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RESEARCH ARTICLE

# Fuzzy Optimal Guaranteed Cost Control of a Single Specie Bio-Economic Singular Systems in Algae Environment

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**ABSTRACT** In this article, the problem of optimal guaranteed cost control for a single specie bio-economic singular system in an algal environment is investigated using fuzzy methods. Based on the aquaculture of whitebait in algae environment, a class of non-linear bio-economic model is established. In addition, owing to the practical problem of a finite production cycle and the need to control biomass within a certain range, a cost-guaranteed controller based on a fuzzy model was developed to ensure that the biomass concentration reaches a certain range in a limited time and at a minimum cost. Finally, the feasibility of the developed controller is demonstrated through a practical example.

**INDEX TERMS** Algae environment, bio-economic singular systems, finite-time bounded, guaranteed cost control, Takagi-Sugeno (T-S) fuzzy model.

## I. INTRODUCTION

How to optimize the yields in agriculture is a constant concern for people and a hot topic for scientists. Loads of related research has been done in recent years. A lot of bio-economic systems are discussed. In 2019, Meng and Li proposed a delayed phytoplankton-zooplankton system with Allee effect and linear harvesting, the optimal harvesting strategy of the system is obtained [1]. Two years later, in the paper published by Nadjah Kerioui and others, a differential-algebraic bioeconomic system with predator harvesting is studied [2]. A mathematical model of harvesting strategy for fish population was presented to calculate the optimal economic return of sustainable fishing fish stocks by Chen [3]. Based on the classical modern control theory, Hasan and others studied the application of optimal control in biological economic system [4]. In recent years, research methods have been continuously innovated. In 2021, Shao introduced stochastic processes into environmental variables and generalized them to the study of continuous-time Mahalanobis decision-making processes [5]. Subsequently, Chinese scholar Zhang Lin discussed the finite time contraction stability and optimal control of the mosquito population inhibition model [6]. Meanwhile, the method can also be used for researching the conditions of the symbiosis of herring and mysis shrimp [7]. Nowadays, the optimal control of population dynamical systems with age structure has been widely discussed [8], [9]. In order to save costs and maximize economic returns, De Feo and others studied the economics of optimal advertising and investment portfolios in 2024 [10]. As the situation with various toxins and pollution become more and more serious, researchers are increasingly focusing on how to optimise harvesting, particularly in the face of toxins and pollution.

Using applied mathematical model to study how the pollution affects biological systems began with Hallam and colleagues [11], [12], [13]. Since then, there has been a steady stream of related research. In the following years, some related research results were emerging. Zhang and

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others have investigated the optimal harvesting of farmed fish under the prymnesiaceae toxin environment and fuzzy optimal guaranteed cost control of a single species model with stage-structure in toxic environment [14], [15]. Agmour and others discussed the effects of harmful phytoplankton population on the stability of bio-economic system model and the dynamic profit of phytoplankton species in 2021 [16]. For the problem of population competition in polluted environments, Li studied the optimal harvesting problem of the scale-structure competitive population system in the environment with toxin pollution by controlling the harvest effort [17]. Then, Gunathilaka discussed the global issue of pollution and summarized the recent studies on the response of plankton community to antibiotics [18]. It is worth noting that Wang Quan his colleague sused probability density function to study the effects of environmental toxins and time delays on the population dynamics of the model [19]. They found that concentrations of the toxin could lead to extinction.In the actual breeding process, farmers tend to ignore the existing age characteristics of the organisms and the age structure of natural reproduction when releasing population, which is equivalent to the organisms have the same growth cycle. And according to the theory of economics of shared resources, when the economic benefit of fishing is zero ($m = 0$), the economic profit maintains a certain equilibrium, this phenomenon is also known as overfishing, which we do not want to happen. Therefore, we should take measures to view this system as a bio-economic singular system and introduce differential algebraic equations to study it, so that the biological system can be stabilized and access to sustainable benefits. Simultaneously, in order to make the system have stronger stability, PDC, Non-PDC and $H_\infty$ controllers can be designed. And the method has gradually become mainstream. Shen constructs a nonlinear compensation composite controller for the fuzzy Markov jump system, studies the $H_\infty$ output feedback problem, and finally obtains the stochastic and stable impulse condition [20], [21]. The main highlighted contributions of the article are:

1) Based on the traditional biological economic model, this paper introduces Differential algebraic equation which can ensure the system is stable and the sustainable benefits can be maximized.

2) A PDC controller is designed by combining the biological economic system with the fuzzy control theory, and the Linear matrix inequality method is used to ensure the innovation and effectiveness of the method, at last, the relevant parameters of the system are obtained.

In this paper, we focus on the algae effect and its influences on the species. By fuzzy methods, the complex nonlinear model is first changed into fuzzy model. Second, we show that the system is regular and pulse-free. At the same time, in order to maximize the benefits at the lowest cost, a closed-loop system with a PDC controller was discussed. At last, a practical example is carried out to illustrate the feasibility of designed controller.

For the reason that the aquacultural environment is the limited space in general. Then the increase of the farmed population and algae concentration produced by the growth and reproduction of them can be performed by growth model with limited space

$$\begin{aligned}\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} &= x_1(t)(r_1 - a_1x_1(t)),\\ \frac{\mathrm{d}x_2(t)}{\mathrm{d}t} &= x_2(t)(r_2 - bx_2(t)),\end{aligned}$$

where $x_1(t)$ is the concentrate of the farmed population and $x_2(t)$ is the algae concentrate at the time $t$, $r_1$ and $r_2$ are the natural growth rate of farmed population and the algae, respectively. $a_1$ and $b$ respectively stand for the restrictive coefficient of the farmed population and algae in certain aquaculture space. At the same time, the excess algae growth will make the farmed population died of anoxic, and the coefficient of it is $a_2$.

Let $E(t)$ is the harvesting effort of $x_1(t)$ at the time $t$. $\alpha$ is the harvesting coefficient of $E(t)$. Then $\alpha E(t)$ is the capture rate of the farmed population. The the harvesting model in algae environment is built as follow

$$\begin{aligned}\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} &= x_1(t)(r_1 - a_1x_1(t) - a_2x_2 - \alpha E(t)),\\ \frac{\mathrm{d}x_2(t)}{\mathrm{d}t} &= x_2(t)(r_2 - bx_2(t)).\end{aligned}$$

The bio-economic singular model system in this article is built by considering the economic interest of harvesting effort on the farmed population which is first proposed by Zhang et al. [22]

$$\begin{aligned}\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} &= x_1(t)(r_1 - a_1x_1(t) - a_2x_2 - \alpha E(t)),\\ \frac{\mathrm{d}x_2(t)}{\mathrm{d}t} &= x_2(t)(r_2 - bx_2(t)),\\ 0 &= E(t)(\rho\alpha x_1(t) - c) - m.\end{aligned} \tag{1}$$

where $E(t)$ is the harvesting effort of the farmed population, $\rho$ is a price coefficient of per unit farmed population, $c$ is the cost coefficient, $cE(t)$ is the total cost, and $m$ is the economic interest of harvesting.

In the next section, the equilibria and fuzzy model of system (1) will be discussed.

## II. THE EQUILIBRIA AND FUZZY MODEL

According to the economic theory of a common-property resource [23], there is a phenomenon of bio-economic equilibrium when the economic interest of harvesting is zero ($m = 0$), i.e, total revenue is equal to total cost, which is also known as the economic overfishing. Then, this will cause the ecological imbalance, there must be an ecological disaster. But some artificial means can be used to control it. For example, in order to protect resources and promote the economic development by adjusting the amount of the tax to increase or decrease the cost for harvesting, the government implements control to the bio-economic system such that the system can develop continuously and continue to profit [24].

 

Firstly, the equilibrium points of system (1) will be discussed when $m = 0$ in this section. Then, the considered system is converted into T-S fuzzy models and we write it in the general form.

*Lemma 1:* There is a positive equilibrium point of system (1), if

$$a_1bc + a_2\alpha r_2\rho - \alpha br_1\rho < 0.$$

*proof*: The points of equilibrium of the parametric model (1) are given by the equations of the steady state

$$\begin{aligned} x_1(t)(r_1 - a_1x_1(t) - a_2x_2(t) - \alpha E(t)) &= 0,\\ x_2(t)(r_2 - bx_2(t)) &= 0,\\ E(t)(\rho\alpha x_1(t) - c) &= 0, \end{aligned} \tag{2}$$

After algebraic calculation, we obtain the trivial and the non-trivial points of equilibrium: $P_1(0, 0, 0)$, $P_2(0, \frac{r_2}{b}, 0)$, $P_3(\frac{r_1}{a_1}, 0, 0)$, $P_4(-\frac{a_1r_2-br_1}{a_1b}, \frac{r_2}{b}, 0)$, $P_5(\frac{c}{\alpha\rho}, 0, -\frac{a_1c-\alpha r_1\rho}{\alpha^2\rho})$, $P_6(\frac{c}{\alpha\rho}, \frac{r_2}{b}, -\frac{a_1bc+a_2\alpha r_2\rho-\alpha br_1\rho}{\alpha^2\rho})$

Clearly, $P_1$ and $P_2$ are extinction equilibrium points. $P_3$, $P_4$ are the non-harvesting equilibrium points. And $P_5$ is the non-algae equilibrium point (indeed it is an unpractical equilibrium). And if

$$a_1bc + a_2\alpha r_2\rho - \alpha br_1\rho < 0, \tag{3}$$

then $-\frac{a_1bc+a_2\alpha r_2\rho-\alpha br_1\rho}{\alpha^2\rho} > 0$, $P_6$ is the positive equilibrium point.

This completes the proof of the theorem.

For the system (1), $P_6$ is the positive equilibrium point. Let $Z(t) = (z_1(t), z_2(t), z_3(t))^T$, where $z_1(t) = x_1(t) - \frac{c}{\alpha\rho}$, $z_2(t) = x_2(t) - \frac{r_2}{b}$ and $z_3(t) = E(t) + \frac{a_1bc+a_2\alpha r_2\rho-\alpha br_1\rho}{\alpha^2\rho}$.

Then system (1) can be transformed into the following equivalent form

$$\begin{aligned} \frac{\mathrm{d}z_1(t)}{\mathrm{d}t} &= -a_1z_1^2(t) - a_2z_1(t)z_2(t) - \frac{a_1c}{\alpha\rho}z_1(t)\\ &\quad - \alpha z_1(t)z_3(t) - \frac{a_2c}{\alpha\rho}z_2(t) - \frac{c}{\rho}z_3(t),\\ \frac{\mathrm{d}z_2(t)}{\mathrm{d}t} &= -bz_2^2(t) - r_2z_2(t),\\ 0 &= \alpha\rho z_1(t)z_3(t) - \frac{a_1bc + a_2\alpha r_2\rho - \alpha br_1\rho}{\alpha\rho}z_1(t)\\ &\quad + u(t), \end{aligned} \tag{4}$$

where $u(t)$ is the control input representing regulation control for a biological resource, such as the increase or decrease in tax.

The system (4) is then transformed into a fuzzy T-S model and the model is described in its general form. Finally, considering the general form of the system, the main results of this paper are presented.

Since $z_1(t)$ and $z_2(t)$ represent the aquaculture population density and algal concentration, respectively, it can be assumed that $z_1(t) \in [l_1, l_2]$, and $z_2(t) \in [d_1, d_2]$, $l_i, d_i \in \mathbb{R}$, $i = 1, 2$. $H_i$ and $N_i$ $(i = 1, 2)$ is a fuzzy set for system (4). Then the fuzzy state model can be written as follows, which is suitable for the description of the model system (4) as $z_1(t) \in [l_1, l_2]$, and $z_2(t) \in [d_1, d_2]$.

Rule 1: If $z_1(t)$ is $H_1$ and $z_2(t)$ is $N_1$, then

$$E\frac{\mathrm{d}Z(t)}{\mathrm{d}t} = A_1Z(t) + B_1u(t).$$

Rule 2: If $z_1(t)$ is $H_2$ and $z_2(t)$ is $N_1$, then

$$E\frac{\mathrm{d}Z(t)}{\mathrm{d}t} = A_2Z(t) + B_2u(t).$$

Rule 3: If $z_1(t)$ is $H_1$ and $z_2(t)$ is $N_2$, then

$$E\frac{\mathrm{d}Z(t)}{\mathrm{d}t} = A_3Z(t) + B_3u(t).$$

Rule 4: If $z_1(t)$ is $H_2$ and $z_2(t)$ is $N_2$, then

$$E\frac{\mathrm{d}Z(t)}{\mathrm{d}t} = A_4Z(t) + B_4u(t),$$

where

$$A_1 = \begin{bmatrix} -\frac{a_1c}{\alpha\rho} - a_1l_2 & -\frac{a_2c}{\alpha\rho} - a_2l_2 & -\frac{c}{\rho} - \alpha l_2\\ 0 & -r_2 - bd_2 & 0\\ -\frac{a_1bc+a_2\alpha r_2\rho-\alpha br_1\rho}{\alpha\rho} & 0 & \alpha\rho l_2 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -\frac{a_1c}{\alpha\rho} - a_1l_1 & -\frac{a_2c}{\alpha\rho} - a_2l_1 & -\frac{c}{\rho} - \alpha l_1\\ 0 & -r_2 - bd_2 & 0\\ -\frac{a_1bc+a_2\alpha r_2\rho-\alpha br_1\rho}{\alpha\rho} & 0 & \alpha\rho l_1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -\frac{a_1c}{\alpha\rho} - a_1l_2 & -\frac{a_2c}{\alpha\rho} - a_2l_2 & -\frac{c}{\rho} - \alpha l_2\\ 0 & -r_2 - bd_1 & 0\\ -\frac{a_1bc+a_2\alpha r_2\rho-\alpha br_1\rho}{\alpha\rho} & 0 & \alpha\rho l_2 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} -\frac{a_1c}{\alpha\rho} - a_1l_1 & -\frac{a_2c}{\alpha\rho} - a_2l_1 & -\frac{c}{\rho} - \alpha l_1\\ 0 & -r_2 - bd_1 & 0\\ -\frac{a_1bc+a_2\alpha r_2\rho-\alpha br_1\rho}{\alpha\rho} & 0 & \alpha\rho l_1 \end{bmatrix},$$

$$E = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0\\ 0\\ 1 \end{bmatrix},\quad B_2 = \begin{bmatrix} 0\\ 0\\ 1 \end{bmatrix}, B_3 = \begin{bmatrix} 0\\ 0\\ 1 \end{bmatrix},\quad B_4 = \begin{bmatrix} 0\\ 0\\ 1 \end{bmatrix},$$

$$Z(t) = \begin{bmatrix} z_1(t)\\ z_2(t)\\ z_3(t) \end{bmatrix}.$$

Denote: $m_1[z_1(t)] = \frac{z_1-l_1}{l_2-l_1}$, $m_2[z_1(t)] = \frac{l_2-z_1}{l_2-l_1}$, $n_1[z_2(t)] = \frac{z_2-d_1}{d_2-d_1}$, $n_2[z_2(t)] = \frac{d_2-z_2}{d_2-d_1}$. Then $h_1 = m_1[z_1(t)] \times n_1[z_2(t)]$, $h_2 = m_2[z_1(t)] \times n_1[z_2(t)]$, $h_3 = m_1[z_1(t)] \times n_2[z_2(t)]$, $h_4 = m_2[z_1(t)] \times n_2[z_2(t)]$.

Fuzzy blending is used to obtain a overall fuzzy model as follows:

$$E\frac{\mathrm{d}Z(t)}{\mathrm{d}t} = \sum_{i=1}^{4} h_i[A_iZ(t) + B_iu(t)]. \tag{5}$$

*Remark 1:* In the above process, we transform the single-species algae environment model into a generalized fuzzy economic system, which is convenient for us to maximize the performance benefit of the guaranteed cost, and treat the nonlinear model approximately into several linear models, it's a huge efficiency boost.

In order to improve the readability of the article, we have written (5) as a standard way to describe fuzziness, and have changed it to (6). In the following, the related definitions and results will be introduced for the general form of T-S fuzzy singular systems.

$$E\frac{\mathrm{d}X(t)}{\mathrm{d}t} = \sum_{i=1}^{r} h_i[A_iX(t) + B_iu(t)], \tag{6}$$

where $X(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input. $r$ is the number of If-Then rules, $A_i$ and $B_i$ are real known matrices with appropriate dimensions.

## III. OPTIMAL GUARANTEED COST CONTROL

This section begins with introducing the definitions and results related to the general form of biological fuzzy systems, then focused on researching mode-dependent control law for system (6) and the following cost function has been designed for the system in order to meet certain requirements:

$$J_{T_0} = \int_0^{T_0} [X^{\mathrm{T}}(t)M_1X(t) + u^{\mathrm{T}}(t)M_2u(t)]dt, \tag{7}$$

where $M_1$ and $M_2$ are two symmetric positively defined matrices of appropriate dimensions.

*Remark 2:* In system (6), $\int_0^{T_0} X^{\mathrm{T}}(t)M_1X(t)dt$ represents the average cost per unit volume of biomass storage in period $[0, T_0]$, and $\int_0^{T_0} u^{\mathrm{T}}(t)M_2u(t)dt$ represents the average cost per unit volume of biomass and algae control in period $[0, T_0]$. In this paper, $T_0$ is the length of the production cycle.

The state feedback fuzzy control rule

$$u(t) = \sum_{i=1}^{r} h_i(\xi)K_iX(t) = K_\xi X(t) \tag{8}$$

based on parallel distributed compensation, is adopted, resulting in the following closed-loop system:

$$E\frac{\mathrm{d}X(t)}{\mathrm{d}t} = \sum_{i=1}^{r}\sum_{j=1}^{r} h_ih_j(A_i + B_iK_j)X(t). \tag{9}$$

First, we introduce the definitions necessary to develop the main results of this paper. System (9) can be transformed into an equivalent system

$$E\frac{\mathrm{d}X(t)}{\mathrm{d}t} = (A_\xi + B_\xi K_\xi)X(t), \tag{10}$$

for simplicity, where $A_\xi = \sum_{i=1}^{r} h_iA_i$, $B_\xi = \sum_{i=1}^{r} h_iB_i$.

*Definition 1 ([25]):* The system (10) is said to be regular in the time interval $[0, T_0]$, if the characteristic polynomial $det(sE-(A_\xi+B_\xi K_\xi))$ is not identically zero for all $t \in [0, T_0]$.

2) The system (10) is said to be impulse free in the interval $[0, T_0]$, if deg (det $(sE - (A_\xi + B_\xi K_\xi))$) = rank $E$ for all $t \in [0, T_0]$.

*Definition 2 ((SFTB) [26]):* The closed-loop system (10) is said to be singular finite-time boundedness (SFTB) with respect to $(c_1, c_2, T_0, R)$, if

$$x^{\mathrm{T}}(t_0)E^{\mathrm{T}}REx(t_0) \le c_1 \Rightarrow x^{\mathrm{T}}(t)E^{\mathrm{T}}REx(t) < c_2,$$

$t_0 \in [-\tau, 0]$, $t \in [0, T_0]$. $R$ is positive definite matrix with proper dimension. $c_1$ and $c_2$ are real numbers, and $c_2 > c_1 > 0$.

*Lemma 2 ([27]):* If the following conditions hold

$$M_{ii} < 0, 1 \le i \le r,$$
$$\frac{2}{r-1}M_{ii} + \frac{1}{2}(M_{ij} + M_{ji}) < 0, 1 \le i \ne j \le r,$$

then the following parameterized matrix inequality holds

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i(t)\alpha_j(t)M_{ij} < 0,$$

where $\alpha_i(t) \ge 0$ and $\sum_{i=1}^{r}\alpha_i(t) = 1$.

*Theorem 1:* The system (10) is SFTB with respect to $(c_1, c_2, T_0, R)$ and the cost function (7) has an upper bound in the time interval $[0, T_0]$, if there exists a scalar $\varrho \ge 0$, matrices $P$, $Q > 0$ such that

$$E^{\mathrm{T}}P = P^{\mathrm{T}}E \ge 0 \tag{11}$$
$$\begin{aligned}&(A_\xi + B_\xi K_\xi)^{\mathrm{T}}P^{\mathrm{T}} + P(A_\xi + B_\xi K_\xi) - \varrho E^{\mathrm{T}}P\\ &\quad + M_1 + K_\xi^{\mathrm{T}}M_2K_\xi < 0\end{aligned} \tag{12}$$
$$E^{\mathrm{T}}P = E^{\mathrm{T}}R^{1/2}QR^{1/2}E \tag{13}$$
$$e^{\varrho T_0}c_1\lambda_{max}(Q) - c_2\lambda_{min}(Q) < 0, \tag{14}$$

And the upper guaranteed cost bound is

$$J^* = \lambda_{max}(Q)c_1e^{\varrho T_0}$$

*proof:* Firstly, it can be proved that singular system (10) is regular and impulse free in the time interval $[0, T_0]$. From(12), $M_1 > 0$ and $M_2 > 0$, we have

$$(A_\xi + B_\xi K_\xi)P^{\mathrm{T}} + P^{\mathrm{T}}(A_\xi + B_\xi K_\xi) - \varrho E^{\mathrm{T}}P < 0 \tag{15}$$

We know $E = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$. Supposing that there exist matrices $P$ such that (11) holds, $P$ can be constructed as $P = \begin{bmatrix} P_1 & 0 \\ P_3 & P_4 \end{bmatrix}$. Accordingly, take $A_\xi + B_\xi K_\xi = \begin{bmatrix} A_{1\xi} & A_{2\xi} \\ A_{3\xi} & A_{4\xi} \end{bmatrix}$. Then, by (15), it follows that

$$\begin{bmatrix} \star & \star \\ \star & A_{4\xi}^{\mathrm{T}}P_4 + P_4^{\mathrm{T}}A_{4\xi} \end{bmatrix} < 0 \tag{16}$$

which implies $A_{4\xi}$ is nonsingular. According to Definition 3.1, the system (10) is regular and impulse free in the time interval $[0, T_0]$.

Choose the Lyapunov functional candidate as

$$V(X(t)) = X^{\mathrm{T}}(t)E^{\mathrm{T}}PX(t). \tag{17}$$

Then, from (15), it can be obtained

$$\frac{dV(X(t))}{dt} - \varrho V(X(t)) = X^{\mathrm{T}}(t)((A_\xi + B_\xi K_\xi)P^{\mathrm{T}} + P^{\mathrm{T}}(A_\xi + B_\xi K_\xi) - \varrho E^{\mathrm{T}} P)X(t) < 0 \tag{18}$$

Multiplying (18) by $e^{-\varrho t}$, we can obtain

$$\frac{d}{dt}(e^{-\varrho t}V(X(t))) < 0. \tag{19}$$

Integrating (19) from 0 to $t$ with $t \in [0, T_0]$, we have

$$V(X(t)) < V(X(0))e^{\rho t}. \tag{20}$$

Then

$$\begin{aligned} V(X(t)) < V(X(0))e^{\varrho t} &= [X^{\mathrm{T}}(0)E^{\mathrm{T}}PX(0)]e^{\varrho t} \\ &= [X^{\mathrm{T}}(0)E^{\mathrm{T}}R^{\frac{1}{2}}QR^{\frac{1}{2}}EX(0)]e^{\varrho t} \\ &\le \lambda_{max}(Q)X^{\mathrm{T}}(0)E^{\mathrm{T}}REX(0) \\ &\le c_1\lambda_{max}(P)e^{\varrho t} \\ &\le c_1\lambda_{max}(P)e^{\varrho T}. \end{aligned} \tag{21}$$

On the other hand,

$$\begin{aligned} V(X(t)) &> X^{\mathrm{T}}(t)E^{\mathrm{T}}R^{\frac{1}{2}}QR^{\frac{1}{2}}EX(t) \\ &\ge \lambda_{min}(Q)X^{\mathrm{T}}(t)E^{\mathrm{T}}REX(t). \end{aligned} \tag{22}$$

Combining (21) and (22) leads to

$$X^{\mathrm{T}}(t)E^{\mathrm{T}}REX(t) < \frac{c_1\lambda_{max}(Q)}{\lambda_{min}(Q)}e^{\varrho T_0}. \tag{23}$$

Then, by (14), we have

$$X^{\mathrm{T}}(t)RX(t) < c_2 \tag{24}$$

for all $t \in [0, T_0]$. Once again from (12) and (19), it can be easily seen

$$\frac{dV(X(t))}{dt} < \varrho V(X(t)) - X(t)^{\mathrm{T}}M_1X(t) - u(t)^{\mathrm{T}}M_2u(t). \tag{25}$$

Further, (25) can be represented as

$$\frac{d}{dt}(e^{-\varrho t}V(X(t))) < -e^{-\varrho t}(X(t)^{\mathrm{T}}M_1X(t) + u(t)^{\mathrm{T}}M_2u(t)). \tag{26}$$

Integrating (26) from 0 to $T_0$, we have

$$\begin{aligned} &\int_0^{T_0} e^{-\varrho t}(X(t)^{\mathrm{T}}M_1X(t) + u(t)^{\mathrm{T}}M_2u(t)) \\ &\quad < -\int_0^{T_0} \frac{d}{dt}(e^{-\varrho t}V(X(t))). \end{aligned} \tag{27}$$

Noting that $\varrho \ge 0$, it follows that:

$$\begin{aligned} J_{T_0} &= \int_0^{T_0} (X(t)^{\mathrm{T}}M_1X(t) + u(t)^{\mathrm{T}}M_2u(t)) \\ &\le e^{\varrho T_0}\int_0^{T_0} e^{-\varrho t}(X(t)^{\mathrm{T}}M_1X(t) + u(t)^{\mathrm{T}}M_2u(t)) \\ &< -e^{\varrho T_0}\int_0^{T_0} \frac{d}{dt}(e^{\varrho t}V(X(t))) \\ &\le e^{\varrho T_0}\lambda_{max}(Q)c_1. \end{aligned} \tag{28}$$

This completes the proof of the theorem.

*Theorem 2:* There exists a state feedback controller with $u = \sum_{i=1}^{r} h_i(\xi)Y_iX^{-1}X(t)$ such that the system (10) is SFTB with respect to $(c_1, c_2, T_0, R)$ and the cost function (7) has an upper bound in the time interval $[0, T_0]$, if there exist scalars $\rho > 0$, $\eta_1 > 0$, $\eta_2 > 0$ and matrices $X = \begin{bmatrix} X_1 & 0 \\ X_3 & X_4 \end{bmatrix}$, $V = \begin{bmatrix} X_1 & 0 \\ 0 & \Psi \end{bmatrix}$, $X_1 \in \mathbb{R}^{r\times r}$, $X_1 > 0$, $X_4 \in \mathbb{R}^{(n-r)\times(n-r)} > 0$, $\Psi \in \mathbb{R}^{(n-r)\times(n-r)} > 0$, $Y_i$, such that

$$\Omega_{ii} < 0, 1 \le i \le r, \tag{29}$$

$$\frac{2}{r-1}\Omega_{ii} + \frac{1}{2}(\Omega_{ij} + \Omega_{ji}) < 0, 1 \le i \ne j \le r, \tag{30}$$

$$\eta_1 I < R^{-\frac{1}{2}}VR^{-\frac{1}{2}} < \eta_2 I \tag{31}$$

$$e^{-\rho T_0}c_1\eta_2 - c_2\eta_1 < 0 \tag{32}$$

and the upper guaranteed bound is

$$J_{T_0} < J^* = \frac{1}{\eta_1}c_1e^{\varrho T_0},$$

where

$$\Omega_{ij} = \begin{bmatrix} A_iX + B_iY_j + (A_iX + B_iY_j)^{\mathrm{T}} - \rho X^{\mathrm{T}}E & X^{\mathrm{T}}M_1 & Y_j^TM_2 \\ M_1^{\mathrm{T}}X & -M_1 & 0 \\ M_2Y_j & 0 & -M_2 \end{bmatrix},$$

*proof:* From (29), (30) and **Lemma 1**, we get

$$\begin{bmatrix} \lambda X + X^T\lambda^T - \rho X^TE & X^{\mathrm{T}}M_1 & Y_\xi{}^TM_2 \\ M_1{}^{\mathrm{T}}X & -M_1 & 0 \\ M_2Y_\xi & 0 & -M_2 \end{bmatrix} < 0 \tag{33}$$

where $\lambda = A_\xi + B_\xi K_\xi$

Similar the proof of **Theorem 1**, we can obtain that $X$ is nonsingular. So, $X_1$ and $X_4$ are also nonsingular. Then, there exists $P = X^{-1}$ such that

$$\begin{aligned} E^{\mathrm{T}}P &= E^{\mathrm{T}}X^{-1} \\ &= \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} X_1^{-1} & 0 \\ -X_4^{-1}X_3X_1^{-1} & X_4^{-1} \end{bmatrix} \\ &= \begin{bmatrix} X_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} = P^{\mathrm{T}}E \end{aligned} \tag{34}$$

Now pre and postmultiply (III) by diag $\{X^{-1}, I, I\}$ and its transposition. Then using the Schur complement Lemma, it follows that (29) and (30) imply (12).

Further, letting $Q = R^{-\frac{1}{2}}V^{-1}R^{-\frac{1}{2}}$, it can be obtained that

$$\begin{aligned} E^{\mathrm{T}}R^{\frac{1}{2}}QR^{\frac{1}{2}}E &= E^{\mathrm{T}}R^{\frac{1}{2}}R^{-\frac{1}{2}}V^{-1}R^{-\frac{1}{2}}R^{\frac{1}{2}}E \\ &= E^{\mathrm{T}}VE = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} X_1^{-1} & 0 \\ 0 & \Psi^{-1} \end{bmatrix}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

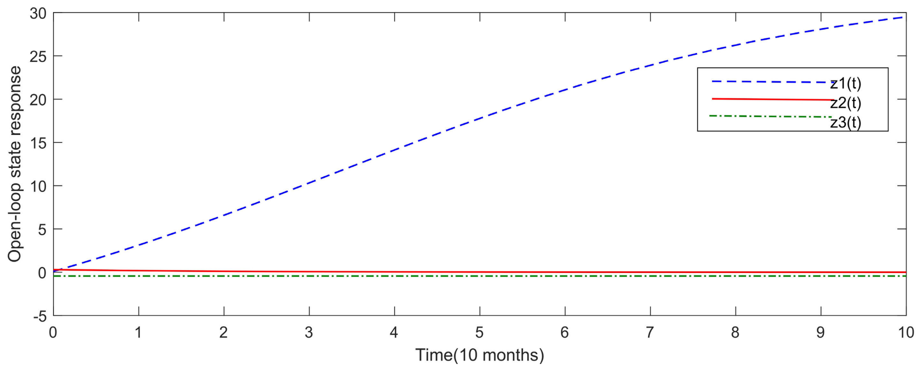


**FIGURE 1.** Closed-loop system state response diagram.

$$= \begin{bmatrix} X_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} = E^{\mathrm{T}} P \tag{35}$$

Therefore (13) holds. By (31), we can obtain

$$\frac{1}{\eta_2} \leq Q = V^{-1} \leq \frac{1}{\eta_1} I, \tag{36}$$

So, from (32), combining (36), it follows (14) holds. Then

$$J_{T_0} < J^* = \frac{1}{\eta_1} c_1 e^{\tilde{\alpha} T_0}.$$

This completes the proof of the theorem.

*Remark 3:* Here, we first present a finite-time stability analysis of a single population using fuzzy technique, and solve the controller parameters by strict Linear matrix inequality.

## IV. A PRACTICAL EXAMPLE

In this section, we illustrate the main conclusions of this paper with practical examples.

Whitebait is very famous for its high nutritional value and economic value. The protein content of whitebait is 72.1 percent, and the amino acid content of whitebait is also rich. What's more, whitebaits are rich in calcium and very little in fat [28], which are popular because of their high nutritional value and flavor. Whitebaits are popular because of their high nutritional value and flavor. Furthermore the whitebait has the special nutritious function to the undernourished people. However, whitebait is extremely sensitive to algae. Flourishing algae caused by the continuing high temperature, too much nutrition in the water, and a shortage of running water, is said to have contributed to the whitebait dying from hypoxia. The effects of algae cause significant economic damage to aquaculture (especially whitebait aquaculture) because of their high concealment capacity, short time from hypoxia to death, and high mortality rate. When algal blooms occur, whitebait quickly die if no control measures are taken. As a result, agriculturists lose their entire initial investment.

The Shangsantai reservoir, situated in Siping, Jilin Province, China, produces around 400 tons of fish and prawns annually, generating a total income of approximately 6.4 million RMB (The data above is from the 2021 fishery production statistics in Siping City, Jilin Province).

The cultivation of Shangsantai is a thriving industry in the area, with 30 ponds covering an area of 3 hectares and an initial concentration of 20,000 fish per hectare. The whitebaits produced by these farmers are of high quality and command an price of 120 yuan per kilogram. According to the 2022 fishery production statistics of Siping City, Jilin Province, the net profit per pond during the rearing period is approximately 10,800 RMB. Nearly 20 ponds died due to algae blooms in 2022. (The above data was taken from the 2021 fishery production statistics of Siping City, Jilin Province.)

We built the model on the basis of data from the city of Siping, includes key parameters such as the initial concentration of whitebait $r_1$ is 0.60. The internal control coefficient of whitebait $a_1$ is 0.008. The natural growth rate of algae in algae blooms condition $r_2$ is 0.5. The internal control coefficient of algae $b$ is 0.01. Additionally, the mortality rate of whitebait $a_2$ in an algae environments is 0.005. The coefficients of the whitebait-harvesting effort $\alpha$ is 0.6, and the coefficient of price per unit of whitebait $\rho$ is 0.8. Using these parameters, we can specially determine the model (4) as the following form:

$$\begin{aligned} \frac{\mathrm{d}z_1(t)}{\mathrm{d}t} &= -0.008 z_1^2(t) - 0.005 z_1(t) z_2(t) - 0.083 z_1(t) \\ &\quad - 0.60\, z_1(t) z_3(t) - 0.052 z_2(t) - 8.333 z_3(t), \\ \frac{\mathrm{d}z_2(t)}{\mathrm{d}t} &= -0.01 z_2^2(t) - 0.50 z_2(t), \\ 0 &= 0.48\, z_1(t) z_3(t) - 0.245 z_1(t) + u(t), \end{aligned}$$

When $T_0 = 5$, $c = 5$, $c_1 = 0.4$, $c_2 = 10$, $l_1 = 1$, $l_2 = 10$, $d_1 = 1$, $d_2 = 10$, $M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$; $M_2 = 1$, $R = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$, by **Theorem 2**, the controller gains are given by $\eta_1 = 0.0197$, $\eta_2 = 0.0337$

$K_1 = \begin{bmatrix} -69.4143 & 3.1422 & 387.6187 \end{bmatrix}$,
$K_2 = \begin{bmatrix} -68.7910 & 3.1132 & 386.3748 \end{bmatrix}$,
$K_3 = \begin{bmatrix} -67.8841 & 3.0726 & 378.9678 \end{bmatrix}$,

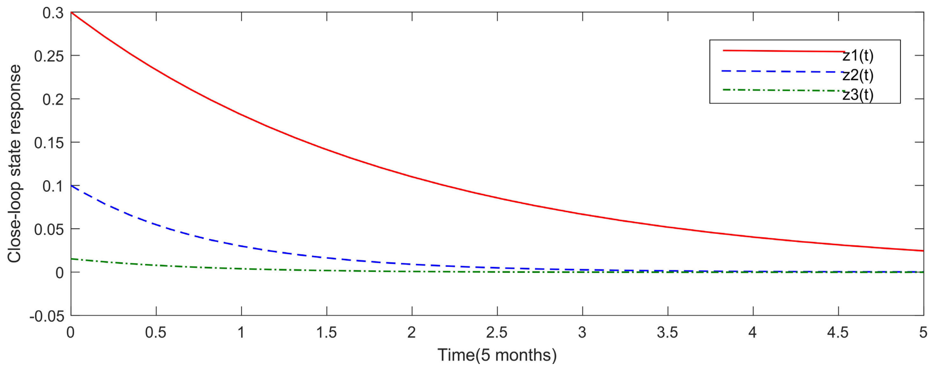


**FIGURE 2.** System state response diagram with PDC-controller.

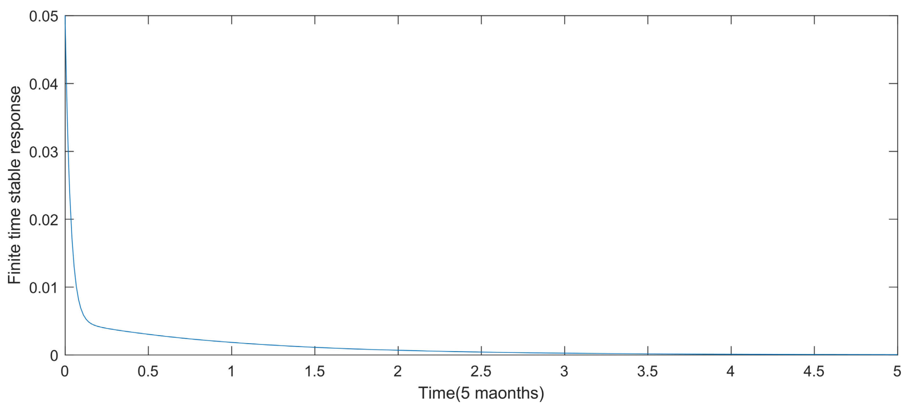


**FIGURE 3.** Finite time stable response diagram.

$$K_4 = \left[-68.0379\ 3.0792\ 382.1225\right],$$
$$X = \begin{bmatrix} 0.3486 & 0.0006 & 0 \\ 0.0006 & 0.3773 & 0 \\ 0.0619 & -0.0029 & -0.0012 \end{bmatrix},$$
$$V = \begin{bmatrix} 0.3486 & 0.0006 & 0 \\ 0.0006 & 0.3773 & 0 \\ 0 & 0 & 0.4269 \end{bmatrix}$$

The optimal guaranteed cost's upper bound is 248 RMB yuan, which represents the optimal average cost per unit volume of biomass storage and control over the period [0, 5].

To demonstrate the effectiveness, Fig.1 and Fig.2 display the state response of an open-loop system and a closed-loop system controlled by (8) under the initial condition $Z(0) = [0.1\ \ 0.3\ \ 0.1]^{\mathrm{T}}$.

To further verify the finite time stability of the system, we give a trajectory plot of time $t$ as a function of $x^T(t)E^TREx(t)$ as shown in Fig. 3.

*Remark 4:* As illustrated in Fig.1, if system (4) can not be effectively controlled, the algae concentration will continue to increase. Under the influence of algae, the density of the aquaculture population will decrease. If the methods presented in this paper are used, algae can be managed to reduce the density of the cultured population. The harvesting effect will also reduce the density of the growing population and the labor involved in harvesting. Both the concentration of algae and the density of cultured populations reached the specified range in a limited time and at minimum cost, as shown in Fig.2.

## V. CONCLUSION

This article investigates fuzzy optimal guaranteed cost control for single-species model in algal environment. Firstly, a harvesting model of aquaculture in algae environment is built based on actual aquaculture. Secondly, considering the limited duration of the culture, a controller for this system was developed to bring the concentration of the cultured stock to the specified range for a limited period of time at minimum cost. The presented approach is exemplified to demonstrate its feasibility.

In the follow-up research, we can combine the fuzzy method with the multi-species model. At the same time, we can study how to minimize the time when the system reaches a certain target profit.

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