$	Universe
$\mathscr{R}$	Eqivalence relation
$\mathscr{U}/\mathscr{R}$	Eqivalence class
e	Identity element of the rough group.
$\overline{\mathsf{H}}$	Upper Approximation of H
$\mathcal{T}$	Topology on $\overline{\mathsf{H}}$
$\mathcal{T}_{\mathsf{H}}$	Relative topology on H
$Cl(A)$	Closure of A
$Int(A)$	Semi closure of A
$SO(A)$	Set of all semi open subsets of A
$SC(A)$	Set of all semi closed subsets of A
$sCl(A)$	Semi closure of A
$sInt(A)$	Semi interior of A
$\mathcal{M}_C$	Correlation Matrix
$\{e\}^c_{\mathcal{T}}$	Closure of $\{e\}$ in $\mathcal{T}$

II. RELATED WORKS

Polish mathematician Pawlak [20] introduced rough sets in 1982, which are the mathematical theory for representing incomplete and inadequate data. The motive of rough set theory is to use the known imprecise data to approximately deal with the entire problem. Significant advancements and diversification in rough set theory have emerged. Similar to how rough sets have become increasingly important in recent years, they are now integrated with mathematical theories like algebra and topology and are used in diverse domains such as pattern recognition, decision-making, data mining, etc. The algebraic structures of rough sets are investigated by Iwinski [12], Biswas and Nanda [3], Bonikowski [5], and Pomykala et al. [21]. The idea of rough groups and rough subgroups are developed by Biswas and Nanda [3], which merely depend on upper approximation. Miao et. al., [17] introduced and analysed the structure normal subgroups in the context of rough set theory. Few flaws remain in the preliminary rough group definition, which Wu and Huang [25] modified in 2011. Numerous authors have updated the concepts of rough groups, rough subgroups and explored their features. The idea of topological rough group, is an extension of the concept of topological group by adopting Biswas's rough group structure, is presented by N. Bağırmaz et al [2].

Moreover, they investigated the characteristics of topological rough groups with examples. Based on Wu and Huang's updated definition of rough groups, Lin et al. [15] examined the idea of topological rough groups and described some of its topological features and morphisms.

Levine [13] popularized the idea of semi-open sets in 1963 by using closure and interior. The characteristics of semi-topological spaces were researched by Gene Crossley [9]. Maheswari [16], who later coined semi-compactness, examined the characteristics of separation axioms. In 1965, Bohn [4] introduced the semi-topological group. It was later implemented as a s-topological group. Similarly, Bosan et al. [6] introduced the S-topological group in 2014. In his PhD dissertation, Bosan [7] and examined the characteristics of the S-topological group and the s-topological group with essential examples and counterexamples.

The correlation coefficient for fuzzy sets was derived by D. Dumitrescu [8], which influenced us to develop a measure for the subsets of the symmetric group S_n. The correlation coefficient for Atanassov's intuitionistic fuzzy sets was extended by T. Gerstenkorn [10]. Xu et al. [26], who proposed a clustering algorithm for Atanassov's intuitionistic fuzzy sets, P. Singh, who extended the clustering algorithm for picture fuzzy sets, are the sources of inspiration for the idea of defining the clustering algorithm for the set of all permutations of S_n.

III. PROPOSED WORK: GENERALIZATION OF TOPOLOGICAL ROUGH GROUPS

The generalization of topological rough groups using semi-continuity results in the (S,s)-topological rough group structure, whose properties are analyzed with illustrations.

A. (S,s)-TOPOLOGICAL ROUGH GROUPS AND SUBGROUPS

Let $(\mathsf{H}, \circledast)$ be a rough group, $\mathcal{T}$ be a topology on $\overline{\mathsf{H}}$ and $\mathcal{T}_{\mathsf{H}}$ be a topology on H induced by $\mathcal{T}$. Then H is said to be a S-topological rough group (S-TRG) if the product map $\mathfrak{m} : \mathsf{H} \times \mathsf{H} \to \overline{\mathsf{H}}$ such that $\mathfrak{m}(h_1, h_2) = h_1 h_2$ and the inverse map $\mathfrak{i} : \mathsf{H} \to \mathsf{H}$ such that $\mathfrak{i}(h_1) = h_1^{-1}$ are semi continuous. Similarly $(\mathsf{H}, \circledast)$ be a rough group, $\mathcal{T}$ be a topology on $\overline{\mathsf{H}}$ and $\mathcal{T}_{\mathsf{H}}$ be a topology on H induced by $\mathcal{T}$. Then H is said to be s-topological rough group (s-TRG) if $\forall\, h_1, h_2 \in \mathsf{H}$ and $\forall\, \mathsf{D}_3 \in \mathcal{T}$ containing $h_1 h_2^{-1}$, there exist $\mathsf{D}_1, \mathsf{D}_2 \in SO(\mathsf{H})$ containing h_1 and h_2 respectively $\ni$ $\mathsf{D}_1 \circledast \mathsf{D}_2^{-1} \subseteq \mathsf{D}_3$. Throughout the paper $\mathsf{D}_1 \circledast \mathsf{D}_2^{-1} = \mathsf{D}_1\mathsf{D}_2^{-1}$

From above concept its clear that every topological rough group is both s-TRG and S-TRG, as it is evident from the definition. Every s-TRG will be a S-TRG, whereas the illustrations 1(1) and 1(2) demonstrates that the converse is false. From Illustration 1(1) and 1(2) its clear that the belongingness of the identity element of the approximation space in S-TRG does not influence the statement that every S-TRG need not be a s-TRG.

Let H be a S-TRG and K_1 be a rough subgroup of H. Then, K_1 is called a S-topological rough subgroup of H if K_1 is a rough subgroup and the maps $\mathfrak{m}_{\mathsf{K}_1}(h_1, h_2) = h_1 h_2$, $\mathfrak{i}_{\mathsf{K}_1}(h_1) = h_1^{-1}$ are semi-continuous. Similarly H be a s-TRG and K_1 be a rough subgroup of H. Then, K_1 is called a s-topological rough subgroup of H $(\mathsf{K}_1 \leqslant \mathsf{H})$ if for each $h_1, h_2 \in \mathsf{K}_1$ and for each neighborhood $\mathsf{D}_3 \in \mathcal{T}$ containing $h_1 h_2^{-1}$, $\exists\, \mathsf{D}_1, \mathsf{D}_2 \in SO(\mathsf{K}_1)$ containing h_1 and h_2 such that $\mathsf{D}_1\mathsf{D}_2^{-1} \subseteq \mathsf{D}_3$.

Let H be a s-TRG and $\mathsf{N} \leqslant \mathsf{H}$. Then N is called a s-topological rough normal subgroup of H $(\mathsf{N} \trianglelefteq \mathsf{H})$ if $\forall h_1 \in \mathsf{H}$, $h_1 N = N h_1$.

Let H be a rough group then H is said to be a quasi s-topological rough group if for each $h_1, h_2 \in \mathsf{H}$, the map $\mathcal{L}_{h_1} : \mathsf{H} \to \overline{\mathsf{H}}$ defined by $\mathcal{L}_{h_1}(h_2) = h_1 h_2$, the map $\mathcal{R}_{h_1} : \mathsf{H} \to \overline{\mathsf{H}}$ defined by $\mathcal{R}_{h_1}(h_2) = h_2 h_1$ and the inverse map $\mathfrak{i}$ are semi continuous.

Illustration 1:

1) *Let* $\mathscr{U} = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \tilde{5}, \tilde{6}, \tilde{7}\}$ *be the set of congruence classes obtained by integers mod 8 and* $\circledast$ *be the addition mod 8. A classification of* $\mathscr{U}$ *is* $\mathscr{U}/\mathscr{R} = \{F_1, F_2\}$, *where* $F_1 = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}\}$, $F_2 = \{\tilde{4}, \tilde{5}, \tilde{6}, \tilde{7}\}$. *Let* $\mathsf{H} = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{4}, \tilde{6}, \tilde{7}\}$, *then* $\overline{\mathsf{H}} = \mathscr{U}$. *Let* $\mathcal{T} = \{\emptyset, \overline{\mathsf{H}}, \{\tilde{0}\}\}$ *be a topology on* $\overline{\mathsf{H}}$, *then* $\mathcal{T}_{\mathsf{H}} = \{\emptyset, \mathsf{H}, \{\tilde{0}\}\}$. $(\mathsf{H}, \mathcal{T}_{\mathsf{H}})$ *is a S-TRG but not a S-topological group, since* H *is not a group. Not a topological rough group, since* $\mathfrak{m}^{-1}(\{\tilde{0}\})$ *is not open in* $\mathsf{H}\times\mathsf{H}$. *Not a s-TRG, since* $\tilde{1}, \tilde{7} \in \mathsf{H}$ *and* $\{\tilde{0}\} \in \mathcal{T}$ *containing* $\tilde{0}$ *but* $\nexists\, \mathsf{D}_1, \mathsf{D}_2 \in SO(\mathsf{H})$ *containing* $\tilde{1}$ *and* $\tilde{7}$ *such that* $\mathsf{D}_1\mathsf{D}_2 \subset \{\tilde{0}\}$.
2) *Let* $\mathscr{U} = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \tilde{5}, \tilde{6}, \tilde{7}\}$ *be the set of congruence classes obtained by integers mod 8 and* $\circledast$ *be the addition mod 8. A classification of* $\mathscr{U}$ *is* $\mathscr{U}/\mathscr{R} = \{F_1, F_2\}$, *where* $F_1 = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}\}$, $F_2 = \{\tilde{4}, \tilde{5}, \tilde{6}, \tilde{7}\}$. *Let* $\mathsf{H} = \{\tilde{1}, \tilde{2}, \tilde{4}, \tilde{6}, \tilde{7}\}$, *then* $\overline{\mathsf{H}} = \mathscr{U}$. *Let* $\mathcal{T} = \{\emptyset, \overline{\mathsf{H}}, \{\tilde{0}\}, \{\tilde{2}\}, \{\tilde{4}\}, \{\tilde{6}\}, \{\tilde{0}, \tilde{2}\}, \{\tilde{0}, \tilde{4}\}, \{\tilde{0}, \tilde{6}\}, \{\tilde{2}, \tilde{4}\}, \{\tilde{2}, \tilde{6}\}, \{\tilde{4}, \tilde{6}\}, \{\tilde{0}, \tilde{2}, \tilde{4}\}, \{\tilde{0}, \tilde{2}, \tilde{6}\}, \{\tilde{0}, \tilde{4}, \tilde{6}\}, \{\tilde{2}, \tilde{4}, \tilde{6}\}, \{\tilde{0}, \tilde{2}, \tilde{4}, \tilde{6}\}\}$, *then* $\mathcal{T}_{\mathsf{H}} = \{\emptyset, \mathsf{H}, \{\tilde{2}\}, \{\tilde{4}\}, \{\tilde{6}\}, \{\tilde{2}, \tilde{4}\}, \{\tilde{2}, \tilde{6}\}, \{\tilde{4}, \tilde{6}\}, \{\tilde{2}, \tilde{4}, \tilde{6}\}\}$ *be a relative topology on* H. $(\mathsf{H}, \mathcal{T}_{\mathsf{H}})$ *is a S-TRG but not a S-topological group, since* H *is not a group. Not a topological rough group, since* $\mathfrak{m}^{-1}(\{\tilde{0}\})$ *is not open in* $\mathsf{H}\times\mathsf{H}$. *Not a s-TRG, since* $\tilde{1}, \tilde{7} \in \mathsf{H}$ *and* $\{\tilde{0}\} \in \mathcal{T}$ *containing* $\tilde{0}$ *but* $\nexists\, \mathsf{D}_1, \mathsf{D}_2 \in SO(\mathsf{H})$ *containing* $\tilde{1}$ *and* $\tilde{7}$ *such that* $\mathsf{D}_1\mathsf{D}_2 \subset \{0\}$.
3) *Let* $\mathscr{U} = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \tilde{5}, \tilde{6}, \tilde{7}\}$ *be the set of congruence classes obtained by integers mod 8 and* $\circledast$ *be the addition mod 8. A classification of* $\mathscr{U}$ *is* $\mathscr{U}/\mathscr{R} = \{F_1, F_2\}$, *where* $F_1 = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}\}$, $F_2 = \{\tilde{4}, \tilde{5}, \tilde{6}, \tilde{7}\}$. *Let* $\mathsf{H} = \{\tilde{1}, \tilde{3}, \tilde{5}, \tilde{7}\}$, *then* $\overline{\mathsf{H}} = \mathscr{U}$. *Let* $\mathcal{T} = \{\emptyset, \overline{\mathsf{H}}, \{\tilde{3}\}, \{\tilde{5}\}, \{\tilde{3}, \tilde{5}\}, \{\tilde{0}, \tilde{4}, \tilde{6}\}, \{\tilde{0}, \tilde{3}, \tilde{4}, \tilde{6}\}, \{\tilde{0}, \tilde{4}, \tilde{5}, \tilde{6}\}, \{\tilde{0}, \tilde{3}, \tilde{4}, \tilde{5}, \tilde{6}\}\}$ *be a topology on* $\overline{\mathsf{H}}$, *then* $\mathcal{T}_{\mathsf{H}} = \{\emptyset, \mathsf{H}, \{\tilde{3}\}, \{\tilde{5}\}, \{\tilde{3}, \tilde{5}\}\}$. $(\mathsf{H}, \mathcal{T}_{\mathsf{H}})$ *is a quasi s-topological rough group but not a s-TRG, since* $\mathfrak{m}^{-1}(\{\tilde{0}, \tilde{4}, \tilde{6}\}) = \{(\tilde{1}, \tilde{3}), (\tilde{3}, \tilde{1}), (\tilde{5}, \tilde{7}), (\tilde{7}, \tilde{5}), (\tilde{1}, \tilde{7}), (\tilde{7}, \tilde{1}), (\tilde{3}, \tilde{5}), (\tilde{5}, \tilde{3}), (\tilde{1}, \tilde{5}), (\tilde{5}, \tilde{1}), (\tilde{3}, \tilde{3}), (\tilde{7}, \tilde{7})\}$ *which cannot be written as the union of product of semi open subsets of* H.

B. PROPERTIES OF PROPOSED STRUCTURES

Result 1: Let H *be a S-TRG and* $\mathsf{D}_1 \subseteq \mathsf{H}$. *Then* $\mathsf{D}_1 \in SO(\mathsf{H})$ *if and only if* $\mathsf{D}_1^{-1} \in SO(\mathsf{H})$ *and* $\mathsf{H} = \mathsf{H}^{-1}$

Proof: Since $\mathsf{D}_1 \in SO(\mathsf{H})$ $\exists\, \mathsf{D}_2 \in \mathcal{T}_{\mathsf{H}} \ni \mathsf{D}_2 \subset \mathsf{D}_1 \subset Cl(\mathsf{D}_2)$. The conclusion follows since $\mathsf{D}_2^{-1} \subset \mathsf{D}_1^{-1} \subset (Cl(\mathsf{D}_2))^{-1} = Cl((\mathsf{D}_2)^{-1})$ and $\mathsf{H} = \mathsf{H}^{-1}$ follows directly from the definition of S-TRG. □

Let H be a s-TRG and $\forall\, \mathsf{D}_1 \in SO(\mathsf{H})$, $\mathsf{D}_2 \subset \mathsf{H}$, $\mathsf{D}_1\mathsf{D}_2$ need not be in $SO(\mathsf{H})$. In general $\mathsf{D}_1\mathsf{D}_2$ need not be a subset of H.

Illustration 2: Let $\mathscr{U} = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \tilde{5}, \tilde{6}, \tilde{7}\}$ *be the set of congruence classes obtained by integers mod 8 and* $\circledast$ *be the addition mod 8. A classification of* $\mathscr{U}$ *is*

$\mathscr{U}/\mathscr{R} = \{F_1, F_2\}$, where $F_1 = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}\}$, $F_2 = \{\tilde{4}, \tilde{5}, \tilde{6}, \tilde{7}\}$. Let $\mathsf{H} = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{6}, \tilde{7}\}$, then $\overline{\mathsf{H}} = \mathscr{U}$. Let $\mathcal{T} = \{\emptyset, \overline{\mathsf{H}}, \{\tilde{4}\}, \{\tilde{2}, \tilde{4}\}, \{\tilde{4}, \tilde{6}\}, \{\tilde{2}, \tilde{4}, \tilde{6}\}\}$, then $\mathcal{T}_{\mathsf{H}} = \{\emptyset, \mathsf{H}, \{\tilde{2}\}, \{\tilde{6}\}, \{\tilde{2}, \tilde{6}\}\}$. $(\mathsf{H}, \mathcal{T}_{\mathsf{H}})$ is a s-TRG and $\mathsf{D}_1 = \{\tilde{6}\} \in SO(\mathsf{H})$, $\mathsf{D}_2 = \{\tilde{7}\} \subset \mathsf{H}$ but $\mathsf{D}_1\mathsf{D}_2 = \{\tilde{5}\} \not\subset \mathsf{H}$ and there is no semi open neighborhood of identity which is symmetric. Hence H be a s-TRG with e and $\mathsf{D}_3 \subseteq \overline{\mathsf{H}}$ be a neighborhood with $e \in \mathsf{D}_3$. Then there need not exist $\mathsf{S} \in SO(\mathsf{H})$, which is symmetric containing e such that $\mathsf{SS} \subseteq \mathsf{D}_3$.

Result 2: Let H be an extremally disconnected s-TRG containing e and $e \in \mathsf{D}_3 \subseteq \overline{\mathsf{H}}$ be a neighborhood of e. Then there exist $e \in \mathsf{S} \in SO(\mathsf{H})$ such that $\mathsf{S} = \mathsf{S}^{-1}$ and $\mathsf{SS} \subseteq \mathsf{D}_3$.

Proof: Since $\mathfrak{m} : \mathsf{H} \times \mathsf{H} \to \overline{\mathsf{H}}$ is semi continuous, $\mathfrak{m}^{-1}(\mathsf{D}_3) \in SO(\mathsf{H} \times \mathsf{H})$ and $ee = e \in \mathsf{D}_3 \in \mathcal{T}$. Hence, there exist semi open sets $\mathsf{D}_1, \mathsf{D}_2 \in \mathcal{T}_{\mathsf{H}}$ containing e such that $\mathsf{D}_1\mathsf{D}_2 \subseteq \mathsf{D}_3$. By the Result 1, $\mathsf{D}_1^{-1}, \mathsf{D}_2^{-1} \in SO(\mathsf{H})$, hence, $\mathsf{S} = (\mathsf{D}_1 \cap \mathsf{D}_2 \cap \mathsf{D}_1^{-1} \cap \mathsf{D}_2^{-1}) \in SO(\mathsf{H})$, $e \in \mathsf{S}$, $\mathsf{S} = \mathsf{S}^{-1}$ and $\mathsf{SS} \subseteq \mathsf{D}_1\mathsf{D}_2 \subseteq \mathsf{D}_3$. □

Illustration 3: $\mathscr{U} = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \tilde{5}, \tilde{6}, \tilde{7}\}$ be the set of congruence classes obtained by integers mod 8 and ✻ be the addition mod 8. A classification of $\mathscr{U}$ is $\mathscr{U}/\mathscr{R} = \{F_1, F_2\}$, where $F_1 = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}\}$, $F_2 = \{\tilde{4}, \tilde{5}, \tilde{6}, \tilde{7}\}$. Let $\mathsf{H} = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{6}, \tilde{7}\}$, then $\overline{\mathsf{H}} = \mathscr{U}$. Let $\mathcal{T} = \{\emptyset, \overline{\mathsf{H}}, \{\tilde{0}, \tilde{2}\}\}$ be a topology on $\overline{\mathsf{H}}$, then $\mathcal{T}_{\mathsf{H}} = \{\emptyset, \mathsf{H}, \{\tilde{0}, \tilde{2}\}\}$. $(\mathsf{H}, \mathcal{T}_{\mathsf{H}})$ is a S-TRG, its clear that if H is a S-TRG then Result 2 need not be true.

Result 3: Let H be a s-TRG then

1) *The map $\mathcal{L}_{h_1} : \mathsf{H} \to \overline{\mathsf{H}} \ni \mathcal{L}_{h_1}(h_2) = h_1h_2$ is semi continuous and one-to-one, $\forall\, h_2 \in \mathsf{H}$.*
2) *The map $\mathcal{R}_{h_1} : \mathsf{H} \to \overline{\mathsf{H}} \ni \mathcal{R}_{h_1}(h_2) = h_2h_1$ is semi continuous and one-to-one, $\forall\, h_2 \in \mathsf{H}$.*
3) *The map $\mathfrak{i} : \mathsf{H} \to \mathsf{H} \ni \mathfrak{i}(h_2) = h_2^{-1}$ is a semi-homeomorphism, $\forall h_2 \in \mathsf{H}$.*

Proof:

1) For every $h_1, h_2 \in \mathsf{H}$, if $\mathcal{L}_{h_1}(h_2) = \mathcal{L}_{h_1}(h_3)$ then $h_2 = h_3$. Since $h_1 \in \mathsf{H}$, $h_1^{-1} \in \mathsf{H} \subseteq \overline{\mathsf{H}}$. Thus $h_1^{-1}(h_1h_2) = h_1^{-1}(h_1h_3) \implies h_2 = h_3$. Hence $\mathcal{L}_{h_1}$ is one-to-one. For every $h_2 \in \mathsf{H}$, $\mathcal{L}_{h_1}(h_2) = h_1h_2$. Let $h_1h_2 \in \mathsf{D}_3 \in \mathcal{T}$. Then, from the definition of s-topological rough group, there exist $\mathsf{D}_1, \mathsf{D}_2 \in SO(\mathsf{H})$ containing h_1 and h_2 such that $\mathsf{D}_1\mathsf{D}_2 \subseteq \mathsf{D}_3$. Since, $h_1\mathsf{D}_2 \subseteq \mathsf{D}_1\mathsf{D}_2 \subseteq \mathsf{D}_3$, $\mathcal{L}_{h_1}(\mathsf{D}_2) = h_1\mathsf{D}_2 \subseteq \mathsf{D}_3$. Therefore $\mathcal{L}_{h_1}$ is semi-continuous on H.
2) Injectiveness and semi-continuity of $\mathcal{R}_{h_1}$ is similar to the proof of $\mathcal{L}_{h_1}$.
3) From Result 1 the map $\mathfrak{i}$ is irresolute. Pre semi open and bijectiveness of $\mathfrak{i}$ follows from the existence of inverse of rough group.

□ From Illustration 2, its clear that $\mathcal{L}_{\tilde{1}}$ is neither onto nor semi open. Since $\mathcal{L}_{\tilde{1}}(\{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{6}, \tilde{7}\}) = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}, \tilde{7}\}$ is not onto and $\mathcal{L}_{\tilde{1}}(\{\tilde{2}\}) = \{\tilde{3}\} \notin SO(\overline{\mathsf{H}})$.

From Illustration 1(1), its clear that the Result 3 is not true for S-TRG. Since $\{\tilde{0}\}$ is open in $\overline{\mathsf{H}}$ but $\mathcal{L}_{\tilde{1}}^{-1}(\{\tilde{0}\}) = \{\tilde{7}\}$ which is not semi open in H. Thus $\mathcal{L}_{\tilde{1}}$ is not semi-continuous.

Result 4: Let H be a S-TRG and $\mathsf{H}, \{e\} \in \mathcal{T}$. If $e \in \mathsf{H}$, then $\{e\} \in \mathcal{T}_{\mathsf{H}}$.

Proof: Since $e \in \mathsf{H}$ and $\{e\} \in \mathcal{T}$, By the definition of $\mathcal{T}_{\mathsf{H}}$, $\{e\} \in \mathcal{T}_{\mathsf{H}}$. □

Result 5: Let H be a S-TRG(s-TRG). If $\mathsf{H} = \overline{\mathsf{H}}$, then H is a S-topological group(s-topological group).

Proof: Proof is trivial from the definition of topological rough group, S-topological rough group and s-topological rough group. □

Result 6: Let $(\mathsf{H}, \mathcal{T}')$ be a S-topological group and $(\overline{\mathsf{H}}, \mathcal{T})$ be a topological space. Then H is a S-TRG if and only if the topology $\mathcal{T}_1$ and the topology $\mathcal{T}_{\mathsf{H}}$ on H induced by $\mathcal{T}$ are same topologies.

Proof: Proof follows from the definition S-topological rough group and S-topological group. □

Result 7: If H is a s-TRG with $\overline{\mathsf{H}}$ being T_0, then H is semi-T_1.

Proof: It is enough to show that $\forall h_1 \in \mathsf{H}, \{h_1\} \in SC(\mathsf{H})$. Suppose $\exists\, h_1 \in G \ni \{h_1\} \notin SC(\mathsf{H})$, then $\exists\, h_2 \in G\backslash\{h_1\} \ni h_2 \in \{h_1\}^c$. $\mathsf{D}_3 \in SO(\mathsf{H})$ containing h_2, $h_1 \in \mathsf{D}_3$, since the inverse map $\mathfrak{i}$ is a semi-homeomorphism, $h_1^{-1} \in \mathsf{D}_3^{-1} \in SO(\mathsf{H})$ containing h_2^{-1}. Clearly, $h_2^{-1}h_1 \neq e$ and for each $\mathsf{D}_1 \in SO(\mathsf{H} \times \mathsf{H})$ containing (h_2^{-1}, h_1) must contain (h_1^{-1}, h_1). Since $\mathfrak{m} : \mathsf{H} \times \mathsf{H} \to \overline{\mathsf{H}}$ is semi-continuous at (h_2^{-1}, h_1), for each $\mathsf{D}_2 \in \mathcal{T}$ containing $h_2^{-1}h_1$ contains e, hence, $h_2^{-1}h_1 \in \{e\}^c_{\mathcal{T}}$. In addition $\mathfrak{m} : \mathsf{H} \times \mathsf{H} \to \overline{\mathsf{H}}$ is semi-continuous at (h_1^{-1}, h_1), for each $\mathsf{D}_2 \in \mathcal{T}$ containing e contains $h_2^{-1}h_1$, thus, $e \in \{h_2^{-1}h_1\}^c_{\mathcal{T}}$. Hence, $h_2^{-1}h_1 \in \{e\}^c_{\mathcal{T}}$ and $e \in \{h_2^{-1}h_1\}^c_{\mathcal{T}}$. □

Illustration 4:

1) *Let $\mathscr{U} = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \tilde{5}\}$ be the set of congruence classes obtained by integers mod 6 and ✻ be the addition mod 6. A classification of $\mathscr{U}$ is $\mathscr{U}/\mathscr{R} = \{F_1, F_2\}$, where $F_1 = \{\tilde{0}, \tilde{1}, \tilde{2}\}$, $F_2 = \{\tilde{3}, \tilde{4}, \tilde{5}\}$. Let $\mathsf{H} = \{\tilde{2}, \tilde{3}, \tilde{4}\}$, then $\overline{\mathsf{H}} = \mathscr{U}$. Let $\mathcal{T} = \{\emptyset, \overline{\mathsf{H}}, \{\tilde{2}, \tilde{4}\}, \{\tilde{2}, \tilde{3}, \tilde{4}\}\}$ be a topology on $\overline{\mathsf{H}}$, then $\mathcal{T}_{\mathsf{H}} = \{\emptyset, \mathsf{H}, \{\tilde{2}, \tilde{4}\}\}$ which is not semi-T_0.*
2) *From Illustration 1(1) its clear that there exist a semi-T_0 S-TRG $(\mathsf{H}, \mathcal{T}_{\mathsf{H}})$ containing e but $\mathcal{T}_{\mathsf{H}}$ is not semi-T_1.*
3) *Let $\mathscr{U} = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \tilde{5}\}$ be the set of congruence classes obtained by integers mod 6 and ✻ be the addition mod 6. A classification of $\mathscr{U}$ is $\mathscr{U}/\mathscr{R} = \{F_1, F_2\}$, where $F_1 = \{\tilde{0}, \tilde{1}, \tilde{2}\}$, $F_2 = \{\tilde{3}, \tilde{4}, \tilde{5}\}$. Let $\mathsf{H} = \{\tilde{2}, \tilde{3}, \tilde{4}\}$, then $\overline{\mathsf{H}} = U$. Let $\mathcal{T} = \{\emptyset, \overline{\mathsf{H}}, \{\tilde{2}\}, \{\tilde{4}\}, \{\tilde{2}, \tilde{4}\}, \{\tilde{2}, \tilde{3}, \tilde{4}\}\}$ be a topology on $\overline{\mathsf{H}}$, then $\mathcal{T}_{\mathsf{H}} = \{\emptyset, G, \{\tilde{2}\}, \{\tilde{4}\}, \{\tilde{2}, \tilde{4}\}\}$ forms a s-TRG which is both semi-T_1 and semi-T_2 but neither T_1 nor T_2.*

Result 8: Let H be a semi-T_0 s-TRG. If $e \in \mathsf{H}$ then $\{e\} \in SC(\mathsf{H})$.

Proof: Suppose $\{e\} \notin SC(\mathsf{H})$, then $\exists\, h_1 \in \mathsf{H}\backslash\{e\} \ni h_1 \in \{e\}^c$. Since $\mathfrak{i}$ is a semi-homeomorphism, then $h_1^{-1} \in \{e\}^c$. Thus, $e \in \mathsf{D}_1$ and $e \in \mathsf{D}_2\ \forall\ \mathsf{D}_1, \mathsf{D}_2 \in SO(\mathsf{H})$ containing h_1 and h_1^{-1}, respectively. Since $\mathfrak{m} : \mathsf{H} \times \mathsf{H} \to \overline{\mathsf{H}}$ is semi-continuous at (h_1, h_1^{-1}), $\{h_1, h_1^{-1}\} \subset \mathsf{D}_3$ for any $\mathsf{D}_3 \in SO(\mathsf{H})$ containing e. Thus, $\{h_1, h_1^{-1}\} \subset \{e\}^c$, $\exists\, \mathsf{D}_1 \in SO(\mathsf{H})$ containing $e \ni \mathsf{D}_1 \cap \{h_1, h_1^{-1}\} \neq \emptyset$. □

Result 9: Let H *be a semi-*T_0 *s-TRG,* $e \in H$*. Then* $e \notin \{h_1\}^c$, $\forall\, h_1 \in H\backslash\{e\}$.

Proof: Suppose $\exists\, h_1 \in H\backslash\{e\} \ni e \in \{h_1\}^c$, then each $D_1 \in SO(H)$ containing e contains h_1 and h_1^{-1}. Since $\mathfrak{m} : H \times H \to \overline{H}$ is semi-continuous at (e, h_1), each $D_2 \in SO(H)$ must contain e. Hence, $h_1 \in \{e\}^c$, the set $\{e\}$ is not semi closed in H. □

Result 10: Let H *be a semi-*T_0 *s-TRG. If* H *containing e is finite then* $\{e\} \in SO(H)$ *and* $\{e\} \in SC(H)$.

Proof: From Results 8 and 9, if H is a finite s-TRG, then the set $\{e\}$ is both semi open and semi closed in H. □

Result 11: A s-topological rough group H *is semi-*T_2 *whenever* $\{e\}$ *is semi closed in* $\overline{H}$.

Proof: Suppose H is not semi-T_2, then $\exists\, h_1 \neq h_2 \in H$ such that h_1 and h_2 cannot be separated by $D_1, D_2 \in SO(H)$. Clearly, $h_1h_2^{-1} \in H$. Consider $D_3 \in \mathcal{T}$ containing $h_1h_2^{-1}$. Since $\mathfrak{m} : H \times H \to \overline{H}$ is semi-continuous (h_1, h_2^{-1}), $D_1, D_2 \in SO(H)$ containing h_1 and h_2^{-1} respectively, $\ni\; D_1D_2 \subset D_3$. Since $\mathfrak{i}$ is a semi-homeomorphism, $D_2^{-1} \in SO(H)$ containing h_2. Since intersection of D_1 and D_2^{-1} is non empty, $\exists\, h_3 \in D_1 \cap D_2^{-1}$, then, $(h_3, h_3^{-1}) \in D_1 \times D_2$, thus, $e \in D_3$. Hence, $h_1h_2^{-1} \in \{e\}^c_{\mathcal{T}}$. □

Result 12: If H *is a s-TRG with upper approximation of* H *being* T_1*, then* H *is semi-*T_2.

Proof: Since every singleton set is closed in T_1 space. Thus from the Result 11, H is semi-T_2. □

Result 13: Let H *be a semi-*T_1 *extremally disconnected s-TRG. If* $e \in G$*, then* $\forall h_1 \in H\backslash\{e\}\ \exists\, D_1, D_2 \in SO(H)$ *containing e and* h_1 *respectively, such that intersection of* D_1 *and* D_2 *is empty.*

Proof: Assume that $\exists\, h_1 \in H\backslash\{e\} \ni D_1 \cap D_2$ is non empty $\forall\, D_1, D_2 \in SO(H)$ containing e and h_1 respectively, D_1 is symmetric. Then, $B \cap C \neq \emptyset\ \forall\, D_1, D_2 \in SO(H \times H)$ containing (e, h_1) with the set $\{(h_2, h_2^{-1}) : h_2 \in H\}$. Since the map $\mathfrak{m} : H \times H \to \overline{H}$ is semi continuous at (e, h_1), $\forall\, D_3 \in SO(H)$ containing h_1, $e \in D_3$, $h_1 \in \{e\}^c$. Thus, $\{e\}$ is not closed in H, which is a contradiction to the assumption that H is T_1. □

Result 14: Let H *be a s-TRG, the base at e of* $\overline{H}$ *be* $\mathfrak{B}_e$ *and* $e \in H$*. Then,*

1) $\forall\, D_1 \in \mathfrak{B}_e, \exists\, D_2 \in SO(H)$ *containing* $e \ni D_2^{-1} \subset D_1$
2) $\forall\, D_1 \in \mathfrak{B}_e, \exists\, D_2 \in SO(H)$ *containing* $e \ni D_2h_1 \subset D_1$*, and* $h_1D_2 \subset D_1$*, for each* $h_1 \in H$
3) $\forall\, D_1 \in \mathfrak{B}_e, \exists\, D_2 \in SO(H, e) \ni h_1D_2h_1^{-1} \subset D_1$
4) $\forall\, D_1, D_2 \in \mathfrak{B}_e, \exists\, D_3 \in SO(H, e) \ni D_3 \subset D_1 \cap D_2$

Proof:

1) Let $D_1 \in \mathfrak{B}_e$, H is a s-TRG, then $D_2 \in SO(H)$ containing $e \ni \mathfrak{i}(D_2) = D_2^{-1} \subset D_1$ since $\mathfrak{i}$ is semi continuous.
2) Proof directly follows from the semi-continuity of R_{h_1}.
3) Proof is similar to statement (2).
4) It is trivial from the definition of base and s-TRG.

□

Result 15: Let H *be a s-TRG. Then* H *is a discrete space if* $\{e\} \in \mathcal{T}$.

Proof: It is enough to show that $\{h_2\}$ is open in $\overline{H}$ for any $h_2 \in H$. Consider $h_2 \in H$. Since $\mathfrak{m} : H \times H \to \overline{H}$ is semi-continuous at (h_2, h_2^{-1}), $\exists\, h_2 \in D_2 \in SO(H)\ \ni D_2D_2^{-1} \subset \{e\}$, then $D_2D_2^{-1} = \{e\}$, since intersection of D_1 and D_2^{-1} is non empty. Since the rough inverse of elements of H is unique, $D_2 = \{h_1\} \in SO(G)$. Since any singleton semi open set is open, H is discrete. □

For any S-TRG, Result 15 need not be true, shown in illustration 1(1).

Result 16: Let H *be a s-TRG. If every rough subgroup* K_1 *and its approximation space* $\overline{K_1}$ *are open then* $K_1 \leqslant H$.

Proof: Since $\overline{K_1}$ is open in $\overline{H}$, D_3 is an open subset of $\overline{H}$. Since H is a s-topological group there are semi open neighbourhoods B of h_1 and C of h_2 such that $BC^{-1} \subset D_3$. Since K_1 is open, the sets $D_1 = B \cap K_1$ and $D_2 = C \cap K_1$ are semi open subsets of K_1. Also, $D_1D_2^{-1} \subset BC^{-1} \subset D_3$, K_1 is a s-TRG. □

In general every s-topological rough subgroup need not be open. In Illustration 2 consider $K_1 = \{\tilde{1}\}$, $\overline{K_1} = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}\}$, $K_1 \leqslant H$ but K_1 and $\overline{K_1}$ is not open in H and $\overline{H}$ respectively. If the approximation space $\overline{H}$ is trivial, then any rough subgroup K_2 of H, $K_2 \leqslant H$.

$\mathscr{U} = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \tilde{5}, \tilde{6}, \tilde{7}, \tilde{8}\}$ be the set of congruence classes obtained by integers mod 9 and $\circledast$ be the addition mod 9. A classification of $\mathscr{U}$ is $\mathscr{U}/\mathscr{R} = \{F_1, F_2, F_3\}$, where $F_1 = \{\tilde{0}, \tilde{1}, \tilde{2}\}$, $F_2 = \{\tilde{3}, \tilde{4}, \tilde{5}\}$ $F_3 = \{\tilde{6}, \tilde{7}, \tilde{8}\}$. $K_1 = \{\tilde{0}, \tilde{2}, \tilde{5}, \tilde{4}, \tilde{7}\}$, $K_2 = \{\tilde{2}, \tilde{3}, \tilde{6}, \tilde{7}\}$, then K_1, K_2 are rough groups but $K_1 \cap K_2$ is not a rough group since $\overline{K_1 \cap K_2} \neq \overline{K_1 \cap K}$.

Result 17: Let H *be a s-TRG and* $K_1, K_2 \leqslant H$*. Then* $K_1 \cap K_2$ *is a s-topological rough subgroup if* $\overline{K_1} \cap \overline{K_2} = \overline{K_1 \cap K_2}$ *and* $K_1 \cap K_2$ *is open in* H*,* $\overline{K_1 \cap K_2}$ *is open in* $\overline{H}$.

Proof: Suppose $K_1, K_2 \leqslant H$. It is obvious that $K_1 \cap K_2 \subset H$. Consider $h_1, h_2 \in K_1 \cap K_2$. Since K_1 and K_2 are rough subgroups, $h_1h_2 \in \overline{K_1}$, $h_1h_2 \in \overline{K_2}$, and $h_1^{-1} \in \overline{K_1}$, $h_1^{-1} \in \overline{K_2}$, i.e. $h_1h_2 \in \overline{K_1} \cap \overline{K_2}$ and $h_1^{-1} \in \overline{K_1} \cap \overline{K_2}$. By hypothesis $h_1h_2 \in \overline{K_1 \cap K_2}$ and $h_1^{-1} \in \overline{K_1 \cap K_2}$. Hence by Result 16, $K_1 \cap K_2 \leqslant H$. □

Result 18: Let H *be a s-TRG and* $K_2 \leqslant H$*,* $K_1 \subset H$*,* $K_1 \neq \emptyset$*. If* $K_1 \leqslant K_2$ *and* K_1 *is open in* H*, then* $K_1 \leqslant H$.

Proof: By the Definition of rough subgroup and Result 16, $K_1 \leqslant H$ follows. □

Illustration 5: Let $n = \{1, 2, 3, 4\}$*,* $\mathscr{U}$ *be the set of all bijective function on n and* $\circledast$ *be the composition of elements of* $\mathscr{U}$*. A classification of* $\mathscr{U}$ *is* $\mathscr{U}/\mathscr{R} = \{F_1, F_2, F_3, F_4\}$*, where* $F_1 = \big\{(1), (12), (13), (14), (23), (24), (34)\big\}$*,* $F_2 = \big\{(123), (132), (124), (142), (134), (143), (234), (243)\big\}$*,* $F_3 = \big\{(1234), (1243), (1324), (1342), (1423), (1432)\big\}$*,* $F_4 = \big\{(12)(34), (13)(24), (14)(23)\big\}$*. Let* $K_1 = \big\{(12)\big\}$*,* $K_2 = \big\{(13)\big\}$ *then* K_1*,* K_2 *are rough groups but* K_1K_2 *is not a rough group since* $\overline{K_1K_2} \neq \overline{K_1}\ \overline{K_2}$ *and* $K_1K_2 \neq K_2K_1$*. Similarly, Let* $K_1 = \big\{(12), (123), (132)\big\}$*,* $K_2 = \big\{(34), (1234), (1432)\big\}$*, then* K_1*,* K_2 *are rough groups and* $\overline{K_1K_2} = \overline{K_1}\ \overline{K_2}$ *but* K_1K_2 *is not a rough group since* $K_1K_2 \neq K_2K_1$.

Result 19: Let H *be a s-TRG and* $K_1, K_2 \leqslant H$ *such that the product* $\overline{K_1K_2}$*,* K_1K_2 *are open and* $\overline{K_1}\ \overline{K_2} = \overline{K_1K_2}$*. Then* $K_1K_2 \leqslant H$ *if and only if* $K_1K_2 = K_2K_1$.

Proof: The proof of the result follows directly from the definition of topological rough groups and Result 16. □

In Result 19 if H is abelian then product of s-topological rough subgroups K_1 and K_2 are s-topological rough subgroup for $\overline{\mathsf{K}_1\mathsf{K}_2}$, $\mathsf{K}_1\mathsf{K}_2$ open and $\overline{\mathsf{K}_1}\ \overline{\mathsf{K}_2} = \overline{\mathsf{K}_1\mathsf{K}_2}$.

Result 20: Let H *be a s-TRG,* $\mathsf{K}_1, \mathsf{K}_2 \trianglelefteq \mathsf{H}$ *and* $\overline{\mathsf{K}_1}\ \overline{\mathsf{K}_2} = \overline{\mathsf{K}_1\mathsf{K}_2}$, $\mathsf{K}_1\mathsf{K}_2$ *is open. Then* $\mathsf{K}_1\mathsf{K}_2 \trianglelefteq \mathsf{H}$.

Proof: From Result 19, $\mathsf{K}_1\mathsf{K}_2$ is a s-topological rough subgroup of H. Its enough to show that $\mathsf{K}_1\mathsf{K}_2$ is normal. Let h_1 be any elements of H. Since $h_1(\mathsf{K}_1\mathsf{K}_2) = (h_1\mathsf{K}_1)\mathsf{K}_2 = (\mathsf{K}_1 h_1)\mathsf{K}_2 = \mathsf{K}_1(h_1\mathsf{K}_2) = \mathsf{K}_1(\mathsf{K}_2 h_1) = (\mathsf{K}_1\mathsf{K}_2)h_1$, then from the definition of s-topological rough normal subgroup, $\mathsf{K}_1\mathsf{K}_2$ is a s-topological rough normal subgroup of H. □

Result 21: The following results are true

1) *For any quasi s-topological rough group* H. *If* $\mathsf{B} \in \mathcal{T}_{\mathsf{H}}$, *then* $\mathsf{B}^{-1} \in SO(\mathsf{H})$.
2) *Every s-TRG is a quasi s-topological rough group.*
3) *Let* H *be a quasi s-topological rough group,* $\mathsf{B} \subset \mathsf{H}$. *Then* $(sCl(\mathsf{B}))^{-1} \subseteq Cl(\mathsf{B}^{-1})$
4) *Let* H *be a s-TRG,* $\mathsf{B}, \mathsf{C} \subset \mathsf{H}$. *Then* $sCl(\mathsf{B})\ sCl(\mathsf{C}) \subseteq Cl(\mathsf{BC})$ *and* $(sCl(\mathsf{B}))^{-1} \subset Cl(\mathsf{B}^{-1})$.
5) *Let* H *be a s-TRG,* $\mathsf{K}_1 \neq \emptyset$, $\mathsf{K}_1 \leqslant \mathsf{H}$ *is semi open if and only if semi interior of* K_1 *is non-empty.*
6) *Let* H *be a S-TRG and* $\mathsf{D}_1 \in SO(\mathsf{H})$. *Then* $\mathsf{L} = \bigcup_{n=1}^{\infty} \mathsf{D}_1^n \in SO(\mathsf{H})$ *if* $\mathsf{D}_1^n \in SO(\mathsf{H})\ \forall n \in \mathbb{N}$.
7) *Let* C *be any subset of a S-TRG* H. *Then* $(sInt(\mathsf{C}))^{-1} = sInt(\mathsf{C}^{-1})$.
8) *Let* H *be a s-TRG,* $\mathsf{D}_1, \mathsf{D}_2 \in SO(\mathsf{H})$ *containing e such that* $\mathsf{D}_2^4 \subset \mathsf{D}_1$ *and* $\mathsf{D}_2^{-1} = \mathsf{D}_2$. *If a subset* B *of* H *is* D_1*-semi disjoint, then* $C_s = \{a\mathsf{D}_2 : a \in \mathsf{B}\}$, $C_s \subset SO(\mathsf{H})$ *is semi discrete in* H.

Result 22: Let e be a neutral element in $\overline{\mathsf{H}}$, H *be a s-TRG and* $\mathfrak{B}_e$ *be a base of* $(\overline{\mathsf{H}}, \mathcal{T})$ *at e,*

1) *If* $e \in \mathsf{D}_1 \in SO(\mathsf{H})$, *then* $\mathsf{D}_1 \subset sCl(\mathsf{D}_1) \subset \mathsf{D}_1^2$ *need not be true.*
2) *For each* $\mathsf{C} \subset \mathsf{H}$ *and* $\mathsf{D}_1 \in \mathcal{T}$ *of e,* $sCl(\mathsf{C}) \subseteq \mathsf{CD}_1$ *need not be true.*
3) *For each* $\mathsf{C} \subset \mathsf{H}$, $sCl(\mathsf{C}) = \cap\{\mathsf{CD}_1 : \mathsf{D}_1 \in \mathfrak{B}_e\}$ *need not be true.*

whereas all these statements are true if H *is a s-topological group.*

Illustration 6:

1) *In Illustration 1(3), consider* $\mathsf{B} = \{\tilde{1}, \tilde{5}\}$ *then* $(sCl(\mathsf{B}))^{-1} = \{\tilde{3}, \tilde{7}\}$ *and* $Cl(\mathsf{B}^{-1}) = \{\tilde{1}, \tilde{3}, \tilde{7}\}$ *thus* $(sCl(\mathsf{B}))^{-1} \subset Cl(\mathsf{B}^{-1})$.
2) *Let* $\mathscr{U} = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \tilde{5}\}$ *be the set of congruence classes obtained by integers mod 8 and* $\circledast$ *be the addition mod 8. A classification of* $\mathscr{U}$ *is* $\mathscr{U}/\mathscr{R} = \{F_1, F_2\}$, *where* $F_1 = \{\tilde{0}, \tilde{1}, \tilde{2}\}$, $F_2 = \{\tilde{3}, \tilde{4}, \tilde{5}\}$. *Let* $\mathsf{H} = \{\tilde{0}, \tilde{1}, \tilde{5}\}$, *then* $\overline{\mathsf{H}} = \mathscr{U}$. *Let* $\mathcal{T} = \{\emptyset, \overline{\mathsf{H}}, \{\tilde{3}\}, \{\tilde{1}, \tilde{2}, \tilde{3}\}, \{\tilde{3}, \tilde{4}, \tilde{5}\}, \{\tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \tilde{5}\}\}$ *be a topology on* $\overline{\mathsf{H}}$, *then* $\mathcal{T}_{\mathsf{H}} = \{\emptyset, \mathsf{H}, \{\tilde{1}\}, \{\tilde{5}\}, \{\tilde{1}, \tilde{5}\}\}$. $(\mathsf{H}, \mathcal{T}_{\mathsf{H}})$ *forms a s-TRG. Let* $\mathsf{B} = \{\tilde{0}, \tilde{1}\}$, $\mathsf{C} = \{\tilde{1}, \tilde{5}\}$, *Result 21(4) is true.*
3) *If* H *is a S-TRG then the Result 21(4) is not true. In Illustration 1(1), consider* $\mathsf{B} = \{\tilde{0}, \tilde{2}\}$, $\mathsf{C} = \{\tilde{1}, \tilde{5}\}$, *then* $sCl(\mathsf{B})\ sCl(\mathsf{C}) \nsubseteq Cl(\mathsf{BC})$.
4) *Let* $\mathscr{U} = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \tilde{5}, \tilde{6}, \tilde{7}\}$ *be the set of congruence classes obtained by integers mod 8 and* $\circledast$ *be the addition mod 8. A classification of* $\mathscr{U}$ *is* $\mathscr{U}/\mathscr{R} = \{F_1, F_2\}$, *where* $F_1 = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}\}$, $F_2 = \{\tilde{4}, \tilde{5}, \tilde{6}, \tilde{7}\}$. *Let* $\mathsf{H} = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{4}, \tilde{6}, \tilde{7}\}$, *then* $\overline{\mathsf{H}} = \mathscr{U}$. *Let* $\mathcal{T} = \{\emptyset, \overline{\mathsf{H}}, \{\tilde{0}\}, \{\tilde{0}, \tilde{2}\}, \{\tilde{0}, \tilde{4}\}, \{\tilde{0}, \tilde{5}, \tilde{6}, \tilde{7}\}, \{\tilde{0}, \tilde{2}, \tilde{4}\}, \{\tilde{0}, \tilde{2}, \tilde{5}, \tilde{6}, \tilde{7}\}, \{\tilde{0}, \tilde{4}, \tilde{5}, \tilde{6}, \tilde{7}\}, \{\tilde{0}, \tilde{2}, \tilde{4}, \tilde{5}, \tilde{6}, \tilde{7}\}\}$ *be a topology on* $\overline{\mathsf{H}}$, *then* $\mathcal{T}_{\mathsf{H}} = \{\emptyset, \mathsf{H}, \{\tilde{0}\}, \{\tilde{0}, \tilde{2}\}, \{\tilde{0}, \tilde{4}\}, \{\tilde{0}, \tilde{2}, \tilde{4}\}, \{\tilde{0}, \tilde{6}, \tilde{7}\}, \{\tilde{0}, \tilde{2}, \tilde{6}, \tilde{7}\}, \{\tilde{0}, \tilde{4}, \tilde{6}, \tilde{7}\}, \{\tilde{0}, \tilde{2}, \tilde{4}, \tilde{6}, \tilde{7}\}\}$. $(\mathsf{H}, \mathcal{T}_{\mathsf{H}})$ *is a s-TRG but the statements are not true. Since* $\mathsf{D}_1 = \{\tilde{0}, \tilde{2}\}$ *be a semi open neighborhood but* $\mathsf{D}_1 \subseteq sCl(\mathsf{D}_1) \nsubseteq \mathsf{D}_1^2$. *Let* $\mathsf{C} = \{\tilde{0}, \tilde{2}\}$, $\mathsf{D}_1 = \{\tilde{0}\}$ *then* $sCl(\mathsf{C}) \nsubseteq \mathsf{CD}_1$.
5) *From Illustration 2, Result 21(5) is clear, since* $\forall \mathsf{D}_1 \in SO(\mathsf{H})$, $sInt(\mathsf{D}_1) \neq \emptyset$.
6) *Let* $\mathscr{U} = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}\}$ *be the set of congruence classes obtained by integers mod 4 and* $\circledast$ *be the addition mod 4. A classification of* $\mathscr{U}$ *is* $\mathscr{U}/\mathscr{R} = \{F_1, F_2\}$, *where* $F_1 = \{\tilde{0}, \tilde{1}\}$, $F_2 = \{\tilde{2}, \tilde{3}\}$. *Let* $\mathsf{H} = \{\tilde{0}, \tilde{1}, \tilde{3}\}$, *then* $\overline{\mathsf{H}} = \mathscr{U}$. *Let* $\mathcal{T} = \{\emptyset, \overline{\mathsf{H}}, \{\tilde{2}\}, \{\tilde{2}, \tilde{3}\}, \{\tilde{1}, \tilde{2}\}, \{\tilde{1}, \tilde{2}, \tilde{3}\}\}$ *be a topology on* $\overline{\mathsf{H}}$, *then* $\mathcal{T}_{\mathsf{H}} = \{\emptyset, \mathsf{H}, \{\tilde{1}\}, \{\tilde{3}\}, \{\tilde{1}, \tilde{3}\}\}$ *is a relative topology on* H *forms a s-TRG. Let* $D_1 = \{1\}$, $D_1^2 \not\subset \mathsf{H}$. *Thus* $\bigcup_{n=1}^{\infty} \mathsf{D}_1^n \notin SO(\mathsf{H})$.
7) *In Illustration 1(2), arbitrary union of semi open subsets of* H *is again a semi open subset of* H.
8) *In Illustration 6(6), consider* $\mathsf{C} = \{\tilde{0}, \tilde{3}\}$, *then* $(sInt(\mathsf{C}))^{-1} = \{\tilde{0}, \tilde{5}\} = sInt(\mathsf{C}^{-1})$.
9) *In Illustration 1(2), consider* $\mathsf{C} = \{\tilde{1}, \tilde{4}, \tilde{6}\}$, *then* $(sInt(\mathsf{C}))^{-1} = \{\tilde{2}, \tilde{4}, \tilde{7}\} = sInt((\mathsf{C})^{-1})$.
10) *From Illustration 6(4), its clear that the closure of any symmetric subset of s-TRG* H, *need not be symmetric. Consider a subset* $\mathsf{D}_1 = \{\tilde{4}\}$, $Cl(\mathsf{D}_1) = \{\tilde{1}, \tilde{4}\}$ *is not symmetric but* D_1 *is symmetric whereas this statement is true for S-topological group.*
11) *In Illustration 6(4) let* $\mathsf{D}_1 = \{\tilde{0}, \tilde{2}, \tilde{4}\}, \mathsf{D}_2 = \{\tilde{0}, \tilde{4}\}$, $\mathsf{B} = \{\tilde{4}, \tilde{7}\}$. *Then* $C_s = \{\{\tilde{2}, \tilde{6}\}, \{3, \tilde{7}\}\}$ *is semi discrete.*

IV. ALGORITHM FOR CLASSIFICATION OF S_N ON S-TOPOLOGICAL ROUGH GROUP STRUCTURE

A measure on the subsets of symmetric group have been defined by using the characteristics of the subset based on belongingness within the alternating group and an algorithm is proposed for categorizing the similarity of open subsets of S-TRG on S_n. In addition the implementation of the proposed algorithm is provided at the end of the section.

A. MEASURE ON SYMMETRIC GROUP

Measure $Q(D_1, D_2)$ have been defined for subsets of symmetric group S_n in this subsection through correlation.

Let S_n be a symmetric group of n elements and A_n be a alternating group and $a, b \in S_n$, $a \circ b = ab = c \in S_n$

$$P(ab) = P(c) = \begin{cases} 0, & \text{if } c \in A_n \\ 1, & \text{otherwise} \end{cases} \tag{1}$$

$D_1 = \{a_1, a_2, \cdots a_p\}, D_2 = \{b_1, b_2, \cdots, b_q\}, a_i, b_j \in S_n, i \in 1, 2, \cdots, p, j \in 1, 2, \cdots, q$.

$$Q(D_1, D_2) = \begin{cases} \dfrac{\sum_{j=1}^{q}\sum_{i=1}^{p}(P(a_i b_j))}{pq}, & \text{if } D_1, D_2 \neq \emptyset \\ \dfrac{\sum_{i=1}^{p}(P(a_i))}{p}, & \text{if } D_1 \neq \emptyset, D_2 = \emptyset \\ \dfrac{\sum_{j=1}^{q}(P(b_j))}{q}, & \text{if } D_1 = \emptyset, D_2 \neq \emptyset \\ 0, & \text{otherwise} \end{cases} \quad (2)$$

Let $D_j (j = 1, 2, \cdots, q)$ be m open subsets of S-TRG, and $C = (Q_{ij})_{p\times p}$ be a correlation matrix, where $Q_{ij} = Q(D_i, D_j)$ denotes the measure of two open sets D_i and D_j and satisfies:

1) $0 \leq Q_{ij} \leq 1$;
2) $Q_{ij} = Q_{ji}$.

B. ALGORITHM

The following algorithm helps to categorize the similarity of open subsets of S-TRG on S_n based on their degree of belongingness. For this, we follow the following steps, see the below diagram given in Figure 1.

Step 1 (Compute Equivalence Classes and Construct Topology): Input the finite universe $\mathscr{U} = S_n$. Compute the equivalence classes of $\mathscr{U}$, $\mathscr{U}/\mathscr{R} = \{F_1, F_2, F_3, F_4\}$. Consider the rough group $\mathsf{H} \subset \mathscr{U}$ such that $\overline{\mathsf{H}} = \mathscr{U}$. Construct the topology $\mathcal{T}$ of $\overline{\mathsf{H}}$ such H forms a S-TRG.

For each element in the universe $\mathscr{U}$ (with n elements), the equivalence relation $\mathscr{R}$ is computed once, resulting in a linear time complexity of $O(n)$. Computing the equivalence classes therefore requires $O(n)$ time. Constructing the topology involves iterating through all equivalence classes to check for non-empty intersections with H. In the worst-case scenario, this requires examining all equivalence classes and their intersections, which can be quadratic in terms of the number of elements if there are numerous equivalence classes. Therefore, the time complexity of Algorithm 1 and 2 is $O(n^2)$.

Step 2 (Calculate Measure and Construct Correlation Matrix): Let $\{D_1, D_2, \cdots, D_q\}$ be a open subsets of S-TRG in Using Equation (1) and Equation (2), calculate the measure of subsets of S-TRG, and then construct a correlation matrix $C = (Q_{ij})_{q\times q}$, where $Q_{ij} = Q(D_i, D_j)$ by using the definition of measure.

The measure function is assumed to be a constant time operation $O(1)$ for simplicity. However, if the measure calculation is complex, its time complexity should be considered. Constructing the correlation matrix involves iterating over all pairs of open sets D_i and D_j, where q is the number of open sets. For each pair, a measure is calculated, which leads to a quadratic time complexity with respect to the number of open sets. Therefore, the time complexity of Algorithm 3 and 4 is $O(q^2)$.

Algorithm 1 Compute Equivalence Classes Based on Transpositions

```
Functiontransposition_class(perm, n)
Input: A permutation perm of length n
Output: The equivalence class based on transpositions

perm ← list(perm)
count ← 0
for i ← 0 to len(perm) - 1 do
  for j ← i + 1 to len(perm) do
    if perm[i] > perm[j] then
      count ← count+ 1
    end if
  end for
end for
if count = 0 or count = n - 1 then
  return 'Class 0/(n-1)'
else
  return 'Class ' + count
end if

Function compute_equivalence_classes(U, R)
Input: A list U and a relation R
Output: List of equivalence classes

equivalence_classes ← empty dictionary
for each element in U do
  class_repr ← R(element)
  if class_repr not in equivalence_classes then
    equivalence_classes[class_repr] ← empty list
  end if
  equivalence_classes[class_repr].append(element)
end for
return list(equivalence_classes.values())

Main Procedure
n ← 5
elements ← list(permutations(range(1, n + 1)))
relation ← lambda perm: transposition_class(perm, n)
equivalence_classes ← compute_equivalence_classes(elements, relation)
print("Equivalence Classes:", equivalence_classes)
```

Step 3 (Check Transitive Closure): Check whether $\mathcal{M}_C^2 = \mathcal{M}_C$, where $\mathcal{M}_C^2 = \mathcal{M}_C \circ \mathcal{M}_C = (\overline{Q}_{ij})_{m\times m} = max_n\left\{min\{Q_{in}, Q_{nj}\}\right\} = Q_{ij}$ where $i, j = 1, 2, \cdots, m$. Construct the equivalent correlation matrix if it is false. $\mathcal{M}_C^{2^k}$: $\mathcal{M}_C \to \mathcal{M}_C^2 \to \mathcal{M}_C^4 \to \cdots \to \mathcal{M}_C^{2^k} \to \cdots$, until $\mathcal{M}_C^{2^k} = \mathcal{M}_C^{2^{k+1}}$. The transitive closure algorithm involves iterating over all pairs of matrix elements and updating them based on the maximum of minimum values. This operation is performed in a nested loop structure over the matrix, resulting in $O(q^3)$ time complexity where q is the size of the matrix

Algorithm 2 Check If a Collection of Subsets Forms a Topology

```
Input: Set U (universe), List of sets S (subsets)
Output: True if S forms a topology on U, False otherwise
Check for Empty Set and Universal Set:
if {} ∉ S or U ∉ S then
    Return False
end if
Check for Closure Under Arbitrary Unions:
for each set A in S do
    for each set B in S do
        if A ∪ B ∉ S then
            Return False
        end if
    end for
end for
Check for Closure Under Finite Intersections:
for each set A in S do
    for each set B in S do
        if A ∩ B ∉ S then
            Return False
        end if
    end for
end for
Return True
```

Algorithm 3 Calculate Measure

```
Input: Open sets D1, D2
Output: Measure between D1 and D2
Implement measure calculation using given equations
```

Algorithm 4 Construct Correlation Matrix

```
Input: Set of open subsets D_set
Output: Correlation matrix C
q ← length(D_set)
C ← [[0] * q for in range(q)]
for i in range(q) do
    for j in range(q) do
        C[i][j] ← calculate_measure(D_set[i], D_set[j])
    end for
end for
return C
```

Algorithm 5 Transitive Closure

```
Input: Correlation matrix C
Output: Transitive closure matrix C_closure
transitive ← False
while not transitive do
    transitive ← True
    new_C ← [[0] * q for in range(q)]
    for i in range(q) do
        for j in range(q) do
            max_min ← max(min(C[i][n], C[n][j]) for n in range(q))
            if new_C[i][j] not equal max_min then
                transitive ← False
                new_C[i][j] ← max_min
            end if
        end for
    end for
    C ← new_C
end while
return C
```

Algorithm 6 Alpha-Cutting Matrix

```
Input: Correlation matrix C, confidence level α
Output: Alpha-cutting matrix M_Cα
q ← length(C)
M_Cα ← [[0] * q for in range(q)]
for i in range(q) do
    for j in range(q) do
        M_Cα[i][j] ← 1 if C[i][j] ≥ α else 0
    end for
end for
return M_Cα
```

Algorithm 7 Categorize Open Sets

```
Input: Alpha-cutting matrix M_Cα
Output: Categories of open sets
categories ← {}
q ← length(M_Cα)
for i in range(q) do
    for j in range(i+1, q) do
        if M_Cα[i] == M_Cα[j] then
            if i in categories then
                categories[i].append(j)
            else
                categories[i] ← [j]
            end if
        end if
    end for
end for
return categories
```

(number of open sets). Therefore, the time complexity of Algorithm 5 is $O(q^3)$.

Step 4 (Construct α-Cutting Matrix and Categorize Open Sets): In order to categorize the open sets $D_j (j = 1, 2, \cdots, q)$ construct a α - cutting matrix $\mathcal{M}_{C_\alpha} = ({}_\alpha Q_{ij})_{m\times m}$ by the defintion of α-cutting matrix for confidence level α, the open sets D_i and D_j are of the same type if all entries of the ith column in $\mathcal{M}_{C_\alpha}$ are identical to the corresponding entries of the jth column.

Creating the α-cutting matrix involves checking each element of the correlation matrix C and comparing it to α, which is a quadratic operation in terms of the number of open sets. Similarly, categorizing involves comparing each pair of rows in the α-cutting matrix to check if they are identical. This operation is quadratic with respect to the number of open sets. Hence, the time complexity of Algorithm 6 and 7 is $O(q^2)$.

Algorithm 8 Main Function

Input: Universe U, rough group H, equivalence relation R, confidence level α
Output: Results of all steps
$T \leftarrow$ construct_topology(U, H, R)
Print "Topology T:", T
$D_{set} \leftarrow$ list(T)
$C \leftarrow$ construct_correlation_matrix(D_{set})
Print "Correlation Matrix C:", C
$C_{closure} \leftarrow$ transitive_closure(C)
Print "Transitive Closure Matrix $C_{closure}$:", $C_{closure}$
$M_{C_\alpha} \leftarrow$ alpha_cutting_matrix$(C_{closure}, \alpha)$
Print "Alpha-Cutting Matrix M_{C_α}:", M_{C_α}
categories $\leftarrow$ categorize_open_sets(M_{C_α})
Print "Categories of Open Sets:", *categories*

Step 5 (Final Computation):

The main function integrates all the steps of the algorithm. Its overall time complexity is determined by the most computationally expensive steps involved:

- Computing equivalence classes and constructing the topology: $O(n^2)$
- Constructing the correlation matrix: $O(q^2)$
- Computing the transitive closure: $O(q^3)$
- Creating the α-cutting matrix and categorizing open sets: $O(q^2)$

Since the transitive closure step has the highest time complexity, the overall time complexity of the main function is $O(q^3)$, where q is the number of open sets.

The following experimental result provides the application of the proposed algorithm.

C. EXPERIMENTAL RESULT OF PROPOSED ALGORITHM ON S_N

For a practical illustration, consider a cipher text in which the encrypted text is obtained by transforming the vowels by some other vowels in plain text. Any form of cipher text with above features is equivalent to the subset of the set of permutations of S_5. Therefore the above mentioned algorithm can be applied to the S-TRG on the permutations S_5 under the composition operation. Let $n = \{1, 2, 3, 4, 5\}$, $\mathscr{U}$ be the set of all bijective function on n and $\circledast$ be the composition of elements of $\mathscr{U}$. The classification of $\mathscr{U}$ is $\mathscr{U}/\mathscr{R} = \{F_1, F_2, F_3, F_4\}, F_1$ is the set of permutations with either number of transposition 4 or 0, F_2 is the set of permutations with either number of transposition 3, F_3 is the set of permutations with either number of transposition 2, F_4 = set of permutations with either number of transposition 1, $\mathsf{H} = \{(145)(23), (154)(23), (123), (132), (24), (12345), (15432)\}$, $\overline{G} = \mathscr{U}$ and $\mathcal{T} = \{\emptyset, \overline{\mathsf{H}}, \{(24)\}, \{(24), (145)(23)\}, \{(123), (15432), (132), (12345)\}, \{(24), (123), (15432), (132), (12345)\}, \{(24), (145)(23), (123), (15432), (132), (12345)\}\}$ and let $\mathcal{T}_{\mathsf{H}} = \{\emptyset, \{(24)\}, \{(24), (145)(23)\}, \{(123), (15432), (132), (12345)\}, \{(24), (123), (15432), (132), (12345)\}, \{(24), (145)(23), (123), (15432), (132), (12345)\}, \mathsf{H}\}$. For simple notation, the elements of $\mathcal{T}_{\mathsf{H}}$ are denoted as follows $D_1 = \emptyset$, $D_2 = \{(24)\}$, $D_3 = \{(24), (145)(23)\}$, $D_4 = \{(123), (15432), (132), (12345)\}$, $D_5 = \{(24), (123), (15432), (132), (12345)\}$, $D_6 = \{(24), (145)(23), (123), (15432), (132), (12345)\}$, $D_7 = \mathsf{H}$.

Procedure for Classification of Open Subsets:

Step 1: The measure of open subsets of S-TRG $D_j (j = 1, 2, 3, 4, 5, 6, 7)$ can be computed by using Equation 1 and 2 and the correlation matrix $\mathcal{M}_C$ is constructed:

$$\mathcal{M}_C = \begin{pmatrix} 0 & 1 & 1 & 0 & 0.25 & 0.40 & 0.50 \\ 1 & 0 & 0 & 1 & 0.75 & 0.60 & 0.50 \\ 1 & 0 & 0 & 1 & 0.75 & 0.60 & 0.67 \\ 0 & 1 & 1 & 0 & 0.33 & 0.40 & 0.50 \\ 0.25 & 0.75 & 0.75 & 0.33 & 0.38 & 0.45 & 0.50 \\ 0.40 & 0.60 & 0.60 & 0.40 & 0.45 & 0.48 & 0.50 \\ 0.50 & 0.50 & 0.67 & 0.50 & 0.50 & 0.50 & 0.50 \end{pmatrix}$$

Step 2: Construct equivalent correlation matrix:

$$\mathcal{M}_C^2 = \begin{pmatrix} 1 & 0.50 & 0.50 & 1 & 0.75 & 0.60 & 0.67 \\ 0.50 & 1 & 1 & 0.50 & 0.50 & 0.50 & 0.50 \\ 0.50 & 1 & 1 & 0.50 & 0.50 & 0.50 & 0.50 \\ 1 & 0.50 & 0.50 & 1 & 0.75 & 0.60 & 0.67 \\ 0.75 & 0.50 & 0.50 & 0.75 & 0.75 & 0.60 & 0.67 \\ 0.60 & 0.50 & 0.50 & 0.60 & 0.60 & 0.60 & 0.60 \\ 0.67 & 0.50 & 0.50 & 0.67 & 0.67 & 0.60 & 0.67 \end{pmatrix}$$

$$\mathcal{M}_C^4 = \begin{pmatrix} 1 & 0.50 & 0.50 & 1 & 0.75 & 0.60 & 0.67 \\ 0.50 & 1 & 1 & 0.50 & 0.50 & 0.50 & 0.50 \\ 0.50 & 1 & 1 & 0.50 & 0.50 & 0.50 & 0.50 \\ 1 & 0.50 & 0.50 & 1 & 0.75 & 0.60 & 0.67 \\ 0.75 & 0.50 & 0.50 & 0.75 & 0.75 & 0.60 & 0.67 \\ 0.60 & 0.50 & 0.50 & 0.60 & 0.60 & 0.60 & 0.60 \\ 0.67 & 0.50 & 0.50 & 0.67 & 0.67 & 0.60 & 0.67 \end{pmatrix}$$

Therefore $\mathcal{M}_C^4 = \mathcal{M}_C^2$. Hence $\mathcal{M}_C^2$ is an equivalent matrix. Thus the implementation of algorithm leads to the classification of cipher text based on their transposition similarity and the classification is provided in Table 2.

TABLE 2. Classification of Cipher text using cluster algorithm.

	Class	Confidence level	Clustering result
	1	$0 \leq \alpha \leq 0.50$	$\{D_1, D_2, D_3, D_4, D_5, D_6, D_7\}$
Open	2	$0.50 \leq \alpha \leq 0.60$	$\{D_2, D_3\}, \{D_1, D_4, D_5, D_6, D_7\}$
sets	3	$0.60 \leq \alpha \leq 0.67$	$\{D_6\}, \{D_2, D_3\}, \{D_1, D_4, D_5, D_7\}$
	4	$0.67 \leq \alpha \leq 0.75$	$\{D_6, D_7\}, \{D_2, D_3\}, \{D_1, D_4, D_5\}$
	5	$0.75 \leq \alpha \leq 1$	$\{D_5, D_6, D_7\}, \{D_2, D_3\}, \{D_1, D_4\}$

D. RESULTS AND DISCUSSIONS OF THE EXPERIMENTAL RESULT

The classification of cipher text is obtained by the algorithm at the second stage of the iteration. The S-topological rough structure minimizes the number of iterations in classifying the cipher text based on their similarity. Federated learning emphasizes collaborative model training while safeguarding

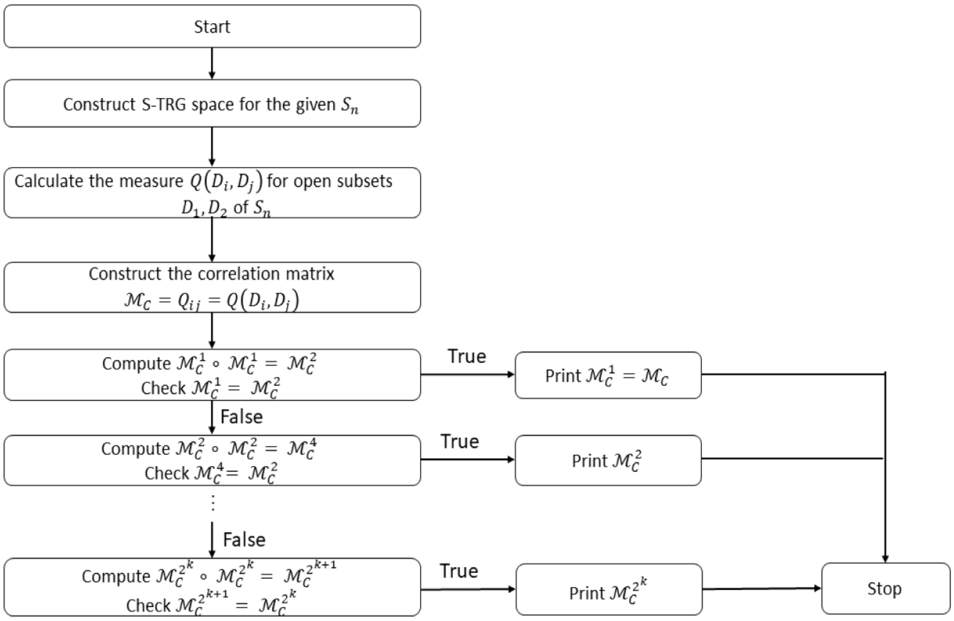

FIGURE 1. Flow chart of proposed algorithm.

data privacy, whereas rough sets excel in extracting crucial features for precise gastric cancer detection. Integrating these methodologies can yield robust, privacy-preserving diagnostic tools that effectively harness distributed healthcare data. The GBP-CS algorithm introduces a constrained gradient-based optimizer designed to select subsets of devices within factories to form homogeneous federated learning super nodes. GBP-CS demonstrates efficient selection strategies within a short timeframe, applicable beyond healthcare, such as in ciphertext classification within symmetric groups. In the context of ciphertext classification, the GBP-CS federated algorithm facilitates the selection of subsets within the S-TRG, showcasing its versatility in various practical scenarios.

V. ADVANTAGES AND LIMITATION OF THE PROPOSED STRUCTURE

Topological rough groups offer powerful tools for analyzing the properties and behaviours of group elements under conditions of uncertainty. This integration of rough set theory and topology can lead to new insights and a deeper understanding of underlying structures. It is particularly valuable in fields where data may be imprecise or incomplete, such as medical diagnostics or economic forecasting. By combining rough sets and topological properties, the algebraic structure is enriched, enabling the exploration of new mathematical properties and relationships.

The use of lower and upper approximations within the proposed structure is especially useful when exact computation is infeasible or unnecessary. The structure can accommodate various sizes of input sets and can be adapted to different equivalence relations, allowing it to scale with the data and be extended to more complex operations or additional steps if needed. This structure provides a comprehensive approach for analyzing the rough group structure and its associated topological properties.

However, the effectiveness of topological rough groups can be highly dependent on the chosen topology and equivalence relation. Selecting the appropriate structures may require domain-specific knowledge and can be a challenging task. Additionally, the categorization of open sets could result in redundant categories if not handled carefully, especially if the open sets are not sufficiently distinct. Furthermore, the lack of standardized tools, algorithms, or software for working with topological rough groups may hinder practical implementation and experimentation. The continued development of methods and tools in this area is essential to enhance their applicability and utility across various domains.

VI. CONCLUSION

In this paper, (S, s)-topological rough groups are defined and their properties are studied with illustrations. S-topological rough groups have the fundamental benefit of allowing one to analyse the entire structure using any suitable subset that satisfies both algebraic and topological characteristics. Additionally, its upper approximation equals the entire space through an equivalence relation. To investigate the algebraic and topological characteristics of the whole space, such appropriate subsets are enough. Rough set theory therefore aids in our analysis of the entire space through subspace. This study examines the different characteristics of the S-topological rough group, influenced by rough set theory.

By applying rough set theory to the analysis of ciphertexts transmitted over a network, it is possible to detect unusual patterns that might indicate a security breach. For instance, if an attacker attempts to inject malicious data into the communication stream, rough set-based models can help in identifying such anomalies. The algorithm for classifying S_n based on S-topological rough groups is discussed, along with a practical application. MAPLE and PYTHON are used to compute the correlation matrix and to generate pseudo code for the proposed algorithm. Experimental results demonstrate that the process of classifying ciphertexts based on their transposition similarity is streamlined by using the S-topological rough group structure.

The structure contributes for the advancement of topological rough groups by extending classical group and topological group theories with rough set approximations. This integration enhances theoretical understanding of uncertain systems and provides practical tools for data analysis and decision-making. It proves particularly valuable in domains where data is imprecise or incomplete, such as in medical diagnostics and economic forecasting. For economic data, such as market indicators and consumer behavior, the algorithm helps identify patterns and relationships that are not immediately apparent with precise data, leading to more robust predictions. Similarly, in geographical information systems, the framework effectively manages uncertainty in spatial data, improving the analysis of geographic patterns and relationships, which supports urban planning and environmental monitoring.

Moreover, the algorithm for S-topological rough groups demonstrate its applicability in text classification tasks. For instance, consider the text "Sun rises in the east" By applying permutations from S_5, where 5 represents the cardinality of vowels in the text, the algorithm can classify permutations of the vowels. The algorithm helps in the classification of encrypted text by grouping permutations based on their transposition properties. For any S-topological rough group structure, where the upper approximation is the universe and the group operation generates this universe, the algorithm efficiently classifies the encrypted text by identifying the minimal structure that can generate the universal structure.

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