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THEORY

Bounded Real Lemma for Singular Caputo Fractional-Order Systems

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ABSTRACT In this paper, we introduce an innovative generalized Lyapunov theorem and a novel bounded real lemma designed for continuous-time linear singular systems with Caputo fractional derivative of order α, with the constraint $1 \leq \alpha < 2$. We initially present a condition that is both necessary and sufficient for establishing the admissibility of singular fractional-order systems (SFOSs). This condition is articulated through strict linear matrix inequalities (LMIs). Following this, we demonstrate that a SFOS satisfies $H_{\infty}-$norm requirement if and only if two strict LMIs are feasible. The key advantage of the presented LMI conditions is that only one matrix variable needs to be solved. Ultimately, this paper concludes by presenting illustrative examples that highlight the practical effectiveness of our theoretical findings.

INDEX TERMS Generalized Lyapunov theorem, Caputo fractional-order singular systems, generalized bounded real lemma, linear matrix inequality.

I. INTRODUCTION

Over the past few decades, fractional-order differential systems have attracted significant attention, as evidenced by numerous studies [1], [2], [3], [4], [5], [6], [7], [8]. The history of fractional calculus, spanning over three centuries [9], [10], [11], [12], shows that it provides a more precise representation of many real-world systems. This field extends derivatives and integrals beyond integer orders, offering models that are more flexible and capable of capturing memory properties effectively. Recent research has particularly focused on Riemann-Liouville and Caputo fractional derivatives [9], applying fractional differential equations widely in areas like biology, finance, quantum mechanics, material science, and medicine [10]. Significant advancements have been made in stability analysis and control synthesis for linear fractional-order systems (FOSs), largely using the LMI approach [13], [14], [15], [16].

The architecture of the systems discussed in this paper builds on foundational concepts from previous works, highlighting the distinct stable regions for pole locations within the ranges $1 \leq \alpha < 2$ and $0 \leq \alpha < 1$. For certain practical systems, the fractional-order derivative within $1 \leq \alpha < 2$ has proven effective in describing dynamics, as evinced in circuit systems [7] and diffusion issues associated with super diffusion mechanisms [21]. Despite the similarity in techniques used for both ranges, the $1 \leq \alpha < 2$ range has been less explored, prompting further research into this class of fractional-order derivatives. Furthermore, while numerous analytical techniques are available to solve fractional differential equations, especially within the $0 \leq \alpha < 1$ range, the complexity of conditions often necessitates numerical methods [17], [18], [19], [20]. Despite ongoing progress, the $1 \leq \alpha < 2$ range continues to represent a significant gap in research, impacting both theoretical advances and practical applications. Recently, there has been increasing interest in this range, with studies like those in references [5], [7] examining fractional derivatives of order α within it.

Additionally, singular systems, also known as descriptor, implicit, or generalized state space systems, have attracted considerable research interest [21], [22], [23]. These systems are preferred for their enhanced ability to accurately depict various physical systems and phenomena, especially those involving impulses, compared to regular systems. They find widespread applications in fields such as economic systems,

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electrical networks, and power systems. The substantial body of literature dedicated to the synthesis and analysis of singular systems reflects the diligent efforts of researchers. Notably, significant progress has been made in admissibility analyses for descriptor systems [24]. The initial admissibility analysis utilized generalized Lyapunov equations [25], which evolved to include novel equations incorporating both equality and non-strict matrix inequality constraints [26]. Another approach, employing non-strict generalized Lyapunov inequalities with an equality constraint, was introduced [27], though it was less computationally efficient [28]. More recently, researchers have proposed more manageable and numerically robust conditions based on strict LMIs for the admissibility analysis of descriptor systems [23], [28]. These results have led to further investigations into SFOSs. In the context of traditional singular systems, the concept of admissibility, which encompasses stability, regularity, and the absence of impulses, is foundational, as noted in reference [29]. While recent literature has presented various admissibility conditions for SFOSs based on LMIs [30], [31], many challenges remain unresolved in this class of systems. Moreover, SFOSs have garnered considerable attention lately, primarily due to the significant contributions of fractional-order calculus, as well as emerging results on descriptor systems. Despite notable advancements in understanding FOS, the field of SFOSs still faces numerous challenging and unresolved issues, especially in the areas of stability analysis and controller synthesis [29], [32]. Specifically, the decision matrices used in stability criteria for linear time-invariant SFOSs are often either excessively complex or characterized by an overwhelming number of parameters [31], [33]. This inherent complexity not only hampers the practicality of straightforward controller design but also complicates stability analysis. A significant focus has been placed on examining the impacts of robust stabilization and admissibility of SFOSs, as explored in [31]. Further, the research in [33] has established sufficient conditions for addressing these problems, focusing on a specific class of SFOSs equipped with state and static output feedback stabilizing controllers.

Furthermore, the classical bounded real lemma, crucial for $H_\infty-$norm analysis, has been extended to continuous-time descriptor systems [23], [27], [28], [34], [35], [36]. These extensions include [34] and other studies obtaining non-strict LMI-based conditions and [23], [28] obtaining strict LMI-based conditions. These lemmas, essential in characterizing system performance and admissibility, vary in matrix variable requirements and contribute significantly to establishing necessary criteria for singular systems. Very recently, Wu and Yung [37] introduced a novel generalized Lyapunov theorem, incorporating a fresh generalized bounded real lemma for addressing singular systems. However, research specifically addressing bounded real lemmas in SFOSs remains limited, underlining the importance of these lemmas in characterizing system performance and ensuring system admissibility. In the context of establishing the necessary and sufficient criteria for bounded real lemmas, the SFOSs incorporating the H_∞ norm was formulated by [7]. In [5], a strict LMI-based bounded real lemma for SFOSs was presented. In [5], there are two matrix variables in the LMIs to be solved. Inspired by the aforementioned discussions and facts, this brief delves into the challenges surrounding bounded real lemmas in the context of H_∞ control for SFOSs. Although the focus of this paper is on the case of $1 \leq \alpha < 2$, the results obtained can be easily extended to the case of $0 < \alpha < 1$ with appropriate modifications. The key accomplishments of this study comprise:

1. **Development of Novel Stability Conditions:** Our research represents a significant advancement in the stability analysis of SFOSs. We have developed novel stability conditions and extended $H_\infty-$norm conditions utilizing strict LMIs with a single indefinite matrix variable. Additionally, we have introduced a full real number matrix version of these strict LMIs, designed for direct application in existing toolboxes. This approach provides a precise tool for stability analysis and effectively supports a practical bounded real lemma, making a substantial contribution to the field.
2. **Comparison and Efficiency of Verification Process:**Compared to existing SFOS approaches detailed in [29], [30], [31], and [32], our work introduces an innovative generalized Lyapunov theorem and a novel bounded real lemma, both specifically tailored for SFOSs. We emphasize the advantages of univariate LMIs, as discussed in [37], and bridge traditional singular systems with modern research through our focus on bounded real lemmas for SFOSs [5], [7]. While there are challenges associated with synthesizing H_∞ controller using equality-constrained LMI conditions highlighted in [7], the strict LMI conditions in [5] provide greater applicability. Our approach significantly streamlines the verification process, demonstrated through numerical examples, reduces verification time, and leverages the practicality of the MATLAB toolbox, thus simplifying implementation and managing complexities in real-world scenarios. Unlike the results obtained in [5] and [7], our conditions are both necessary and sufficient, and are expressed using strict LMIs. The previous studies involved two or more matrix variables in the LMI-based conditions, with at least one required to be positive definite. Our approach simplifies this by using only one matrix variable in the LMI-based conditions, which can be indefinite. Consequently, our LMI-based conditions feature the smallest number of decision variables compared to the strict LMI-based conditions proposed in [5] and [7]. For high-order descriptor linear systems, solving the LMIs proposed in this study is considerably more efficient than those presented in previous studies [5], [7].

The structure of this paper is as follows: Section II provides background information on linear Caputo SFOSs and related preliminaries. In Section III, we present the main results, which include a generalized Lyapunov theorem and a generalized bounded real lemma for SFOCs, represented in terms of strict LMIs involving a single matrix variable. Section IV offers numerical examples to validate our theoretical findings and compare them with other studies. Finally, Section V summarizes the conclusions drawn from this work.

Notation:

In this paper, $\mathbb{N}$ represents the set of natural numbers. The spectrum of a matrix M is denoted by $\sigma(M)$, the trace of matrix M is represented $tr(M)$, M^T is the transpose of M, and $\|M\|$ denotes the Frobenius norm of M; the complex conjugate of $\lambda \in \mathbb{C}$ is denoted by $\overline{\lambda}$, and the real part of λ is denoted as $Re(\lambda)$. The Kronecker product is denoted by $\otimes$. For any square complex matrix X, $Her(X)$ represents to $X+X^*$. The argument of the complex number z is denoted as $arg(z)$ [38].

II. PROBLEM FORMULATION AND PRELIMINARIES

In this section, we introduce the definition of the fractional-order derivative utilized, present the problem formulation, and provide necessary preliminaries. Various definitions of fractional-order derivatives exist, including the Grunwald–Letnikov, Riemann–Liouville and Caputo fractional-order derivatives (refer to [9], [10], [11], and [12] for detailed explanations). This study specifically adopts the Caputo fractional-order derivative, as articulated in the following discussion.

Definition 1 ([9], [10], [11], [12]): Let $f(t)$ be an integrable piecewise continuous function on any finite subinterval within the range $[0, \infty)$. The fractional integral of $f(t)$ with a positive order $\alpha > 0$ is expressed as follows:

$$J^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, t > 0 \tag{1}$$

where $\Gamma(\cdot)$ represents the gamma function.

Definition 2 ([9], [10], [11], [12]): The Caputo fractional-order derivative of order $\alpha > 0$ is defined as:

$$D^{\alpha} f(t) = \frac{1}{\Gamma(M-\alpha)} \int_0^t \frac{f^{(M)}(\tau)}{(t-\tau)^{\alpha+1-M}} d\tau \tag{2}$$

where $f^{(M)}(\tau) = \frac{d^M f(\tau)}{d\tau^M}$ with $M-1 \le \alpha < M,\ M \in \mathbb{N}$.

Considering the aforementioned issue, the following equation illustrates a singular fractional-order system with a disturbance signal:

$$\begin{aligned} ED^{\alpha}x &= Ax + Bw \\ z &= Cx + Dw \end{aligned} \tag{3}$$

where $x \in R^n$ denote the state, $w \in R^m$ represent the exogenous input, $z \in R^r$ is the output; The matrices $E, A \in R^{n\times n}$, where $\text{rank}(E) = r \le n$, $B \in R^{n\times m}$, $C \in R^{r\times n}$, and $D \in R^{r\times m}$ are constant with appropriate dimensions, and $1 \le \alpha < 2$.

The transfer function of system (3) is represented by

$$G(s) = C\left(s^{\alpha}E - A\right)^{-1}B. \tag{4}$$

Definition 3 ([5]): The H_∞ norm of the transfer function $G(s)$ in (4) is defined by

$$\|G(s)\|_\infty = \sup_{Re(s)\ge 0} \overline{\sigma}(G(s)). \tag{5}$$

The unforced SFOS of (3) is written as

$$ED^{\alpha}x = Ax, 1 \le \alpha < 2. \tag{6}$$

Definition 4 ([5], [39]):

1. A system, as denoted by (6), is considered regular if there exists $s \in \mathbb{C}$ such that $det\,(s^{\alpha}E - A)$ is not identically zero.
2. The system (6) is impulse-free if $deg\left(det\left(\lambda E - A\right)\right) = rank(E)$, where $\lambda \in \mathbb{C}$.
3. The system (6) is stable if $\lambda \in \sigma(E, A)$ and $\left|\arg(\lambda)\right| > \frac{\alpha\pi}{2}$, where $\sigma(E, A) := \{\lambda\ |\lambda \in \mathbb{C}$ and finite, $\det\left(\lambda E - A\right) = 0\}$.
4. The system (6) is admissible if it is regular, impulse-free, and stable.

The following lemma presented is a complex version Schur complement.

Lemma 1 ([40]): A complex Hermitian matrix satisfies the inequality $\begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix} < 0$ if and only if $R < 0$ and $P - QR^{-1}Q^* < 0$.

Remark 1 ([40]): A Hermitian matrix $X_1 + jX_2 = (X_1 + jX_2)^* > 0$ if and only if $\begin{bmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{bmatrix}^T > 0$, where $X_1,\ X_2 \in R^{n\times n}$ and $j = \sqrt{-1}$.

Let $\theta = \left(1 - \frac{\alpha}{2}\right)\pi$ where α is the fractional-order. The following theorem provides an important theoretical foundation for extending the Lyapunov stability theorem to SFOSs.

Theorem 1 ([30]): The unforced SFOS (6) is admissible if and only if the following equivalent conditions hold:

(i) There exists a positive definite matrix $X \in R^{n\times n}$ and a matrix $Y \in R^{(n-r)\times n}$ such that the following LMI holds:

$$Her\left\{e^{-j\theta}A\left(XE^T + SY\right)\right\} < 0, \tag{7}$$

or, equivalently,

$$Her\left\{\Theta \otimes A\left(XE^T + SY\right)\right\} < 0, \tag{8}$$

where $\Theta = \begin{bmatrix} \sin(\theta_1) & -\cos(\theta_1) \\ \cos(\theta_1) & \sin(\theta_1) \end{bmatrix}$ with $\theta_1 = \frac{\alpha\pi}{2}$ and $S \in R^{n\times(n-r)}$ is a full column rank matrix satisfying $ES = 0$.

(ii) There exists a positive definite matrix $X \in R^{n\times n}$ such that

$$EX = X^TE^T \ge 0,\ Her\left\{e^{-j\theta}AX\right\} < 0. \tag{9}$$

Theorem 2 ([5]): Given a real constant $\gamma > 0$. The SFOS (3) is admissible and $\|G(s)\|_\infty < \gamma$ if and only if the following equivalent conditions hold:

(i) There exist matrices $P \in C^{n\times n}$ and $Q \in C^{(n-r)\times n}$ such that the following LMI holds:

$$\begin{bmatrix} \mathrm{Her}\left(e^{-j\theta}A\left(PE^T+SQ\right)\right) & * & * \\ e^{-j\theta}C\left(PE^T+SQ\right) & -\gamma^2 I_p & * \\ B^T & 0 & -I_m \end{bmatrix} < 0. \quad (10)$$

(ii) There exist matrices $P_0 \in C^{n\times n}$, $Q_0 \in C^{(n-r)\times n}$, $\mathcal{V} \in R^{(n+p)\times(n+p)}$, and $\mathcal{U} \in R^{(n+p)\times(n+p)}$ such that the following LMI holds

$$\begin{bmatrix} e^{-j\theta}\mathcal{A}_c\mathcal{U}+e^{j\theta}\mathcal{U}^T\mathcal{A}_c^T+\mathcal{B}\mathcal{B}^T & * \\ \mathcal{P}_c\mathcal{E}_c^T+\mathcal{S}\mathcal{Q}_c+e^{j\theta}\mathcal{V}^T\mathcal{A}_c^T-\mathcal{U} & -\mathcal{V}-\mathcal{V}^T \end{bmatrix} < 0 \quad (11)$$

where $\mathcal{S}$ satisfied $\mathcal{E}_c\mathcal{S} = 0$, and

$$\mathcal{E}_c = \begin{bmatrix} E & 0_{n\times p} \\ 0_{p\times n} & \frac{\gamma^2}{2}I_p \end{bmatrix}, \mathcal{A}_c = \begin{bmatrix} A & 0_{n\times p} \\ C & -I_p \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} B \\ 0_{p\times m} \end{bmatrix},$$

$$\mathcal{P}_c = \begin{bmatrix} P & 0_{n\times p} \\ 0_{p\times n} & e^{j\theta}I_p \end{bmatrix}, \ \mathcal{Q}_c = \begin{bmatrix} Q_0 & 0_{(n-r)\times p} \end{bmatrix}.$$

A novel form of generalized bounded real lemma for continuous-time linear descriptor systems with $\alpha = 1$ has been introduced in recent research [37]. In the following, define $Q_E = qq^T$, where $q \in R^{n\times(n-r)}$ is a basis matrix of $kerE^T$.

Lemma 2 ([37]): The system (3) with $\alpha = 1$ is admissible and satisfies

$$\left\| C(sE-A)^{-1}B + D \right\|_\infty < \gamma \quad (12)$$

if and only if there is a real symmetric matrix P of size $n \times n$ that satisfies the following LMIs:

$$E^T PE + A^T Q_E A > 0 \quad (13)$$

and (14), as shown at the bottom of the next page.

III. MAIN RESULTS

This section presents the main results, which can be divided into two parts. The first part is a generalized Lyapunov theorem, and the second part is a generalized bounded real lemma for SFOSs.

A. FRACTIONAL GENERALIZED LYAPUNOV THEOREM

The following is a generalized Lyapunov theorem for SFOSs.

Theorem 3: The SFOS (6) is admissible if and only if the following equivalent conditions hold:

(i) There exists a Hermitian matrix $P \in C^{n\times n}$ such that the following LMI holds:

$$E^T PE + A^T Q_E A > 0 \quad (15)$$

and

$$e^{-j\theta}E^T PA + e^{j\theta}A^T PE - A^T Q_E A < 0. \quad (16)$$

(ii) There exist $n \times n$ real matrices $P_r = P_r^T$ and $P_i = -P_i^T$ satisfying

$$\begin{bmatrix} E^T P_r E + A^T Q_E A & E^T P_i E \\ -E^T P_i E & E^T P_r E + A^T Q_E A \end{bmatrix} > 0, \quad (17)$$

and

$$\begin{bmatrix} \Gamma_1 & \Gamma_2 \\ -\Gamma_2 & \Gamma_1 \end{bmatrix} < 0; \quad (18)$$

where

$$\Gamma_1 = cos(\theta)\left(E^T P_r A + A^T P_r E\right) + sin(\theta)\left(E^T P_i A + A^T P_i E\right) - A^T Q_E A,$$

$$\Gamma_2 = cos(\theta)\left(E^T P_i A - A^T P_i E\right) + sin(\theta)\left(A^T P_r E - E^T P_r A\right).$$

Proof: First, we prove the sufficiency of part (i). To demonstrate that the system (6) is regular and impulse-free, we assume the validity of inequalities (15) and (16). By employing singular value decomposition specifically on matrix E, we identify invertible and orthogonal matrices $U \in R^{n\times n}$ and $V \in R^{n\times n}$ such that

$$U^T EV = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix} \quad (19)$$

where E_{11} is a diagonal matrix, whose diagonal entries are the positive singular values of E. We then decompose U as $U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$, with U_2 forming a basis matrix for $\ker E^T$. Setting $q = U_2$. In this case, $U^T q = U^T U_2 = \begin{bmatrix} 0 \\ I \end{bmatrix}$. Next, we decompose $U^T AV$ as

$$U^T AV = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \quad (20)$$

Assuming $P = P^T$ and that it satisfies inequalities (15) and (16), we obtain

$$U^T PU = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}.$$

Following a method similar to that in [37], we deduce that A_{22} is invertible when the inequality (15) holds. According to Theorem 2 in [30] this means the system (6) is regular and impulse-free.

Next, to demonstrate the stability of system (6), we consider a finite eigenvalue λ and its corresponding eigenvector v of the system (6), as detailed in reference [41]. Specifically, we have

$$\lambda^\alpha Ev = Av. \quad (21)$$

We pre-multiply and post-multiply inequality (15) and (16) with v^* and v, respectively, to derive

$$\begin{aligned} v^*E^T PEv + v^*A^T qq^T Av &= v^*E^T PEv + \left|\lambda^\alpha\right|^2 v^*E^T qq^T Ev \\ &= v^*E^T PEv > 0; \end{aligned} \quad (22)$$

and

$$\begin{aligned} & v^*\left(e^{-j\theta}E^T PA\right)v + v^*\left(e^{j\theta}A^T PE\right)v - v^*\left(A^T qq^T A\right)v \\ &= v^*E^T PEv\left(e^{-j\theta}\lambda^\alpha + e^{j\theta}\overline{\lambda^\alpha}\right) - \left|\lambda^\alpha\right|^2 v^*E^T qq^T Ev \\ &= v^*E^T PEv\left(e^{-j\theta}\lambda^\alpha + e^{j\theta}\overline{\lambda^\alpha}\right) < 0. \end{aligned}$$

From this, we deduce: $\left(e^{-j\theta}\lambda^{\alpha}+e^{j\theta}\overline{\lambda^{\alpha}}\right) < 0$. Consequently, this implies that $Re\left(e^{-j\theta}\lambda^{\alpha}\right) < 0$. Therefore, for any finite eigenvalues $\lambda \in \sigma(E, A)$, we have $\left|\arg\left(\lambda\right)\right| > \frac{\alpha\pi}{2}$ and the system (6) is deemed stable [39].

Second, we establish the necessity of condition (i). Given that system (6) is admissible, without loss of generality, we assume it is in Weierstrass form as follows:

$$E=\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, A=\begin{bmatrix} \Lambda & 0 \\ 0 & I_{n-r} \end{bmatrix} \tag{23}$$

where $\Lambda \in R^{r\times r}$, and it holds that $\left|\arg\left(\lambda\right)\right| > \frac{\alpha\pi}{2}$ for each $\lambda \in \sigma\left(\Lambda\right)$. We set $q=\begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix}$ and consequently $Q_E=\begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix}$. According to reference [5], there exists $P_{11} > 0$ satisfying

$$e^{-j\theta}\Lambda^T P_{11}+e^{j\theta}P_{11}\Lambda < 0. \tag{24}$$

Let $P=\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$. Assuming the admissibility of system (6), the expressions on the left-hand sides of inequalities (15) and (16) can be represented as follows:

$$\begin{bmatrix} P_{11} & 0 \\ 0 & I_{n-r} \end{bmatrix}, \tag{25}$$

and

$$\begin{bmatrix} e^{-j\theta}\Lambda^T P_{11}+e^{j\theta}P_{11}\Lambda & e^{-j\theta}P_{12} \\ e^{j\theta}P_{12}^T & -I \end{bmatrix}. \tag{26}$$

Based on (25), we confirm that inequality (15) indeed holds due to $P_{11} > 0$ and $I_{n-r} > 0$. On the other hand, since (24) holds and P_{12} can be chosen arbitrarily, inequality (16) is also satisfied.

Finally, we demonstrate that conditions (i) and (ii) are equivalent. Consider P_r and P_i as real-valued $n\times n$ matrices such that $P = P_r + jP_i$. The inequality (15) can be equivalently expressed as:

$$\begin{aligned} E^T PE+A^T Q_E A &= E^T\left(P_r+jP_i\right)E+A^T QA \\ &= \left(E^T P_r E+A^T Q_E A\right)+jE^T P_i E > 0; \end{aligned} \tag{27}$$

On the other hand, by applying Euler's formula and substituting P_r and P_i into inequality (16), it transforms as follows:

$$\begin{aligned} &\Big(\cos\left(\theta\right)\left(E^T P_r A+A^T P_r E\right) \\ &+\sin\left(\theta\right)\left(E^T P_i A+A^T P_i E\right)-A^T Q_E A\Big) \\ &+j\Big(\cos\left(\theta\right)\left(E^T P_i A-A^T P_i E\right) \\ &+\sin\left(\theta\right)\left(A^T P_r E-E^T P_r A\right)\Big) < 0. \end{aligned} \tag{28}$$

In accordance with Remark 1, inequality (27) is equivalent to condition (17) and inequality (28) is equivalent to condition (18). ■

Recently, the stability analysis of singular fractional-order systems has been a subject of study, with methods presented using LMIs that involve more than one matrix variable, as referenced in [5] and [30]. Theorem 3 marks a significant advancement in this field. This progression is achieved through the introduction of a novel stability condition that utilizes strict LMIs. Notably, this method employs a single indefinite matrix variable, which, as discussed in [37], offers a practical solution for enhanced computational efficiency and user-friendliness. Conversely, to tackle the computational limitations encountered by existing toolboxes when handling complex LMIs, Theorem 3 introduces an equivalent condition using strict LMIs composed entirely of real numbers. This provides a pragmatic approach to overcoming the computational challenges in the stability analysis of these advanced systems.

B. FRACTIONAL GENERALIZED BOUNDED REAL LEMMA

The following theorem provides a bounded real lemma for SFOSs.

Theorem 4: The system (3) is admissible under $w = 0$, and satisfies

$$\left\|C\left(s^{\alpha}E-A\right)^{-1}B+D\right\|_{\infty} < \gamma \tag{29}$$

if and only if the following equivalent conditions hold:

(i) There exists a Hermitian matrix P of size $n\times n$ that fulfills:

$$E^T PE+A^T Q_E A > 0 \tag{30}$$

and

$$\begin{bmatrix} \Pi_0+C^T C & E^T PB+C^T D \\ B^T PE+D^T C & -\gamma^2 I+D^T D+B^T Q_E PQ_E B \end{bmatrix} < 0; \tag{31}$$

where $\Pi_0=e^{-j\theta}E^T PA+e^{j\theta}A^T PE-A^T Q_E PQ_E A$.

(ii) There exists a Hermitian matrix P of size $n\times n$ that fulfills:

$$E^T PE+A^T Q_E A > 0 \tag{32}$$

and

$$\begin{bmatrix} \Pi_0 & E^T PB & C^T \\ B^T PE & -\gamma^2 I+B^T Q_E PQ_E B & D^T \\ C & D & -I \end{bmatrix} < 0; \tag{33}$$

where $\Pi_0=e^{-j\theta}E^T PA+e^{j\theta}A^T PE-A^T Q_E PQ_E A$.

(iii) There exist $n\times n$ real matrices $P_r=P_r^T$ and $P_i=-P_i^T$ such that

$$\begin{bmatrix} E^T P_r E+A^T Q_E A & E^T P_i E \\ -E^T P_i E & E^T P_r E+A^T Q_E A \end{bmatrix} > 0, \tag{34}$$

$$\begin{bmatrix} E^T PA+A^T PE+C^T C-A^T Q_E PQ_E A & E^T PB+C^T D \\ B^T PE+D^T C & -\gamma^2 I+D^T D+B^T Q_E PQ_E B \end{bmatrix} < 0. \tag{14}$$

and

$$\begin{bmatrix} \Pi_1 & \Pi_2 \\ -\Pi_2 & \Pi_1 \end{bmatrix} < 0 \tag{35}$$

where

$$\Pi_1 = \begin{bmatrix} \Upsilon_1 & \Upsilon_2 \\ \Upsilon_2^T & \Upsilon_3 \end{bmatrix}, \ \Pi_2 = \begin{bmatrix} \Phi_1 & \Phi_2 \\ \Phi_2^T & \Phi_3 \end{bmatrix},$$

with

$$\begin{aligned}
\Upsilon_1 &= \cos(\theta)\left(E^T P_r A + A^T P_r E\right) \\
&\quad + \sin(\theta)\left(E^T P_i A + A^T P_i E\right) \\
&\quad + A^T Q_E P_r Q_E A + C^T C, \\
\Upsilon_2 &= E^T P_r B + C^T D, \\
\Upsilon_3 &= -\gamma^2 I + D^T D + B^T Q_E P_r Q_E B, \\
\Phi_1 &= \cos(\theta)\left(E^T P_i A - A^T P_i E\right) \\
&\quad + \sin(\theta)\left(A^T P_r E - E^T P_r A\right) + A^T Q_E P_i Q_E A, \\
\Phi_2 &= E^T P_i B, \\
\Phi_3 &= B^T Q_E P_i Q_E.
\end{aligned}$$

Proof: Initially, to confirm the sufficiency of condition (i), we deduce that under condition (30), system (3) is regular and impulse-free. When expressed in Weierstrass form, the system (3) is characterized by:

$$E = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, A = \begin{bmatrix} \Lambda & 0 \\ 0 & I_{n-r} \end{bmatrix} \tag{36}$$

with corresponding matrices $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ and $C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$.

Consequently, under these conditions, the transfer function (4) is expressed as:

$$C_1\left(s^\alpha I - \Lambda\right)^{-1} B_1 + D - C_2 B_2 \tag{37}$$

Suppose a Hermitian matrix P satisfies (30) and (31) and is expressed as:

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix}$$

According to Theorem 3, and drawing from findings in [37], we find that inequality (30) implies $P_{11} > 0$. Building on this, expression (31) can be equivalently transformed as follows:

$$\Psi_1 \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{12}^* & \delta_{22} & \delta_{23} \\ \delta_{13}^* & \delta_{23}^* & \delta_{33} \end{bmatrix} \Psi_2 = \begin{bmatrix} \delta_{11} & \zeta_{12} & \delta_{12} \\ \zeta_{12}^* & \zeta_{22} & \zeta_{32} \\ \delta_{12}^* & \zeta_{32}^* & \delta_{22} \end{bmatrix} < 0 \tag{38}$$

where

$$\begin{aligned}
\delta_{11} &= e^{j\theta}\Lambda^T P_{11} + e^{-j\theta} P_{11}\Lambda + C_1^T C_1, \\
\delta_{12} &= e^{-j\theta} P_{12} + C_1^T C_2, \\
\delta_{13} &= P_{11} B_1 + P_{12} B_2 + C_1^T D, \\
\delta_{22} &= C_2^T C_2 - P_{22}, \\
\delta_{23} &= C_2^T D, \\
\delta_{33} &= -\gamma^2 I + D^T D + B_2^T P_{22} B_2, \\
\zeta_{12} &= P_{11} B_1 + C_1^T\left(D - C_2 B_2\right), \\
\zeta_{22} &= -\gamma^2 I + \left(D - C_2 B_2\right)^T\left(D - C_2 B_2\right), \\
\zeta_{32} &= C_2^T\left(D - C_2 B_2\right) + B_2,
\end{aligned}$$

with

$$\Psi_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & -B_2^T & I \\ 0 & I & 0 \end{bmatrix} \text{ and } \Psi_2 = \begin{bmatrix} I & 0 & 0 \\ 0 & -B_2 & I \\ 0 & I & 0 \end{bmatrix}.$$

By (38), we obtain

$$\begin{bmatrix} \delta_{11} & \zeta_{12} \\ \zeta_{12}^* & \zeta_{22} \end{bmatrix} < 0. \tag{39}$$

This implies that $e^{j\theta}\Lambda^T P_{11} + e^{-j\theta} P_{11}\Lambda + C_1^T C_1 < 0$, demonstrating the stability of Λ, given that P_{11} is positive definite. Using inequality (39), we can then infer that $\|C_1\left(s^\alpha I - \Lambda\right) B_1 + D - C_2 B_2\|_\infty < \gamma$ as shown in [42]. Consequently, the system (3) is admissible, and $\|G(s)\|_\infty < \gamma$. In demonstrating the necessity of condition (i) we apply a methodology analogous to that used in [37]. Specifically, we assume the system (3) to be in the Weierstrass form as delineated in (36) and establish its admissibility with $\|G(s)\|_\infty < \gamma$. This results in determining a positive define matrix P_{11} in accordance with inequality (39). Extending this approach, we ascertain the existence of matrices P_{12} and P_{22} thereby constructing the matrix $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix}$. This matrix complies with conditions (30) and (31), mirroring the logic applied in the proof of Theorem 3.

Finally, we demonstrate that conditions (i), (ii), and (iii) are equivalent. Following a similar approach to the proof of Theorem 3, let's presume that $P = P_r + jP_i$ with $P_r, P_i \in R^{n\times n}$ and apply Euler's formula to substitute them into inequality (31). This leads to the following LMI:

$$\Pi_1 + j\Pi_2 = \begin{bmatrix} \Upsilon_1 & \Upsilon_2 \\ \Upsilon_2^T & \Upsilon_3 \end{bmatrix} + j\begin{bmatrix} \Phi_1 & \Phi_2 \\ \Phi_2^T & \Phi_3 \end{bmatrix} < 0.$$

According to Remark 1, the above LMI is equivalent to

$$\begin{bmatrix} \Pi_1 & \Pi_2 \\ -\Pi_2^T & \Pi_1 \end{bmatrix} < 0.$$

Hence, conditions (i) and (iii) are equivalent. Subsequently, we demonstrate the equivalency of conditions (i) and (ii). The LMIs (31) and (33) are equivalent by employing the Schur complement. Hence, conditions (i) and (iii) are equivalent, completing the proof of Theorem 4. ■

Remark 2: According to references [13], [14], [15], [16], [42], the results established in Theorems 3 and 4 can also be extended to the $0 \leq \alpha < 1$ range case. Numerous methods are available to achieve this extension, each necessitating extensive derivations and yielding different outcomes. However, as a result of the concise scope of this study and the specific concentration on the $1 \leq \alpha < 2$ range, the results for the $0 \leq \alpha < 1$ case are not included.

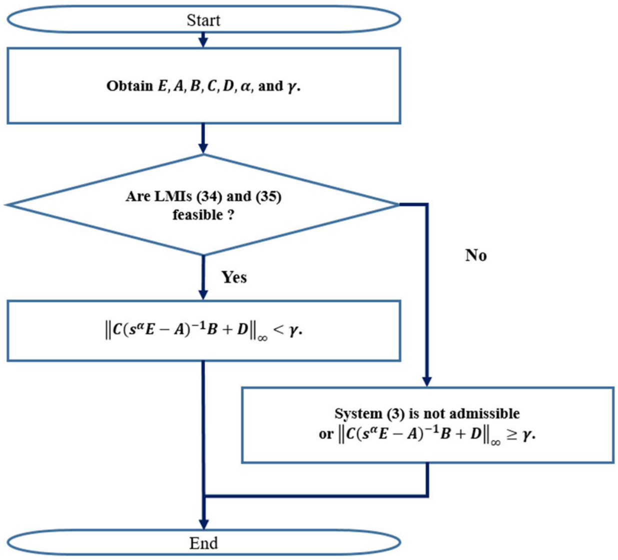

FIGURE 1. The flow chart for verification process using Theorem 4.

Remark 3: In practice, the inherent complexity of fractional-order differentials complicates the simulation and validation of systems. This difficulty is further exacerbated by the relatively undeveloped state of current simulation tools and their corresponding toolboxes, particularly when validating high-dimensional systems. The application of the methods described in references [5], [7] introduces the challenge of managing complex LMIs. This paper builds on the work presented in [37], which showcases the advantages of univariate LMIs, and develops purely real LMIs. This advancement offers a more streamlined and accessible method for the subsequent validation of the stability of fractional-order descriptor systems. Figure 1 displays the solution flowchart.

Remark 4: We provide brief comparisons between the new generalized bounded real lemma (Theorem 4) and those obtained in [5], [7], and [43] as shown in Table 1. Table 1 systematically compares various bounded real lemmas, highlighting differences in types, the number of matrix variables, and the differential orders of the systems they address. Theorem 3 in reference [5] and Theorem 4 in this paper provide detailed strict LMIs for fractional-order systems, underscoring their broad applicability to diverse mathematical challenges. Moreover, the LMIs in Theorem 4 involve only one matrix variable to be solved, enhancing the high-order fractional descriptor linear systems, solving the LMIs in Theorem 4 is more efficient than those presented in references [5], [7], [43]. However, recent findings by some researchers indicate that, in certain practical situations, the limited number of variable matrices can pose challenges in finding suitable solutions for controller design. This realization highlights a crucial area for potential improvement in our methodology, especially under diverse practical conditions where controller optimization is paramount.

TABLE 1. Comparisons with existing generalized bounded real lemmas.

Name	Type	Number of matrix variables	Differential order α
Theorem 1 in [43]	Non-strict LMIs with an equality constraint	1	$0 \leq \alpha < 1$
Theorem 1 in [7]	Non-strict LMIs with an equality constraint	1	$1 \leq \alpha < 2$
Theorem 3 in [5]	Strict LMIs	2	$1 \leq \alpha < 2$
Theorem 4	Strict LMIs	1	$1 \leq \alpha < 2$

IV. ILLUSTRATIVE EXAMPLE

Example 1: The descriptor electrical circuit depicted in [7] includes resistances R_1, R_2, R_3, inductances L_1, L_2, L_3, and source voltages e_1, e_2. By applying Kirchhoff's laws, the dynamic equation of the circuit can be derived:

$$\begin{aligned} ED^{\alpha}x(t) &= Ax(t) + Bw(t) \\ z(t) &= Cx(t) \end{aligned}$$

where

$$\mathrm{E} = \begin{bmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathrm{A} = \begin{bmatrix} -R_1 & 0 & -R_3 \\ 0 & -R_2 & -R_3 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -4 & 0 & -2 \\ 0 & -2 & -5 \\ 1 & 1 & -1 \end{bmatrix},$$

$C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ with $x(t) = \begin{bmatrix} i_{L_1}(t) \\ i_{L_2}(t) \\ i_{L_3}(t) \end{bmatrix}, w(t) = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ and $\alpha = 1.5$, given $\gamma = 0.5$. After solving the LMIs (34) and (35) in Theorem 4, we obtain

$$P = \begin{bmatrix} 0.01 & -0.01 & 4416 \\ -0.01 & 0.04 & 1129 \\ 4416 & 1129 & 2.52 \end{bmatrix} + j \begin{bmatrix} 0 & 0.06 & 0.03 \\ -0.06 & 0 & 0.24 \\ -0.03 & -0.24 & 0 \end{bmatrix}.$$

The solution is indefinite, given the eigenvalues of P as $\{\pm 4556.8,\ 0.043\}$. The presence of a solution P satisfying LMIs (17) and (18) indicates the admissibility of the pair (E, A). The finite eigenvalues of (E, A) are $\{-1.6, -0.4\}$. Moreover, the maximum singular value of $G(jw)$ indicates that the peak value is around 0.45, a fact that is also revealed and illustrated in **Figure 2**.

Example 2: In this study, we have transformed Theorem 2 in [5] into various formats to facilitate a comparison with the results of Theorem 4, our own findings. Assuming that $P = P_r + jP_i$, $Q = Q_r + jQ_i$, with $P_r, P_i, \in R^{n\times n}$ and $Q_r, Q_i \in R^{(n-r)\times n}$, we substitute them into LMI (10) and obtain the

TABLE 2. Comparative analysis of average execution time for LMIs.

(n, m)	$(3, 2)$	$(6, 4)$	$(15, 12)$	$(30, 20)$	$(45, 30)$	$(51, 30)$	$(60, 40)$
Theorem 2 [4]	0.018 s	0.100 s	20.87 s	76.10 s	830.80 s	1810.71 s	4548.49 s
Theorem 4	0.017 s	0.031 s	1.59 s	6.28 s	57.33 s	172.58 s	540.50 s
Number of test sets	100	100	50	50	15	15	15

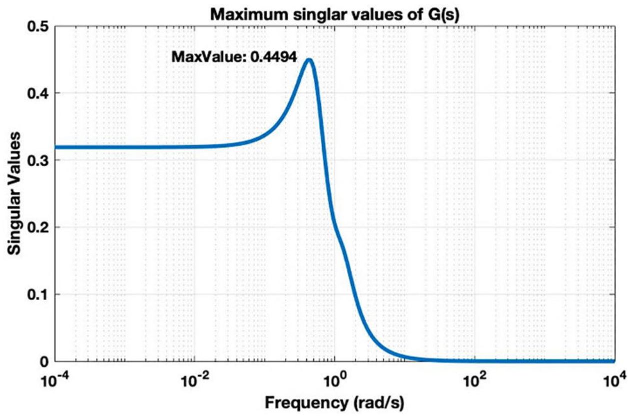

FIGURE 2. Maximum singular values of $G(jw)$ illustrated in Example 1.

following expression using the Euler formula:

$$\Omega_r + j\Omega_i \equiv \begin{bmatrix} \Omega_1 & \Omega_2 & B \\ \Omega_2^T & -\gamma^2 I_p & 0 \\ B^T & 0 & -I_m \end{bmatrix} + j \begin{bmatrix} \Omega_3 & \Omega_4 & 0 \\ -\Omega_4^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} < 0 \tag{40}$$

where

$$\Omega_1 = cos(\theta)\left(AP_rE^T + ASQ_r + EP_r^TA^T + S^TQ_r^TA^T\right) + sin(\theta)\left(AP_iE^T + ASQ_i + EP_i^TA^T + S^TQ_i^TA^T\right),$$

$$\Omega_2 = cos(\theta)\left(EP_r^TC^T + S^TQ_r^TC^T\right) + sin(\theta)\left(EP_i^TC^T + S^TQ_i^TC^T\right),$$

$$\Omega_3 = cos(\theta)\left(AP_iE^T + ASQ_i - EP_i^TA^T - S^TQ_i^TA^T\right) + sin(\theta)\left(EP_i^TA^T + S^TQ_i^TA^T - AP_rE^T - ASQ_r\right),$$

and

$$\Omega_4 = sin(\theta)\left(EP_r^TC^T + S^TQ_r^TC^T\right) - cos(\theta)\left(EP_i^TC^T + S^TQ_i^TC^T\right).$$

The LMI (40) is equivalent to $\begin{bmatrix} \Omega_r & \Omega_i \\ -\Omega_i & \Omega_r \end{bmatrix} < 0$, which serves as our basis for comparison.

The experiment's design revolves around generating system matrices of predefined dimensions to assess the computational efficiency of two distinct LMI solving methods—namely, our Theorem 4 and Theorem 2 from [5]. The primary steps of the experimental process are outlined below:

1: **Initialization:** The dimensions for the state vectors n, input vectors m, are defined as presented in Table 1. The rank of E is set as a random integer between 1 and $\frac{n}{2}$, and $\gamma = 20$.

2: **Number of Test Systems:** To effectively manage the computational load, especially in consideration of hardware limitations, the experiment strategically adjusts the number of tested systems based on the system dimensions (n, m). For systems with dimensions of 3 and 6, we conduct a comprehensive set of 100 tested systems to ensure robust data collection. As dimensions increase to 15 and 30—the number of tested systems is adjusted to 50, balancing the need for detailed analysis with the computational intensity of larger systems. For dimensions surpassing 30, recognizing the significantly increased computational demands and mitigating the strain on our hardware resources, the number of tested systems is further reduced to 15. This tiered approach allows for a thorough investigation across varying system sizes while accommodating the constraints imposed by our available hardware infrastructure. The differences in the number of test sets are presented in Table 1.

3: **Matrix Generation and Computation:**
   - (i) The rank of E is set to r, achieved by multiplying two randomly generated matrices $U \in R^{n\times r}$ and $V \in R^{r\times n}$.
   - (ii) Matrix A is a randomly generated stable matrix. Matrices B and C are also randomly generated.
   - (iii) With the generated system matrices, we solve the LMIs in Theorem 2 [5] and Theorem 4, respectively, and record the computation times. If the H_∞ norm of (E, A, B, C, D) is greater than γ, the computation is disregarded, and another random set is selected.

4: **Average Computation Time Calculation**: After completing all tested systems for a dimension set, the average computation times for both LMI solving methods are calculated and recorded.

The simulation results are shown in Table 2. Table 2 concludes with an evaluation of the computational performance of Theorem 4 (our theorem) and Theorem 2 (as described in [5]) across a range of system dimensions. The findings highlight the computational efficiency of Theorem 4, particularly in systems with higher dimensions. This experiment not only showcases the practical applications of our theoretical discoveries but also illuminates paths for future research and optimization in the LMI solving processes for singular Caputo fractional-order systems.

V. CONCLUSION

This article introduces a novel generalized Lyapunov theorem and a generalized bounded real lemma, tailored for SFOSs characterized by a fractional-order of α. By showcasing the feasibility of strict LMIs employing only a single matrix variable, the proposed paper achieves the outlined objectives. Examples that illustrate the concept are provided to juxtapose the differences and benefits of the results herein with the extant literature. In subsequent studies, we intend to delve deeper into these findings, applying them to the challenge of designing H_∞ controllers for fractional-order descriptor systems.

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