IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received 27 June 2024, accepted 23 July 2024, date of publication 29 July 2024, date of current version 7 August 2024.

Digital Object Identifier 10.1109/ACCESS.2024.3434701

THEORY

Inverse Optimality of Regulation Design for Korteweg-De Vries-Burgers Equation

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This work was supported by the National Natural Science Foundation of China under Grant 62173131.

ABSTRACT In optimal control, it is often necessary to solve Hamilton-Jacobi-Isaacs (HJI) partial differential equation, but it is not only difficult to solve, sometimes even impossible to solve. It is possible to avoid solving the HJI equation by using inverse optimal methods. We investigate inverse optimality of regulation design for Korteweg-de Vries-Burgers (KdVB) equation. Two kinds of boundary control laws are achieved to regulate the state of closed-loop system to the set point from any initial value. In order to regulate the convergent speed of the closed-loop system, one or two parameters are designed in the boundary control laws. We proved that boundary control laws are optimal for two meaningful functionals, respectively. The effectiveness of the proposed design has been shown through simulations, and the convergence speed of the closed-loop system accelerates with increase of adjustable parameters.

INDEX TERMS Korteweg-de Vries-Burgers equation, inverse optimality, regulation design, boundary control.

I. INTRODUCTION

From a physical point of view, Korteweg-de Vries-Burgers (KdVB) equation represents a model for the motion of long water waves in channels of shallow depth, in which three different phenomena are presented, namely, nonlinear convection, dispersion and dissipation [1], [2]. Therefore, controlling the height of long water waves through the two boundaries of the channel is a meaningful task. This is the reason why regulation design of KdVB is investigated in this paper.

In optimal control, it is often necessary to solve Hamilton-Jacobi-Isaacs (HJI) partial differential equations (PDEs). But a simple example is given to illustrate that the HJI equation is not only difficult to solve, but also impossible to solve in [3]. It is possible to avoid solving the HJI equation by using inverse optimal methods. Inverse optimal control is originated by Kalman and introduced into robust nonlinear control via Freeman [4] based on robust control Lyapunov functions [5]. It has been proven that for a class of input unmodeled dynamics, the inverse optimal controller has a margin of stability in [3].

For linear systems with time-varying input delay and additive disturbances, inverse optimal control are presented in [6] and [7]. Inverse optimal control for strict-feedforward nonlinear systems with input delays can be found in [8]. For a class of nonlinear systems with dynamic uncertainties, the problem of inverse optimal adaptive control is solved and the proposed methods are successfully applied to industrial robots in [9]. An observer-based fuzzy adaptive inverse optimal output feedback control is developed for a class of nonlinear systems in strict-feedback form with unknown dynamics by using the inverse optimal principle and adaptive backstepping design theory in [10]. Global stabilization of Burgers' equation is established, most of important, inverse optimal control of Burgers' equation is presented in [11]. How to extend the inverse optimal control method in [11] to KdVB equation will be an interesting work.

In recent decades, many significant achievements have been made in the stability analysis and control synthesis for KdVB equation. A boundary feedback for the KdVB is achieved and global boundary stabilization of the closed-loop system is established in [12]. In [13], a strength boundary

The associate editor coordinating the review of this manuscript and approving it for publication was Xiwang Dong.

control of KdVB equation is presented, further results on stabilization for KdVB equation and well-posedness of the closed-loop system are drawn. Composite disturbance rejection control for KdVB equation under event-triggering mechanism is found in [14]. Robust stabilization [15], finite-time boundary control [16], passivity-based boundary stabilization [17] are also appeared. For a generalized KdVB equation with variable dissipation parameter, stability analysis is achieved in [18]. Observer-based dissipative saturation control is presented for KdVB equation with stochastic noise and incomplete measurable information in [19]. Well-posedness and stability analysis are established for the KdVB equation with infinite memory in [20], with timely delay in [21]. Optimal control for KdVB equation are given in [22] and [23]. However, to the author's knowledge, inverse optimal regulation design of KdVB has never been published.

In this paper, we consider inverse optimality of regulation design for KdVB equation, the main contributions are as follows:

(1) For a given set point, two kinds of boundary control laws are established to regulate the state of closed-loop system to the set point from any initial value. In order to regulate the convergent speed of the closed-loop system, one or two parameters are designed in the boundary control laws.

(2) It is proved that boundary control laws are optimal for two meaningful functionals, respectively.

(3) The effectiveness of the proposed design has been shown through simulations, and the convergence speed of the closed-loop system accelerates with increase of adjustable parameters.

This paper is organized as follows: system description and problem statement are in Section II. Regulation design is in III. Inverse optimality of regulation design is in IV, and simulation results are shown in V. Concluding remarks are in VI.

Notation. For a scalar function $u(x,t) \in C^3([0,1] \times [0,\infty), R)$, denote with $u_t(x,t) = \frac{\partial u(x,t)}{\partial t}$, $u_x(x,t) = \frac{\partial u(x,t)}{\partial x}$, $u_{xx}(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}$, $u_{xxx}(x,t) = \frac{\partial^3 u(x,t)}{\partial x^3}$.

II. SYSTEM DESCRIPTION AND PROBLEM STATEMENT

Consider KdVB equation given by

$$w_t(x,t) = \varepsilon w_{xx}(x,t) - \delta w_{xxx}(x,t) - w_x(x,t)w(x,t), \quad (1)$$

$$w(x,0) = w_0(x), \quad (2)$$

where $t \geq 0$, $0 \leq x \leq 1$ and $w(x,t)$ is the system state, and $\varepsilon > 0, \delta > 0$ denote dissipation and dispersion coefficients, respectively. Our objective is to achieve set regulation:

$$\lim_{t\to\infty} w(x,t) = H, \quad (3)$$

in $L_2[0,1]$, where H is a constant by boundary controls. Without loss the generality, we assume $H \geq 0$. Denote the error as

$$u(x,t) = w(x,t) - H, \quad (4)$$

then (1)–(2) can be rewritten as

$$u_t(x,t) = \varepsilon u_{xx}(x,t) - \delta u_{xxx}(x,t) - u_x(x,t)(u(x,t) + H), \quad (5)$$

$$u(x,0) = w_0(x) - H, \quad (6)$$

In order to get (3), the problem is transferred to guarantee that $u(x,t)$ is convergent to zero in $L_2[0,1]$ by boundary controls.

The following Lemma from [11] is needed in this paper.

Lemma 1 (Poincare's Inequality): For any $u \in C^1[0,1]$, the following inequalities hold

$$\int_0^1 u(x)^2 dx \leq 2u(0)^2 + 4\int_0^1 u_x(x)^2\, dx, \quad (7)$$

$$\int_0^1 u(x)^2 dx \leq 2u(1)^2 + 4\int_0^1 u_x(x)^2\, dx. \quad (8)$$

III. REGULATION DESIGN

In this section, boundary controls are designed to globally exponentially stabilize system (5)–(6) in $L_2[0,1]$. Further, globally exponentially regulate design to system (1)–(2) is achieved.

Theorem 1: Boundary controls

$$u(0,t) = 0, \quad (9)$$

$$u_x(1,t) = -\frac{\varepsilon}{2\delta}u(1,t), \quad (10)$$

$$u_{xx}(1,t) = \frac{a_1}{2\delta}u(1,t) + \frac{1}{18a_1\delta}u(1,t)^3, \quad (11)$$

with $a_1 > 0$, globally exponentially stabilize system (5)–(6) in $L_2[0,1]$.

Proof. Let

$$V(t) = \frac{1}{2}\int_0^1 u(x,t)^2\, dx, \quad (12)$$

and compute its time derivative along trajectory of (5)–(6), it holds

$$\begin{aligned}
\dot{V}(t) &= \varepsilon u(1,t)u_x(1,t) - \varepsilon u(0,t)u_x(0,t) \\
&\quad - \varepsilon\int_0^1 u_x(x,t)^2 dx - \delta u(1,t)u_{xx}(1,t) \\
&\quad + \delta u(0,t)u_{xx}(0,t) + \delta\int_0^1 u_x(x,t)u_{xx}(x,t)dx \\
&\quad - \frac{1}{3}u(1,t)^3 + \frac{1}{3}u(0,t)^3 - \frac{H}{2}u(1,t)^2 + \frac{H}{2}u(0,t)^2 \\
&= \frac{\delta}{2}u_x(1,t)^2 + u(1,t)(\varepsilon u_x(1,t) - \delta u_{xx}(1,t) \\
&\quad + \frac{a_1}{2}u(1,t) + \frac{1}{18a_1}u(1,t)^3) \\
&\quad - u(0,t)\,(\varepsilon u_x(0,t) - \delta u_{xx}(0,t) \\
&\quad - \frac{H}{2}u(0,t) - \frac{a_0}{2}u(0,t) - \frac{1}{18a_0}u(0,t)^3\Big) \\
&\quad - \frac{a_0}{2}(u(0,t) - \frac{1}{3a_0}u(0,t)^2)^2 \\
&\quad - \frac{a_1}{2}(u(1,t) - \frac{1}{3a_1}u(1,t)^2)^2
\end{aligned}$$

$$-\varepsilon\int_0^1 u_x(x,t)^2dx-\frac{H}{2}u(1,t)^2-\frac{\delta}{2}u_x(0,t)^2. \quad (13)$$

Using boundary controls (9)–(11), we have

$$\begin{aligned}\dot{V}(t)=&-\frac{3\varepsilon^2}{8\delta}u(1,t)^2-\frac{a_1}{2}(u(1,t)-\frac{1}{3a_1}u(1,t)^2)^2\\&-\varepsilon\int_0^1 u_x(x,t)^2dx-\frac{H}{2}u(1,t)^2-\frac{\delta}{2}u_x(0,t)^2\\\leq&-\varepsilon\int_0^1 u_x(x,t)^2dx-\frac{H}{2}u(1,t)^2-\frac{\delta}{2}u_x(0,t)^2, \quad (14)\end{aligned}$$

thus

$$\dot{V}(t)\leq-\varepsilon\int_0^1 u_x(x,t)^2\,dx. \quad (15)$$

By Lemma 1, we get

$$\dot{V}(t)\leq-\frac{\varepsilon}{2}V. \quad (16)$$

Thus the closed-loop system is globally exponential stable in $L_2[0,1]$. □

Remark 1: In view of $(u(1,t)-\frac{1}{3a_1}u(1,t)^2)^2\geq 0$, and $a_1>0$, from (14), it can be deduced that the convergent speed of the closed-loop system to the equilibrium point accelerates as a_1 increases.

From Theorem 1, we are easy to deduce the Corollary 1.

Corollary 1: Consider system (1)–(2), boundary controls

$$w(0,t)=H, \quad (17)$$

$$w_x(1,t)=-\frac{\varepsilon}{2\delta}(w(1,t)-H), \quad (18)$$

$$w_{xx}(1,t)=\frac{a_1}{2\delta}(w(1,t)-H)+\frac{1}{18a_1\delta}(w(1,t)-H)^3, \quad (19)$$

with $a_1>0$, globally exponentially regulate state of system (1)–(2) to H in $L_2[0,1]$.

Proof. It can be deduced from Theorem 1.

In what follows, another boundary controls are presented to globally exponentially stabilize system (5)–(6) in $L_2[0,1]$.

Theorem 2: Boundary controls

$$\begin{aligned}u_{xx}(0,t)=&\frac{\varepsilon}{\delta}u_x(0,t)-\frac{H+a_0}{2\delta}u(0,t)\\&-\frac{1}{18a_0\delta}u(0,t)^3, \quad (20)\end{aligned}$$

$$u_x(1,t)=0, \quad (21)$$

$$u_{xx}(1,t)=\frac{a_1}{2\delta}u(1,t)+\frac{1}{18a_1\delta}u(1,t)^3, \quad (22)$$

with $a_0>0, a_1>0$, globally exponentially stabilize system (5)–(6) in $L_2[0,1]$.

Proof. Let $V(t)$ be given by (12), and compute its time derivative along trajectory of (5)–(6), it holds (13). With the help of (20)–(22), from (13), we get

$$\begin{aligned}\dot{V}(t)=&-\frac{a_0}{2}(u(0,t)-\frac{1}{3a_0}u(0,t)^2)^2\\&-\frac{a_1}{2}(u(1,t)-\frac{1}{3a_1}u(1,t)^2)^2\\&-\varepsilon\int_0^1 u_x(x,t)^2dx-\frac{H}{2}u(1,t)^2-\frac{\delta}{2}u_x(0,t)^2, \quad (23)\end{aligned}$$

thus

$$\dot{V}(t)\leq-min\left(\frac{\varepsilon}{4},\frac{H}{4}\right)\left(4\int_0^1 u_x(x,t)^2dx+2u(1,t)^2\right). \quad (24)$$

By Lemma 1, we get

$$\dot{V}(t)\leq-min\left(\frac{\varepsilon}{2},\frac{H}{2}\right)V. \quad (25)$$

Thus the closed-loop system is globally exponential stable in $L_2[0,1]$. □

Remark 2: In view of $(u(i,t)-\frac{1}{3a_1}u(i,t)^2)^2\geq 0$, and $a_i>0, i=0,1$, from (23), it can be deduced that the convergent speed of the closed-loop system to the equilibrium point accelerates as a_0 or a_1 increases.

Corollary 2: Consider system (1)–(2), boundary controls

$$\begin{aligned}w_{xx}(0,t)=&\frac{\varepsilon}{\delta}w_x(0,t)-\frac{H+a_0}{2\delta}(w(0,t)-H)\\&-\frac{1}{18a_0\delta}(w(0,t)-H)^3, \quad (26)\end{aligned}$$

$$w_x(1,t)=0, \quad (27)$$

$$w_{xx}(1,t)=\frac{a_1}{2\delta}(w(1,t)-H)+\frac{1}{18a_1\delta}(w(1,t)-H)^3, \quad (28)$$

with $a_0>0$, $a_1>0$, globally exponentially regulate state of system (1)–(2) to H in $L_2[0,1]$.

Proof. It can be deduced from Theorem 2.

IV. INVERSE OPTIMALITY OF REGULATION DESIGN

Follow the boundary control design in Theorem 1, an inverse optimal design is achieved in Theorem 3.

Theorem 3: Under boundary condition

$$u(0,t)=0, \quad (29)$$

boundary controls

$$u_x(1,t)=-\frac{\varepsilon}{\delta}u(1,t), \quad (30)$$

$$u_{xx}(1,t)=\frac{a_1}{\delta}u(1,t)+\frac{1}{9a_1\delta}u(1,t)^3, \quad (31)$$

with $a_1>0$, minimize the cost functional

$$\begin{aligned}J_1=\int_0^\infty\Bigg(L(t)+&\frac{\varepsilon^2u(1,t)^2}{\delta}\\&+\frac{\delta^2u_{xx}(1,t)^2}{a_1+\frac{1}{9a_1}u(1,t)^2}\Bigg)dt, \quad (32)\end{aligned}$$

where $L(t)$ is given as

$$\begin{aligned}L(t)=&\varepsilon\int_0^1 u_x(x,t)^2dx+\frac{H}{2}u(1,t)^2+\frac{\delta}{2}u_x(0,t)^2\\&+\frac{a_1}{2}\left(u(1,t)-\frac{1}{3a_1}u(1,t)^2\right)^2. \quad (33)\end{aligned}$$

Proof. Under boundary condition (29), and boundary controls (30)–(31), from (13), we know

$$\begin{aligned}\dot{V}(t)&\\ =& -\frac{\varepsilon^2}{2\delta}u(1,t)^2 + u(1,t)\left(-\frac{a_1}{2}u(1,t) - \frac{1}{18a_1}u(1,t)^3\right)\\ &-\frac{a_1}{2}\left(u(1,t) - \frac{1}{3a_1}u(1,t)^2\right)^2\\ &-\varepsilon\int_0^1 u_x(x,t)^2dx - \frac{H}{2}u(1,t)^2 - \frac{\delta}{2}u_x(0,t)^2\\ \le& -min\left(\frac{\varepsilon}{2},\frac{H}{2}\right)V. \end{aligned} \tag{34}$$

Thus system (5)–(6) is globally exponentially stable in $L_2[0, 1]$ under (29) and boundary controls (30)–(31). Next, with the help of (29), we have

$$\begin{aligned} L(t) =& 2\varepsilon\int_0^1 u_x(x,t)^2dx + Hu(1,t)^2 + \delta u_x(0,t)^2\\ &+a_1\left(u(1,t) - \frac{1}{3a_1}u(1,t)^2\right)^2\\ &+a_0\left(u(0,t) - \frac{1}{3a_0}u(0,t)^2\right)^2\\ &-\delta u_x(1,t)^2 - 2u(1,t)(\varepsilon u_x(1,t) - \delta u_{xx}(1,t)\\ &+\frac{a_1}{2}u(1,t) + \frac{1}{18a_1}u(1,t)^3)\\ &+2u(0,t)\,(\varepsilon u_x(0,t) - \delta u_{xx}(0,t)\\ &-\frac{H}{2}u(0,t) - \frac{a_0}{2}u(0,t) - \frac{1}{18a_0}u(0,t)^3\Big)\\ &-a_0\left(u(0,t) - \frac{1}{3a_0}u(0,t)^2\right)^2\\ &+\delta u_x(1,t)^2 + 2u(1,t)(\varepsilon u_x(1,t) - \delta u_{xx}(1,t)\\ &+\frac{a_1}{2}u(1,t) + \frac{1}{18a_1}u(1,t)^3)\\ &-2u(0,t)\,(\varepsilon u_x(0,t) - \delta u_{xx}(0,t)\\ &-\frac{H}{2}u(0,t) - \frac{a_0}{2}u(0,t) - \frac{1}{18a_0}u(0,t)^3\Big)\\ =& -2\dot{V}(t) + \delta u_x(1,t)^2\\ &+2\varepsilon u(1,t)u_x(1,t) - 2\delta u(1,t)u_{xx}(1,t)\\ &+a_1u(1,t)^2 + \frac{1}{9a_1}u(1,t)^4. \end{aligned} \tag{35}$$

Substitute $L(t)$ to J_1 in (32), and with the help of (35), we obtain

$$\begin{aligned} J_1 =& 2V(0) - 2V(\infty)\\ &+\int_0^\infty \delta\left(u_x(1,t) + \frac{\varepsilon}{\delta}u(1,t)\right)^2dt\\ &+\int_0^\infty \frac{\left(\delta u_{xx}(1,t) - u(1,t)\left(a_1 + \frac{u(1,t)^2}{9a_1}\right)\right)^2}{a_1 + \frac{u(1,t)^2}{9a_1}}dt. \end{aligned} \tag{36}$$

Since system (5)–(6) is globally exponentially stable in $L_2[0, 1]$ under (29) and boundary controls (30)–(31), so $V(\infty) = 0$. From (36), we know that boundary controls (30)–(31) minimize the cost functional (32), and the minimum value is $2V(0)$.□

Remark 3: Inverse optimal control means that the boundary control is such that the closed-loop system is asymptotically stable, and minimizes a meaningful functional.

Remark 4: The criterion for a meaningful function in inverse optimal control is that each term is non negative and can penalize the control law.

From Theorem 3, we have the following Corollary 3.

Corollary 3: Under boundary condition

$$w(0,t) = H, \tag{37}$$

boundary controls

$$w_x(1,t) = -\frac{\varepsilon}{\delta}(w(1,t) - H), \tag{38}$$

$$w_{xx}(1,t) = \frac{a_1}{\delta}(w(1,t) - H) + \frac{1}{9a_1\delta}(w(1,t) - H)^3, \tag{39}$$

with $a_1 > 0$, minimize the cost functional

$$\bar{J}_1 = \int_0^\infty\left(\bar{L}(t) + \frac{\varepsilon^2}{\delta}(w(1,t) - H)^2 + \frac{\delta^2 w_{xx}(1,t)^2}{a_1 + \frac{1}{9a_1}(w(1,t) - H)^2}\right)dt, \tag{40}$$

where

$$\begin{aligned}\bar{L}(t) =& \varepsilon\int_0^1 w_x(x,t)^2dx + \frac{H}{2}(w(1,t) - H)^2\\ &+\frac{\delta}{2}w_x(0,t)^2\\ &+\frac{a_1}{2}\left(w(1,t) - H - \frac{1}{3a_1}(w(1,t) - H)^2\right)^2. \end{aligned} \tag{41}$$

Theorem 4: Under the boundary condition

$$u_x(1,t) = 0, \tag{42}$$

boundary controls

$$\begin{aligned} u_{xx}(0,t) =& \frac{\varepsilon}{\delta}u_x(0,t) - \frac{H + a_0}{\delta}u(0,t)\\ &-\frac{1}{9a_0\delta}u(0,t)^3, \end{aligned} \tag{43}$$

$$u_{xx}(1,t) = \frac{a_1}{\delta}u(1,t) + \frac{1}{9a_1\delta}u(1,t)^3, \tag{44}$$

with $a_0 > 0$, $a_1 > 0$, minimize the cost functional

$$J_2 = \int_0^\infty\left(l(t) + \frac{\delta^2 u_{xx}(1,t)^2}{a_1 + \frac{1}{9a_1}u(1,t)^2} + \frac{(\varepsilon u_x(0,t) - \delta u_{xx}(0,t))^2}{H + a_0 + \frac{1}{9a_0}u(0,t)^2}\right)dt, \tag{45}$$

where

$$l(t) = \varepsilon \int_0^1 u_x(x,t)^2 dx + \frac{H}{2} u(1,t)^2 + \frac{\delta}{2} u_x(0,t)^2 + \frac{a_0}{2}\left(u(0,t) - \frac{1}{3a_0} u(0,t)^2\right)^2 + \frac{a_1}{2}\left(u(1,t) - \frac{1}{3a_1} u(1,t)^2\right)^2. \quad (46)$$

Proof. With the help of (42) and boundary controls (26)–(28), from (13), it holds

$$\begin{aligned}\dot{V}(t) = &-u(1,t)\left(\frac{a_1}{2}u(1,t) + \frac{1}{18a_1}u(1,t)^3\right)\\ &- u(0,t)\left(\frac{H}{2}u(0,t) + \frac{a_0}{2}u(0,t) + \frac{1}{18a_0}u(0,t)^3\right)\\ &- \frac{a_0}{2}(u(0,t) - \frac{1}{3a_0}u(0,t)^2)^2\\ &- \frac{a_1}{2}(u(1,t) - \frac{1}{3a_1}u(1,t)^2)^2\\ &- \varepsilon\int_0^1 u_x(x,t)^2dx - \frac{H}{2}u(1,t)^2 - \frac{\delta}{2}u_x(0,t)^2\\ \le &-min\left(\frac{\varepsilon}{2}, \frac{H}{2}\right)V. \end{aligned} \quad (47)$$

Thus system (5)–(6) is globally exponentially stable in $L_2[0, 1]$ under (42) and boundary controls (26)–(28). Next, using (42), we have

$$\begin{aligned} l(t) = &\, 2\varepsilon\int_0^1 u_x(x,t)^2dx + Hu(1,t)^2 + \delta u_x(0,t)^2\\ &+ a_0(u(0,t) - \frac{1}{3a_0}u(0,t)^2)^2\\ &+ a_1(u(1,t) - \frac{1}{3a_1}u(1,t)^2)^2\\ &- \delta u_x(1,t)^2 - 2u(1,t)(\varepsilon u_x(1,t) - \delta u_{xx}(1,t)\\ &+ \frac{a_1}{2}u(1,t) + \frac{1}{18a_1}u(1,t)^3)\\ &+ 2u(0,t)\left(\varepsilon u_x(0,t) - \delta u_{xx}(0,t)\right.\\ &\left.-\frac{H}{2}u(0,t) - \frac{a_0}{2}u(0,t) - \frac{1}{18a_0}u(0,t)^3\right)\\ &+ \delta u_x(1,t)^2 + 2u(1,t)(\varepsilon u_x(1,t) - \delta u_{xx}(1,t)\\ &+ \frac{a_1}{2}u(1,t) + \frac{1}{18a_1}u(1,t)^3)\\ &- 2u(0,t)\left(\varepsilon u_x(0,t) - \delta u_{xx}(0,t)\right.\\ &\left.-\frac{H}{2}u(0,t) - \frac{a_0}{2}u(0,t) - \frac{1}{18a_0}u(0,t)^3\right)\\ = &-2\dot{V}(t) - 2u(1,t)(\delta u_{xx}(1,t)\\ &- \frac{a_1}{2}u(1,t) - \frac{1}{18a_1}u(1,t)^3)\\ &- 2u(0,t)\left(\varepsilon u_x(0,t) - \delta u_{xx}(0,t)\right.\\ &\left.-\frac{H}{2}u(0,t) - \frac{a_0}{2}u(0,t) - \frac{1}{18a_0}u(0,t)^3\right). \end{aligned} \quad (48)$$

Substitute $l(t)$ to J_2 in (45), and by (48), it holds

$$\begin{aligned} J_2 = &\, 2V(0) - 2V(\infty)\\ &+ \int_0^\infty \frac{\left(\delta u_{xx}(1,t) - a_1u(1,t) - \frac{u(1,t)^2}{9a_1}\right)^2}{a_1 + \frac{u(1,t)^2}{9a_1}}dt\\ &+ \int_0^\infty \frac{1}{H + a_0 + \frac{1}{9a_0}u(0,t)^2}\\ &\times \left(u(0,t)\left(H + a_0 + \frac{1}{9a_0}u(0,t)^2\right)\right.\\ &\left.- \varepsilon u_x(0,t) + \delta u_{xx}(0,t)\right)^2 dt. \end{aligned} \quad (49)$$

Since system (5)–(6) is globally exponentially stable in $L_2[0, 1]$ under (42) and boundary controls (26)–(28), it yields $V(\infty) = 0$. From (49), it can be seen that boundary controls (26)–(28) minimize the cost functional (45), and the minimum value is $2V(0)$. □

From Theorem 4, we get the following result.

Corollary 4: Under the boundary condition

$$w_x(1,t) = 0, \quad (50)$$

boundary controls

$$w_{xx}(0,t) = \frac{\varepsilon}{\delta}w_x(0,t) - \frac{H + a_0}{\delta}(w(0,t) - H) - \frac{1}{9a_0\delta}(w(0,t) - H)^3, \quad (51)$$

$$w_{xx}(1,t) = \frac{a_1}{\delta}(w(1,t) - H) + \frac{1}{9a_1\delta}(w(1,t) - H)^3, \quad (52)$$

with $a_0 > 0$, $a_1 > 0$, minimize the cost functional

$$\bar{J}_2 = \int_0^\infty \left(\bar{l}(t) + \frac{\delta^2 w_{xx}(1,t)^2}{a_1 + \frac{1}{9a_1}(w(1,t) - H)^2} + \frac{(\varepsilon w_x(0,t) - \delta w_{xx}(0,t))^2}{H + a_0 + \frac{1}{9a_0}(w(0,t) - H)^2}\right)dt, \quad (53)$$

where

$$\begin{aligned}\bar{l}(t) = &\, \varepsilon\int_0^1 w_x(x,t)^2dx + \frac{H}{2}(w(1,t) - H)^2\\ &+ \frac{\delta}{2}w_x(0,t)^2dx + \frac{a_0}{2}(w(0,t) - H\\ &- \frac{1}{3a_0}(w(0,t) - H)^2)^2\\ &+ \frac{a_1}{2}(w(1,t) - H - \frac{1}{3a_1}(w(1,t) - H)^2)^2. \end{aligned} \quad (54)$$

V. SIMULATION RESULTS

Example 1. Consider the KdVB equation in [16] given by

$$w_t(x,t) = 0.5\, w_{xx}(x,t) - 1.2w_{xxx}(x,t) - w_x(x,t)w(x,t), \quad (55)$$

$$w(x,0) = 0.03sin(1.49\pi x) - 0.5cos(1.51\pi x). \quad (56)$$

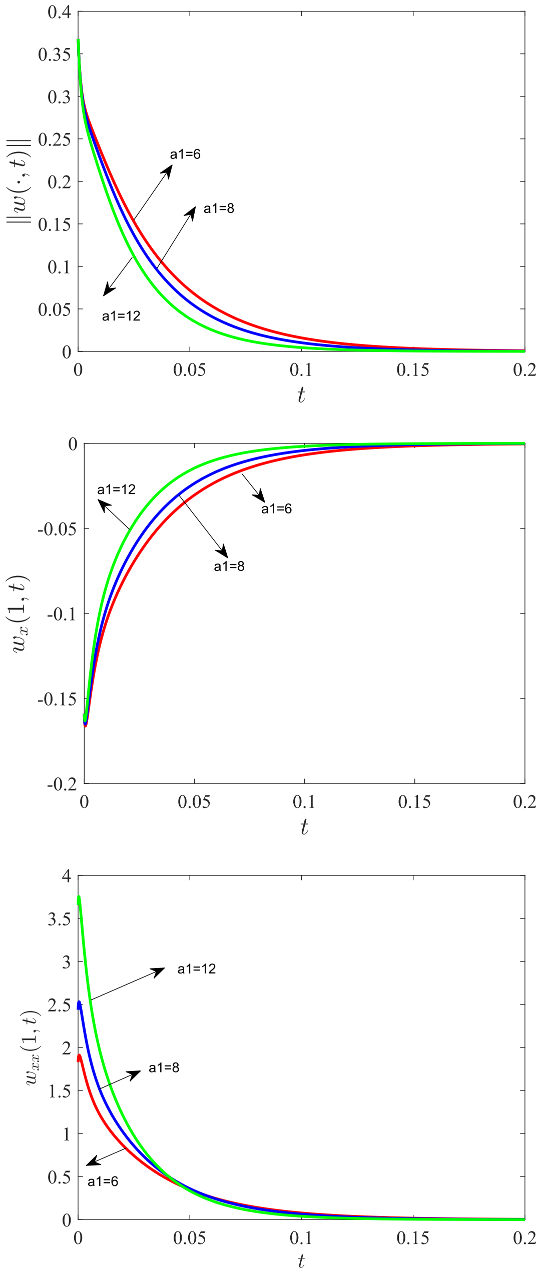

FIGURE 1. Responses of state norm $\|w(\cdot, t)\|$ under boundary condition (57), boundary controls (58)–(59) and boundary controls (58)–(59) with $H = 0$, $a_1 = 6$, $a_1 = 8$, $a_1 = 12$, respectively.

Using Corollary 3, for a given set point H, under boundary condition

$$w(0, t) = H, \tag{57}$$

boundary controls

$$w_x(1, t) = -\frac{5}{12}(w(1, t) - H), \tag{58}$$

$$w_{xx}(1, t) = \frac{a_1}{1.2}(w(1, t) - H) + \frac{1}{10.8a_1}(w(1, t) - H)^3, \tag{59}$$

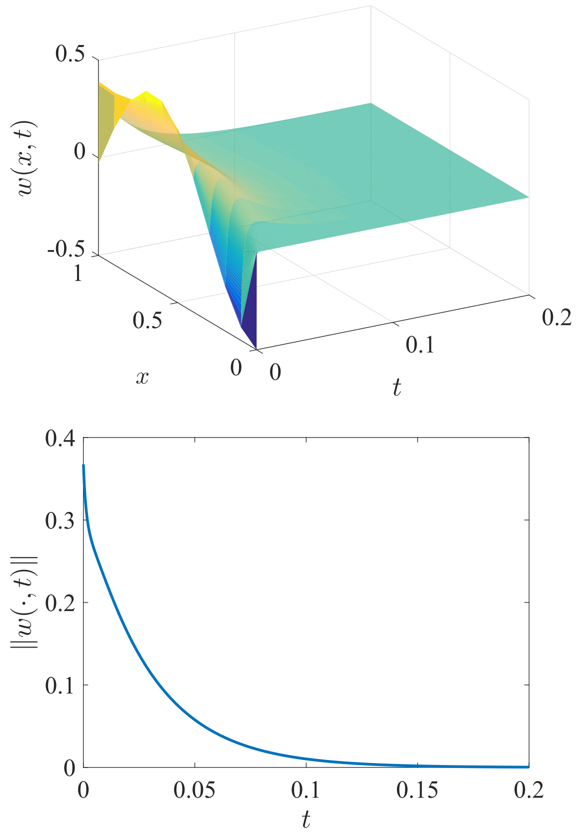

FIGURE 2. Response of state $w(x, t)$ and its norm $\|w(\cdot, t)\|$ under boundary condition (57), boundary controls (58)–(59).

with $a_1 > 0$, regulate the state of closed-loop system (55)–(56) and (57), (58)–(59) to the set point H and minimize the cost functional (40).

In boundary control (59), $a_1 > 0$ is an adjustable parameter, responses of state norm $\|w(\cdot, t)\|$ under boundary condition (57), boundary controls (58)–(59) and boundary controls (58)–(59) with $H = 0$, $a_1 = 6$, $a_1 = 8$, $a_1 = 12$, respectively, are shown in Fig.1.

We can see that as a_1 increases, the convergence speed of the state norm $\|w(\cdot, t)\|$ of the closed-loop system towards zero becomes faster, and the boundary controls (58)–(59) also changes similarly. The simulation result is consistent with remark 1.

To compare the results with those of [16], we display responses of the PDE state $w(x, t)$ and its norm $\|w(\cdot, t)\|$ under boundary condition (57), boundary controls (58)–(59) where $H = 0$, $a_1 = 8$ in Fig. 2.

In [16], for system (55)– (56), boundary controls are designed as

$$w(0, t) = 0, \tag{60}$$

$$w_x(1, t) = 0, \tag{61}$$

$$w_{xx}(1, t) = u(t), \tag{62}$$

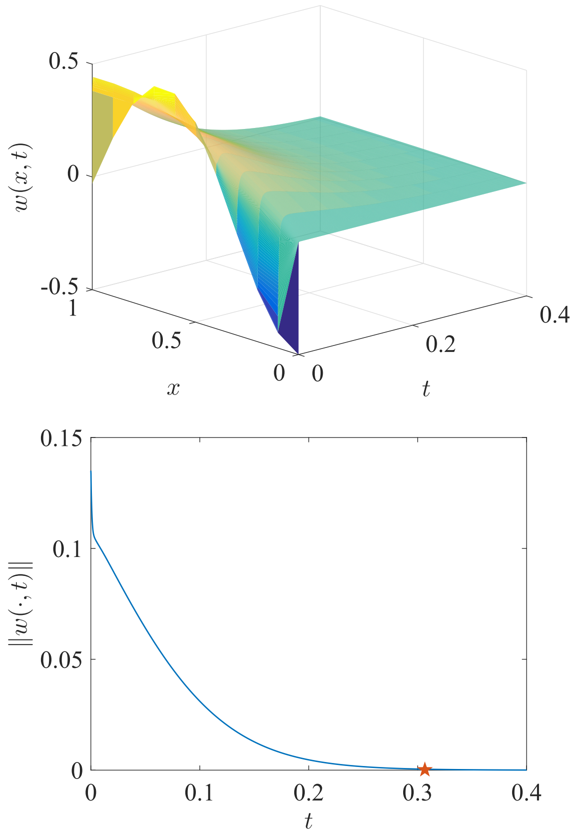

FIGURE 3. Response of state $w(x,t)$ and its norm $\|w(\cdot,t)\|$ under boundary controls (60)–(62) in [16].

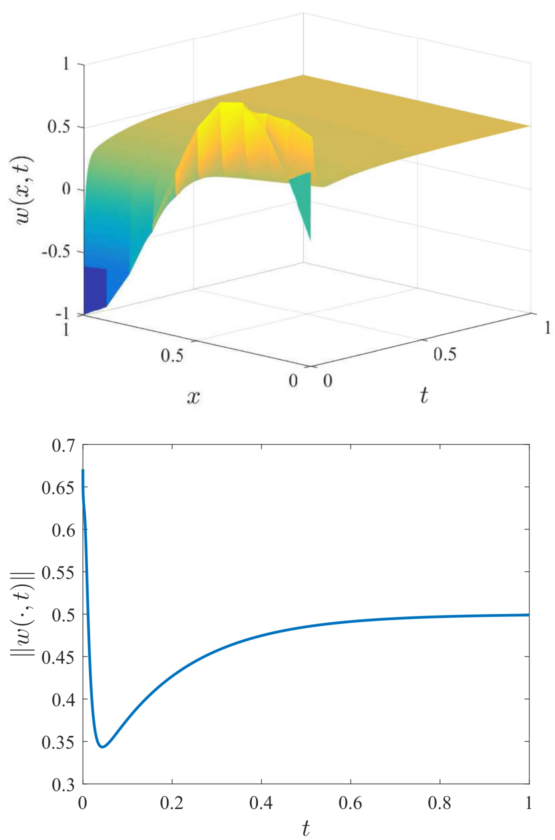

FIGURE 4. Response of state $w(x,t)$ and its norm $\|w(\cdot,t)\|$ under boundary condition (67), and boundary controls (68), (69) with $a_0 = 0.5$, $a_1 = 0.7$.

where

$$u(t) = -4w(1,t)^2 + \frac{0.05}{w(1,t)}\left(\int_0^1 w(x,t)^2 dx\right)^2, \tag{63}$$

if $w(1,t) \neq 0$, and

$$u(t) = 0, \tag{64}$$

if $w(1,t) = 0$. Responses of the PDE state $w(x,t)$ and its norm $\|w(\cdot,t)\|$ under boundary controls (60)–(62) are in Fig.3.

From Fig. 2, one can see that the state $w(x,t)$ and its norm $\|w(\cdot,t)\|$ of system (55)–(56) tend to zero at $t = 0.15$ under the proposed boundary condition (57), boundary controls (58) and (59), while in Fig. 3, the state $w(x,t)$ and its norm $\|w(\cdot,t)\|$ of the same system go to zero at $t = 0.3$ under boundary controls (60)–(62) in [16]. Obviously, under the proposed boundary condition and boundary controls, the convergence speed of the closed-loop system is faster than that of [16].

Example 2. Consider the KdVB equation

$$w_t(x,t) = 1.2\, w_{xx}(x,t) - 0.5w_{xxx}(x,t) - w_x(x,t)w(x,t), \tag{65}$$

$$w(x,0) = sin(1.5\pi x). \tag{66}$$

For a given set point H, using Corollary 4, boundary condition

$$w_x(1,t) = 0, \tag{67}$$

and boundary controls

$$w_{xx}(0,t) = \frac{12}{5}w_x(0,t) - \frac{H+a_0}{0.5}(w(0,t)-H) - \frac{1}{4.5a_0}(w(0,t)-H)^3, \tag{68}$$

$$w_{xx}(1,t) = \frac{a_1}{0.5}(w(1,t)-H) + \frac{1}{4.5a_1}(w(1,t)-H)^3, \tag{69}$$

with $a_0 > 0, a_1 > 0$, can regulate the state of system (65)–(66) to the set point H and minimize the cost functional (53).

Let $H = 0.5$, in order to observe how parameters a_0 and a_1 impact on the convergent speed of the closed-loop system, we first set $a_0 = 0.5, a_1 = 0.7$. Responses of the PDE state $w(x,t)$ and its norm $\|w(\cdot,t)\|$ under boundary condition (67), and boundary controls (68)–(69) are in Fig.4. Boundary controls (68), (69) are in Fig.5. The state of closed-loop system tends towards the set point $H = 0.5$ when almost $t = 1$, and boundary controls (68), (69) also tends to zero when $t = 1$.

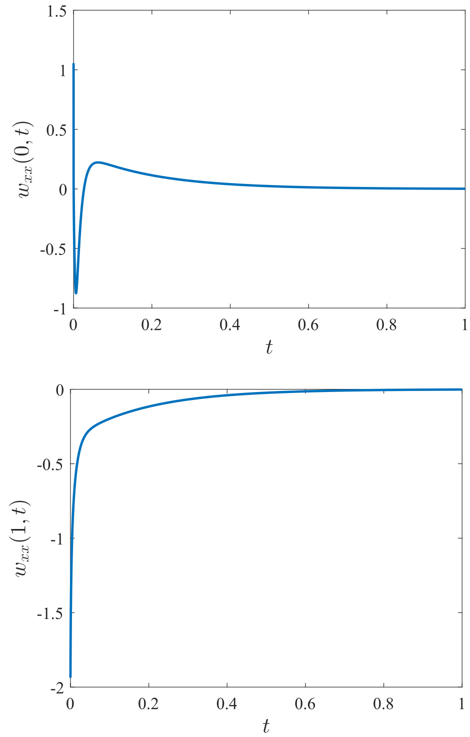

FIGURE 5. Boundary controls (68), (69) with $a_0 = 0.5$, $a_1 = 0.7$.

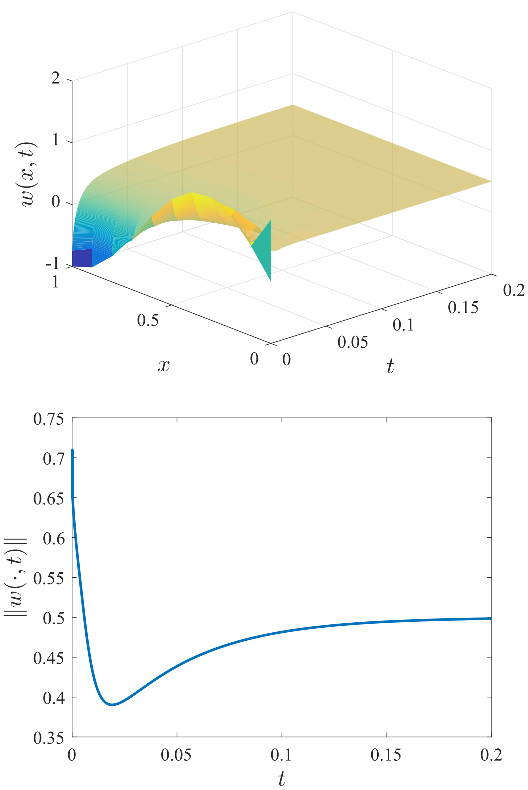

FIGURE 6. Response of state $w(x, t)$ and its norm $\|w(\cdot, t)\|$ under boundary condition (67), and boundary controls (68), (69) with $a_0 = 5$, $a_1 = 7$.

Next, let $H = 0.5$ again, we set $a_0 = 5$, $a_1 = 7$. Responses of the PDE state $w(x, t)$ and its norm $\|w(\cdot, t)\|$ under boundary

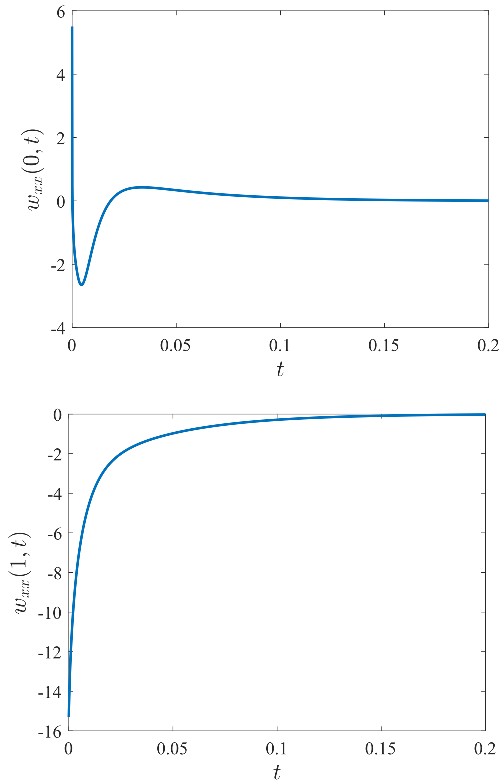

FIGURE 7. Boundary controls (68), (69) with $a_0 = 5$, $a_1 = 7$.

condition (67), and boundary controls (68)–(69) are in Fig.6. Boundary controls (68), (69) are in Fig.7. The state of closed-loop system tends towards the set point $H = 0.5$ when almost $t = 0.2$, and boundary controls (68), (69) also tends to zero when $t = 0.2$.

It is obvious that the convergence speed of the closed-loop system accelerates as the parameters a_0 and a_1 increase, When $a_0 = 0.5, a_1 = 0.7$ to $a_0 = 5, a_1 = 7$, the convergence speed is almost five times faster than before. The result is consistent with remark 2.

VI. CONCLUSION

We investigate inverse optimality of regulation design for KdVB equation. We design two kinds of boundary control laws to regulate the state of closed-loop system to the set point from any initial value. In order to regulate the convergent speed of the closed-loop system, one or two parameters are designed in the boundary control laws. It is shown that boundary control laws are optimal for two meaningful functionals, respectively. The effectiveness of the proposed design has been shown through simulations, and the convergence speed of the closed-loop system accelerates with increase of parameters.

ACKNOWLEDGMENT

The authors would like thank the reviewers for their good suggestions on improving the article and also would like thank Shuang Liang and Kaining Wu for sharing the computer code of the article [16] with them.

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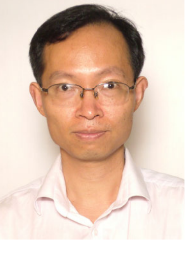

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