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RESEARCH ARTICLE

Exploration of Offshore Drilling for Oil and Gas Operations Based on the Probabilistic Linguistic Dombi Aggregation Decision Model

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ABSTRACT This study focuses on optimizing offshore drilling operations through Multi-Criteria Decision-Making (MCDM). It recognizes MCDM as a crucial framework for evaluating alternatives in the complex offshore drilling landscape. The objective is to identify the most suitable alternative among four approaches: Advanced Drilling Technologies, Drilling Process Optimization, Human Factors and Safety Enhancement, and Environmental Impact Mitigation. Evaluation criteria include Operational Efficiency, Safety Performance, Environmental Impact, and Cost-effectiveness. The research aims to contribute insights and recommendations for stakeholders in the offshore oil and gas industry. In the field of information aggregation and fusion, there is a growing interest among researchers in the domain of probabilistic linguistic expression sets, which are particularly effective in consolidating uncertain data. This article aims to explore various methodologies for information aggregation using probabilistic linguistic expressions. To achieve this goal, we have introduced procedural principles based on the Dombi (D) framework, specifically designed for managing probabilistic linguistic term elements (PLTEs). These principles are firmly grounded in both the product and sum of Dombi operations. As a result, we have developed a range of techniques for probabilistic linguistic aggregation, including entities such as the Probabilistic Linguistic Dombi Average (PLDA) and the Probabilistic Linguistic Dombi Geometric (PLDG). Additionally, we have created weighted aggregation operators (AOs) such as PLDWA and PLDWG, along with ordered AOs like PLDOWA and PLDOWG. By utilizing the D τ-norm and τ-conorm, we have designed versatile aggregation tools that support information reinforcement in both ascending and descending directions. Furthermore, we provide a comparative analysis between our proposed methodologies and the MARCOS approach. Additionally, we offer a detailed explanation of the distinctive attributes associated with these operators. Through the application of PLDA, PLDG, PLDWA, PLDWG, PLDOWA, and PLDOWG, we present strategies for effectively integrating probabilistic linguistic term sets (PLTs) into the realm of MCDM.

INDEX TERMS Probabilistic linguistic sets, multi criteria decision-making problem, Dombi operations.

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I. INTRODUCTION

This research endeavors to address the intricate challenges faced by the offshore oil and gas industry in optimizing drilling operations, emphasizing the need for a comprehensive strategy to enhance efficiency, safety, environmental sustainability, and economic viability. At the core of this study is the recognition of Multi-Criteria Decision-Making (MCDM) as an instrumental framework, offering a structured approach for evaluating and comparing alternative strategies in the complex domain of offshore drilling. Novel techniques have been demonstrated in a number of applied sciences and engineering domains in recent research. To forecast the rate of penetration in the Halahatang oil field, Jiao et al. [2] introduced a unique hybrid physics-machine learning model. Similar to this, Yin et al. [3] provided a strong prediction framework for mining safety and a technique for using low-grade thermal energy during drilling was proposed by Xiao et al. [4], and it offers the potential to improve drilling operations' energy efficiency. Furthermore, Zhu [5],Yu et al. [6], Zheng et al. [7] created adaptive decision models that combines energy and its management issues. The primary objective of this study is to explore and identify the most suitable alternative among four distinct approaches: Advanced Drilling Technologies (ADT), Drilling Process Optimization (DPO), Human Factors and Safety Enhancement (HFSE), and Environmental Impact Mitigation (EIM). The evaluation will be conducted against four critical criteria: Operational Efficiency (OE), Safety Performance (SP), Environmental Impact (EI), and Cost-effectiveness (CE). The research methodology involves a rigorous application of the MCDM framework, incorporating data collection methods and a comprehensive literature review covering offshore drilling technologies, safety protocols, environmental considerations, and decision-making methodologies. The study aims to present and analyze findings derived from the evaluation, culminating in a discussion that offers valuable insights and practical recommendations for stakeholders in the offshore oil and gas industry navigating the complexities of optimizing drilling operations. Given the global nature of these challenges, international collaboration and collective efforts are imperative. In this context, the decision-making process involves adaptable criteria, posing a challenge for decision-makers to pinpoint an optimal solution for each criterion in a specific context. Consequently, decision-makers are directing their endeavors towards crafting more appropriate and efficient methods for selecting the optimal choices.

The integration of PLTs enriches the decision-making process by accommodating the inherent uncertainties and subjectivity in offshore drilling optimization. By allowing decision-makers to express their assessments in terms of probabilities and linguistic terms, the MCDM framework becomes more adaptive to the dynamic nature of the offshore environment, where factors such as technological advancements and environmental conditions introduce inherent uncertainties. This approach enhances the robustness of the decision-making process, enabling a more realistic and nuanced evaluation of alternatives in the context of optimizing offshore drilling operations.

A. REVIEW OF LITERATURE'S

In the realm of expressing preferences through linguistic data, decision-makers often grapple with challenges marked by ambiguity and vagueness, a concern emphasized by Pang et al. [1]. To address this, Zadeh [8] introduced the application of fuzzy sets (FS) for handling such challenges in decision-making. Subsequently, Torra [9] introduced hesitant fuzzy sets (HFSs) as an advancement of FS to manage situations where defining an element's membership involves a range of potential values. Due to limitations in HFSs, Rodriguez et al. [10] recommended HF linguistic term sets (HFLTSs) to improve managing vague information, particularly when dealing with multiple sources of vagueness simultaneously. Additionally, Batool et al. [11] introduced the concept of PHF.

Rodriguez et al. [10] highlighted the limitations of traditional fuzzy set modeling tools, primarily focusing on quantitative issues. Liao et al. [12] introduced the concept of distance for HFLTs. Given that uncertainty in qualitative problems often stems from vagueness in expert explanations, adopting a fuzzy linguistic approach is deemed appropriate for more concrete results. Recent research on HFLTSs has expressed concerns that values proposed by decision-makers may lack adherence to a realistic pattern, as underscored by Pang et al. [1]. To address this, Pang et al. [1] introduced the notion of Perception of PLTs. PLTs were introduced as an extension of HFLTSs, incorporating probabilities without compromising genuine linguistic data. PLTs evolved through the extension of existing models, including HFLTSs and HFSs, by introducing probabilities and hesitations. In the domain of decision-making, PLTs prove valuable for representing qualitative judgments made by experts [1]. Their integration aims to enhance versatility and precision in decision-making procedures, gaining significance in group decision-making. Various fundamental AOs, such as the PLWA operator and the PLWG operator, have emerged for combining PLTEs. Bai et al. [13] introduced more suitable comparison methods, and Gou and Xu [14] formulated novel functional principles related to probabilistic information, while He et al. [15] proposed an MCDM algorithm accommodating uncertainty in range preference rankings.

In the context of the linguistic probabilistic environment, Kobina et al. [16] introduced probabilistic linguistic dominance operators for aggregations to address MCDM challenges. The precision of final outcomes is significantly influenced by the information aggregation stage. Scholars have dedicated efforts to create AOs tailored for handling PLTs-related information, such as analogies of probabilistic linguistic term sets [13] and innovative operational principles for linguistic expressions [14]. Many other PLTs applications are studied in [34] and [35] and PLTs normalization in [36] while Xiao et al. [37] discussed novel method to estimate incomplete PLTS information.

Algebraic operational laws represent one set of operational laws for knowledge synthesis; mathematical operations, as proposed by Dombi, also serve as valuable alternatives [17]. Ashraf et al. [18] introduced the Dombi product as a τ-norm and the Dombi for instance a τ-conorm. Liu and Rong [19] devised accumulation operators for decision-making situations with multiple attributes in fuzzy environments. Waqar et al. [20] introduced the IVIFDWA operator, demonstrating its practicality. Seikh and Mandal [21] analyzed the operator known as the range-associated IF Dombi integral of Choquet in geometry and [39] explored Dombi in neutrosophic cubic sets. The proposed PLTs AO, utilizing the Dombi triangular norm and triangular conorm, aims to enhance the effectiveness of decision-making processes within fuzzy systems. These Dombi operations based on triangular functions have proven effective for handling ambiguity and lack of clarity in various applications.

B. MOTIVATIONS OF THE STUDY

Our research is motivated by the immense potential of probabilistic linguistic dombi aggregation decision models to transform decision-making processes in complex and uncertain environments. The study is driven by the desire to unlock the full capabilities of this advanced analytics approach, which offers a robust and flexible framework for handling linguistic variables and probabilistic uncertainties with unprecedented precision. By harnessing the Dombi model's ability to capture suitable nuances in linguistic assessments and merge them with probabilistic information, the aim is to develop a more accurate, reliable, and efficient decision-making tool. Furthermore, we seek to demonstrate the versatility and applicability of the Dombi model in various fields, contributing to its advancement, promoting its adoption in real-world applications, and paving the way for new research avenues in the feild of decision-making under uncertainty.

C. RESEARCH GAPS OF THE STUDY

By applying the Dombi triangular norm and triangular conorm, the suggested PLTs AO seeks to improve the efficacy of decision-making procedures in fuzzy systems. These triangular function-based Dombi operations have shown to be useful in managing ambiguity and uncertainty in a range of applications. Although a lot of work has been done on PLTs originally, Dombi operators are not yet present in the framework. Although many decision-based methodologies and frameworks have employed Dombi aggregation operations, their implementation in PLTS has not yet been addressed in the literature.

Dombi operators capture intricate interactions between inputs and outputs since they are non-linear. They are modifiable and adaptable to fit particular domains and applications. Dombi operators are appropriate for real-world applications because they are resistant to noise and unpredictability. Dombi operators are also computationally efficient, which facilitates quick processing and judgment. To make decisions that are more accurate, the operators take into account the context in which the inputs are applied.

This research gap emphasizes the necessity of including Dombi operators into PLTS frameworks because of their special abilities, which can greatly improve decision-making, especially when dealing with complicated, ambiguous, and uncertain settings. Closing this gap may result in the creation of fuzzy systems decision-making tools that are more reliable and effective.

D. CONTRIBUTIONS OF THE STUDY

The primary goals of the research proposal are as follows:

- The research introduces and explores the integration of a Strategic Decision Support System with PLTS under Dombi AOs specifically tailored for offshore drilling for oil and gas operations.
- This novel integration addresses the need for advanced decision support tools that can handle linguistic uncertainty and complex risk scenarios.
- The research article contributes to the academic literature by expanding the theoretical understanding of decision-making methodologies under uncertainty.
- It fills a gap in existing literature by proposing and validating a novel approach that integrates linguistic preferences and probabilistic reasoning within the Dombi AOs framework.

The primary objective of this study is to explore various PL operators based on the Dombi PLTD operational principles. Developed PL operators include PLDA, PLDG, PLDWA, PLDWG, PLDOWA, and PLDOWG AOs, offering an innovative approach for MCDM using PLTs materials.

E. ORGANIZATION OF THE STUDY

The structure of this paper involves introducing key concepts related to PLTs and D operations, exploring the application of D operations to transformed PLTs, developing a suite of aggregation operators for Probabilistic linguistic operators using the Dombi norm, outlining the methodology for MCDM, featuring an illustrative example, conducting a comparative analysis, and summarizing findings and insights for future research in Section VIII.

II. PRELIMINARIES

In this segment, we delve into the foundational principles associated with probabilistic and linguistic terms. This includes an examination of their scoring methodologies and essential algebraic operations, among other distinguished subjects.

Definition 1 [1]: Examine a collection of linguistic terms denoted as $\vartheta = \{\upsilon_\sigma | \sigma = 0, 1, \ldots, \beth\}$, *where each term* $\upsilon_\beth$ *represents a potential value for a linguistic variable within a singular dimension of* $\beth + 1$. *The Linguistic Term Set (LTS) must adhere to the following conditions:*

(1) Orderliness: The arrangement of the set is such that $\upsilon_\sigma \geq \upsilon_\flat$ *when* $\sigma \geq \flat$.

(2) Negation Operator: The operator for negation, represented as $neg(\upsilon_\sigma)$, *should yield* $\upsilon_{\beth-\sigma}$.

Definition 2 [1] : To determine a PLTs consider $\vartheta = \{\upsilon_\sigma \mid \sigma = 0, 1, \ldots, \gimel\}$ *be a LTS. Then,*

$$\daleth(\rho) = \{\daleth^\kappa((\rho)^\kappa) \mid \daleth^\kappa \in \vartheta, \gamma^\kappa \in \sigma, \sum_{\kappa}^{\blacklozenge\daleth(\rho)} (\rho)^\kappa \le 1,$$
$$(\rho)^\kappa \ge 0, \kappa = 1, 2, 3, \ldots, \blacklozenge\daleth(\rho)\} \tag{1}$$

In which $(\ell^\kappa((\rho)^\kappa))$ *is the appropriate linguistic term* $\daleth^\kappa$ *linked to the likelihood* $(\rho)^\kappa$, $\gamma^\kappa)$ *is the underscript of* $\daleth^\kappa$ *and* $\blacklozenge\daleth(\rho)$ *is the total count of appropriate linguistic term in* $\daleth(\rho)$.

Definition 3 [1]: When considering the lower index of linguistic designation $\daleth^\kappa$, *imagine* γ^κ *in PLTs, defined as* $\daleth(\rho) = \daleth^\kappa(\rho^\kappa), \kappa = 1, 2, 3, \ldots \blacklozenge\daleth(\rho)$. *The score of* $\daleth(\rho)$ *can be determined as follows*:

$$\Im(\daleth(\rho)) = \upsilon_{\tilde{o}} \tag{2}$$

where $\tilde{o} = \frac{\sum_{\kappa=1}^{\blacklozenge\daleth(\rho)} \gamma^\kappa \rho^\kappa}{\sum_{\kappa=1}^{\blacklozenge\daleth(\rho)} \rho^\kappa}$. *The degree of deviation of* $\daleth(\rho)$ *is*:

$$\zeta(\daleth(\rho)) = \frac{\sum_{\kappa=1}^{\blacklozenge\daleth(\rho)} \left((\rho^\kappa(\gamma^\kappa - \tilde{o}))^2\right)^{0.5}}{\sum_{\kappa=1}^{\blacklozenge\daleth(\rho)} \rho^\kappa} \tag{3}$$

Definition 4 [1]: When dealing with two PLTs, namely $\daleth_1(\rho)$ *and* $\daleth_2(\rho)$, $\Im(\daleth_1(\rho))$ *and* $\Im(\daleth_2(\rho))$ *represent the scores associated with* $\daleth_1(\rho)$ *and* $\daleth_2(\rho)$ *in each instance.*

1) *If* $\Im(\daleth_1(\rho)) > \Im(\daleth_1(\rho))$, *then* $\daleth_1(\rho)$ *is* $\daleth_2(\rho)$, *represented as* $\daleth_1(\rho) > \daleth_2(\rho)$.
2) $\Im(\daleth_1(\rho)) < \Im(\daleth_1(\rho))$ *leads to* $\daleth_1(\rho)$ *is lower* $\daleth_2(\rho)$, *represented as* $\daleth_1(\rho) < \daleth_2(\rho)$.
3) *If* $\Im(\daleth_1(\rho)) = \Im(\daleth_1(\rho))$, *then we must assess their respective degrees of deviation.*

(a). $\zeta(\daleth_1(\rho)) = \zeta(\daleth_1(\rho))$*leads to* $\daleth_1(\rho)$ *is equated to* $\daleth_2(\rho)$, *depicted by* $\daleth_1(\rho) \sim \daleth_2(\rho)$

(b). *If* $\zeta(\daleth_1(\rho)) > \zeta(\daleth_1(\rho))$, *subsequently* $\daleth_1(\rho)$ *is of lower rank to* $\daleth_2(\rho)$, *represented as* $\daleth_1(\rho) < \daleth_2(\rho)$

(c). $\zeta(\daleth_1(\rho)) < \zeta(\daleth_1(\rho))$ *leads to* $\daleth_1(\rho)$ *surpasses the* $\daleth_2(\rho)$, *denoted by* $\daleth_1(\rho) > \daleth_2(\rho)$

Definition 5 [12] : Consider a PLTs $\daleth(\rho)$ *and let* $\vartheta = \{\upsilon_\sigma | \sigma = -\gimel, \ldots, -1, 0, \ldots, \gimel\}$ *be any LTS, then correspondent modification process of* $\daleth(\rho)$ *is given as*:

$$\wp(\daleth(\rho)) = \left\{\left[\frac{\gamma^\kappa}{2\gimel} + \frac{1}{2}\right](\rho^\kappa)\right\} \tag{4}$$

where $\wp : [-\gimel, \gimel] \rightarrow [0, 1]$ *and* $\acute{\gamma} = \wp(\daleth(\kappa)), \acute{\gamma} \in [0, 1]. \wp(\daleth)^\kappa = \left[\frac{\gamma^\kappa}{2\gimel} + \frac{1}{2}\right] = \acute{\gamma}$.

Definition 6 [38]: Let $\vartheta = \{\upsilon_\sigma \mid \sigma = 0, 1, \ldots, \gimel\}$ *be a LTS. And let* $\daleth_1(\rho) = \{\daleth_1^\kappa(p_1^{(\kappa)}), \kappa = 1, 2, 3, \ldots, \blacklozenge\daleth_1(\rho)\}$ *and* $\daleth_2(\rho) = \{\daleth_2^\kappa(p_1^{(\kappa)}), \kappa = 1, 2, , 3 \ldots, \blacklozenge\daleth_2(\rho)\}$ *be two PLTs with* $\blacklozenge\daleth_1(\rho) = \blacklozenge\daleth_2(\rho)$, *then Hamming Distance* $d(\daleth_1(\rho), \daleth_2(\rho))$*between* $\daleth_1(\rho)$*and*$\daleth_2(\rho)$ *is defined as follows*:

$$d(\daleth_1(\rho), \daleth_2(\rho)) = \frac{\sum_{\kappa=1}^{\blacklozenge\daleth_1(\rho)} |p_1^\kappa \wp(\daleth_1^\kappa) - p_1^\kappa \wp(p\daleth_1^\kappa)|}{\blacklozenge\daleth_1(\rho)} \tag{5}$$

When normalizing Probabilistic Linguistic Terms (PLTs) using the expansion designation method, it becomes apparent, based on the laws of PLT comparison, that distinct levels of Linguistic Designation (LD) exist. These levels are elucidated as follows:

Definition 7 [1]: Consider two probabilistic LD sets with different quantities of LDs as:

$$\daleth_1(\rho) = \{\daleth_1^\kappa(\rho_1^{(\kappa)}), \kappa = 1, 2, 3, \ldots, \blacklozenge\daleth_1(\rho)\}$$
$$\daleth_2(\rho) = \{\daleth_2^\kappa(\rho_1^{(\kappa)}), \kappa = 1, 2, , 3 \ldots, \blacklozenge\daleth_2(\rho)\}$$

When $\blacklozenge\daleth_1(\rho) > \blacklozenge\daleth_2(\rho)$, *we augment* $\daleth_2(\rho)$ *with the smallest linguistic designations having zero probability to match the numeral of designations in* $\daleth_1(\rho)$. *A similar approach can be applied when*$\blacklozenge\daleth_1(\rho) < \blacklozenge\daleth_2(\rho)$.

Definition 8 [1]: Let $\vartheta = \{\upsilon_\sigma | \sigma = 0, 1, 2, \ldots, \gimel\}$ *be a LTS. When provided with three PLTs,* $\daleth(\rho)$, *and* $\daleth_2(\rho)$,*the fundamental operations they entail as outlined below*:

1) $\daleth_1(\rho) \bigoplus \daleth_2(\rho) = \sqcup_{\daleth_1^{(\kappa)} \in \daleth_1(\rho), \daleth_2^{(\kappa)} \in \daleth_2(\rho)} \{\rho_1^{(\kappa)} \daleth_1^{(\kappa)} \bigoplus \rho_2^{(\kappa)} \daleth_2^{(\kappa)}\}$
2) $\daleth_1(\rho) \bigotimes \daleth_2(\rho) = \sqcup_{\daleth_1^{(\kappa)} \in \daleth_1(\rho), \daleth_2^{(\kappa)} \in \daleth_2(\rho)} \{\rho_1^{(\kappa)} \daleth_1^{(\kappa)} \bigotimes \rho_2^{(\kappa)} \daleth_2^{(\kappa)}\}$
3) *For* $\curlywedge \ge 0, \curlywedge\daleth(\rho) = \sqcup_{\daleth^{(\kappa)} \{\in \daleth(\rho)} \{\curlywedge(\rho)^{(\kappa)} \daleth^{(\kappa)}\}$
4) *For* $\curlywedge \ge 0, (\daleth(\rho))^\curlywedge = \sqcup_{\daleth^{(\kappa)} \{\in \daleth(\rho)} \{(\daleth^{(\kappa)})^{\curlywedge(\rho)^{(\kappa)}}\}$

III. DOMBI OPERATION

Definition 9 [39]: Suppose that $(t, s) \in (0, 1) \times (0, 1)$ *for any real numbers with* $\psi \ge 1$. *Then, Dombi norms are defined as*

$$\hat{T}(t, s) = \frac{1}{1 + \{(\frac{1-t}{t})^\psi + (\frac{1-s}{s})^\psi\}^{\frac{1}{\psi}}}, \tag{6}$$

$$\hat{S}(t, s) = 1 - \frac{1}{1 + \{(\frac{t}{1-t})^\psi + (\frac{s}{1-s})^\psi\}^{\frac{1}{\psi}}}. \tag{7}$$

Definition 10: Consider a LTS $\vartheta = \{\upsilon_\sigma | \sigma = -\gimel, \ldots, -1, 0, 1, \ldots, \gimel\}$ *and* $\curlywedge > 0$.

When presented with three modified PLTs $\wp(\daleth(\rho))$, $\wp(\daleth_1(\rho))$ *and* $\wp(\daleth_2(\rho))$ *then*

1) $\wp(\daleth_1(\rho)) \oplus \wp(\daleth_2(\rho))$
$$= \bigsqcup_{\wp(\daleth_1^{(\kappa)}) \in \wp(\daleth_1(\rho)), \wp(\daleth_2^{(\kappa)}) \in \wp(\daleth_2(\rho))} \left\{1 - \frac{1}{1 + \left\{(\frac{\rho_1^{(\kappa)} \wp(\daleth_1^{(\kappa)})}{1-\rho_1^{(\kappa)} \wp(\daleth_1^{(\kappa)})})^\psi + (\frac{\rho_2^{(\kappa)} \wp(\daleth_2^{(\kappa)})}{1-\rho_2^{(\kappa)} \wp(\daleth_2^{(\kappa)})})^\psi\right\}^{\frac{1}{\psi}}}\right\}$$
2) $\wp(\daleth_1(\rho)) \otimes \wp(\daleth_2(\rho))$
$$= \bigsqcup_{\wp(\daleth_1^{(\kappa)}) \in \wp(\daleth_1(\rho)), \wp(\daleth_2^{(\kappa)}) \in \wp(\daleth_2(\rho))} \left\{\frac{1}{1 + \left\{(\frac{1-\rho_1^{(\kappa)} \wp(\daleth_1^{(\kappa)})}{\rho_1^{(\kappa)} \wp(\daleth_1^{(\kappa)})})^\psi + (\frac{1-\rho_2^{(\kappa)} \wp(\daleth_2^{(\kappa)})}{\rho_2^{(\kappa)} \wp(\daleth_2^{(\kappa)})})^\psi\right\}^{\frac{1}{\psi}}}\right\}$$

3) $\curlywedge.\wp(\daleth(\rho)) = \bigsqcup_{\wp(\daleth^{(\kappa)})\in\wp(\daleth(\rho))}\left\{1-\frac{1}{1+\left\{\curlywedge(\frac{\rho^{(\kappa)}\wp(\daleth^{(\kappa)})}{1-\rho^{(\kappa)}\wp(\daleth^{(\kappa)})})^{\psi}\right\}^{\frac{1}{\psi}}}\right\}$

4) $\wp(\daleth(\rho))^{\curlywedge} = \bigsqcup_{\wp(\daleth^{(\kappa)})\in\wp(\daleth(\rho))}\left\{\frac{1}{1+\left\{\curlywedge(\frac{1-\rho^{(\kappa)}\wp(\daleth^{(\kappa)})}{\rho^{(\kappa)}\wp(\daleth^{(\kappa)})})^{\psi}\right\}^{\frac{1}{\psi}}}\right\}$

IV. PROBABILISTIC LINGUISTIC OPERATORS FOR AGGREGATION

Within this section, we explore the realm of PL Operators, concentrating specifically on their utilization in aggregation processes. Our objective is to offer a comprehensive and persuasive explanation of the fundamental theorems governing these operators. Furthermore, we will conduct an in-depth analysis of the intriguing features linked to these operators, elucidating their distinct characteristics and potential ramifications in decision-making scenarios.

A. AVERAGE AGGREGATION OPERATORS OF PROBABILISTIC LINGUISTIC DOMBI (PLDA)

In this section, we will furnish an elucidation of the Average AOS of Probabilistic Linguistics (PL), utilizing the Dombi norm and denoted as PLDA, along with the corresponding theorem.

Definition 11: Let $\daleth(\rho) = \{\daleth_i^{(\kappa)}(p_i^{(\kappa)})|\kappa = 1,2,3,\ldots,\blacklozenge\daleth_i(\rho)\}$ $(\upsilon = 1,2,3,\ldots,n)$ constitute a family of PLTs and $\wp(\daleth(\rho))$ its correspondent modification. An operator in PLs, rooted in the D Weighted Average (PLDA) is a mapping $\wp(l^n(\rho)) \to \wp(\daleth(\rho))$, is

$$PLDA\,(\wp(\daleth_1(\rho)),\wp(\daleth_2(\rho)),\ldots,\wp(\daleth_n(\rho))) = \bigoplus_{\upsilon=1,2,3,\ldots,n}\left\{\frac{1}{n}(\wp(\daleth_n(\rho)))\right\} \tag{8}$$

Theorem 1: If $\daleth(\rho) = \daleth^{\kappa}((\rho)^{\kappa})|\daleth^{\kappa}\in\vartheta,\gamma^{\kappa}\in\sigma$, $\sum_{\kappa}^{\blacklozenge\daleth(\rho)}(\rho)^{\kappa} \leq 1, (\rho)^{\kappa} \geq 0, \kappa = 1,2,3,\ldots,\blacklozenge\daleth(\rho)$, $(\upsilon = 1,2,3,\ldots,n)$ represents a set of PLTs and $\wp(\daleth(\rho))$ is its corresponding transformation, then the resulting value, when aggregated using the PLDA operator, also belongs to the category of PLTsW.

$$PLDA(\wp(\daleth_1(\rho)),\wp(\daleth_2(\rho)),\ldots,\wp(\daleth_n(\rho))) = \bigsqcup_{\wp(\daleth_i^{(\kappa)})\in\wp(\daleth_i(\rho))}\left\{1-\frac{1}{1+\left\{\sum_{i=1}^{n}\frac{1}{n}(\frac{\rho_i^{(\kappa)}\wp(\daleth_i^{(\kappa)})}{1-\rho_1^{(\kappa)}\wp(\daleth_1^{(\kappa)})})^{\psi}\right\}^{\frac{1}{\psi}}}\right\} \tag{9}$$

Proof: We established (7) through the application of inductive reasoning applied to the variable n. When n=2, as per the operational guidelines (3) defined in Definition 3.3, we possess

$$\frac{1}{2}\bigotimes\wp(\daleth_1(\rho)) = \bigsqcup_{\wp(\daleth_1^{(\kappa)})\in\wp(\daleth_1(\rho))}\left\{1-\frac{1}{1+\left\{\frac{1}{2}(\frac{\rho_1^{(\kappa)}\wp(\daleth_1^{(\kappa)})}{1-\rho_1^{(\kappa)}\ddot{\varrho}(\daleth_1^{(\kappa)})})^{\psi}\right\}^{\frac{1}{\psi}}}\right\}$$

$$\frac{1}{2}\bigotimes\wp(\daleth_2(\rho)) = \bigsqcup_{\wp(\daleth_2^{(\kappa)})\in\wp(\daleth_2(\rho))}\left\{1-\frac{1}{1+\left\{\frac{1}{2}(\frac{\rho_2^{(\kappa)}\wp(\daleth_2^{(\kappa)})}{1-\rho_2^{(\kappa)}\wp(\daleth_2^{(\kappa)})})^{\psi}\right\}^{\frac{1}{\psi}}}\right\}$$

then

$$\begin{aligned}&PLDA(\wp(\daleth_1(\rho)),\wp(\daleth_2(\rho)))\\ &=\frac{1}{2}\bigotimes\wp(\daleth_1(\rho))\quad\bigoplus\quad\frac{1}{2}\bigotimes\wp(\daleth_2(\rho))\\ &=\left\{1-\frac{1}{1+\{\frac{1}{2}(\frac{\rho_i^{(\kappa)}\wp(\daleth_i^{(\kappa)})}{1-\rho_i^{(\kappa)}\wp(\daleth_i^{(\kappa)})})^{\psi}+\frac{1}{2}(\frac{\rho_2^{(\kappa)}\wp(\daleth_2^{(\kappa)})}{1-\rho_2^{(\kappa)}\wp(\daleth_2^{(\kappa)})})^{\psi}\}^{\frac{1}{\psi}}}\right\}.\\ &=\bigsqcup_{\wp(\daleth_i^{(\kappa)})\in\wp(\daleth_i(\rho))}\left\{1-\frac{1}{1+\left\{\sum_{i=1}^{2}\frac{1}{2}(\frac{\rho_i^{(\kappa)}\wp(\daleth_i^{(\kappa)})}{1-\rho_i^{(\kappa)}\wp(\daleth_i^{(\kappa)})})^{\psi}\right\}^{\frac{1}{\psi}}}\right\}\end{aligned}$$

For n=2, equation (7) is valid. If eq. (7) is true for n=m, which means,

$$\begin{aligned}&P\daleth DA(\wp(\daleth_1(\rho)),\wp(\daleth_2(\rho)),\ldots,\wp(\daleth_m(\rho)))\\ &=\bigoplus_{i=1}^{m}\left\{\frac{1}{m}\wp(\daleth_i(\rho))\right\}\\ &=\bigsqcup_{\wp(\daleth_i^{(\kappa)})\in\wp(\daleth_i(\rho))}\left\{1-\frac{1}{1+\left\{\sum_{i=1}^{m}\frac{1}{m}(\frac{\rho_i^{(\kappa)}\wp(\daleth_i^{(\kappa)})}{1-\rho_i^{(\kappa)}\wp(\daleth_i^{(\kappa)})})^{\psi}\right\}^{\frac{1}{\psi}}}\right\}\end{aligned}$$

Subsequently, for n=m+1, as per Definition 4.6 and the operational guidelines defined in Definition 4.1, we obtain

$$\begin{aligned}&\bigoplus_{i=1}^{m+1}\left\{\frac{1}{n}\wp(\daleth_i(\rho))\right\}\\ &=\bigoplus_{i=1}^{m}\left\{\frac{1}{n}\wp(\daleth_i(\rho))\right\}\bigoplus\left\{\frac{1}{n}\wp(\daleth_{m+1}(\rho))\right\}\\ &=PLDA(\wp(\daleth_1(\rho)),\wp(\daleth_2(\rho)),\ldots,\wp(\daleth_{m+1}(\rho)))\\ &=\bigsqcup_{\wp(\daleth_i^{(\kappa)})\in\wp(\daleth_i(\rho))}\left\{1-\frac{1}{1+\left\{\sum_{i=1}^{m+1}\frac{1}{m+1}(\frac{\rho_i^{(\kappa)}\wp(\daleth_i^{(\kappa)})}{1-\rho_i^{(\kappa)}\wp(\daleth_i^{(\kappa)})})^{\psi}\right\}^{\frac{1}{\psi}}}\right\}\\ &=\bigsqcup_{\wp(\daleth_i^{(\kappa)})\in\wp(\daleth_i(\rho))}\left\{1-\frac{1}{1+\left\{\sum_{i=1}^{n}\frac{1}{n}(\frac{\rho_i^{(\kappa)}\wp(\daleth_i^{(\kappa)})}{1-\rho_i^{(\kappa)}\wp(\daleth_i^{(\kappa)})})^{\psi}\right\}^{\frac{1}{\psi}}}\right\}\end{aligned}$$

i.e., for n=m+1, (3) endures, which complete the proving the theorem 1. □

1) **Property:**
Idempotency. Let $\wp(\daleth_i(\rho))$ $(\upsilon = 1,2,3,\ldots,n)$ constitute a set of transformed PLTs. If all $\wp(\daleth_i(\rho))$ $(\upsilon = 1,2,3,\ldots,n)$ are the same, i.e., $\wp(\daleth_i(\rho)) = \wp(\daleth(\rho))$ then

$$PLDA(\wp(\daleth_1(\rho)),\wp(\daleth_2(\rho)),\ldots,\wp(\daleth_n(\rho))) = \wp(\daleth(\rho))$$

Proof: If $\wp(\daleth_i(\rho) = \wp(\daleth(\rho))\ \forall i$, then $PLDA(\wp(\daleth_1(\rho),\wp(\daleth_2(\rho),\ldots,\wp(\daleth_n(\rho))$ is established as follows:

$$PLDA(\wp(\daleth_1(\rho),\wp(\daleth_2(\rho),\ldots,\wp(\daleth_n(\rho))$$

$$
\begin{aligned}
&= \bigoplus_{i=1}^{n} \frac{1}{n} \wp(\daleth_i(\rho)) \\
&= \left\{1 - \frac{1}{1 + \left\{ \sum_{i=1}^{n} \frac{1}{n} \left(\frac{\rho_i^{(\kappa)} \wp(\daleth_i^{(\kappa)})}{1 - \rho_i^{(\kappa)} \wp(\daleth_i^{(\kappa)})}\right)^{\psi} \right\}^{\frac{1}{\psi}}}\right\} \\
&= \left\{1 - \frac{1}{1 + \left\{ \left(\frac{\rho_i^{(\kappa)} \wp(\daleth_i^{(\kappa)})}{1 - \rho_i^{(\kappa)} \wp(\daleth_i^{(\kappa)})}\right)^{\psi} \right\}^{\frac{1}{\psi}}}\right\} \\
&= \left\{1 - \frac{1}{1 + \left\{ \frac{\rho_i^{(\kappa)} \wp(\daleth_i^{(\kappa)})}{1 - \rho_i^{(\kappa)} \wp(\daleth_i^{(\kappa)})} \right\}}\right\} \\
&= \wp(\daleth(\rho))
\end{aligned}
$$

Thus $PLDA(\wp(\daleth_1(\rho)), \wp(\daleth_2(\rho)), \ldots, \wp(\daleth_n(\rho))) = \wp(\daleth(\rho))$, holds. □

2) **Property:**
Boundedness.
Let $\wp(\daleth_i(\rho))$ $(\upsilon = 1, 2, 3, \ldots, n)$ constitute a set of PLTs, So, we can deduce:

$$\wp(\daleth(\rho))^{-} \leq \wp(\daleth(\rho)) \leq \wp(\daleth(\rho))^{+}$$

where $\wp(\daleth(\rho)) \in PLDA(\wp(\daleth_1(\rho)), \wp(\daleth_2(\rho)), \ldots, \wp(\daleth_n(\rho))), \wp(\daleth(\rho))^{-} \in min(\wp(\daleth_1(\rho)), \wp(\daleth_2(\rho)), \ldots, \wp(\daleth_n(\rho)))$ and $\wp(\daleth(\rho))^{+} \in max(\wp(\daleth_1(\rho)), \wp(\daleth_2(\rho)), \ldots, \wp(\daleth_n(\rho)))$.

Proof: In accordance with the outcome of Theorem 1, $PLDA(\wp(\daleth_1(\rho)), \wp(\daleth_2(\rho)), \ldots, \wp(\daleth_n(\rho)))$ is computed as:

$$
\begin{aligned}
&PLDA(\wp(\daleth_1(\rho)), \wp(\daleth_2(\rho)), \ldots, \wp(\daleth_n(\rho))) \\
&= \bigoplus_{i=1}^{n} \frac{1}{n} \wp(\daleth_i(\rho)).
\end{aligned}
$$

In that case, we are able to infer the following relations:

$$
\begin{aligned}
&\left\{1 - \frac{1}{1 + \left\{ \sum_{i=1}^{n} \frac{1}{n} \left(\frac{(\rho_i^{(\kappa)} \wp(\daleth_i^{(\kappa)}))^{-}}{1 - (\rho_i^{(\kappa)} \wp(\daleth_i^{(\kappa)}))^{-}}\right)^{\psi} \right\}^{\frac{1}{\psi}}}\right\} \\
&\leq \left\{1 - \frac{1}{1 + \left\{ \sum_{i=1}^{n} \frac{1}{n} \left(\frac{\rho_i^{(\kappa)} \wp(\daleth_i^{(\kappa)})}{1 - \rho_i^{(\kappa)} \wp(\daleth_i^{(\kappa)})}\right)^{\psi} \right\}^{\frac{1}{\psi}}}\right\} \\
&\leq \left\{1 - \frac{1}{1 + \left\{ \sum_{i=1}^{n} \frac{1}{n} \left(\frac{(\rho_i^{(\kappa)} \wp(\daleth_i^{(\kappa)}))^{+}}{1 - (\rho_i^{(\kappa)} \wp(\daleth_i^{(\kappa)}))^{+}}\right)^{\psi} \right\}^{\frac{1}{\psi}}}\right\}
\end{aligned}
$$

By leveraging the outcome of Theorem 1, we can readily establish the proving of Property. □

3) **Property:**
Consistency.
Suppose $\wp(\daleth_i(\rho))$ and $\wp(\daleth_i(\rho))^{*}$ constitute a pair of sets of PLTs and the quantities of LDs in $\wp(\daleth_i(\rho))$ and $\wp(\daleth_i(\rho)^{*})$ are comparable $(\upsilon = 1, 2, 3, \ldots, n)$. If $\wp(\daleth_i(\rho))(p_i^{(\kappa)}) < \wp(\daleth_i(\rho))(p_i^{(\kappa)})^{*}$ $\forall i$, i.e.,$\wp(\daleth_i(\rho))$ and $\wp(\daleth_i(\rho)^{*})$, following that

$$
\begin{aligned}
&PLDA(\wp(\daleth_1(\rho)), \wp(\daleth_2(\rho)), \ldots, \wp(\daleth_n(\rho))) \\
&\quad \leq PLDA(\wp(\daleth_1(\rho))^{*}, \wp(\daleth_2(\rho))^{*}, \ldots, \wp(\daleth_n(\rho))^{*}).
\end{aligned}
$$

B. PROBABILISTIC LINGUISTIC DOMBI WEIGHTED AVERAGING (PLDWA)

Within this segment, we will provide an exposition of the Weighted Average AO for Probabilistic Linguistic values, employing the Dombi norm and referred to as PLDWA. Additionally, the associated theorem will be presented.

Definition 12: Let $\daleth_i(\rho)$ constitute a set of PLTs, $\triangle = (\triangle_1, \triangle_2, \ldots, \triangle_n)^T$ signifies the weighting vector of $\daleth_i(\rho)$ and $\triangle_i \in [0, 1]$, $\sum_{i=1}^{n} \triangle_i = 1$. Considering the weight value $\triangle = (\triangle_1, \triangle_2, \ldots, \triangle_n)^T$, We establish the weighted PLDWA as follows:

$$
\begin{aligned}
&PLDWA(\wp(\daleth_1(\rho)), \wp(\daleth_2(\rho)), \ldots, \wp(\daleth_n(\rho))) \\
&= \bigoplus_{i=1}^{n} \triangle_i \wp(\daleth_i(\rho))
\end{aligned} \tag{10}
$$

Especially, if $\triangle = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the PLDWA operator becomes the PLDA operator:

$$
\begin{aligned}
&PLDA(\daleth_1(\rho), \daleth_2(\rho), \ldots, \daleth_n(\rho)) \\
&= \frac{1}{n}\wp(\daleth_1(\rho)) \bigoplus_{\varepsilon} \frac{1}{n}\wp(\daleth_2(\rho)) \bigoplus_{\varepsilon} \cdots \bigoplus_{\varepsilon} \frac{1}{n}\wp(\daleth_n(\rho))
\end{aligned} \tag{11}
$$

Theorem 2: Assume $\wp(\daleth_i(\rho))(\upsilon = 1, 2, 3, \ldots, n)$, comprises a set of modified PLTs, following that, the accumulated values through the use of the PLDWA operator is also modified PLTs, and:

$$
\begin{aligned}
&PLDWA(\wp(\daleth_1(\rho)), \wp(\daleth_2(\rho)), \ldots, \wp(\daleth_n(\rho))) \\
&= \bigsqcup_{\wp(\daleth_i^{(\kappa)}) \in \wp(\daleth_i(\rho))} \left\{1 - \frac{1}{1 + \left\{ \sum_{i=1}^{n} \triangle_i \left(\frac{\rho_i^{(\kappa)} \wp(\daleth_i^{(\kappa)})}{1 - \rho_i^{(\kappa)} \wp(\daleth_i^{(\kappa)})}\right)^{\psi} \right\}^{\frac{1}{\psi}}}\right\}
\end{aligned} \tag{12}
$$

where $\triangle_i = (\triangle_1, \triangle_2, \ldots, \triangle_n)^T$ is the weight vector of $\wp(\daleth_i(\rho))(\upsilon = 1, 2, 3, \ldots, n)$ with $\triangle_i \in [0, 1]$ and $\sum_{i=1}^{n} \triangle_i = 1$.

1) **Property:**
Idempotency. Let $\wp(\daleth_i(\rho))$ $(\upsilon = 1, 2, 3, \ldots, n)$ constitute a set of PLTs. If all $\wp(\daleth_i(\rho))$ $(\upsilon = 1, 2, 3, \ldots, n)$ are equated, i.e., $\wp(\daleth_i(\rho)) = \wp(\daleth(\rho))$ then

$$PLDWA(\wp(\daleth_1(\rho)), \wp(\daleth_2(\rho)), \ldots, \wp(\daleth_n(\rho))) = \wp(\daleth(\rho))$$

2) **Property:**
Boundedness. Let $\wp(\daleth_i(\rho))$ $(\upsilon = 1, 2, 3, \ldots, n)$ constitute a set of PLTs, subsequently, we possess:

$$
\begin{aligned}
&\min_{i=1}^{n} \min_{\kappa=1}^{\#\daleth_i(\rho)} (p_i^{(\kappa)}) \wp(\daleth_i^{(\kappa)}) \leq \wp(\daleth) \\
&\leq \max_{i=1}^{n} \max_{\kappa=1}^{\#\daleth_i(\rho)} (p_i^{(\kappa)}) \wp(\daleth_i^{(\kappa)})
\end{aligned}
$$

where $\wp(L) \in PLDWA(\wp(\daleth_1(\rho)), \wp(\daleth_2(\rho)), \ldots, \wp(\daleth_n(\rho)))$.

3) **Property:**
Consistency.
Let $\wp(\urcorner_i(\rho))$ and $\wp(\urcorner_i(\rho))^*$ are two sets of PLTs and the quantity of LDs in $\wp(\urcorner_i(\rho))$ and $\wp(\urcorner_i(\rho)^*)$ are equated $(\upsilon = 1,2,3,\ldots,n)$. If $\wp(\urcorner_i(\rho))(p_i^{(\kappa))} < \wp(\urcorner_i(\rho))(p_i^{(\kappa)*}\ \forall i$, i.e.,$\wp(\urcorner_i(\rho))$ and $\wp(\urcorner_i(\rho)^*)$, following

$$PLDWA(\wp(\urcorner_1(\rho)),\wp(\urcorner_2(\rho)),\ldots,\wp(\urcorner_n(\rho))) \leq PLDWA(\wp(\urcorner_1(\rho))^*,\wp(\urcorner_2(\rho))^*,\ldots,\wp(\urcorner_n(\rho))^*).$$

C. PROBABILISTIC LINGUISTIC DOMBI ORDERED WEIGHTED AVERAGING (PLDOWA)

In this section, we will provide a clarification and theorem concerning the Ordered Weighted Average AO for Probabilistic Linguistic values. This operator utilizes the Dombi norm and is designated as PLDOWA.

Definition 13: Let$\urcorner_i(\rho)$ constitute a set of PLTs, $\triangle = (\triangle_1,\triangle_2,\ldots,\triangle_n)^T$ indicates the vector used for assigning weights of $\urcorner_i(\rho)$ and $\triangle_i \in [0,1]$, $\sum_{i=1}^{n}\triangle_i = 1$. With the provided weight value $\triangle = (\triangle_1,\triangle_2,\ldots,\triangle_n)^T$, we define PLDOWA as follows:

$$PLDOWA(\wp(\urcorner_1(\rho)),\wp(\urcorner_2(\rho)),\ldots,\wp(\urcorner_n(\rho))) = \bigoplus_{i=1}^{n}\triangle_i\wp(\urcorner_{\delta i}(\rho)) \quad (13)$$

Especially, if $\triangle = (\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n})^T$, then the PLDOWA operator become the PLDA operator:

$$PLDA(\urcorner_1(\rho),\urcorner_2(\rho),\ldots,\urcorner_n(\rho)) = \frac{1}{n}\wp(\urcorner_1(\rho))\bigoplus_{\varepsilon}\frac{1}{n}\wp(\urcorner_2(\rho))\bigoplus_{\varepsilon}\cdots\bigoplus_{\varepsilon}\frac{1}{n}\wp(\urcorner_{\delta i}(\rho)) \quad (14)$$

Theorem 3: Suppose that $\wp(\urcorner_i(\rho))(\upsilon = 1,2,3,\ldots,n)$ is a compilation of modified PLTs, following that aggregated values by the use of PLDOWA operator is also modified PLTs, and:

$$PLDOWA(\wp(\urcorner_1(\rho)),\wp(\urcorner_2(\rho)),\ldots,\wp(\urcorner_n(\rho))) = \bigsqcup_{\wp(\urcorner_i^{(\kappa)})\in\wp(\urcorner_i(\rho))}\left\{1-\frac{1}{1+\left\{\sum_{i=1}^{n}\triangle_i(\frac{\rho_{\delta i}^{(\kappa)}\wp(\urcorner_{\delta i}^{(\kappa)})}{1-\rho_{\delta i}^{(\kappa)}\wp(\urcorner_{\delta i}^{(\kappa)})})^{\psi}\right\}^{\frac{1}{\psi}}}\right\} \quad (15)$$

where $\triangle_i = (\triangle_1,\triangle_2,\ldots,\triangle_n)^T$ is the weight vector of $\wp(\urcorner_{\delta i}(\rho))(\upsilon=1,2,3,\ldots,n)$ with $\triangle_i \in [0,1]$ and $\sum_{i=1}^{n}\triangle_i=1$.

1) **Property:**
Idempotency. Let $\wp(\urcorner_{\delta i}(\rho))\quad(\upsilon = 1,2,3,\ldots,n)$ constitute a set of PLTs. If all $\wp(\urcorner_{\delta i}(\rho))\quad(\upsilon = 1,2,3,\ldots,n)$ are equated, i.e., $\wp(\urcorner_{\delta i}(\rho)) = \wp(\urcorner(\rho))$ then

$$PLDOWA(\wp(\urcorner_{\delta 1}(\rho)),\wp(\urcorner_{\delta 2}(\rho)),\ldots,\wp(\urcorner_{\delta n}(\rho))) = \wp(\urcorner(\rho))$$

2) **Property:**
Boundedness. Let $\wp(\urcorner_{\delta i}(\rho))\quad(\upsilon = 1,2,3,\ldots,n)$ constitute a set of PLTs, subsequently, we possess

$$\min_{i=1}^{n}\min_{\kappa=1}^{\blacklozenge\urcorner_{\delta i}(\rho)}(p_{\delta i}^{(\kappa)})\wp(\urcorner_{\delta i}^{(\kappa)}) \leq \wp(\urcorner) \leq \max_{i=1}^{n}\max_{\kappa=1}^{\blacklozenge\urcorner_{\delta i}(\rho)}(p_{\delta i}^{(\kappa)})\wp(\urcorner_{\delta i}^{(\kappa)})$$

where $\wp(L) \in PLDWA(\wp(\urcorner_{\delta 1}(\rho)),\wp(\urcorner_{\delta 2}(\rho)),\ldots,\wp(\urcorner_{\delta n}(\rho)))$.

3) **Property:**
Consistency.
Assume $\wp(\urcorner_{\delta i}(\rho))$ and $\wp(\urcorner_{\delta i}(\rho))^*$ couple of sets of PLTs and the quantity of LDs in $\wp(\urcorner_{\delta i}(\rho))$ and $\wp(\urcorner_{\delta i}(\rho)^*)$ are equated $(\upsilon = 1,2,3,\ldots,n)$. If $\wp(\urcorner_{\delta i}(\rho))(p_{\delta i}^{(\kappa))} < \wp(\urcorner_{\delta i}(\rho))(p_{\delta i}^{(\kappa)*}\ \forall i$, i.e.,$\wp(\urcorner_{\delta i}(\rho))$ and $\wp(\urcorner_{\delta i}(\rho)^*)$, then

$$PLDOWA(\wp(\urcorner_{\delta 1}(\rho)),\wp(\urcorner_{\delta 2}(\rho)),\ldots,\wp(\urcorner_{\delta n}(\rho))) \leq PLDOWA(\wp(\urcorner_{\delta 1}(\rho))^*,\wp(\urcorner_{\delta 2}(\rho))^*,\ldots,\wp(\urcorner_{\delta n}(\rho))^*).$$

D. PROBABILISTIC LINGUISTIC DOMBI GEOMETRIC (PLDG) AGGREGATION OPERATORS

This subsection serves as an initial exploration of geometric aggregation operators designed for Probabilistic Linguistic Trees (PLTs). We will explore the mathematical representation of these operators and clarify several essential characteristics they possess. Furthermore, detailed explanations will be provided to support the precision and dependability of these operators when aggregating information from PLTs.

Definition 14: Let $L(\rho) = \{\urcorner_i^{(\kappa)}(p_i^{(\kappa)})|\kappa = 1,2,3,\ldots,\blacklozenge\urcorner_i(\rho)\}$ $(\upsilon = 1,2,3,\ldots,n)$ constitute a set of PLTs and $\wp(\urcorner_i(\rho))$ its Correspondent modification. PLDG is a mapping $\wp(L^n(\rho)) \rightarrow \wp(\urcorner(\rho))$, in such a manner that

$$PLDG(\wp(\urcorner_1(\rho)),\wp(\urcorner_2(\rho)),\ldots,\wp(\urcorner_n(\rho))) = \bigotimes_{i=1}^{n}(\wp(\urcorner_i(\rho)))^{\frac{1}{n}} \quad (16)$$

Theorem 4: Suppose that $L(\rho) = \urcorner i^{(\kappa)}(pi^{(\kappa)})|\kappa = 1,2,3,\ldots,\blacklozenge\urcorner_i(\rho)(\upsilon = 1,2,3,\ldots,n)$ represents a collection of PLTs, with $\wp(\urcorner(\rho))$ as their corresponding modifications. The combined outcome, when determined using the PLDG operator, also falls within the category of PLT.

$$PLDG(\wp(\urcorner_1(\rho)),\wp(\urcorner_2(\rho)),\ldots,\wp(\urcorner_n(\rho))) = \bigsqcup_{\wp(\urcorner_i^{(\kappa)})\in\wp(\urcorner_i(\rho))}\left\{\frac{1}{1+\left\{\sum_{i=1}^{n}\frac{1}{n}(\frac{1-\rho_i^{(\kappa)}\wp(\urcorner_i^{(\kappa)})}{\rho_1^{(\kappa)}\wp(\urcorner_1^{(\kappa)})})^{\psi}\right\}^{\frac{1}{\psi}}}\right\} \quad (17)$$

Proof: We demonstrated the validity of (15) through the application of inductive mathematical argument on n. For the case when n=2, following the operational guidelines (4) *outlined in Definition 3.3, we obtain*

$$\wp(\urcorner_1(\rho))^{\frac{1}{2}} = \bigsqcup_{\wp(\urcorner_1^{(\kappa)})\in\wp(\urcorner_1(\rho))} \left\{\frac{1}{1+\left\{\frac{1}{2}(\frac{1-\rho_1^{(\kappa)}\wp(\urcorner_1^{(\kappa)})}{\rho_1^{(\kappa)}\wp(\urcorner_1^{(\kappa)})})^{\psi}\right\}^{\frac{1}{\psi}}}\right\}$$

$$\wp(\urcorner_2(\rho))^{\frac{1}{2}} = \bigsqcup_{\wp(\urcorner_2^{(\kappa)})\in\wp(\urcorner_2(\rho))} \left\{\frac{1}{1+\left\{\frac{1}{2}(\frac{1-\rho_2^{(\kappa)}\wp(\urcorner_2^{(\kappa)})}{\rho_2^{(\kappa)}\wp(\urcorner_2^{(\kappa)})})^{\psi}\right\}^{\frac{1}{\psi}}}\right\}$$

Then

$$\begin{aligned}
&PLDG(\wp(\urcorner_1(\rho)),\wp(\urcorner_2(\rho)))\\
&= \wp(\urcorner_1(\rho))^{\frac{1}{2}} \bigotimes \wp(\urcorner_2(\rho))^{\frac{1}{2}}\\
&= \left\{1-\frac{1}{1+\{\frac{1}{2}(\frac{1-\rho_i^{(\kappa)}\wp(\urcorner_i^{(\kappa)})}{\rho_i^{(\kappa)}\wp(\urcorner_i^{(\kappa)})})^{\psi}+\frac{1}{2}(\frac{1-\rho_2^{(\kappa)}\wp(\urcorner_2^{(\kappa)})}{\rho_2^{(\kappa)}\wp(\urcorner_2^{(\kappa)})})^{\psi}\}^{\frac{1}{\psi}}}\right\}.\\
&= \bigsqcup_{\wp(\urcorner_i^{(\kappa)})\in\wp(\urcorner_i(\rho))} \left\{\frac{1}{1+\left\{\sum_{i=1}^{2}\frac{1}{2}(\frac{1-\rho_i^{(\kappa)}\wp(\urcorner_i^{(\kappa)})}{\rho_1^{(\kappa)}\wp(\urcorner_1^{(\kappa)})})^{\psi}\right\}^{\frac{1}{\psi}}}\right\}
\end{aligned}$$

That is, for n=2, Equation (15) is true. Assuming n=m, Equation (15) endures, meaning that,

$$\begin{aligned}
&PLDG(\wp(\urcorner_1(\rho)),\wp(\urcorner_2(\rho)),\ldots,\wp(\urcorner_m(\rho)))\\
&= \bigsqcup_{\wp(\urcorner_i^{(\kappa)})\in\wp(\urcorner_i(\rho))} \left\{\frac{1}{1+\left\{\sum_{i=1}^{m}\frac{1}{m}(\frac{1-\rho_i^{(\kappa)}\wp(\urcorner_i^{(\kappa)})}{\rho_1^{(\kappa)}\wp(\urcorner_1^{(\kappa)})})^{\psi}\right\}^{\frac{1}{\psi}}}\right\}
\end{aligned}$$

Subsequently, for n=m+1, in accordance with Definition 4.6 and the operational guidelines outlined in Definition 4.1, we obtain

$$\begin{aligned}
&\bigotimes_{i=1}^{m+1}(\wp(\urcorner_i(\rho)))^{\frac{1}{m+1}}\\
&= \bigotimes_{i=1}^{m+1}(\wp(\urcorner_i(\rho)))^{\frac{1}{m}} \bigotimes (\wp(\urcorner_{m+1}(\rho)))^{\frac{1}{m+1}}\\
&= \bigsqcup_{\wp(\urcorner_i^{(\kappa)})\in\wp(\urcorner_i(\rho))} \left\{\frac{1}{1+\left\{\sum_{i=1}^{m+1}\frac{1}{m+1}(\frac{1-\rho_i^{(\kappa)}\wp(\urcorner_i^{(\kappa)})}{\rho_1^{(\kappa)}\wp(\urcorner_1^{(\kappa)})})^{\psi}\right\}^{\frac{1}{\psi}}}\right\}
\end{aligned}$$

Thus prove is completed. □

1) **Property:**
Idempotency.
Let $\wp(\urcorner_i(\rho))$ $(\upsilon = 1,2,3,\ldots,n)$ constitute a set of modified PLTs. If all $\wp(\urcorner_i(\rho))$ $(\upsilon = 1,2,3,\ldots,n)$ are equated, i.e., $\wp(\urcorner_i(\rho)) = \wp(\urcorner(\rho))$ then

$$PLDG(\wp(\urcorner_1(\rho)),\wp(\urcorner_2(\rho)),\ldots,\wp(\urcorner_n(\rho)))=\wp(\urcorner(\rho))$$

Proof: If $\wp(\urcorner_i(\rho) = \wp\ (\urcorner(\rho))\ \forall i$, then $PLDA(\wp(\urcorner_1(\rho),\wp(\urcorner_2(\rho),\ldots,\wp(\urcorner_n(\rho))$ is established as follows:

$$\begin{aligned}
&PLDA(\wp(\urcorner_1(\rho),\wp(\urcorner_2(\rho),\ldots,\wp(\urcorner_n(\rho))\\
&= \bigotimes_{i=1}^{n}(\wp(\urcorner_i(\rho)))^{\frac{1}{n}}\\
&= \left\{\frac{1}{1+\left\{\sum_{i=1}^{n}\frac{1}{n}(\frac{1-\rho_i^{(\kappa)}\wp(\urcorner_i^{(\kappa)})}{\rho_1^{(\kappa)}\wp(\urcorner_1^{(\kappa)})})^{\psi}\right\}^{\frac{1}{\psi}}}\right\}\\
&= \left\{\frac{1}{1+\left\{(\frac{1-\rho_i^{(\kappa)}\wp(\urcorner_i^{(\kappa)})}{\rho_1^{(\kappa)}\wp(\urcorner_1^{(\kappa)})})^{\psi}\right\}^{\frac{1}{\psi}}}\right\}\\
&= \left\{\frac{1}{1+\frac{1-\rho_i^{(\kappa)}\wp(\urcorner_i^{(\kappa)})}{\rho_1^{(\kappa)}\wp(\urcorner_1^{(\kappa)})}}\right\}\\
&= \wp(\urcorner(\rho))
\end{aligned}$$

Thus $PLDA(\wp(\urcorner_1(\rho)),\wp(\urcorner_2(\rho)),\ldots,\wp(\urcorner_n(\rho))) = \wp(\urcorner(\rho))$, holds. □

2) **Property:**
Boundedness.
Let $\wp(\urcorner_i(\rho))$ $(\upsilon = 1,2,3,\ldots,n)$ constitute a set of PLTs, So, we can deduce:

$$\wp(\urcorner(\rho))^{-} \leq \wp(\urcorner(\rho)) \leq \wp(\urcorner(\rho))^{+}$$

where $\wp(\urcorner(\rho)) \in PLDA(\wp(\urcorner_1(\rho)),\wp(\urcorner_2(\rho)),\ldots,\wp(\urcorner_n(\rho)))$, $\wp(\urcorner(\rho))^{-} \in min(\wp(\urcorner_1(\rho)),\wp(\urcorner_2(\rho)),\ldots,\wp(\urcorner_n(\rho)))$ and $\wp(\urcorner(\rho))^{+} \in max(\wp(\urcorner_1(\rho)),\wp(\urcorner_2(\rho)),\ldots,\wp(\urcorner_n(\rho)))$.

Proof: In accordance with the outcome of Theorem 4.17, $PLDA(\wp(\urcorner_1(\rho)),\wp(\urcorner_2(\rho)),\ldots,\wp(\urcorner_n(\rho)))$ is computed as:

$$\begin{aligned}
&PLDA(\wp(\urcorner_1(\rho)),\wp(\urcorner_2(\rho)),\ldots,\wp(\urcorner_n(\rho)))\\
&= \bigotimes_{i=1}^{n}(\wp(\urcorner_i(\rho)))^{\frac{1}{n}}.
\end{aligned}$$

In that case, we are able to infer the following relations:

$$\begin{aligned}
&\left\{\frac{1}{1+\left\{\sum_{i=1}^{n}\frac{1}{n}(\frac{1-(\rho_i^{(\kappa)}\wp(\urcorner_i^{(\kappa)}))^{-}}{(\rho_1^{(\kappa)}\wp(\urcorner_1^{(\kappa)}))^{-}})^{\psi}\right\}^{\frac{1}{\psi}}}\right\}\\
&\leq \left\{\frac{1}{1+\left\{\sum_{i=1}^{n}\frac{1}{n}(\frac{1-\rho_i^{(\kappa)}\wp(\urcorner_i^{(\kappa)})}{\rho_1^{(\kappa)}\wp(\urcorner_1^{(\kappa)})})^{\psi}\right\}^{\frac{1}{\psi}}}\right\}\\
&\leq \left\{\frac{1}{1+\left\{\sum_{i=1}^{n}\frac{1}{n}(\frac{1-(\rho_i^{(\kappa)}\wp(\urcorner_i^{(\kappa)}))^{+}}{(\rho_1^{(\kappa)}\wp(\urcorner_1^{(\kappa)}))^{+}})^{\psi}\right\}^{\frac{1}{\psi}}}\right\}
\end{aligned}$$

By leveraging the outcome of Theorem 4.17, we can readily establish the proving of Property. □

3) **Property:**
Consistency. Let $\wp(\urcorner_i(\rho))$ and $\wp(\urcorner_i(\rho))^{*}$ constitute a couple of sets of PLTs, and the quantities of linguistic labels within $\wp(\urcorner_i(\rho))$ and $\wp(\urcorner_i(\rho)^{*})$ are equated $(\upsilon = 1,2,3,\ldots,n)$. If $\wp(\urcorner_i(\rho))(p_i^{(\kappa))} < \wp(\urcorner_i(\rho))(p_i^{(\kappa)*}\ \forall i$, i.e.,$\wp(\urcorner_i(\rho))$ and $\wp(\urcorner_i(\rho)^{*})$, then

$$\begin{aligned}
&PLDG(\wp(\urcorner_1(\rho)),\wp(\urcorner_2(\rho)),\ldots,\wp(\urcorner_n(\rho)))\\
&\leq PLDA(\wp(\urcorner_1(\rho))^{*},\wp(\urcorner_2(\rho))^{*},\ldots,\wp(\urcorner_n(\rho))^{*}).
\end{aligned}$$

E. PROBABILISTIC LINGUISTIC DOMBI WEIGHTED GEOMETRIC (PLDWG)

In this section, we will offer an explanation and theorem related to the Weighted Geometric AO for Probabilistic Linguistic values, using the Dombi norm, which is denoted as PLDWG.

Definition 15: Let $\daleth i(\rho)$ represent a set of PLTs, with $\wp(\daleth i(\rho))$ denoting their corresponding modifications. The weight vector $\triangle = (\triangle_1, \triangle_2, \ldots, \triangle_n)^T$ signifies the weights assigned to $\daleth i(\rho)$, where $\triangle i$ ranges from 0 to 1, and the sum of all $\triangle_i$ values equals 1. With the weight vector $\triangle = (\triangle_1, \triangle_2, \ldots, \triangle_n)^T$, we define the PLDWG operator as follows:

$$PLDWG(\wp(\daleth_1(\rho)), \wp(\daleth_2(\rho)), \ldots, \wp(\daleth_n(\rho))) = \bigotimes_{i=1}^{n} \wp(\daleth_i(\rho))^{\triangle_i} \tag{18}$$

Theorem 5: Assume a collection of PLTs represented as $L(\rho) = \daleth i^{(\kappa)}(pi^{(\kappa)})|\kappa = 1, 2, 3, \ldots, \blacklozenge\daleth_i(\rho)(\upsilon = 1, 2, 3, \ldots, n)$, and their corresponding modifications denoted as $\wp(\daleth(\rho))$. When these are combined using the PLDWG operator, the resulting value is also a PLTE.

$$PLDWG(\wp(\daleth_1(\rho)), \wp(\daleth_2(\rho)), \ldots, \wp(\daleth_n(\rho))) = \bigsqcup_{\wp(\daleth_i^{(\kappa)})\in\wp(\daleth_i(\rho))} \left\{ \frac{1}{1 + \left\{ \sum_{i=1}^{n} \triangle_i \left(\frac{1-\rho_i^{(\kappa)}\wp(\daleth_i^{(\kappa)})}{\rho_1^{(\kappa)}\wp(\daleth_1^{(\kappa)})}\right)\psi \right\}^{\frac{1}{\psi}}} \right\} \tag{19}$$

1) **Property:**
Idempotency.
Assume $\wp(\daleth_i(\rho))$ $(\upsilon = 1, 2, 3, \ldots, n)$ constitute a set of PLTs and $\wp(\daleth_i(\rho))$ its Correspondent modification. If all $\wp(\daleth_i(\rho))$ $(\upsilon = 1, 2, 3, \ldots, n)$ are equated, i.e., $\wp(\daleth_i(\rho)) = \wp(\daleth(\rho))$ following that
$$PLDWG(\wp(\daleth_1(\rho)), \wp(\daleth_2(\rho)), \ldots, \wp(\daleth_n(\rho))) = \wp(\daleth(\rho))$$

2) **Property:**
Boundedness.
Let $\wp(\daleth_i(\rho))$ $(\upsilon = 1, 2, 3, \ldots, n)$ constitute a set of PLTs and $\wp(\daleth_i(\rho))$ its Correspondent modification, subsequently, we possess
$$\min_{i=1}^{n} \min_{\kappa=1}^{\blacklozenge\daleth_i(\rho)} (p_i^{(\kappa)})\wp(\daleth_i^{(\kappa)}) \leq \wp(\daleth) \leq \max_{i=1}^{n} \max_{\kappa=1}^{\blacklozenge\daleth_i(\rho)} (p_i^{(\kappa)})\wp(\daleth_i^{(\kappa)})$$

3) **Property:**
Consistency.
Let $\wp(\daleth_i(\rho))$ and $\wp(\daleth_i(\rho))^*$ constitute two groups of PLTs and the quantity of LDs in $\wp(\daleth_i(\rho))$ and $\wp(\daleth_i(\rho)^*)$ are equated $(\upsilon = 1, 2, 3, \ldots, n)$. If $\wp(\daleth_i(\rho))(p_i^{(\kappa))} < \wp(\daleth_i(\rho))(p_i^{(\kappa)*}$ $\forall i$, i.e.,$\wp(\daleth_i(\rho))$ and $\wp(\daleth_i(\rho)^*)$, then
$$PLDWG(\wp(\daleth_1(\rho)), \wp(\daleth_2(\rho)), \ldots, \wp(\daleth_n(\rho))) \leq PLDWG(\wp(\daleth_1(\rho))^*, \wp(\daleth_2(\rho))^*, \ldots, \wp(\daleth_n(\rho))^*).$$

F. PROBABILISTIC LINGUISTIC DOMBI ORDERED WEIGHTED GEOMETRIC (PLDOWG)

In this section, we will present the explanation and theorem related to the Ordered Weighted Geometric AO for Probabilistic Linguistic values, utilizing the Dombi norm, which is represented as PLDOWG.

Definition 16: Let $\daleth_i(\rho)$ constitute a set of PLTs and $\wp(\daleth_i(\rho))$ its correspondent modification $\triangle = (\triangle_1, \triangle_2, \ldots, \triangle_n)^T$ represents the vector of weights of $\daleth_i(\rho)$, $\triangle_i \in [0, 1]$ and $\sum_{i=1}^{n} \triangle_i = 1$. In light of the provided value of the weight vector $\triangle = (\triangle_1, \triangle_2, \ldots, \triangle_n)^T$, we defined PLDWG as stated below:

$$PLDOWG(\wp(\daleth_1(\rho)), \wp(\daleth_2(\rho)), \ldots, \wp(\daleth_n(\rho))) = \bigotimes_{i=1}^{n} \wp(\daleth_{\delta i}(\rho))^{\triangle_i} \tag{20}$$

Theorem 6: Let $L(\rho) = \daleth_i^{(\kappa)}(p_i^{(\kappa)})|\kappa = 1, 2, 3, \ldots, \blacklozenge\daleth_i(\rho)(\upsilon = 1, 2, 3, \ldots, n)$ constitute a set of PLTs and $\wp(\daleth(\rho))$ its Correspondent modification, then their combined value can be determined through the utilization of PLDOWG is also a PLT and

$$PLDOWG(\wp(\daleth_{\delta 1}(\rho)), \wp(\daleth_{\delta 2}(\rho)), \ldots, \wp(\daleth_{\delta n}(\rho))) = \bigsqcup_{\wp(\daleth_{\delta i}^{(\kappa)})\in\wp(\daleth_{\delta i}(\rho))} \left\{ \frac{1}{1 + \left\{ \sum_{i=1}^{n} \triangle_i \left(\frac{1-\rho_{\delta i}^{(\kappa)}\wp(\daleth_{\delta i}^{(\kappa)})}{\rho_{\delta i}^{(\kappa)}\wp(\daleth_{\delta i}^{(\kappa)})}\right)\psi \right\}^{\frac{1}{\psi}}} \right\} \tag{21}$$

1) **Property:**
Idempotency.
Let $\wp(\daleth_{\delta i}(\rho))$ $(\upsilon = 1, 2, 3, \ldots, n)$ constitute a set of PLTs and $\wp(\daleth_i(\rho))$ its Correspondent modification. If all $\wp(\daleth_{\delta i}(\rho))$ $(\upsilon = 1, 2, 3, \ldots, n)$ are equated, i.e., $\wp(\daleth_{\delta i}(\rho)) = \wp(\daleth(\rho))$ then
$$PLDOWG(\wp(\daleth_{\delta 1}(\rho)), \wp(\daleth_{\delta 2}(\rho)), \ldots, \wp(\daleth_{\delta n}(\rho))) = \wp(\daleth(\rho))$$

2) **Property:**
Boundedness.
Let $\wp(\daleth_{\delta i}(\rho))$ $(\upsilon = 1, 2, 3, \ldots, n)$ constitute a set of PLTs and $\wp(\daleth_{\delta i}(\rho))$ its Correspondent modification, subsequently, we possess:
$$\min_{i=1}^{n} \min_{\kappa=1}^{\blacklozenge\daleth_{\delta i}(\rho)} (p_{\delta i}^{(\kappa)})\wp(\daleth_{\delta i}^{(\kappa)}) \leq \wp(\daleth) \leq \max_{i=1}^{n} \max_{\kappa=1}^{\blacklozenge\daleth_{\delta i}(\rho)} (p_{\delta i}^{(\kappa)})\wp(\daleth_{\delta i}^{(\kappa)})$$

3) **Property:**
Consistency.
Consider $\wp(\daleth_{\delta i}(\rho))$ and $\wp(\daleth_{\delta i}(\rho))^*$ constitute two classes of PLTs and the quantity of LDs in $\wp(\daleth_{\delta i}(\rho))$ and $\wp(\daleth_{\delta i}(\rho)^*)$ are equated $(\upsilon = 1, 2, 3, \ldots, n)$. If $\wp(\daleth_{\delta i}(\rho))(p_{\delta i}^{(\kappa))} < \wp(\daleth_{\delta i}(\rho))(p_{\delta i}^{(\kappa)*}$ $\forall i$, i.e.,$\wp(\daleth_{\delta i}(\rho))$ and $\wp(\daleth_{\delta i}(\rho)^*)$, leads to
$$PLDOWG(\wp(\daleth_{\delta 1}(\rho)), \wp(\daleth_{\delta 2}(\rho)), \ldots, \wp(\daleth_{\delta n}(\rho))) \leq PLDOWG(\wp(\daleth_{\delta 1}(\rho))^*, \wp(\daleth_{\delta 2}(\rho))^*, \ldots, \wp(\daleth_{\delta n}(\rho))^*).$$

V. OPERATORS FOR PLTS IN MCDM

Algorithm

1) Collective Data by the expert.
2) By using definition 2.7, we derive the normalised PL matrix.
3) The attribute weights were set. The following formula can be used to determine the attribute weights:

$$\aleph\daleth_{ij}(\rho) = \frac{\sum_{\kappa=1}^{\blacklozenge\daleth(\rho)} \rho_{ij}^{\kappa}\wp(\daleth_{ij}^{\kappa})}{\sum_{i=1}^{n}\sum_{\kappa=1}^{\blacklozenge\daleth(\rho)} \rho_{ij}^{\kappa}\wp(\daleth_{ij}^{\kappa})} \tag{22}$$

Then, the entropy $\xi = (\xi_1, \xi_2, , \xi_n)$ is computed in the following way:

$$\xi_j = \frac{-1}{ln(m)}\sum_{i=1}^{n}\aleph\daleth_{ij}(\rho)ln(\aleph\daleth_{ij}(\rho)), j = 1, 2, \ldots, m \tag{23}$$

At last, the attribute weight vector $\triangle = (\triangle_1, \triangle_2, , \triangle_m)$ is found:

$$\triangle_j = \frac{1-\xi_j}{\sum_{j=1}^{m}(1-\xi_j)}, j = 1, 2, \ldots, m \tag{24}$$

4) The MCDM process entails the fusion of attributes for each alternative by employing both arithmetic and geometric aggregation operators designed for Probabilistic Linguistic Dombi (PLD).
5) To determine the scoring values for each alternative as outlined in Definitions 2.4.
6) Choosing the optimal choice after arranging all the alternatives in a descending order of preference.e.

The algorithm's flowchart is illustrated in Figure. 1.

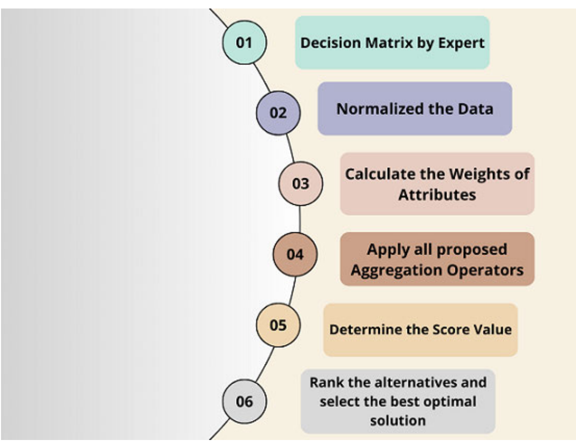

FIGURE 1. Flow chart of algorithm.

VI. CASE STUDY: OPTIMIZING OFFSHORE DRILLING OPERATIONS

The exploration and extraction of hydrocarbons from beneath the Earth's seabed represent a critical phase in the global energy supply chain. The offshore oil and gas industry, characterized by its complex operations in challenging environments, has been pivotal in meeting the world's escalating energy demands. However, as technological advancements push the boundaries of extraction capabilities, the industry is confronted with the imperative to optimize its operations comprehensively. In this context, the application of MCDM emerges as a strategic approach to navigate the intricate landscape of offshore drilling.

The offshore oil and gas sector operates in an environment fraught with challenges. The exploration and extraction of hydrocarbons from beneath the seabed involve intricate processes that demand a delicate balance between efficiency, safety, environmental sustainability, and economic viability. The industry has historically faced scrutiny for its environmental impact, safety concerns, and the need to continually enhance operational efficiency. As global energy demands surge and the push for cleaner energy intensifies, the imperative to address these challenges becomes ever more pressing.

Optimizing offshore drilling operations holds profound significance for multiple stakeholders. From the perspective of oil and gas companies, optimization directly impacts profitability, risk mitigation, and compliance with environmental regulations. For host countries and local communities, the optimization of offshore operations is intricately linked to economic development, job creation, and environmental stewardship. Furthermore, in a broader context, optimizing offshore drilling aligns with the global imperative to ensure energy security, reduce carbon footprints, and transition toward sustainable energy sources.

MCDM has emerged as a powerful tool in the decision-making process, particularly in complex and multifaceted domains such as offshore drilling. MCDM provides a structured framework for evaluating and comparing alternatives based on multiple criteria, allowing decision-makers to make informed choices that align with organizational goals and stakeholder expectations. By incorporating diverse criteria such as operational efficiency, safety, environmental impact, and cost-effectiveness, MCDM facilitates a holistic assessment of potential strategies and technologies.

This study aims to explore the optimization of offshore drilling operations through the application of MCDM. The focus is on identifying the most suitable alternative among four distinct approaches: Advanced Drilling Technologies (ADT), Drilling Process Optimization (DPO), Human Factors and Safety Enhancement (HFSE), and Environmental Impact Mitigation (EIM). These alternatives will be rigorously evaluated against four key criteria: Operational Efficiency (OE), Safety Performance (SP), Environmental Impact (EI), and Cost-effectiveness (CE).

This research is structured to provide a comprehensive understanding of the challenges and opportunities in optimizing offshore drilling operations. Following this introduction, the subsequent sections will delve into a detailed literature review, examining existing research on offshore drilling technologies, safety protocols, environmental considerations, and decision-making methodologies. The methodology section will outline the research design, data collection methods, and the application of the MCDM framework. The findings will be presented and analyzed, leading to a robust discussion and conclusions that contribute to the existing body of knowledge in the field.

As the global energy landscape continues to evolve, this study aspires to offer valuable insights and practical recommendations for stakeholders in the offshore oil and gas industry seeking to navigate the complexities of optimizing their drilling operations.

Criteria for Evaluation:

Operational Efficiency (OE)($\mathfrak{A}_1$): Assessing the effectiveness of each alternative in improving drilling speed, reducing downtime, and optimizing resource utilization.

Safety Performance (SP)($\mathfrak{A}_2$): Evaluating the safety measures and their impact on preventing accidents and ensuring the well-being of personnel.

Environmental Impact (EI)($\mathfrak{A}_3$): Measuring the environmental footprint of each alternative, considering factors such as emissions, waste disposal, and ecological impact.

Cost-effectiveness (CE)($\mathfrak{A}_4$): Analyzing the economic implications of each alternative, including initial investment, operational costs, and potential long-term savings.

Alternatives:

Advanced Drilling Technologies (ADT) ($\mathfrak{D}_1$): Implementation of cutting-edge drilling technologies, including automated drilling systems and real-time data analytics.

Drilling Process Optimization (DPO) ($\mathfrak{D}_2$): Focus on improving traditional drilling processes through enhanced procedural efficiency and reduced downtime.

Human Factors and Safety Enhancement (HFSE)($\mathfrak{D}_3$): Emphasis on training, safety protocols, and crew well-being to optimize human performance and reduce accidents.

Environmental Impact Mitigation (EIM) ($\mathfrak{D}_4$): Implementation of practices and technologies to minimize the environmental footprint of drilling operations.

Offshore drilling is a complex problem that inherently involves MCDM. The decision to drill in a particular location requires the consideration of numerous conflicting factors, including the potential economic benefits, environmental impact, technical feasibility, safety risks, regulatory compliance, and social acceptability. For instance, a drilling site may offer high economic returns but also pose significant environmental risks, such as oil spills or habitat destruction. meanwhile, technical limitations and regulatory requirements must also be considered, alongside the need to ensure social acceptability and minimize the community disruption. The goal is to strike a balance among these competing criteria, optimizing decision-making to maximize overall performance. By acknowledging the inherent multi-criteria nature of offshore drilling, decision-makers can leverage specialized method to navigate this complexity and make more informed, balanced, and sustainable decisions.

To demonstrate the practical application of our proposed study, hypothetical data from offshore drilling operations for oil and gas are utilized in a case study scenario. The proposed algorithm is systematically applied to evaluate and optimize drilling site selection. This process involves integrating the algorithm with the hypothetical data to assess best optimal option for offshore drilling. By applying our methodology, we illustrate its efficacy in decision-making for offshore drilling, showcasing its capability to enhance operational outcomes and mitigate risks in real-world applications.

The data we are using is hypothetical, but in reality, experts in the field have access to real data, which they use to make informed decisions. However, for the purpose of this exercise, we are using hypothetical data to illustrate the concept.

Step 1: Collective Data by the experts In Table 12.

TABLE 1. Collective data by the experts.

Alternatives	$\mathfrak{A}_1$	$\mathfrak{A}_2$
$\mathfrak{D}_1$	$\{s_3(0.7), s_4(0.3)\}$	$\{s_2(0.25), s_3(0.7)\}$
$\mathfrak{D}_2$	$\{s_2(0.6), s_3(0.4)\}$	$\{s_2(0.6), s_3(0.4)\}$
$\mathfrak{D}_3$	$\{s_2(0.8), s_3(0.2)\}$	$\{s_2(0.7), s_3(0.3)\}$
$\mathfrak{D}_4$	$\{s_1(0.9)\}$	$\{s_1(0.8), s_2(0.2)\}$
Alternatives	$\mathfrak{A}_3$	$\mathfrak{A}_4$
$\mathfrak{D}_1$	$\{s_2(0.2), s_3(0.8)\}$	$\{s_2(0.3), s_3(0.5), s_4(0.2)\}$
$\mathfrak{D}_2$	$\{s_2(1)\}$	$\{s_2(0.65), s_3(0.35)\}$
$\mathfrak{D}_3$	$\{s_1(0.3), s_2(0.7)\}$	$\{s_2(0.9), s_3(0.1)\}$
$\mathfrak{D}_4$	$\{s_3(0.5), s_4(0.5)\}$	$\{s_4(0.4)\}$

Step 2: By definition 2.7, we normalize the decision matrix.

TABLE 2. Normalized data.

Alternatives	$\mathfrak{A}_1$	$\mathfrak{A}_2$
$\mathfrak{D}_1$	$\{s_3(0.7), s_4(0.3)\}$	$\{s_2(0.25), s_3(0.7)\}$
$\mathfrak{D}_2$	$\{s_2(0.6), s_3(0.4)\}$	$\{s_2(0.6), s_3(0.4)\}$
$\mathfrak{D}_3$	$\{s_2(0.8), s_3(0.2)\}$	$\{s_2(0.7), s_3(0.3)\}$
$\mathfrak{D}_4$	$\{s_1(0), s_1(0.9)\}$	$\{s_1(0.8), s_2(0.2)\}$
Alternatives	$\mathfrak{A}_3$	$\mathfrak{A}_4$
$\mathfrak{D}_1$	$\{s_2(0.2), s_3(0.8)\}$	$\{s_2(0.3), s_3(0.5), s_4(0.2)\}$
$\mathfrak{D}_2$	$\{s_2(0), s_2(1)\}$	$\{s_2(0), s_2(0.65), s_3(0.35)\}$
$\mathfrak{D}_3$	$\{s_1(0.3), s_2(0.7)\}$	$\{s_2(0), s_2(0.9), s_3(0.1)\}$
$\mathfrak{D}_4$	$\{s_3(0.5), s_4(0.5)\}$	$\{s_4(0), s_4(0), s_4(0.4)\}$

Step 3: We calculate the weights assigned to the attributes. These particular weights for the attributes can be determined using the following mathematical formula:

$$\triangle_j = \frac{1 - \Im_j}{\sum_{j=1}^{m}(1 - \Im_j)}, j = 1, 2, \ldots, m$$

By using above formula, we have weights $\triangle_1 = 0.4$, $\triangle_2 = 0.2149$, $\triangle_3 = 0.2127$ and $\triangle_4 = 0.1724$.

Step 4(a): Incorporate the attributes for each alternative using the PLDWA operator in Table 14 and the PLDWG operator in Table 23, respectively.

TABLE 3. PLDWA operator.

$\mathfrak{D}_1$	$\{0.5039, 0.5144, 0.5035, 0.5964, 0.6012, 0.5962, 0.5552, 0.5621, 0.5549, 0.6221, 0.6259, 0.6219, 0.2457, 0.3068, 0.2423, 0.5286, 0.5372, 0.5282, 0.4428, 0.4592, 0.4421, 0.5718, 0.5778, 0.5715\}$
$\mathfrak{D}_2$	$\{0.3908, 0.4297, 0.4002, 0.3912, 0.4300, 0.4006, 0.3649, 0.4108, 0.3762, 0.3654, 0.4111, 0.3766, 0.3376, 0.3920, 0.3514, 0.3382, 0.3924, 0.3519, 0.2969, 0.3664, 0.3152, 0.2977, 0.3668, 0.3159\}$
$\mathfrak{D}_3$	$\{0.5200, 0.5807, 0.5202, 0.5439, 0.5954, 0.5441, 0.4921, 0.5647, 0.4923, 0.5214, 0.5815, 0.5216, 0.3507, 0.5044, 0.3513, 0.4237, 0.5312, 0.4240, 0.1921, 0.4722, 0.1943, 0.3558, 0.5060, 0.3564\}$
$\mathfrak{D}_4$	$\{0.3695, 0.3695, 0.3933, 0.3954, 0.3954, 0.4152, 0.2690, 0.2690, 0.3153, 0.3190, 0.3190, 0.3524, 0.5006, 0.5006, 0.5098, 0.5106, 0.5106, 0.5191, 0.4716, 0.4716, 0.4831, 0.4841, 0.4841, 0.4945\}$

TABLE 4. PLDWG operator.

$\mathfrak{D}_1$	$\{0.2167, 0.2292, 0.2122, 0.2852, 0.3199, 0.2744, 0.2487, 0.2693, 0.2418, 0.3958, 0.5698, 0.3643, 0.2047, 0.2150, 0.2010, 0.2577, 0.2812, 0.2499, 0.2305, 0.2462, 0.2251, 0.3241, 0.3833, 0.3077\}$
$\mathfrak{D}_2$	$\{0, 0, 0, 0, 0.1474, 0.1461, 0, 0, 0, 0, 0.1466, 0.1453, 0, 0, 0, 0, 0.1459, 0.1447, 0, 0, 0, 0, 0.1452, 0.1440\}$
$\mathfrak{D}_3$	$\{0, 0.3231, 0.1722, 0, 0.5703, 0.1855, 0, 0.2915, 0.1677, 0, 0.4090, 0.1798, 0, 0.2165, 0.1506, 0, 0.2471, 0.1590, 0, 0.2073, 0.1476, 0, 0.2333, 0.1555\}$
$\mathfrak{D}_4$	$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.4776, 0, 0, 0.4930, 0, 0, 0.2626, 0, 0, 0.2644\}$

Step 4(b): Combine the attributes for each alternative using the PLDA operator in Table 5 and the PLDG operator in Table 6, respectively.

TABLE 5. PLDA operator.

$\mathfrak{D}_1$	$\{0.4470, 0.4717, 0.4460, 0.5862, 0.5938, 0.5859, 0.5306, 0.5427, 0.5301, 0.6186, 0.6243, 0.6184, 0.2289, 0.3184, 0.2233, 0.5456, 0.5563, 0.5451, 0.4554, 0.4767, 0.4545, 0.5889, 0.5964, 0.5886\}$
$\mathfrak{D}_2$	$\{0.2903, 0.4282, 0.3173, 0.2913, 0.3860, 0.3182, 0.3287, 0.4057, 0.3495, 0.3295, 0.4061, 0.3501, 0.3287, 0.4057, 0.3495, 0.3295, 0.4061, 0.3501, 0.2758, 0.3786, 0.3056, 0.2769, 0.3791, 0.3065\}$
$\mathfrak{D}_3$	$\{0.4314, 0.5833, 0.4319, 0.4823, 0.5825, 0.4826, 0.4380, 0.5649, 0.4385, 0.4868, 0.5844, 0.4871, 0.3648, 0.5427, 0.3657, 0.4409, 0.5660, 0.4414, 0.1917, 0.5158, 0.1948, 0.3711, 0.5443, 0.3719\}$
$\mathfrak{D}_4$	$\{0.2800, 0.3878, 0.3387, 0.3333, 0.3333, 0.3754, 0.2851, 0.2851, 0.3420, 0.3367, 0.3367, 0.3779, 0.4744, 0.4744, 0.4903, 0.4887, 0.4887, 0.5030, 0.4307, 0.4307, 0.4526, 0.4503, 0.4503, 0.4693\}$

TABLE 6. PLDG operator.

$\mathfrak{D}_1$	$\{0.2011, 0.2156, 0.1960, 0.2636, 0.3038, 0.2517, 0.2301, 0.2538, 0.2224, 0.3581, 0.5519, 0.3267, 0.1950, 0.2080, 0.1904, 0.2493, 0.2815, 0.2393, 0.2208, 0.2412, 0.2140, 0.3210, 0.4192, 0.2988\}$
$\mathfrak{D}_2$	$\{0, 0, 0, 0, 0.1379, 0.1364, 0, 0, 0, 0, 0.1372, 0.1357, 0, 0, 0, 0, 0.1372, 0.1357, 0, 0, 0, 0, 0.1365, 0.1350\}$
$\mathfrak{D}_3$	$\{0, 0.3076, 0.1498, 0, 0.5680, 0.1596, 0, 0.2766, 0.1465, 0, 0.3949, 0.1555, 0, 0.2357, 0.1401, 0, 0.2895, 0.1480, 0, 0.2220, 0.1374, 0, 0.2638, 0.1447\}$
$\mathfrak{D}_4$	$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.4601, 0, 0, 0.4758, 0, 0, 0.2484, 0, 0, 0.2501\}$

Step 4(c): Reorganized decision matrix in Table 7.

Step 4(d): Combine the attributes for each alternative using the PLDOWA operator in Table 8 and the PLDOWG operator in Table 9, respectively.

Step 5: Determine the score values for each alternative in accordance with the specifications provided in Definition 1.2.

By using PLDWA operator $\mathfrak{D}_1 = 0.5130$, $\mathfrak{D}_2 = 0.3696$, $\mathfrak{D}_3 = 0.4642$, $\mathfrak{D}_4 = 0.4218$

TABLE 7. Re-ordered DM.

Alternatives	$\mathfrak{A}_1$	$\mathfrak{A}_2$
$\mathfrak{D}_1$	$\{s_3(0.7), s_4(0.3)\}$	$\{s_2(0.3), s_3(0.5), s_{0.2}\}$
$\mathfrak{D}_2$	$\{s_2(0.6), s_3(0.4)\}$	$\{s_2(0.6), s_3(0.4)\}$
$\mathfrak{D}_3$	$\{s_2(0.7), s_3(0.3)\}$	$\{s_2(0.8), s_3(0.2)\}$
$\mathfrak{D}_4$	$\{s_4(0), s_4(0), s_4(0.4)\}$	$\{s_3(0.5), s_4(0.5)\}$
Alternatives	$\mathfrak{A}_3$	$\mathfrak{A}_4$
$\mathfrak{D}_1$	$\{s_2(0.2), s_3(0.8)\}$	$\{s_2(0.25), s_3(0.7)\}$
$\mathfrak{D}_2$	$\{s_2(0), s_2(0.65), s_3(0.35)\}$	$\{s_2(0), , s_2(0.1)\}$
$\mathfrak{D}_3$	$\{s_2(0.9), s_3(0.1)\}$	$\{s_1(0.3), s_2(0.7)\}$
$\mathfrak{D}_4$	$\{s_1(0.8), s_2(0.2)\}$	$\{s_1(0), s_1(0.9)\}$

TABLE 8. PLDOWA operator.

$\mathfrak{D}_1$	$\{0.5041, 0.5467, 0.5965, 0.6175, 0.5169, 0.5559, 0.6025, 0.6225, 0.5035, 0.5463, 0.5962, 0.6173, 0.2468, 0.4211, 0.5287, 0.5645, 0.3189, 0.4446, 0.5393, 0.5724, 0.2427, 0.4200, 0.5283, 0.5642\}$
$\mathfrak{D}_2$	$\{0.3908, 0.3912, 0.4373, 0.4376, 0.4023, 0.4026, 0.3649, 0.3653, 0.4196, 0.4198, 0.3787, 0.3791, 0.3376, 0.3381, 0.4021, 0.4024, 0.3544, 0.3548, 0.2969, 0.2975, 0.3785, 0.3788, 0.3191, 0.3197\}$
$\mathfrak{D}_3$	$\{0.4976, 0.5210, 0.5795, 0.5917, 0.4979, 0.5212, 0.4160, 0.4571, 0.5442, 0.5605, 0.4165, 0.4574, 0.4243, 0.4632, 0.5472, 0.5632, 0.4248, 0.4636, 0.2087, 0.3423, 0.4984, 0.5216, 0.2110, 0.3432\}$
$\mathfrak{D}_4$	$\{0.3692, 0.4421, 0.2699, 0.3937, 0.3954, 0.4577, 0.3200, 0.4158, 0.3692, 0.4421, 0.2699, 0.3937, 0.3954, 0.4577, 0.3200, 0.4158, 0.4191, 0.4730, 0.3593, 0.4364, 0.4376, 0.4855, 0.3872, 0.4527\}$

TABLE 9. PLDOWG operator.

$\mathfrak{D}_1$	$\{0.2186, 0.2439, 0.2901, 0.3733, 0.2351, 0.2686, 0.3393, 0.5588, 0.2129, 0.2359, 0.2761, 0.3419, 0.2063, 0.2268, 0.2612, 0.3126, 0.2198, 0.2456, 0.2932, 0.3809, 0.2016, 0.2204, 0.2511, 0.2948\}$
$\mathfrak{D}_2$	$\{0, 0, 0, 0.1604, 0, 0.1584, 0, 0, 0, 0.1594, 0, 0.1574, 0, 0, 0, 0.1586, 0, 0.1566, 0, 0, 0, 0.1576, 0, 0.1556\}$
$\mathfrak{D}_3$	$\{0, 0, 0.3418, 0.5608, 0.1619, 0.1704, 0, 0, 0.2566, 0.1626, 0, 0.3030, 0.1513, 0.1581, 0, 0, 0.2811, 0.3502, 0.1552, 0, 0.2299, 0.2597, 0.1458, 0.1518\}$
$\mathfrak{D}_4$	$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.4437, 0, 0.2589, 0, 0.4552, 0, 0.2606\}$

By using PLDWG operator $\mathfrak{D}_1 = 0.2814$, $\mathfrak{D}_2 = 0.0486$, $\mathfrak{D}_3 = 0.1590$, $\mathfrak{D}_4 = 0.0624$

By using PLDA operator $\mathfrak{D}_1 = 0.5072$, $\mathfrak{D}_2 = 0.3455$, $\mathfrak{D}_3 = 0.4544$, $\mathfrak{D}_4 = 0.4006$

By using PLDG operator $\mathfrak{D}_1 = 0.2689$, $\mathfrak{D}_2 = 0.0455$, $\mathfrak{D}_3 = 0.1558$, $\mathfrak{D}_4 = 0.0598$

By using PLDOWA operator $\mathfrak{D}_1 = 0.5091$, $\mathfrak{D}_2 = 0.3737$, $\mathfrak{D}_3 = 0.4613$, $\mathfrak{D}_4 = 0.3991$

By using PLDOWG operator $\mathfrak{D}_1 = 0.2795$, $\mathfrak{D}_2 = 0.0527$, $\mathfrak{D}_3 = 0.1600$, $\mathfrak{D}_4 = 0.0591$

Step 6: Enumerating all the alternatives in Table 10 in a descending sequence.

TABLE 10. Ranking.

Sr.	Operators	Ranking	Scoring
1	PLDWA	$\mathfrak{D}_1 > \mathfrak{D}_3 > \mathfrak{D}_4 > \mathfrak{D}_2$	$0.5130 > 0.4642 > 0.4218 > 0.3696$
2	PLDWG	$\mathfrak{D}_1 > \mathfrak{D}_3 > \mathfrak{D}_4 > \mathfrak{D}_2$	$0.2814 > 0.1590 > 0.0624 > 0.0486$
3	PLDA	$\mathfrak{D}_1 > \mathfrak{D}_3 > \mathfrak{D}_4 > \mathfrak{D}_2$	$0.5072 > 0.4544 > 0.4006 > 0.3455$
4	PLDG	$\mathfrak{D}_1 > \mathfrak{D}_3 > \mathfrak{D}_4 > \mathfrak{D}_2$	$0.2689 > 0.1558 > 0.0598 > 0.0455$
5	PLDOWA	$\mathfrak{D}_1 > \mathfrak{D}_3 > \mathfrak{D}_4 > \mathfrak{D}_2$	$0.5091 > 0.4613 > 0.3991 > 0.3737$
6	PLDOWG	$\mathfrak{D}_1 > \mathfrak{D}_3 > \mathfrak{D}_4 > \mathfrak{D}_2$	$0.2795 > 0.1600 > 0.0591 > 0.0527$

Thus, among the available alternatives, $\mathfrak{D}_1$ i.e., Advanced Drilling Technologies stands out as the optimal solution based on the rankings generated by our proposed methodology. This outcome highlights the robustness and reliability of our approach in effectively assessing and choosing the most suitable option for offshore drilling operations. Advanced Drilling Technologies offers several benefits, including enhanced drilling efficiency through advanced automation and data analytics, improved safety protocols utilizing cutting-edge technologies, minimized environmental impact through efficient resource utilization, and cost-effectiveness achieved by optimizing operational processes. These advantages underscore its suitability as the preferred choice for optimizing offshore drilling operations, aligning with industry standards and regulatory requirements while maximizing overall performance and sustainability.

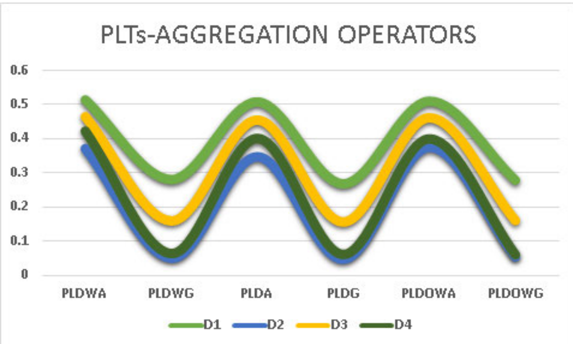

FIGURE 2. **Graphical representation of ranking.**

VII. COMPARISON ANALYSIS

Numerous approaches, including VIKOR [22], TODIM [23], GRA [24], EDAS [25], TWD [26], MABAC [27], MOORA [28], SV [29], PROMETHEE [30], ORESTE [31], LINMAP [32], and TOPSIS [33] have been developed to address MCDM problems. The MARCOS approach has been frequently employed to tackle various MADM problems.

MARCOS Method: MARCOS stands as a decision-making approach facilitating the comparison of alternatives and the identification of the optimal choice according to predefined criteria. This method takes into account both the relative significance of criteria and the performance of alternatives, offering valuable assistance for informed decision-making in various domains, such as business, engineering, and healthcare [40].

The algorithm of MARCOs in PLTs framework is given below:

Algorithm

1) Decision matrix by the expert.
2) The normalized matrix for probabilistic linguistics is derived by definition 2.7
3) Compute the weight values using the MEREC method [41]. First find score matrix then begin by calculating the overall performance of the alternatives ($\mho_i$).

$$\mho_i = ln(1 + (\frac{1}{n}\sum_{j=1}^{n}|n^*_{ij}|)) \tag{25}$$

Compute the performance of the alternatives by removing each criterion.

$$\mho'_{ij} = ln(1 + (\frac{1}{n}\sum_{k,k\neq j}|h^*_{ik}|). \tag{26}$$

Calculate the summation of absolute deviations.

$$\mathfrak{D}_j = \sum_{i=1}|\mho'_{ij} - \mho_i|. \tag{27}$$

Evaluate the final weights of the criteria.

$$\nabla^o_j = \frac{\mathfrak{D}_j}{\sum_{j=1}^{n}\mathfrak{D}_j}. \tag{28}$$

4) Rank the alternative by MARCOS. Evaluate an expanded initial PLTs DM by evaluating the PLT-PIS and PLT-NIS.

$$PIS = max\varepsilon_{ij} \tag{29}$$

$$NIS = min\varepsilon_{ij} \tag{30}$$

5) Compute the distance for PIS and NIS by using Definition 2.6.
6) Closeness coefficient: Utilizing $\xi^+_{ij} and \xi^-_{ij}$, determine the closeness coefficient as follows:

$$\mathfrak{CL}_{ij} = \frac{\xi^-_{ij}}{\xi^-_{ij} + \xi^+_{ij}}. \tag{31}$$

7) Extended decision matrix: Create an expanded decision matrix by incorporating $\mathfrak{CL}ij$, along with the anti-ideal ($\mathfrak{A}^- = \mathfrak{CL}i1^-, \mathfrak{CL}i2^-, \ldots, \mathfrak{CL}in^-$) and ideal ($\mathfrak{A}^+ = \mathfrak{CL}^+_{ij}; j = 1, 2, \ldots, n$) solutions.

$$\mathfrak{A} = \begin{pmatrix} \mathfrak{CL}^-_{i1} & \mathfrak{CL}^-_{i2} & \ldots & \mathfrak{CL}^-_{in} \\ \mathfrak{CL}_{11} & \mathfrak{CL}_{12} & \ldots & \mathfrak{CL}_{1n} \\ \mathfrak{CL}_{21} & \mathfrak{CL}_{22} & \ldots & \mathfrak{CL}_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ \mathfrak{CL}_{m1} & \mathfrak{CL}_{m2} & \ldots & \mathfrak{CL}_{mn} \\ \mathfrak{CL}^+_{i1} & \mathfrak{CL}^+_{i2} & \ldots & \mathfrak{CL}^+_{in} \end{pmatrix} \tag{32}$$

Here

$$\mathfrak{CL}^-_{ij} = min\mathfrak{CL}_{ij} \tag{33}$$

and

$$\mathfrak{CL}^-_{ij} = max\mathfrak{CL}_{ij} \tag{34}$$

8) Normalization: Transform the extended decision matrix E into its normalized form, denoted as $E = [n_{ij}]_{(m+2)\times n}$, utilizing the provided equation.

$$n_{ij} = \frac{\mathfrak{CL}_{ij}}{\mathfrak{CL}^+_{ij}} \tag{35}$$

where $\mathfrak{CL}_{ij}$ and $\mathfrak{CL}_{ij}^{+}$ are the elements in the E matrix.

9) Weighted decision matrix: Construct the ultimate weighted decision matrix F, represented by the equation below:

$$f_{ij} = n_{ij} \times \nabla_j \tag{36}$$

In the context where n_{ij} constitutes an element within the matrix E', and ∇_j represents the weight assigned to the jth criterion.

10) Utility degree of alternatives: Assess the degree of utility for alternatives $\mathfrak{U}_i$ through the application of the following equations:

$$\mathfrak{U}_i^{-} = \frac{\mathfrak{S}_i}{\mathfrak{S}^{-}}, \tag{37}$$

$$\mathfrak{U}_i^{+} = \frac{\mathfrak{S}_i}{\mathfrak{S}^{+}}, \tag{38}$$

where $\mathfrak{S}_i = \sum_{j=1}^{n} f_{(i+1)j}(i = 1, 2, \ldots, m)$, $\mathfrak{S}^{-} = \sum_{j=1}^{n} f_{1j}$ and $\mathfrak{S}^{+} = \sum_{j=1}^{n} f_{(m+2)j}$.

11) Utility function: Calculate the utility function for alternatives $F(\mathfrak{U}_i)$ using the provided equation.

$$F(\mathfrak{U}_i) = \frac{\mathfrak{U}_i^{+} + \mathfrak{U}_i^{-}}{1 + \frac{1-F(\mathfrak{U}_i^{+})}{F(\mathfrak{U}_i^{+})} + \frac{1-F(\mathfrak{U}_i^{-})}{F(\mathfrak{U}_i^{-})}}, \tag{39}$$

In cases where the utility function is defined in terms of the ideal $F(\mathfrak{U}i^{+})$ and anti-ideal $F(\mathfrak{U}i^{-})$, their respective formulations are provided by the following expressions:

$$F(\mathfrak{U}_i^{+}) = \frac{\mathfrak{U}_i^{-}}{\mathfrak{U}_i^{+} + \mathfrak{U}_i^{-}}, \tag{40}$$

$$F(\mathfrak{U}_i^{-}) = \frac{\mathfrak{U}_i^{+}}{\mathfrak{U}_i^{+} + \mathfrak{U}_i^{-}}. \tag{41}$$

12) Ranking: Order the alternatives according to their values in the utility function. It is preferable for an alternative to possess the highest attainable utility function value.

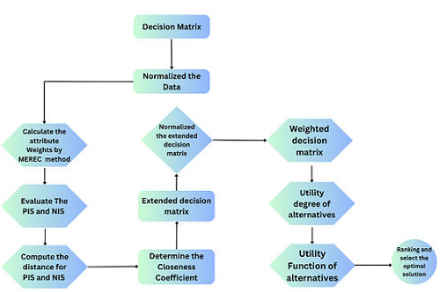

FIGURE 3. Flow chart of MARCOS algorithm.

Flowchart of the algorithm of MARCOS method is given in figure 3. Below, you'll find the sequential computational process outlined for the specified MCDM problem.

Step 1: Decision matrices by the expert in Table 12.

TABLE 11. Decision matrix by the expert.

Alternatives	$\mathfrak{A}_1$	$\mathfrak{A}_2$
$\mathfrak{D}_1$	$\{s_3(0.7), s_4(0.3)\}$	$\{s_2(0.25), s_3(0.7)\}$
$\mathfrak{D}_2$	$\{s_2(0.6), s_3(0.4)\}$	$\{s_2(0.6), s_3(0.4)\}$
$\mathfrak{D}_3$	$\{s_2(0.8), s_3(0.2)\}$	$\{s_2(0.7), s_3(0.3)\}$
$\mathfrak{D}_4$	$\{s_1(0.9)\}$	$\{s_1(0.8), s_2(0.2)\}$
Alternatives	$\mathfrak{A}_3$	$\mathfrak{A}_4$
$\mathfrak{D}_1$	$\{s_2(0.2), s_3(0.8)\}$	$\{s_2(0.3), s_3(0.5), s_4(0.2)\}$
$\mathfrak{D}_2$	$\{s_2(1)\}$	$\{s_2(0.65), s_3(0.35)\}$
$\mathfrak{D}_3$	$\{s_1(0.3), s_2(0.7)\}$	$\{s_2(0.9), s_3(0.1)\}$
$\mathfrak{D}_4$	$\{s_3(0.5), s_4(0.5)\}$	$\{s_4(0.4)\}$

Step 2: Normalized decision matrix In Table 12.

TABLE 12. Normalized decision matrix.

Alternatives	$\mathfrak{A}_1$	$\mathfrak{A}_2$
$\mathfrak{D}_1$	$\{s_3(0.7), s_4(0.3)\}$	$\{s_2(0.25), s_3(0.7)\}$
$\mathfrak{D}_2$	$\{s_2(0.6), s_3(0.4)\}$	$\{s_2(0.6), s_3(0.4)\}$
$\mathfrak{D}_3$	$\{s_2(0.8), s_3(0.2)\}$	$\{s_2(0.7), s_3(0.3)\}$
$\mathfrak{D}_4$	$\{s_1(0), s_1(0.9)\}$	$\{s_1(0.8), s_2(0.2)\}$
Alternatives	$\mathfrak{A}_3$	$\mathfrak{A}_4$
$\mathfrak{D}_1$	$\{s_2(0.2), s_3(0.8)\}$	$\{s_2(0.3), s_3(0.5), s_4(0.2)\}$
$\mathfrak{D}_2$	$\{s_2(0), s_2(1)\}$	$\{s_2(0), s_2(0.65), s_3(0.35)\}$
$\mathfrak{D}_3$	$\{s_1(0.3), s_2(0.7)\}$	$\{s_2(0), s_2(0.9), s_3(0.1)\}$
$\mathfrak{D}_4$	$\{s_3(0.5), s_4(0.5)\}$	$\{s_4(0), s_4(0), s_4(0.4)\}$

Step 3: By using eq's (25,26,27,28), we have weights $\nabla_1 = 0.22612$, $\nabla_2 = 0.2202474$, $\nabla_3 = 0.25637$ and $\nabla_4 = 0.297273$.

Step 4: Arrange the alternatives based on the MARCOS ranking. Evaluate an expanded initial PLTs DM by measuring the PLT-PIS and PLT-NIS by using equations (29,30).

TABLE 13. PLT-PIS.

C_1	C_2	C_3	C_4
$\{0.613, 0.5625\}$	$\{0.525, 0.6125\}$	$\{0.438, 0.7\}$	$\{0.225, 0.675, 0.4\}$

and

TABLE 14. PLT-NIS.

C_1	C_2	C_3	C_4
$\{0, 0.175\}$	$\{0.1875, 0.15\}$	$\{0, 0.075\}$	$\{0, 0, 0.0875\}$

Step 5: Compute the distance for PIS and NIS by using Definition 2.6. We have

TABLE 15. Distance for PIS.

C_1	C_2	C_3	C_4
0.13125	0.16875	0.14375	0.145833
0.1875	0.16875	0.53125	0.16875
0.2	0.175	0.2125	0.179167
0.30625	0.24375	0.1	0.3

TABLE 16. **Distance for NIS.**

C_1	C_2	C_3	C_4
0.36875	0.23125	0.3875	0.258333
0.3125	0.23125	0	0.235417
0.3	0.225	0.31875	0.225
0.19375	0.15625	0.43125	0.104167

And

Step 6: Determine the closeness coefficient by using equation 31.

TABLE 17. **Closeness coefficient.**

	C_1	C_2	C_3	C_4
$\mathfrak{CL}_{1j}$	0.7375	0.578125	0.729412	0.639175
$\mathfrak{CL}_{2j}$	0.625	0.578125	0	0.582474
$\mathfrak{CL}_{3j}$	0.6	0.5625	0.6	0.556701
$\mathfrak{CL}_{4j}$	0.3875	0.390625	0.811765	0.257732

Step 7: Create the expanded decision matrix through the insertion of C_{ij} by using eq.32, eq.33, eq.34.

TABLE 18. **Extended decision matrix.**

	C_1	C_2	C_3	C_4
$\mathfrak{CL}^-$	0.3875	0.390625	0	0.257732
$\mathfrak{CL}_{1j}$	0.7375	0.578125	0.729412	0.639175
$\mathfrak{CL}_{2j}$	0.625	0.578125	0	0.582474
$\mathfrak{CL}_{3j}$	0.6	0.5625	0.6	0.556701
$\mathfrak{CL}_{4j}$	0.3875	0.390625	0.811765	0.257732
$\mathfrak{CL}^+$	0.7375	0.578125	0.811765	0.639175

Step 8: Transform the extended decision matrix E into its normalized representation by using equation 35.

TABLE 19. **Normalized extended decision matrix.**

	C_1	C_2	C_3	C_4
$\mathfrak{CL}^-$	0.5254	0.6757	0	0.4032
$\mathfrak{CL}_{1j}$	1	1	0.8986	1
$\mathfrak{CL}_{2j}$	0.8475	1	0	0.9113
$\mathfrak{CL}_{3j}$	0.8136	0.9730	0.7391	0.8710
$\mathfrak{CL}_{4j}$	0.5254	0.6757	1	0.4032
$\mathfrak{CL}^+$	1	1	1	1

Step 9: Build up the final weighted decision matrix by using equation 36.

TABLE 20. **Weighted decision matrix.**

	C_1	C_2	C_3	C_4
$\mathfrak{CL}^-$	0.1188	0.1488	0	0.1199
$\mathfrak{CL}_{1j}$	0.2261	0.2202	0.2304	0.2973
$\mathfrak{CL}_{2j}$	0.1916	0.2202	0	0.2709
$\mathfrak{CL}_{3j}$	0.1840	0.2143	0.1895	0.2589
$\mathfrak{CL}_{4j}$	0.1188	0.1488	0.2564	0.1199
$\mathfrak{CL}^+$	0.2261	0.2202	0.2564	0.2973

Step 10: Evaluate the utility degree of alternatives $\mathfrak{U}_i$ by leveraging equation 37,38.

Step 11: Derive the utility function of alternatives, denoted as $F(\mathfrak{U}_i)$, utilizing equations 39, 40, and 41.

Step 12: Ranking the alternatives is carried out by assessing and organizing them in order of their utility function values.

TABLE 21. **Utility degree of alternatives.**

Alternatives	$\mathfrak{U}^-$	$\mathfrak{U}^+$
$\mathfrak{D}_1$	2.5136	0.9740
$\mathfrak{D}_2$	1.7620	0.6828
$\mathfrak{D}_3$	2.1850	0.8467
$\mathfrak{D}_4$	1.6616	0.6439

TABLE 22. **Utility function.**

Alternatives	$F(\mathfrak{U}_i)$
$\mathfrak{D}_1$	0.8789
$\mathfrak{D}_2$	0.6161
$\mathfrak{D}_3$	0.7640
$\mathfrak{D}_4$	0.5810

TABLE 23. **Utility function.**

Ranking	$\mathfrak{D}_1 > \mathfrak{D}_3 > \mathfrak{D}_2 > \mathfrak{D}_4$

A. DISCUSSION

We performed a thorough evaluation to measure the efficiency of the algorithms we introduced within the framework of PLTs. While there might be minor discrepancies in the sequence of rankings, all methodologies converge towards the same optimal selection. A detailed breakdown of rankings and visual depictions for different operators, including PLDA, PLDG, PLDWA, PLDWG, PLDOWA, PLDOWG, and the MARCOS approach, can be found in Table 24, complemented by graphical representations in Figure 4.

TABLE 24. **Ranking.**

Sr.	Operators	Ranking	Scoring
1	PLDWA	$\mathfrak{D}_1 > \mathfrak{D}_3 > \mathfrak{D}_4 > \mathfrak{D}_2$	$0.5130 > 0.4642 > 0.4218 > 0.3696$
2	PLDWG	$\mathfrak{D}_1 > \mathfrak{D}_3 > \mathfrak{D}_4 > \mathfrak{D}_2$	$0.2814 > 0.1590 > 0.0624 > 0.0486$
3	PLDA	$\mathfrak{D}_1 > \mathfrak{D}_3 > \mathfrak{D}_4 > \mathfrak{D}_2$	$0.5072 > 0.4544 > 0.4006 > 0.3455$
4	PLDG	$\mathfrak{D}_1 > \mathfrak{D}_3 > \mathfrak{D}_4 > \mathfrak{D}_2$	$0.2689 > 0.1558 > 0.0598 > 0.0455$
5	PLDOWA	$\mathfrak{D}_1 > \mathfrak{D}_3 > \mathfrak{D}_4 > \mathfrak{D}_2$	$0.5091 > 0.4613 > 0.3991 > 0.3737$
6	PLDOWG	$\mathfrak{D}_1 > \mathfrak{D}_3 > \mathfrak{D}_4 > \mathfrak{D}_2$	$0.2795 > 0.1600 > 0.0591 > 0.0527$
7	MARCOS	$\mathfrak{D}_1 > \mathfrak{D}_3 > \mathfrak{D}_2 > \mathfrak{D}_4$	$0.8789 > 0.7640 > 0.6161 > 0.5810$

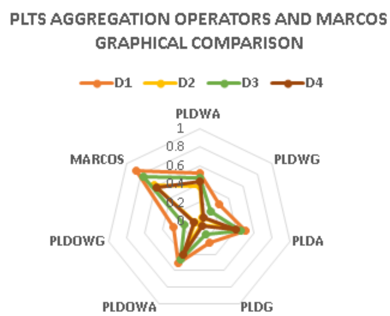

FIGURE 4. **Graphical representation of comparison between the ranking of PLTD and MARCOS method.**

In decision-making processes involving alternative prioritization, it's essential to acknowledge that the choice of methodology can result in distinct determinations of the most favorable solution. In this particular scenario, two separate methodologies lead to their own unique optimal outcomes. Firstly, adopting the Purposed Operators approach aims to maximize the rankings of alternatives, resulting in the identified optimal solution denoted as $\mathfrak{D}_1$ "because it achieves the highest rank among all available alternatives. Conversely, with the MARCOS method, the objective remains the same - to maximize alternative rankings, aligning the optimal solution with the alternative holding the highest rank, which, in this instance, is $\mathfrak{D}_1$." For a visual representation of this comparison, please refer to Figure 4. This figure emphasizes the differences in determining optimal solutions between the proposed AOs and the MARCOS method.

VIII. CONCLUSION

In our study, we conducted an in-depth exploration of MCDM problems, with a specific focus on assessing attributes using modified Probabilistic Linguistic Term Sets (PLTs). We incorporated Dombi operations into the realm of probabilistic linguistics, thereby establishing unique operations based on PLTs and their fundamental principles. Additionally, we introduced innovative operators known as the PLTD Arithmetic and Geometric operators. Within the domain of PLMCDM, we clarified the complexities of decision-making problems and developed analogous methodologies utilizing the PLTD Arithmetic and Geometric operators. This research not only enhances our understanding of Dombi operations but also advances the exploration of PLTs. Furthermore, we conducted a comprehensive analysis of the advantages associated with these operators. Through quantitative experiments, we compared the effectiveness of the PLTD Arithmetic and Geometric operators with the MARCOS method. Our proposed model has demonstrated practicality and distinct advantages. The insights and methodologies uncovered in our study hold promise for providing substantial benefits to both professionals and researchers engaged in the field of PLTs. Future studies will concentrate on creating innovative DM methods for the PLTs SW scenario, employing various approaches such as TOPSIS, EDAS, and ELECTRE methods to enhance the efficacy of MCDM. The key contributions of the study are discussed below:

- The research introduces and explores the integration of a Strategic Decision Support System with PLTS under Dombi aggregation operators Approach specifically tailored for offshore drilling for oil and gas operations.
- This novel integration addresses the need for advanced decision support tools that can handle linguistic uncertainty and complex risk scenarios.
- The research article contributes to the academic literature by expanding the theoretical understanding of decision-making methodologies under uncertainty.
- It fills a gap in existing literature by proposing and validating a novel approach that integrates linguistic preferences and probabilistic reasoning within the Dombi aggregation operators framework.

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