IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received 20 May 2024, accepted 30 June 2024, date of publication 9 July 2024, date of current version 24 July 2024.

Digital Object Identifier 10.1109/ACCESS.2024.3425501

RESEARCH ARTICLE

Optimizing Sustainability and Profitability: A Two−Phase Extended Warranty Framework for Reman and New Products in Monopolistic OEMs

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ABSTRACT Remanufactured products are now a vital part of Original Equipment Manufacturers (OEMs) product portfolio due to the financial benefits associated with them, as well as increased awareness about the need for economic and environmental sustainability on a regional as well as global level. Further, reman products appear to have substantially enhanced dependability when they become available on the product market because they are significantly more reliable. When a product appears (is sold) on the market, providing an extended warranty is one of the most effective methods of highlighting its reliability and standard quality. To determine the optimal pricing and production strategy for a monopolistic OEM who also participates in (re)manufacturing, the author proposes a two−phase Extended Warranty (EW) model for remanufactured (reman) and new products. By using the Karush−Kuhn−Tucker Non−Linear Optimization Programming Model, this study devises a framework for evaluating optimal prices, demands, and profitability of reman and newly produced products that incorporate an EW using a model framework. Based on the findings of our investigation, the OEM can choose to manufacture/produce a reman product, a new product, or a combination of reman and new products. Moreover, a numerical analysis has been conducted to determine the significance of EW, product failure rates, and customer preferences for both reman and brand−new products. The primary objective is to assess the impact of the EW. Through this study, the authors intend to highlight the importance of the EW in the purchasing decisions of customers. The results of the analysis will help to identify the key factors that influence the consumer's willingness to buy a product with an EW. This study is crucial for businesses that want to understand the importance of EW in the market and tailor their strategies accordingly. The research findings indicate that OEMs might benefit by adding an EW to their product line as it could increase their profits.

INDEX TERMS Extended warranty, non−linear optimization, pricing management, profit analysis, (Re)manufacturing strategy, utility theory.

I. INTRODUCTION

The majority of developed countries have successfully completed an environmental stage of advancement. However, most of these developing countries have effectively addressed a significant number of the challenges posed by the environmental devastation caused by waste materials. The vast majority of industries are currently working together to reduce waste. For this reason, (re)manufacturing is one of the waste−reduction solutions proven to be the most successful. According to Bhatia and Kumar Srivastava [1], Supply Chain (SC) management has a greater emphasis on product return processing and recovery strategy resulting in an increasing quantity of returns and the adverse effects of these

The associate editor coordinating the review of this manuscript and approving it for publication was Nikhil Padhi.

product returns on the environment. Furthermore, in a survey published in 2018 by the National Retail Federation [2], the entire worth of product returns in the United States was equivalent to almost 10% of the total sales that occurred in the traditional retail sector, which was $369 billions. The expenditure incurred by United States corporations on an annual basis associated with product returns was higher than $35 billion refer to [3].

In the past decade, the industries manufacturing reman products have experienced substantial revenue growth. (Re)manufacturing denotes a thorough and systematic industrial procedure through which a previously marketed, worn, or Non–Operational product or part is restored to a condition that is 'Like–New' or even surpasses its original state in terms of both quality and performance. This restoration is achieved via a controlled, replicable, and sustainable process. The work process of (re)manufacturing, also known as recycling, is an efficient approach since it reduces the amount of pollution and waste in the surrounding environment. As a result of advancements in manufacturing information, several products now have an extended lifespan, which depends on the development of a new process for their production. The recovery of previously used products is the focus of the collection of operations known as (re)manufacturing. The work process of recovering new products from used ones is referred to as (re)manufacturing, and it encompasses a variety of techniques. According to the definition provided by Ijomah [4], it is a technique for bringing the values of the reman product to a new most effective condition. The procedure of rehabilitation or returning an item must be satisfactory to the consumer, and the product itself must provide an implied warranty preferably comparable to the warranty which comes with a new product. The primary objective of a (re)manufacturer is to restore a product to its original specifications and warranty standards through restoration/(re)conditioning. (Re)conditioning a product entails repairing or substituting any sections or components that have already failed or are on the verge of failure, thereby restoring it to an ideal condition.

(Re)manufacturing has been empirically demonstrated to be an achievable and realistic strategy for dealing with returns for business entities [5], [6], [7]. The (re)manufacturing sector presents a promising opportunity within the equipment industry, with an estimated annual value of $160 billion one can refer to [8]. Recycling is a widely accepted practice lauded by industries for its environmental benefits and its capacity to invigorate the economy. Similarly, consumers are well-versed in the concept of recycling from an early age. However, the advantages of (re)manufacturing are relatively obscure. Arguably, (re)manufacturing lacks the recognition that recycling enjoys. Both recycling and (re)manufacturing play pivotal roles in the transition to a circular economy (refer to Original Engines Co., [9]). A report commissioned by The Australian Council of Recycling provides a comprehensive overview of the economic significance of the recycling industry. This report outlines the socio–economic and environmental advantages of (re)manufacturing, as well as its potential role in Australia. The economic potential of (re)manufacturing in key markets such as the United Kingdom (UK), Europe, and North America is noteworthy. A report by Carbon Trust (visit: https://www.carbontrust.com/en-as) revealed that the (re)manufacturing industry in the UK is valued at $2.4 billion ($4.53 billion AUD), in the USA at $32 billion USD ($48 billion AUD), and €29.8 billion ($48 billion AUD) in Europe. These figures underscore the substantial economic opportunities inherent in (re)manufacturing within these regions. The precise market value of (re)manufacturing in Australia remains unknown. however, based on the manufacturing–to–(re)manufacturing ratio in other markets where (re)manufacturing makes up approximately 1.1% of total manufacturing, it can be estimated that the (re)manufacturing industry in Australia accounts for around $1 billion, given that the manufacturing industry in Australia is valued at $88 billion. Further, the future prospects for (re)manufacturing appear promising. Increasing awareness of the pivotal role that (re)manufacturing plays in the circular economy has piqued the interest of both industries and governments. According to a European market study (visit: https://www.remanufacturing.eu/assets/pdfs/remanufacturing-market-study.pdf) on (re)manufacturing, it is projected that the sector's value will more than double by 2030, reaching €70 billion ($114 billion AUD).According to The European (Re)manufacturing Network (visit: https://www.remanufacturing.eu/), a European market study revealed that reman products retain up to 80% of their original core, resulting in significant cost savings. Moreover, the study indicates that (re)manufacturing practices contribute to an annual reduction of 8.5 billion tonnes of carbon dioxide (CO_2) emissions. The Carbon Trust has reported that (re)manufacturing, on average, consumes 85% less energy compared to new manufacturing. Additionally, a case study involving a French company was conducted to assess the economic and environmental impacts of (re)manufacturing. The study revealed that the CO_2 emissions from the company's reman products were only 27% of those from their newly manufactured counterparts refer to [10].

Even though (re)manufacturing has several financial advantages, businesses are becoming concerned about cannibalization between reman and new products. According to Yenipazarli [11], the acquisition of a reman product has the ability to cannibalize the overall sales of a new product and may result in a reduction in the profit made by the OEM. Because the reman products have lower prices than new products. For this reason, it is essential for OEMs to provide significant thought to the design for their manufacturing and marketing strategies whenever both categories of products are produced. OEMs often to engage in (re)manufacturing operations as a means of enhancing their brand's social image or increasing profitability. For instance, Dell has initiated the Dell–Reconnect project in collaboration with Goodwill to enhance the recycling of electronic waste for (re)manufacturing refer to [12]. Additionally, the company has established a dedicated website for the sale of its (re)manufactured computers [13]. Likewise, BMW has established a (re)manufacturing gearbox factory specializing in steering gear refer to [14]. Some OEMs are unable or unwilling to engage in (re)manufacturing due to a lack of infrastructure and expertise in handling used products, or because they need to prioritize their resources and time on new product production. For instance, Ford, which once acquired multiple parts recycling enterprises and auto salvage yards to capitalize on the potential economic benefits of disposing of used vehicles, had to discontinue its (re)manufacturing business due to its lack of experience in this field [15]. Additionally, Apple has transferred the exclusive rights for (re)manufacturing used iPhones to Foxconn. This strategic move allows Apple to reallocate its resources towards the production and design of new products [16]. Further, Amazon and JD have independently introduced Amazon Renewed and Paipai on their respective platforms. These initiatives encompass the collection, testing, leasing, and transaction of second−hand products, aiming to meet consumer demand for reman products at competitive prices one can refer to [17] and [18].

(Re)manufacturing has been the subject of extensive research, and various sources refer to [19] have attempted to divide the multiple stages of the process, such as the initial assessment, disassembly, comprehensive evaluation, repair, substituting, the reconstruction process, validation, and warranty set. Although the (re)manufacturing process generates a greater profit than usual, there are a few issues preventing it from being promoted in the potential market competition. As an unintended consequence, the reman product may reduce the market share for the new product by benefiting from its advertising. On an individual basis, sales of the new products have a significant margin of profit when contrasted with sales of the reman product, which is a direct result of the cannibalization process. Both of these products are competing in the market, but the reman ones have substantial technological advances in terms of sales because of their reduced selling prices and comparable operational capabilities to those of the new product. In other words, customers will be able to acquire the reman product by comparing the pricing of both products. According to research conducted by marketing and sales professionals at Hewlett−Packard, the OEM experiences a net loss of one brand−new product for every four reman products sold [20].

In today's market, practically all products are available for purchase in addition to an associated warranty. It indicates to the customer the assurance that they will be satisfied with the service of concern. From the perspective point of the customer, product warranty coverage can protect against the possibility of the article being accepted as well as protection against product dissatisfaction, all while simulating the purchase by minimizing risks. It is a strategy used in advertising to entice customers, and it is an essential component in the procedure of increasing demand. According to Shafiee and Chukova [21], an OEM's warranty is meant to represent a legally binding document involving the OEM and the consumer, which specifies that when the product falls apart within a particular period, the OEM would either fix the issue, replace the item that failed, or reimburse the customer. (Re)manufacturers generally provide adaptable warranties to encourage customers to acquire the reman products. The result occurs since the service offered to the OEM indicates to consumers the product's reliability [22]. According to Murthy and Djamaludin [23], the customer receives protection as well as knowledge in the form of a warranty. OEMs can protect themselves from unjustified customer claims by offering service warranties. Because warranties typically come at a slightly higher expenditure, they could be worthwhile for both the OEM and the customers. According to Shafiee and Chukova [21], General Motors' annual warranty expenditures amounted to probably $3 billion in 2019, which accounted for around three percent of the OEM's total sales.

The following studies fail to acknowledge how product EW affects the (re)manufacturing industry's production and pricing decisions. In a two−phase framework, Liu et al. [5] examine the most effective production and pricing strategies for reman and new products with the convex collection. Based on their study, the authors discover that OEM profits are adversely affected by the overwhelming acceptance (extremely high) of the reman product by potential customers. Sun et al. [24] examine how warranty length affects the competition in the market for reman and new products with an OEM, a retailer, and (re)manufacturer. It is imperative that the (re)manufacturer reduces the warranty length if the cost of repairs for the product rises, while the OEM can raise the price of the new product to maximize its profit margins. Liu et al. [25] investigate how warranties for reman and new products affect customers. The most effective manufacturing strategy is found to be influenced heavily by the ratio of the unit production expenditures and the length of the warranty of reman and new products. Liu et al. [26] investigated a dual−channel with CLSC consisting of online and retailer platforms with various pricing policies for reman and new products. Keshavarz-Ghorbani and Arshadi Khamseh [27] examined how the warranty tenure affected the best−selling price. Further, OEMs' (re)manufacturing approach was connected to the (re)manufacturing expenditure and the product quality, as determined by comparison and analysis conducted [28].

A. AIM AND RESEARCH OBJECTIVES

This paper aims to examine the implications of EW in a manufacturing/(re)manufacturing framework and to determine the optimal prices, demands and the profit for both new and reman products. To the best, no previous study has examined the (re)manufacturer's production with pricing decisions for both products when EW is provided on reman and new products and customers have direct exposure to both in the same phase. The purpose of this paper is to cover the knowledge gap by investigating the two−phase Non−Renewing EW strategy for reman and new products: a managerial perspective in a product market. Because of this, and to address a few research questions:

- How does the EW length and customer acceptance level significantly impact the optimal decisions (such as prices, demands, and total profit) for reman and new products?
- To what extent should the OEM (re)manufacturer return products, and when?
- Does an OEM make an investment in the process of increasing the possibility that customers will accept the reman products?
- How should the OEM set reman and new product prices when EW is offered?
- Which variables (factors) will affect the OEM's production and pricing strategies?

The above research questions are addressed through the construction of a two−phase mathematical framework. The model determines how the EW length influences the OEM's decision−making process regarding whether or not to participate in (re)manufacturing. Additionally, numerical experiments demonstrate how EW length and customer acceptance level of reman products affect optimal prices, demands, and total profit. Further, the examination was carried out on how the EW length affected the other influencing parameters, such as the level of customer acceptance, the responsiveness rate of customers, and the failure rate of both reman and new products.

The research adds something new to the existing work of literature in three different ways. Firstly, the authors improve the investigation on the most effective pricing along with the manufacturing strategy for reman and new products in an evolving market by working together to optimize the EW strategy. It will enable us to gain additional insight into the best production and pricing strategies for both reman and new products. OEMs who participate in (re)manufacturing endeavors acquire the opportunity to gain new perspectives and consider the potential consequences of those findings. The authors present an approach that demonstrates how (re)manufacturers can determine the optimal duration of the EW for the products they offer. Secondly, based on the above discussion, the OEM can decide whether to manufacture (produce) new or reman or mix both products. Thirdly, the proposed model differs from previous research in taking into account the scarcity of reman products, the dependent expenditure of EW based on customer demand, the convex interaction between EW length and customer utility theory, and the competitive environment between refurbished and brand−new products in the marketplace.

This paper is structured as: the background information regarding the literary view is presented in Section (II), although Sections (III) explain the model description, and Section (IV) obtains the profit analysis for the proposed two−phase model. Section (V) includes both numerical proofs and sensitivity analysis. Finally, the conclusion and potential future directions of the research are presented in Section (VI). The Appendix Section (VI) contains all of the Lemma and Theorem proofs.

II. LITERATURE REVIEW

This section provides a review of the literature, categorized into three subsections: Warranty Strategies in Product Management, Pricing, and Warranty Strategies in (Re)manufacturing Sector, and Customer Behavior, Pricing Policies, and Recycling in SC. The purpose is to demonstrate that future improvements should focus on enabling customers to choose their warranty contracts by proposing a strategy that allows customers to purchase the EW for both reman and brand-new products directly from the OEM. This paper sets itself apart from previous research efforts by clearly delineating its contributions to the existing literature and highlighting the differences between them. The aim is to clarify the unique contribution of the work and its potential significance in the field.

A. WARRANTY STRATEGIES IN PRODUCT MANAGEMENT

The OEM has the responsibility of selecting the warranty coverage that offers the best protection, because it has such a significant bearing on total profitability by controlling both the selling price and the customer demand for both products.

Regarding previously owned products, Chattopadhyay and Murthy [29] anticipated warranty expenditure including complimentary warranty strategies for both the system and its components. A one−dimensional unbounded warranty that uses reman products to replace defective products can benefit from the conceptual framework provided by [30]. Alqahtani and Gupta [31] focused on developing a method for estimating the price of an embedded washer equipped with a sensor. It compares and contrasts three different kinds of extended warranties: such as free replacement warranty (FRW), Pro−rata warranty (PRW), and combination of both. They came to the conclusion that the lowest claims occurred when OEMs offered an extended period of FRW for embedded sensors. Aksezer [32] looked into consideration the condition, product quality, consumption, and maintenance record of the EW hooked up in the used automobile business to estimate the expenditure of the EW. A free repair and an expenditure−sharing obligations warranty are both compared and contrasted. Also, Liu et al. [25] conducted a study on the impact of the base warranty length on the optimal pricing and retailing decisions of a (re)manufacturer. Their findings suggest that when the warranty period is sufficiently short and the difference in unit manufacturing costs between reman and new products is sufficiently large, reman products may not be viable for sale.

Kuik et al. [33] conducted research on the most effective EW for reman products that were covered by a variety of warranties. They examine an OEM that provides two distinct types of warranties: a Type−I guarantee that addresses all problems for a predetermined period and a Type−II warranty that addresses only particular faults. When compared to the Type$-II$ warranty, the Type$-I$ warranty is regarded by the OEM as being of higher quality. The washing machine was examined, with a particular focus on the warranty. Estimates of the appropriate expenditure for the warranty are made both with and without the adoption associated with the sensor system that ascertains the remaining effective lifespan. In addition to the standard warranty, the OEM provides customers with the option to purchase one of three different types of extended warranties: one that provides a free replacement, one that provides a refund, and one that combines the previous two warranty types. According to the findings, using the sensor can lead to an increase in total revenue as well as profit, with the extended FRW possessing a relatively low average warranty expenditure refer to [34]. From 2001 to 2016, Diallo et al. [35] summarised and explained various types of warranty principles. The research suggests that giving consumers a choice of OEM warranties should be an objective for any strategy for future development.

As per the research above, the standard warranty exclusively pertains to the SC, which is included with the product as a default feature. The study analyzes the impact of the extended warranty (EW) on the SC, which represents a significant revenue stream for the EW service provider. An extended warranty (EW) is stated in addition to the standard warranty. An extended warranty is an ongoing subscription plan that extends the product's servicing or maintenance support for a time after the original warranty has expired [36]. In addition, the market demand for reman products may improve if they are of a higher quality and are sold at lower prices [37]. The effects of EWs on the SC are studied, both in a centralized and decentralized setting [38]. As an illustration, Apple Inc. includes a 2−year of EW coverage (Apple Care+) with every iPad purchase as an added benefit to the traditional 1−year of coverage [39]. Ying et al. [40] look at how the EW provider impacts the choice of reserve channels. Perhaps, both the retailer and the OEM would likely be interested in advertising their extended warranties if allowed. While many companies like Ford, Apple, and JVC sell their EWs directly to customers, certain retailers prefer to sell their EWs in place of the EW offered by the OEM [41]. Due to the fact that it is an add−on, the EW differs from the standard warranty. Its global revenue streams have expanded at a higher rate in the past decade. The profit margin on EW assistance is estimated to be between $50-60\%$ percent or more than eighteen times the profit margin on profits from sales [42]. When the customer makes a purchase decision, price and the impression of quality are the two most significant elements [43]. Afsahi and Shafiee [44] conducted a study to determine the optimal effective warranty term and expenditure in light of the unpredictable nature of repairs for defective returns. In scenarios where a defective product may receive minimal, complete, or imperfect repair, the researchers employed a meta−heuristic Monte Carlo simulation algorithm and a dynamic programming model to address the issue and provide a solution. As per Jin and Zhou [45], implementing a decentralized CLSC with a single supplier and OEM could prove to be an effective approach for optimizing the EW policy for reman products. In the context of reman products, the decision to offer EW to customers lies within the OEM's discretion. Consequently, OEMs and suppliers face challenging situations when they opt not to provide EW for reman products. Additionally, to optimize the OEMs profitability, Ben Mabrouk and Chelbi [46] devised a mathematical algorithm and framework aimed at precisely determining the most effective approach for maintaining leased product equipment, potentially with the assistance of both BW and EW. Further, Jalapathy and Unnissa [47] present a two−phase news−vendor model that integrates EW. This model offers valuable insights into crafting effective manufacturing and inventory strategies within a CLSC framework.

B. PRICING AND WARRANTY STRATEGIES IN (RE)MANUFACTURING SECTOR

The sector of (re)manufacturing, encompassing both new manufacturing and (re)manufacturing of products, implements a range of pricing and warranty strategies with the aim of optimizing profitability and enhancing customer satisfaction.

Steeneck and Sarin [48] analyzed the price difference between reman and brand−new products. Choi [49] conducts an investigation into the most profitable advertising and pricing initiatives for a reman and new retail fashionable retailer. Based on the findings, an increase in both the price of purchase and the (re)manufacturing expenditure results in a rise in optimal pricing despite the corresponding drop in optimum advertising investments. Taleizadeh and Mokhtarzadeh [50] evaluated the Bi−Variate FRW pricing for both online and offline sales channels. Warranty disputes and expenditures are estimated using the Log−Normal and Non−Homogeneous Poisson Process (NHPP) distributions. To resolve this issue framework, they employ the High−Risk High−Reward technique. According to the findings, expanding OEMs' warranty and product use could boost profits and reduce profits, respectively. The product warranty performs a vital role in the SC, as it influences both the pricing of the product as well as consumer preferences regarding the product. Cao and He [51] investigated the extent of rivalry that existed between the pricing and the warranty along the SC, which included the retailer and the OEM. According to the findings, a superior product OEM has the ability to set a higher selling price and offer a warranty that is valid for a more extended period. Further, Cai et al. [52] utilized reverse induction to determine equilibrium profits in the scenario where the supplier actively seeks demand info from the retailer. Li et al. [53] conducted a study on consumer switching behavior and its influence on seller strategies within a competitive market. The research focused on addressing pricing challenges encountered by sellers of both refurbished and new products in a competitive environment.

The pricing strategies that were investigated for reman items in a monopolistic market as well as an SC environment. Liao et al. [54] examined the impacts of warranties on OEM and (re)manufacturer revenue (profit), demand, and pricing. The study compares the expenditures and advantages of three different warranty plans, including no warranty, a warranty for the reman product but not the new product, and a warranty for both products. There are advantages to providing a warranty for products, says the (re)manufacturer. However, the warranty offered by an adversary product cuts into profits. With a separate sales channel, Gan et al. [55] looked into the most effective pricing strategies for both reman as well as brand−new products. The retailers and the OEM each handle sales of their specific products. If the reman product's price is set higher than it would be through a single sales channel, then the entire SC will earn more profits. To better understand how different recycling and warranty options affect the expenditures incurred by the OEMs, retailer, and the entire SC from the consumer's point of view, Ji et al. [56] created four distinct gaming approaches. Chari et al. [57] successfully determined the optimal price approach for the OEM by taking into account the duration of the warranty, the condition of the reman component, and the proportion of the two factors. The authors found that increasing the ratio of parts in the collection had a significant impact on both the duration of the warranty and the price of a product warranty. This relationship highlights the interdependence of these two variables and emphasizes the importance of considering them in the decision−making processes of both OEMs and customers. Qian et al. [58] emphasized the importance of product innovation through the implementation of distinct reusing strategies. Further, Qian et al. [59] developed decision models to evaluate the outsourcing−(re)manufacturing decision for OEMs. Their findings indicate that in−house (re)manufacturing consistently yields superior economic, social, and environmental benefits when compared to outsourcing (re)manufacturing to a retailer. Liu et al. [60] have conducted a study to determine the most effective replacement strategies for products within warranty periods and those subject to time constraints for repairs. The study specifically examines the pricing of both rebate warranty policies and FRW policies to ascertain their respective minimum costs. Chai et al. [61] examined process innovation strategies aimed at promoting the (re)manufacturing of green products in a CLSC, which encompasses an upstream supplier and a downstream OEM. Their findings indicated that the implementation of cooperative innovation mechanisms by OEMs can be incentivized through government subsidies.

C. CUSTOMER BEHAVIOR, PRICING POLICIES, AND RECYCLING IN SC

Understanding consumer behavior is essential for the development of effective pricing strategies that promote sustainable purchasing behaviors. Additionally, implementing robust recycling practices within the SC can contribute to increased environmental accountability and adherence to regulatory requirements.

Feng et al. [62] have conducted a study focusing on the impact of compensation and dynamic pricing policies on a OEM's production and pricing strategies in the context of strategic customer behavior. Wang and Wang [63] have introduced both a two−part tariff and revenue−sharing contract as potential mechanisms to enhance the reverse SCs performance in the presence of strategic customer behavior. The study by Wu et al. [64] presents an analysis of the optimal product pricing during the introduction and clearance phases, taking into account the presence of strategic customers in the market. The research compares three pricing strategies: fixed, strategic high with a small discount in phase−II, and high−low with a large discount in phase−II. Zhou et al. [65] analyzed a SC scenario in which the OEM delegates the production of new products to an external OEM. The two entities collaborate in the (re)manufacturing process as well. Hence, apart from attracting customers, the pricing strategy will also serve to safeguard the company's profits and overall financial performance. A well-considered pricing strategy will contribute to the company's enhanced growth, increased customer appeal, and effective mitigation of financial losses refer to [66].

Recently, Luo and Wu [70] analyzed the product warranty expenditure of product failures caused by hardware, software, or the combination of the two. Liu et al. [71] examined whether OEMs should consider recycling products for renting, sale, and (re)manufacturing under the conditions in which the OEM should use the ORS (outsourcing (re)manufacturing strategy). As described in Hu et al. [69], the impact of both forward and reverse flow of CLSCs is observed when using two different (re)manufacturing design methods. Cao et al. [67] examine government trade−in subsidies and a carbon tax to find the OEM's optimal profit. The OEM must decide whether to provide a trade−in option for both reman and new products. The most effective (optimal) warranty aspects for both cases are the same. According to Tang et al. [43] constructed two warranty frameworks to determine (re)manufacturing product warranty settings and their effects on the CLSC. A mathematical model is suggested by Jauhari and Wangsa [72], in which a CLSC system containing an OEM and a retailer is integrated with an integrated inventory model. This model has the ability to minimize the cost of the total SC incurred by the OEM and retailer. Zhang et al. [68] conducted research to determine whether or not an E−Commerce platform offers EW service. They discovered that the reman and new products associated with EW service strategies are the most effective options. The CLSC is significantly influenced by the expenditure savings that can be achieved through (re)manufacturing, as suggested by refer to [73]. Additionally, Liu et al. [66] examined the most effective pricing and retailing strategies within a CLSC. Their study involved the development of a two−phase model to assess the optimal pricing and retailing

TABLE 1. A comparative study of the related literature with present work.

Name	SC Member	Product	Warranty Type	Time Horizon # Phases	Objective
Aksezer [32]	OEM	Reman	EW	One	Min Cost
Kuik et al. [33]	OEM	Reman	EW	One	Max Profit
Cao and He [51]	OEM	New	BW	One	Max Profit
Zhu et al. [38]	Retailer	Reman	EW	One	Max Profit
Ying et al. [40]	OEM or Retailer	New and Reman	EW	One	Max Profit
Cao et al. [67]	OEM	New and Reman	BW	One	Max Profit
Taleizadeh and Mokhtarzadeh [50]	OEM	New	BW	One	Max Profit
Afsahi and Shafiee [44]	OEM	New	BW and EW	One	Max Profit
Tang et al. [43]	OEM or Retailer	New or Reman	BW	Two	Max Profit
Jin and Zhou [45]	OEM	Reman	EW	One	Max Profit
Liu et al. [60]	OEM	New	BW	One	Min Cost
Zhang et al. [68]	Retailer	New and Reman	EW	One	Max Profit
Wang and Wang [63]	Recycler	Reman	–	One	Max Profit
Hu et al. [69]	OEM	Reman	–	One	Max Profit
Ben Mabrouk and Chelbi [46]	OEM	New	BW and EW	One	Max Profit
Feng et al. [62]	OEM	New and Reman	–	Two	Max Profit
Zhou et al. [65]	OEM	New and Reman	–	One	Max Profit
Cai et al. [52]	Supplier and Retailer	General	–	One	Expected Profit
Jalapathy and Unnissa [47]	OEM	New and Reman	EW	Two	Expected Profit
Liu et al. [66]	OEM	New and Reman	–	Two	Max Profit
This Paper	OEM	Reman and New	BW and EW	Two	Max Profit

Type of Warranty: Base Warranty (BW) and Extended Warranty (EW).

strategies of the OEM in a CLSC, accounting for customer behavior characterized by myopic or strategic tendencies. To highlight the unique contributions and perspectives of the present work and to contextualize the findings with related literature, a comprehensive overview with key distinctions is provided in Table (1).

D. CONTRIBUTION TO THE LITERATURE WORK

The study at hand has yielded valuable insights and contributions that extend the existing body of knowledge in the field. These contributions can be summarized as follows:

- This paper investigates the optimal pricing and production strategies of a reman and new product with a Non−Renewing EW are complex considerations for OEMs. By considering factors such as the limited supply of reman products, the concave relationship between customer utility and EW length, and total EW cost depending on demand, as well as changes in potential market size, OEMs can develop two−phase effective strategies (N, B, and R) to maximize profits and customer satisfaction. The findings of this study can contribute to a better understanding of market dynamics. They can also guide OEMs in making informed decisions regarding pricing and production strategies in competitive markets.
- Engaging in (re)manufacturing can be a strategic decision for OEMs, provided it meets certain conditions. By considering the ratio of unit production costs and EW length for reman and new products, OEMs can determine the economic feasibility and customer acceptance of (re)manufacturing. By adhering to these conditions, OEMs can maximize (re)manufacturing benefits, including environmental benefits and cost savings, while still maintaining customer satisfaction.
- This study provides valuable insights into the sensitivity of optimal prices, demands, and OEM profit to the length of EWs. The findings demonstrate the impact of unit (re)manufacturing and manufacturing costs, product failure rates, and the sensitivity of the customer's utility on optimal EW length. By considering these factors, OEMs can make informed decisions to enhance EW design and achieve long−term profitability.
- Also, the study makes a contribution to the ongoing investigation of the most effective pricing strategies for EWs when they are introduced to customers by the OEM. In addition, the research provides a view into the impacts that the OEM EW possesses, which might or might not influence the decisions made by customers. Finally, it illustrates how the failure rate of the reman and

TABLE 2. Notation for board.

Indices	Description
i	Product type (subscript) : $i = n, n_1$(new product) and $i = r, r_1$(reman product)
j	Planning Phase: $j = I$ (Phase$-I$) and $j = II$ (Phase$-II$)
Parameters	**Description**
c_i	Expenditure of production per unit for reman and new products
λ_i	Product's failure rate
δ_i	Responsiveness of customers towards EW
Q	Market size in the phase$-II$
α	Level of acceptance for customer (reman product)
θ	Fractional returns for reman products
w	Warranty length
w_e	EW length
k_1/k_2	Price coefficients for the new/reman products
z	Customers perceived values on the reman and new products
C_n/C_r	Total unit expenditure for new and reman products
Variables	**Description**
$P_i(j)$	Price for a specific product: phase j and category i
$D_i(j)$	Demand function for product: phase j and category i
π_l	Profit for the individual products
Π	OEM total profit for both products

new product and the utility function rate of the OEM's EWs affect the relationship.

III. MODEL DESCRIPTION, NOTATION, AND RELATED ASSUMPTIONS

A. TABLE OF NOTATION

To ensure coherence and uniformity in the use of parameters and terminology across this paper, the following Table (2) presents a succinct guide of notation along with their respective descriptions.

B. DECISION FRAMEWORK

Consider the possibility of a monopoly producer who creates a new product during phase$-I$ and then produces both the reman and new products during phase$-II$. In phase$-I$, the OEM gathers all of the defective products that have been returned, and these products undergo further examination before the reman process. The supply chain (SC) decision process for the OEM is shown in Figure (1). The products sold in phase$-I$ and phase$-II$ include additional EW (w_e) features with their product specifications. During phase$-II$, the customer makes a decision to purchase either reman or new products. During the EW period, defective products are either replaced or repaired depending on the terms and conditions of the EW. The replacement for the defective component is done with a brand$-$new, or a functionally equivalent one during the phase$-II$.

In addition, the model can be described most effectively by the following assumptions: reman and newly manufactured products with EW taken into account. However, the process of (re)manufacturing requires significant effort and time to create a reman product. This process involves recycling products previously purchased and returned to the OEM. It is imperative to note that a reman product can only be (re)manufactured once. The defective brand$-$new product is returned, gathered, and evaluated to a level of certainty θ. The potential market size is forecast to be 1 for phase$-I$ and Q for phase$-II$, correspondingly. In this scenario, if $Q > 1$ suggests the potential market size is increasing, and if $Q \leq 1$ suggests the potential market size decreases during phase$-II$. Suppose $Q = 1$ refers to the size of the potential market that will remain unchanged in phase$-II$. Customers' perceptions of a new product's value (z) are heterogeneous refer to [74].

C. EXPENDITURE STRUCTURE

The expected total number of product replacements during the warranty period is as follows: for new products $M_n(w) = E[N(t)] = \lambda_n w$ and for reman products $M_r(w) = E[N(t)] = \lambda_r w$ respectively refer to [75]. In a similar manner, the new products failure rate during the EW is $M_{n_1}(w) = E[N(t)] = \lambda_{n_1} w_e$ and for the reman product is $M_{r_1}(w) = E[N(t)] = \lambda_{r_1} w_e$ refer to [76]. In addition, the average expenditure associated with the warranty for both products is as follows: $c_n\lambda_n w$, $c_r\lambda_r w$, and the average expenditure associated with the EW for both products are as follow: $c_{n_1}\lambda_{n_1} w_e$, $c_{r_1}\lambda_{r_1} w_e$. At this stage, the product is sold to the customer, and the OEM is responsible for covering the expenditures associated with the warranty and EW.

D. CUSTOMER PREFERENCE

The selling price of the products and the customer's utility are the primary factors deciding whether or not a (re)manufacturing process will be promoted refer to [77]. It is expected the customer's product utility will be a convex relationship associated with the EW and this function will have a negative trend in the presence of the EW. When there is a decrease in the utility function, there is an increase in the product selling price. Customers' willingness to pay the price for an EW when purchasing a new product is represented by the variable z_n. During phase$-I$, the utility is given by the form,

$$U_n(I) = z_n - k_1 P_n(I) + (\delta_n\sqrt{w} + \delta_{n_1}\sqrt{w_e})$$

where $P_n(I)$ is the new product's selling price during phase$-I$ and δ_n, δ_{n_1} is the sensitivity coefficient for the new product's customer utility towards the length of warranties. Consider the phase$-I$ price sensitivity factor of the new product to be denoted by k_1. During phase$-I$, customers make purchases of the new product, which occurs when the

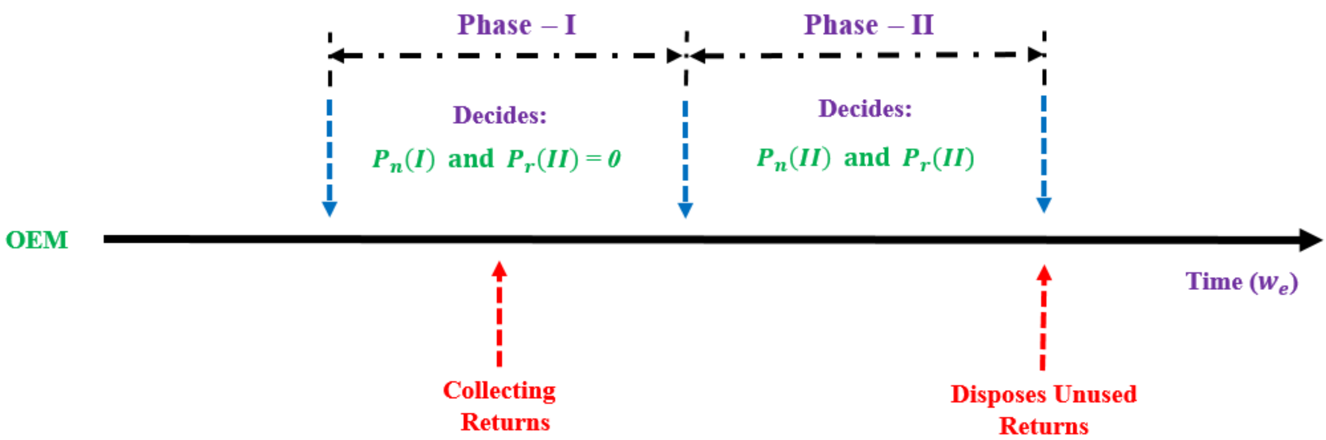


FIGURE 1. SC decision process for the OEM.

utility function is positive $(U_n(I) \geq 0)$, which implies $z_n(I) \geq k_1 P_n(I) - (\delta_n\sqrt{w} + \delta_{n_1}\sqrt{w_e})$, the EW will be purchased by the customer. The demand for new products during phase–I and phase–II are identical in the following respects: $D_n(I) =$

$$\begin{cases} 1 - k_1 P_n(I) + (\delta_n\sqrt{w} + \delta_{n_1}\sqrt{w_e}) \\ \qquad \text{if } P_n(I) \leq \dfrac{1 + \delta_n\sqrt{w} + \delta_{n_1}\sqrt{w_e}}{k_1} \\ 0 \quad \text{if } P_n(I) > \dfrac{1 + \delta_n\sqrt{w} + \delta_{n_1}\sqrt{w_e}}{k_1} \end{cases} \tag{1}$$

During phase–II, $0 < \alpha \leq 1$ represents the level of acceptability of a customer purchasing a reman product. When $\alpha = 1$, customers are unable to differentiate between reman and new products. If $\alpha = 0$, the customers will never select a reman product. During phase–II, the utility is given by the form,

$$U_n(II) = z_n - k_1 P_n(II) + (\delta_n\sqrt{w} + \delta_{n_1}\sqrt{w_e})$$
$$U_r(II) = \alpha z_r - k_2 P_r(II) + (\delta_r\sqrt{w} + \delta_{r_1}\sqrt{w_e})$$

The perceived value that a customer receives is given by,

$$z = z_{nr}(II) \geq \frac{k_1 P_n(II) - k_2 P_r(II) + (\delta_r - \delta_n)\sqrt{w} + (\delta_{r_1} - \delta_{n_1})\sqrt{w_e}}{(1-\alpha)}$$

The customers decide to opt the new product, with the following if and only if conditions satisfying the requirements: $U_n(II) \geq 0$ and $U_n(II) \geq U_r(II)$, which implies $z = z_n(II) \geq k_1 P_n(II) - (\delta_n\sqrt{w} + \delta_{n_1}\sqrt{w_e})$. Suppose the customers decide to purchase the reman product, with the following if and only if conditions satisfying the requirements: $U_r(II) \geq 0$ and $U_r(II) \geq U_n(II)$, which implies

$$z = z_r(II) \geq \frac{k_2 P_r(II) - (\delta_r\sqrt{w} + \delta_{r_1}\sqrt{w_e})}{\alpha}$$

Clearly, $0 \leq z_r(II) \leq z_{nr}(II) \leq 1$. Figure (2) shows three main segments in the potential market.

Based on the size of the potential market Q, the demand for reman and new products during phase–II are:

$$D_n(II) = \begin{cases} Q\left[1 - k_1 P_n(II) + \left[\delta_n\sqrt{w} + \delta_{n_1}\sqrt{w_e}\right]\right] \\ \quad \text{if } P_n(II) < A \\ Q\left[1 - \dfrac{\left[\begin{array}{r} k_1 P_n(II) - k_2 P_r(II) + (\delta_r - \delta_n)\sqrt{w} \\ + (\delta_{r_1} - \delta_{n_1})\sqrt{w_e} \end{array}\right]}{(1-\alpha)}\right] \\ \quad \text{if } A \leq P_n(II) \leq B \\ 0 \quad \text{if } P_n(II) > B \end{cases} \tag{2}$$

$$D_r(II) = \begin{cases} 0 \quad \text{if } P_n(II) < A \\ Q\left[\dfrac{\left[\begin{array}{r} \alpha k_1 P_n(II) + (\delta_r - \alpha\delta_n)\sqrt{w} - k_2 P_r(II) \\ + (\delta_{r_1} - \alpha\delta_{n_1})\sqrt{w_e} \end{array}\right]}{\alpha(1-\alpha)}\right] \\ \quad \text{if } A \leq P_n(II) \leq B \\ Q\left[1 - \dfrac{k_2 P_r(II) - (\delta_r\sqrt{w} + \delta_{r_1}\sqrt{w_e})}{\alpha}\right] \\ \quad \text{if } P_n(II) > B \end{cases} \tag{3}$$

where

$$A = \frac{[\alpha\delta_n - \delta_r]\sqrt{w} + k_2 P_r(II) + \left[\alpha\delta_{n_1} - \delta_{r_1}\right]\sqrt{w_e}}{\alpha k_1} \quad \text{and}$$
$$B = \frac{[1-\alpha] + k_2 P_r(II) + (\delta_n - \delta_r)\sqrt{w} + \left[\delta_{n_1} - \delta_{r_1}\right]\sqrt{w_e}}{k_1}.$$

Here, $P_n(II)$, $P_r(II)$ is the reman and new products sales prices during phase–II and δ_r, δ_{r_1} is the coefficient of sensitivity of the reman products utility towards the warranty length. Consider the phase–II price sensitivity factor of the reman product to be denoted by k_2.

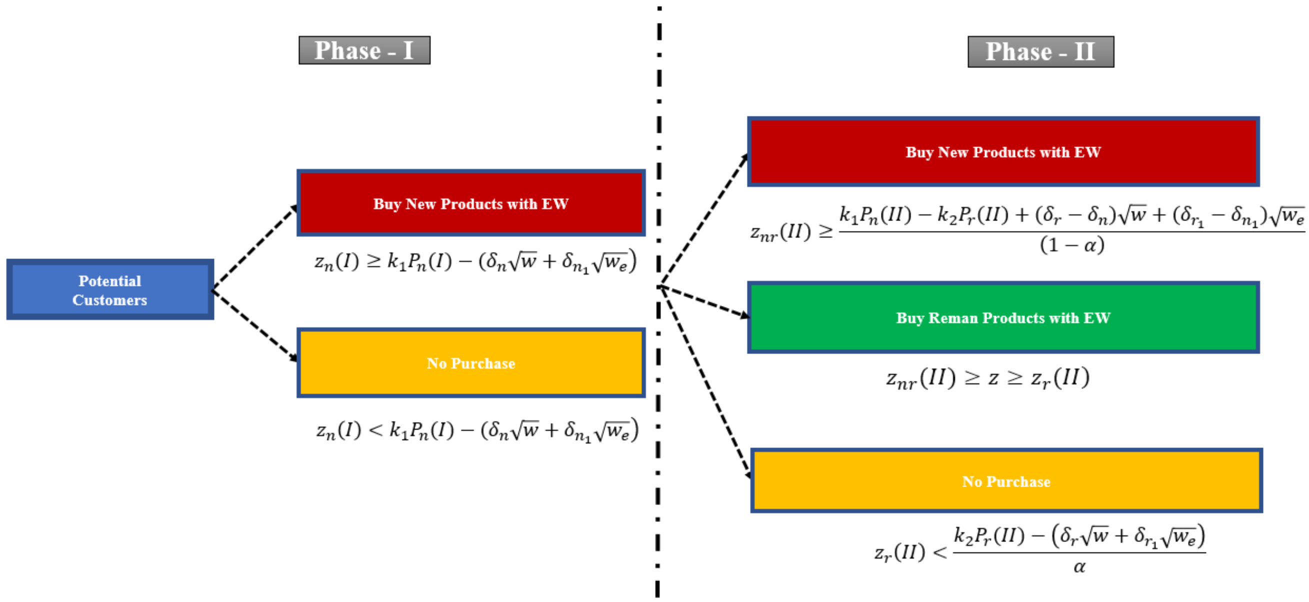


FIGURE 2. Customer purchase segmentation.

IV. PROFIT ANALYSIS FOR TWO−PHASE MODEL

This section discusses the optimal prices for the products, taking into account the following framework in order to achieve maximum profits during two phases:

$$\pi_n(I) = [P_n(I) - C_n]\, D_n(I) \tag{4}$$
$$\pi_n(II) = [P_n(II) - C_n]\, D_n(II) \tag{5}$$
$$\pi_r(II) = [P_r(II) - C_r]\, D_r(II) \tag{6}$$

The optimal total profit for the OEM obtained by substituting the product demand functions (1, 2, and 3) into the mentioned profit Equations (4, 5, and 6).

To express the OEM total profit, by using the following objective function:

$$\underset{Profit}{Max} \quad \Pi\, [P_n(I), P_n(II), P_r(II)] = \sum_{j=I,II} \pi_n(j) + \pi_r(II) \tag{7}$$

Subject to:

$$\theta\left[\lambda_n w + \lambda_{n_1} w_e\right] D_n(I) \geq \left[1 + \lambda_r w + \lambda_{r_1} w_e\right] D_r(II) \tag{8}$$
$$P_n(I), P_n(II), P_r(II) \geq 0 \tag{9}$$

The optimization function is shown by Equation (7), which refers to the sum of the profits acquired from the sales of both products during phase−I and phase−II. It is guaranteed by the constraint Equation (8), that the total number of reman products returned during phase−I will be greater than the number (quantity) of reman products that are sold during phase−II. The total unit expenditure associated with the reman and the new is denoted by $C_n = c_n N_n$ and $C_r = c_r N_r$, respectively. The constraint Equation (9) must have a beneficial (positive) effect.

A. BOTH REMAN AND NEW PRODUCTS ANNOUNCED IN PHASE−II

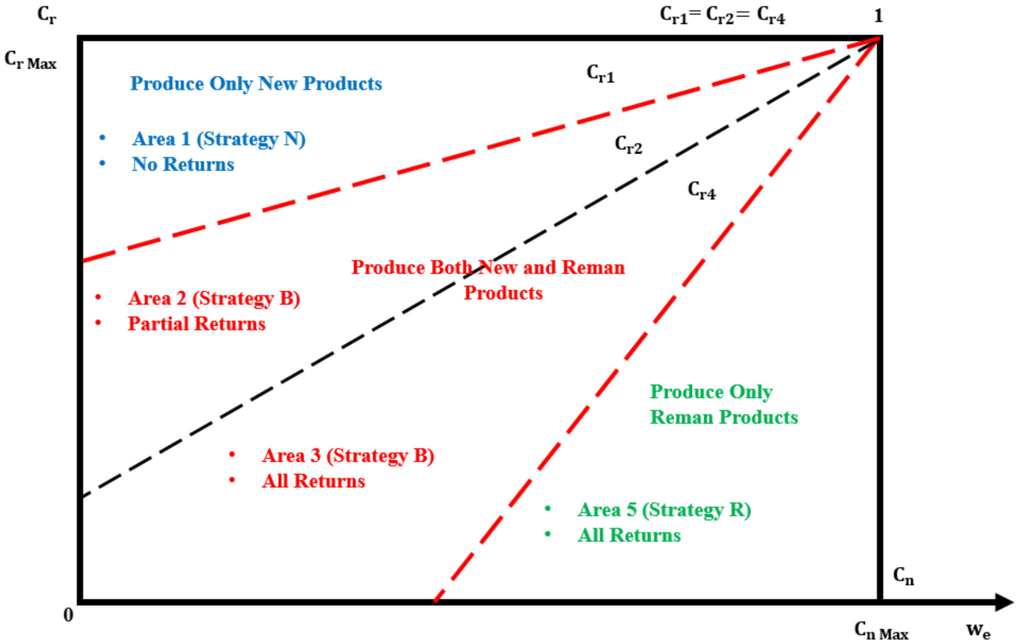
The OEM decides to produce/manufacture both reman and new products during phase−II, the new product demand ($D_n(II)$) and the reman product demand ($D_r(II)$) ought to be non−negative. The following is the proof of Lemma (1) by using the Equations (4) to (9).

Lemma 1: The optimization function (7) in the profit analysis is concave when the demands $D_n(I)$, $D_n(II)$ and $D_r(II)$.

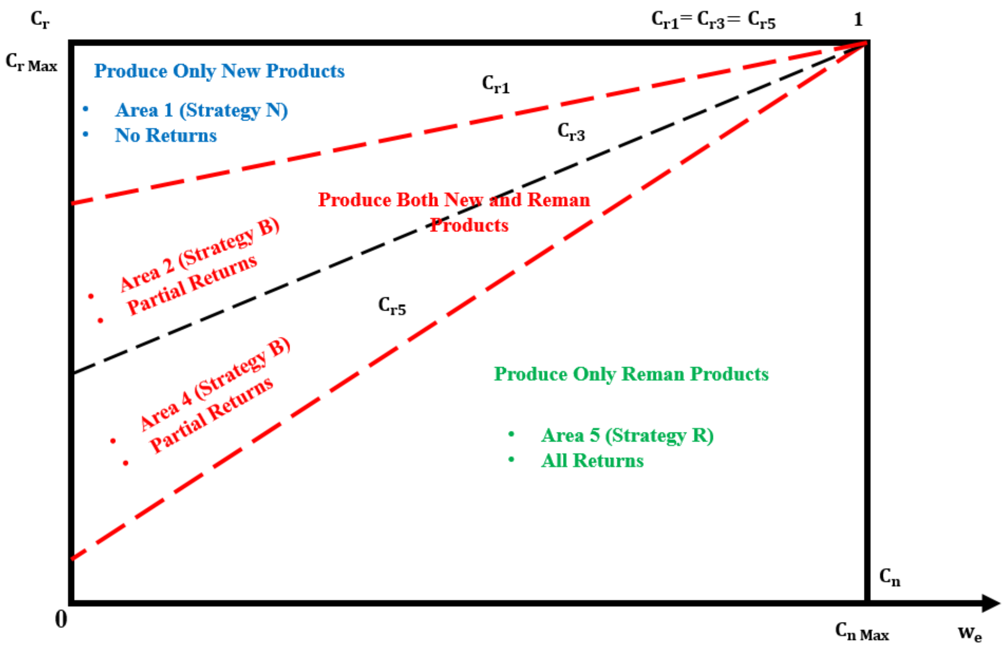
The Proof of Lemma (1) has been presented in the Appendix (VI).

Let $C_n = c_n N_n$ and $C_r = c_r N_r$ be defined by the overall unit expenditures for reman and new products. Using Lemma (1) and the fact that both conditions (8) and (9) are linear with respect to price in both phases and Table (4) shows the optimum prices for both reman and new products during both time phases. Let $C_{r_1} = m_1 - \frac{\alpha E_3\,[QN_r + (1-\alpha)B_1]}{Qk_2N_r}$, $C_{r_2} = \frac{\delta_r\sqrt{w} + \delta_{r_1}\sqrt{w_e}}{k_2} - \frac{\alpha\left[\delta_n\sqrt{w} + \delta_{n_1}\sqrt{w_e}\right]}{k_2}$, $C_{r_3} = m_1 - \frac{\alpha}{k_2}\left[E_3 + \frac{(1-\alpha)B_1}{QN_r}\right]$, $C_{r_4} = m_1 + \frac{E_3\left[N_r - \alpha B_1\left[\lambda_n w + \lambda_{n_1} w_e\right]\right]}{k_2 B_1}$ and

$$C_{r_5} = m_1 + \frac{\left[QN_r^2 + \alpha B_1^2\right]\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]}{k_2 Q^2 N_r^3 B_1^2} \times \left[\frac{B_1E_3}{2} - \frac{\alpha B_1^2 E_3}{2\left[QN_r^2 + \alpha B_1^2\right]} - \frac{QN_r^2 B_1 E_3}{2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]}\right] \times \left[\text{where } m_1 = \frac{\alpha + \delta_r\sqrt{w} + \delta_{r_1}\sqrt{w_e}}{k_2}\right].$$



(a) Case 1: $Q > \dfrac{\alpha B_1}{N_r}$



(b) Case 2: $Q \leq \dfrac{\alpha B_1}{N_r}$

FIGURE 3. **Mapping of the best manufacturing and pricing strategies for the OEM.**

The optimum pricing and manufacturing strategies for an OEM can be obtained by formulating and addressing the Karush−Kuhn−Tucker (KKT) conditions, as demonstrated in Theorem (1), (highlighted in Figure (3) one can refer to [25]).

Theorem 1: The following conditions are the optimal production and pricing approach for the OEM during the phase$-II$:

- To approach N entailing only manufacturing the new product if the value $C_r \geq C_{r_1}$ (Area 1);

TABLE 3. Optimal decisions from manufacturing both reman and new products in phase−*II*.

Strategy	Conditions	Optimal Solutions						
	C_r	$P_n^*(I)$	$P_n^*(II)$	$P_r^*(II)$	$D_n^*(I)$	$D_n^*(II)$	$D_r^*(II)$	Π^*
B	$\max(C_{r_2}, C_{r_3}) \leq C_r < C_{r_1}$	p_1	p_1	p_2	d_1	d_2	d_3	π_1
	$C_r < \min(C_{r_4}, C_{r_5})$	p_3	p_4	p_5	d_4	d_5	d_6	π_2
N	$C_r \geq C_{r_1}$	Only New Product Produced (π_3)						
R	$C_{r_5} < C_r < C_{r_3}$	Only Reman Product Produced (π_4)						

Where the critical value C_{r_t}, p_t, d_t and π_t are presented in Appendix Section (A). It is important to point out that C_{r_t} $(t = 1, \ldots 5)$ are total unit product expenditures are linear functions of C_n.

- To approach B providing both reman and new products if $\max(C_{r_2}, C_{r_3} \leq C_r < C_{r_1})$ (Using partial product returns, Area 2) and $C_{r_4} < C_r < C_{r_2}$ and $Q > \frac{\alpha B_1}{N_r}$ (Using all product returns, Area 3);
- To approach R entailing only reman products if $C_{r_5} < C_r < C_{r_3}$ (Using partial product returns, Area 4) and $C_r < \min(C_{r_4}, C_{r_5})$ and $Q \leq \frac{\alpha B_1}{N_r}$ (Using all product returns, Area 5);

More specifically, Table (3) lists the optimal decisions for the OEM when providing EW.

Both reman and new products must have a lower unit expenditure than their price to be more profitable. When the demand functions in Equations (2) and (3) are adjusted to have a value of zero, the maximums of the unit expenditure of each product that takes the form of $C_{nMax} = c_n N_n$ and $C_{rMax} = c_r N_r$ as depicted in Figure (3). Theorem (1) and Figure (3) show that for the phase−II, the optimum pricing and production framework for the OEM switches from only manufacturing the new product (N) to both products (B), and finally, only manufacturing the reman product (R). From the OEM, reman products can be priced lower to entice too many customers in the product market and increase profits. In addition, as the reman product benefits in profitability, the quantity of new products returns from the phase−I purchase becomes a significant barrier to demand (and output) for the reman product in the phase−II.

The OEM's optimal pricing and manufacturing approaches influence phase−II's market size (Q). If the phase−II market is high enough $\left[Q > \frac{\alpha B_1}{N_r}\right]$, to obtain more profits, when the OEM decides to keep producing both reman and new products or acquire the reman product only in the phase−II. Optionally, to utilize all viable (usable) product returns from the phase−I for the (re)manufacturing the product. In case $\left[Q \leq \frac{\alpha B_1}{N_r}\right]$, all viable product returns are utilized for (re)manufacturing in the phase−II (Area 5). As a result, demand for the reman product rises as phase−II's product market size increases.

The Lemma concludes from the above results.

Lemma 2: The phase−II new product price is not significantly smaller than the phase−I product price and is independent of (re)manufacturing decisions.

According to the Table (4), it is evident that for Strategy approaches N and B (which cover both Areas 2 and 3), where the OEM prefers to manufacture the fresh new product in the phase−II, the optimum price $\left[\frac{E_1}{2k_1}\right]$ for the new product remains unchanged despite the choice to (re)manufacturer and this product price only varies based on a few aspects (production expenditure, the customer's acceptance level (α), and extended warranty (w_e)). Therefore, the optimal price for the new product in phase−I is lower than in phase−II. The main factor for this is that (re)manufacturing the reman product is only Expenditure−effective when it generates a significant profit (Strategy B in Area 3 and R) to acquire additional sources of raw materials from product returns to produce the reman product and enhance the OEM's profitability. Due to this, the OEM significantly reduces the sales price in phase−I.

According to the Theorem (1), only Strategy N requires the OEM to refrain from manufacturing the reman product during phase−II. Furthermore, when $k_2 C_r \leq (\delta_r \sqrt{w} + \delta_{r_1}\sqrt{w_e}) - (\delta_n \sqrt{w} + \delta_{n_1}\sqrt{w_e} + k_1 C_n)\alpha$, the OEM uses (re)manufacturing facilities. Adjusting the above criteria (Using $C_n = c_n N_n$ and $C_r = c_r N_r$), to acquire:

$$\left[\begin{array}{l}(k_2 c_r \lambda_r - \alpha k_1 c_n \lambda_n) w + (k_2 c_r \lambda_{r_1} - \alpha k_1 c_n \lambda_{n_1}) w_e \\ \times (k_2 c_r - \alpha k_1 c_n) + (\alpha\delta_n - \delta_r)\sqrt{w} + (\alpha\delta_{n_1} - \delta_{r_1})\sqrt{w_e}\end{array}\right] \leq 0.$$

TABLE 4. The values of optimal prices and demands for the reman and new products.

	Strategy N (in Area 1)	Strategy B (in Area 2)	Strategy B (in Area 3)	Strategy R (in Area 4)	Strategy R (in Area 5)
$P_n(I)$	$\frac{E_1}{2k_1}$	$\frac{E_1}{2k_1}$	$\frac{E_1}{2k_1}+\frac{\alpha(1-\alpha)B_1^2E_3}{2k_1\left[QN_r^2+\alpha(1-\alpha)B_1^2\right]}+\frac{QN_r\left[\alpha\theta E_3-B_1E_4\right]}{2k_1\left[QN_r^2+\alpha(1-\alpha)B_1^2\right]}$	$\frac{E_1}{2k_1}$	$\frac{E_1}{2k_1}+\frac{B_1\left[\alpha B_1E_3-QN_rE_4\right]}{2k_1\left[QN_r^2+\alpha B_1^2\right]}$
$P_n(II)$	$\frac{E_1}{2k_1}$	$\frac{E_1}{2k_1}$	$\frac{E_1}{2k_1}$	–	–
$P_r(II)$	–	$\frac{E_2}{2k_2}$	$\frac{E_2}{2k_2}+\frac{QN_rE_4}{2k_2\left[QN_r^r+\alpha(1-\alpha)B_1^2\right]}-\frac{\alpha B_1E_3\left[(1-\alpha)B_1+QN_r\right]}{2k_2\left[QN_r^r+\alpha(1-\alpha)B_1^2\right]}$	$\frac{E_2}{2k_2}$	$\frac{E_2}{2k_2}+\frac{QE_4N_r^2}{2k_2\left[QN_r^2+\alpha(1-\alpha)B_1^2\right]}-\frac{\alpha B_1E_3N_r}{2k_2\left[QN_r^2+\alpha(1-\alpha)B_1^2\right]}$
$D_n(I)$	$\frac{E_3}{2}$	$\frac{E_3}{2}$	$\frac{E_3}{2}+\frac{QB_1E_4N_r}{2\left[QN_r^2+\alpha(1-\alpha)B_1^2\right]}-\frac{\alpha B_1E_3((1-\alpha)B_1+QN_r)}{2\left[QN_r^2+\alpha(1-\alpha)B_1^2\right]}$	$\frac{E_3}{2}$	$\frac{E_3}{2}+\frac{B_1\left[QN_rE_4-\alpha B_1E_3\right]}{2\left[QN_r^2+\alpha B_1^2\right]}$
$D_n(II)$	$\frac{E_3}{2}$	$Q\left[\frac{E_3-E_4}{2(1-\alpha)}\right]$	$\frac{\alpha QB_1E_3\left[B_1+QN_r\right]}{2\left[QN_r^2+\alpha(1-\alpha)B_1^2\right]}-\frac{\alpha QB_1\left[E_3N_r+B_1E_4\right]}{2\left[QN_r^2+\alpha(1-\alpha)B_1^2\right]}$	–	–
$D_r(II)$	–	$Q\left[\frac{(E_4-\alpha)-\alpha(E_3-1)}{2\alpha(1-\alpha)}\right]$	$\frac{QB_1\left[E_3N_r+B_1E_4\right]}{2\left[QN_r^2+\alpha(1-\alpha)B_1^2\right]}-\frac{\alpha QB_1^2(\lambda_nw+\lambda_{n_1}w_e)E_3}{2\left[QN_r^2+\alpha(1-\alpha)B_1^2\right]}$	$\frac{QE_4}{2\alpha}$	$\frac{QB_1\left[(1-\alpha)B_1E_4+E_3N_r\right]}{2\left[QN_r^2+\alpha(1-\alpha)B_1^2\right]}$

* Let $B_1=\theta(\lambda_nw+\lambda_{n_1}w_e)$; $E_1=1+(\delta_n\sqrt{w}+\delta_{n_1}\sqrt{w_e})+k_1C_n$; $E_2=\alpha+(\delta_r\sqrt{w}+\delta_{r_1}\sqrt{w_e})+k_2C_r$; $m_1=\frac{\alpha+(\delta_r\sqrt{w}+\delta_{r_1}\sqrt{w_e})}{k_2}$ $E_3=1+(\delta_n\sqrt{w}+\delta_{n_1}\sqrt{w_e})-k_1C_n$; $E_4=\alpha+(\delta_r\sqrt{w}+\delta_{r_1}\sqrt{w_e})-k_2C_r$; $N_n=(1+\lambda_nw+\lambda_{n_1}w_e)$; and $N_r=(1+\lambda_rw+\lambda_{r_1}w_e)$.

By finding a solution to the inequality (Relaxing the standard warranty (w)),

$$w_e=\frac{-2\left[k_2c_r\lambda_{r_1}-\alpha k_1c_n\lambda_{n_1}\right](k_2c_r-\alpha k_1c_n)+\left[\alpha\delta_{n_1}-\delta_{r_1}\right]^2\pm\sqrt{\left[\alpha\delta_{n_1}-\delta_{r_1}\right]^2-4\left[k_2c_r\lambda_{r_1}-\alpha k_1c_n\lambda_{n_1}\right](k_2c_r-\alpha k_1c_n)}}{2\left[k_2c_r\lambda_{r_1}-\alpha k_1c_n\lambda_{n_1}\right]^2}$$

The result shows that the OEM's decision to participate in (re)manufacturing during the phase−II is determined by the length of the EW (w_e) and the ratio of reman and new product expenditure in unit production ($\frac{c_r}{c_n}$).

Lemma 3: The effects of enhancing the customer's responsiveness (δ_i) for reman and new products on the optimum prices and the demand for both phases are illustrated in Table (5).

When a customer's responsiveness (δ_i) to the EW length of a new product increases, the optimal price for that product also increases. This relationship reflects the principle that customer satisfaction and willingness to pay are intertwined. OEMs must adjust prices accordingly to capture the value of their products in the potential market. If the customer's responsiveness to the tenure of the EW for the reman product increases, the optimal price of the new product in phase−I will decrease if the OEM uses all product returns for the (re)manufacturing process. The reman product yields similar results. Furthermore, the optimal price in phase−II of new products remains intact. Even though the EW remains unchanged, the customer's responsiveness towards the EW for a new product increases. This, in turn, leads to an increase in demand for the new product. Hence, the OEM raises the optimal price for the new product to maximize its profitability ($\Pi = 0.7437$). When the customer's responsiveness to the reman product's EW length increases, the phase−II optimal price for the new products remains unaffected by the OEM's optimum decision to either manufacture or not manufacture the reman product (Lemma (2)).

When the reman product is low enough (Strategy N, B in Area 2, and R in Area 4: Utilizing partial product returns), the two−phase optimal pricing strategies for product manufacturing are independent. When it comes to phase−I, the customer's responsiveness towards the reman product's EW duration has no significant effect on the new product's optimal price in terms of its popularity. The purpose of this is to ensure that OEMs are unwilling to sacrifice profit margin during phase−I to receive more returns on products from potential customers. When reman products are highly profitable means (Strategy B in Area 3 and R in Area 5: utilizing all product returns) to maximize the OEM's two−phase profitability. Therefore, the new product's optimal price (optimal demand) in phase−I reflects

TABLE 5. Effects of customer's responsiveness (δ_j) on the optimal prices and demands (↑ for increasing; ↓ for decreasing; − for no effect).

	Strategy N		Strategy B						Strategy R			
	(in Area 1)		(in Area 2)			(in Area 3)			(in Area 4)		(in Area 5)	
	$P_n(I)$	$P_n(II)$	$P_n(I)$	$P_n(II)$	$P_r(II)$	$P_n(I)$	$P_n(II)$	$P_r(II)$	$P_n(I)$	$P_r(II)$	$P_n(I)$	$P_r(II)$
δ_{n_1}	↑	↑	↑	↑	↑	↑	↑	↓	↑	−	↑	↓
δ_{r_1}	−	−	−	−	↑	↓	−	↑	−	↑	↓	↑
	$D_n(I)$	$D_n(II)$	$D_n(I)$	$D_n(II)$	$D_r(II)$	$D_n(I)$	$D_n(II)$	$D_r(II)$	$D_n(I)$	$D_r(II)$	$D_n(I)$	$D_r(II)$
δ_{n_1}	↑	↑	↑	↑	−	↑	↑	↑	−	−	↑	↑
δ_{r_1}	−	−	−	↓	↑	↑	↓	↑	−	↑	↑	↑

phase−II's optimal price (optimal demand) for the reman products, enhancing the responsiveness to the EW length to the reman product increases. To produce sufficient returns for the (re)manufacturing process in phase−II: the phase−I pricing strategy is positive (high) essential.

Although the responsiveness to the EW for the new product becomes more sensitive (increases), the new product's optimal demand increased in phase−I and decreased in phase−II. The reason for the phase−I sales entails more customers being encouraged to purchase the new product that offers the same EW length by increasing the customer's responsiveness to the new product EW length. As a result, Strategies B (in Areas 2), B (in Areas 3), and R (in Area 5) enhance the reman product demand, while Strategy R (in Area 4) remains static, which means that the overall product demand remains the same. Furthermore, all customers acquire reman products (Strategy R in Area 4). This results in a significant impact on the sales of the new product whenever there is an optimal need for the reman product.

To maximize the OEM's profit (Π), strategies B (in Area 3) and R (in Area 5) use all product returns for (re)manufacturing. Consequently, when a customer's response to the EW length of a new product increases, the first−phase optimal demand for it increases, and a higher percentage of product returns are collected, thereby increasing the number of returns. For maximum utilization of all returned products, it is essential to continually increase the demand for reman products.

B. ONLY NEW PRODUCTS ANNOUNCED IN PHASE−II

When the OEM is not announced the reman products in phase−II, the reman products demand $(D_r(II))$ is neutralized to zero. Thus, the constraint Equations (8) to (9) satisfied always and the profit function becomes:

$$\underset{Profit}{Max} \quad \Pi\left[P_n(I), P_n(II)\right] = \pi_n(I) + \pi_n(II) = \sum_{j=I,II} \left[P_n(j) - C_n\right] D_n(j) \quad (10)$$

Lemma 4: The optimization function (10) in the profit analysis is concave when product demands are $D_n(I) = 1 - k_1 P_n(I) + (\delta_n \sqrt{w} + \delta_{n_1}\sqrt{w_e})$ and $D_n(II) = Q\left[1 - k_1 P_n(II) + (\delta_n \sqrt{w} + \delta_{n_1}\sqrt{w_e})\right]$.

Since the optimization function (10) is concave, solving $\frac{\partial \pi_n}{\partial P_n(I)} = 0$ and $\frac{\partial \pi_n}{\partial P_n(II)} = 0$, yields the OEM's optimal product prices and demands for the new products in phase−I and phase−II. The results are summarized in the Theorem (2).

Theorem 2: OEM only produces the new products in phase−II, a distinct optimal product pricing strategy exists, which involves setting the same new products price in both phases:

$$P_n^*(I) = P_n^*(II) = \frac{E_1}{2k_1} \quad (11)$$

Combining the above Equation (11) along with demands (1) to (2) yields the optimal new products demand for both phases $(D_n(I), D_n(II))$ and the total profit $\Pi\,(P_n(I), P_n(II))$:

$$D_n^*(I) = \frac{E_3}{2} \quad (12)$$

$$D_n^*(II) = \frac{QE_3}{2} \quad (13)$$

$$\underset{Profit}{Max} \quad \Pi\left[P_n(I), P_n(II)\right] = \pi_3 = \frac{(1+Q)E_3^2}{4k_1} \quad (14)$$

C. ONLY REMAN PRODUCTS ANNOUNCED IN PHASE−II

When the OEM does not allow the new products in phase−II, the new products demand $(D_n(II))$ is neutralized to zero, and the reman product demand $(D_r(II) \geq 0)$ should be greater than zero. Moreover, using the Equations (1) and (3), the profit function is diminished to:

$$\begin{aligned}\underset{Profit}{Max} \quad & \Pi\left[P_n(I), P_r(II)\right] \\ &= \pi_n(I) + \pi_r(II) \\ &= \left[P_n(I) - C_n\right] D_n(I) + \left[P_r(II) - C_r\right] D_r(II) \end{aligned} \quad (15)$$

Subject to:

$$B_1 D_n(I) \geq N_r D_r(II) \tag{16}$$

$$P_n(I), P_r(II) \geq 0 \tag{17}$$

Lemma 5: The optimization function (15) in the profit analysis is concave when the demands are $D_n(I)$ and $D_r(II)$.

Given the concavity of the optimization function (15) and the linearity of the constraint Equation (16), the optimization problem at hand is concave and possesses optimal product solutions. This observation holds significant implications for businesses and academics alike, as it ensures that the problem can be efficiently solved using established optimization techniques, leading to the identification of the most optimal product solutions. The optimization pricing model is obtained by solving the constraints (16) to (17) by using KKT conditions. Furthermore, the optimal pricing and demand solutions are presented in Appendix Section (VI) in Theorem (1) Proof of Case (2).

Theorem 3: In phase−II, the OEM produces exclusively reman products, there exists an optimal product price depending on the total unit manufacturing expenditure C_n and the (re)manufacturing expenditure C_r.

When the OEM only produces the reman products in phase−II, an different optimal product pricing strategy exists, which involves setting the individual reman and new products price in both phases:

$$P_n(I) = \frac{E_1}{2k_1} + \frac{B_1 \left[\alpha B_1 E_3 - Q N_r E_4\right]}{2k_1 \left[Q N_r^2 + \alpha B_1^2\right]} \tag{18}$$

$$P_r(II) = \frac{E_2}{2k_2} + \frac{N_r \left[Q E_4 N_r - \alpha B_1 E_3\right]}{2k_2 \left[Q N_r^2 + \alpha(1-\alpha) B_1^2\right]} \tag{19}$$

Combining the above Equations (18 and 19) along with demands (1) to (3) yields the optimal reman and new products demand for both phases $(D_n(I), D_r(II))$ and the total profit $\Pi\,(P_n(I), P_r(II))$:

$$D_n(I) = \frac{E_3}{2} + \frac{B_1 \left[Q N_r E_4 - \alpha B_1 E_3\right]}{2\left[Q N_r^2 + \alpha B_1^2\right]} \tag{20}$$

$$D_r(II) = \frac{Q B_1 \left[(1-\alpha) B_1 E_4 + E_3 N_r\right]}{2\left[Q N_r^2 + \alpha(1-\alpha) B_1^2\right]} \tag{21}$$

The optimal total profit for the reman product in phase−II:

$$\begin{aligned}
\underset{Profit}{Max}\quad & \Pi\left[P_n(I), P_r(II)\right] = \pi_4 \\
= & \frac{Q N_r \left[N_r E_3 + B_1 E_4\right]}{4k_1 \left[Q N_r^2 + \alpha B_1^2\right]^2} \\
& \times \begin{bmatrix} \left[E_1 + E_3 - 2k_1 C_n\right] \alpha B_1^2 - Q B_1 N_r E_4 \\ + Q N_r^2 \left[E_1 - 2k_1 C_n\right] \end{bmatrix} \\
& + \frac{Q B_1 \left[(1-\alpha) B_1 E_4 + N_r E_3\right]}{4k_2 \left[Q N_r^2 + \alpha(1-\alpha) B_1^2\right]^2} \\
& \times \begin{bmatrix} \alpha(1-\alpha)\left[E_2 - 2k_2 C_r\right] B_1^2 \\ + Q N_r^2 \left[E_2 + E_4 - 2k_2 C_r\right] \\ - \alpha B_1 N_r E_3 \end{bmatrix}
\end{aligned} \tag{22}$$

Based on Figure (3), when the total unit (re)manufacturing expenditure is high $\left(C_r \geq max(C_{r_1}, C_{r_2})\right)$, the OEM doesn't make any reman products during the phase−II. When the total unit (re)manufacturing expenditure is moderate $\left(max(C_{r_2}, C_{r_5}) < C_r < min(C_{r_1}, C_{r_3})\right)$ the OEM use partial product returns to produce the reman products. When the total unit (re)manufacturing expenditure is low enough $(C_r < min(C_{r_4}, C_{r_5}))$, the OEM use all product returns to make a reman products. The optimal prices and demands for three distinct manufacturing strategies are determined by Theorems (1) to (3), which are as follows: both reman and new products, only new and only reman products in phase−II.

Theorem 4: The most efficient manufacturing strategy for the OEM to implement during phase−II depends on both the total unit manufacturing expenditure C_n and the total unit (re)manufacturing expenditure C_r. The OEMs strategy are same as Theorem (1).

In Theorem (4) and Figure (4), it is fascinating to note that the manufacturing strategy implemented by the OEM in the tinted region $(C_r \leq min\left(C_{r_4}, C_{r_5}\right))$ is the opposite of what one would expect. The analysis suggests that in situations where the total unit manufacturing expenditure surpasses the designated threshold (indicated by the tinted line C_{r_3}), while the total unit expenditure of reman products remains below the corresponding threshold $(C_r \leq min(C_{r_4}, C_{r_5}))$, the optimal production strategy for OEMs is to shift from producing solely reman products to producing both reman and new products. The implications of the aforementioned findings are elucidated as follows.

Firstly, the decrease in the total unit expenditure of (re)manufacturing generates the OEM to attain more profit from the production of reman products. It encourages the OEM to make a significant number of reman products. Fortunately, on the contrary, the total (re)manufacturing expenditure rises as reman product production increases. On the reverse side, collecting and inspection expenditure is convex because of the losses incurred throughout the (re)manufacturing process, implying that it takes more than one returned used product, in general, which is to produce one reman product. Furthermore, the OEM needs to acquire additional returns from phase−I to meet the demands of increased production of reman products. Due to this, the OEM needs to lower the price of the new products that are being sold during phase−I to increase the number of new products that are sold during phase−I. It will allow for a significant number of returns to be acquired after phase−I. The influence of the price reduction is more important than that of the increasing demand. Because of this, the OEM will find itself in a poorer (bad) financial position if it just produces reman products because the average total unit expenditure of (re)manufacturing will be higher.

As a result, the OEM chooses to lower the price of the reman product even though this results in a lower expenditure per unit of (re)manufacturing. This result is carried out so that they can satisfy the rising demand and generate higher profits. On the other hand, the growth rate for reman products of the market cannot expand too quickly because of the negative repercussions discussed earlier. Further, the price of the new products is lowered during phase−I, which pushes the OEM to lower the reman product's price even further as $P_r(II) \leq P_n(I)$. In the phase−II production process, the OEM struggles to make a higher profit by producing reman products only. Therefore, for the OEM to satisfy the demand of the product market, they must manufacture new products in phase−II that will support the reman products. Moreover, if $C_r > 0$, the extent of the tinted region relies on the level of acceptance that customers have for the reman products, assuming that the region's boundary C_{r_5} is a function of α. When the level of customer acceptance for reman products is adequately low, the OEM fails to produce both reman and new products in phase−II for reasons addressed earlier. If the price stays steady, the demand for reman products increases as α increases. This results in a higher price for reman products, which the OEM uses to enhance their profit margins. Because of this, the possibility of the tinted region existing increases when the unit expenditure of the reman products falls.

The following Corollary is the conclusion from the previous discussion.

Corollary 1: When $\dfrac{C_r}{C_n}$, the ratio of the total unit (re)manufacturing expenditure to the total unit manufacturing expenditure, is lesser than

$$\frac{\alpha k_1\left[QN_r+(1-\alpha)B_1\right]}{Qk_2N_r}\times\left[m_1-\frac{\alpha N_n\left[QN_r+(1-\alpha)B_1\right]}{k_2QN_r}\right],$$

the OEM should participate in (re)manufacturing.

Based on Theorem (4), when $C_r \leq C_{r_1} = m_1 - \dfrac{\alpha E_3\left[QN_r+(1-\alpha)B_1\right]}{Qk_2N_r}$ or equivalently $\dfrac{c_r}{c_n} \leq \dfrac{\alpha k_1[QN_r+(1-\alpha)B_1]N_n}{Qk_2N_r^2} \times \left[m_1 - \dfrac{\alpha N_n[QN_r+(1-\alpha)B_1]}{k_2QN_r}\right]$, OEM ready to produce the reman products in phase−II. If the expenditure for the total unit manufacturing is held constant, the OEM will have more chance to earn profit by selling reman products instead of only selling new products in the market. If the gap between the expenditure for manufacturing and the expenditure for (re)manufacturing is significant. The findings indicate that OEMs who are now selling only new products have the potential to boost their profits if they participate in (re)manufacturing in phase−II if $C_r \leq C_{r_1} = m_1 - \dfrac{\alpha E_3\left[QN_r+(1-\alpha)B_1\right]}{Qk_2N_r}$. This is the only way for them to do so. The OEM needs to place a primary emphasis on (re)manufacturing lower−expenditure strategies, such as producing products that are effortless to disassemble, in order to ensure that they maintain profitability.

V. NUMERICAL ILLUSTRATION

This segment presents the significant outcomes obtained from the numerical analyses discussed in the preceding section. Additionally, the study delves into the impact of customer acceptance levels of reman products and the effect of EW on production and pricing strategies. The parameter values utilized include $c_n = 0.3$, $c_r = 0.2$, $\theta = 1.5$, $Q = 1.4$, $k_1 = 1$, $k_2 = 2$ and $w = 0.36$. These values were adopted and analyzed by Liu et al. [78], Liu et al. [25], and Liu et al. [66].

A. THE IMPACT OF EW ON THE OEM OPTIMAL TOTAL PROFITS

This subsection analyzes the impact of EW on the OEM's optimal total profits for the reman products by varying w_e from 0.10 to 5.0 while retaining the values of all remaining parameters. The figure (5) presents a demonstration of the impact of the parameter w_e on the OEM total profit in both phases. The study highlights the influence of w_e and provides valuable insights into the OEM's profitability.

Figure (5) shows that, in particular, enhancing the EW length results in an increase in all profits for both the reman and new products in both phases. As a result, the OEM must charge higher for both reman and new products because of increases in the EW expenditure due to the increase in EW length. This finding may be considered evidence that OEMs are able to set higher prices if they offer effective post−sale support. It's fascinating to notice how EW length impacts total profits for reman and new products in both phases. To be more specific, an increase in the EW length leads to an increase in the profit for both the reman and new products in both phases, while the profit for the new product initially increases and then begins to fall. As a result, the optimal total profit (Π) for both phases begins to increase right after $w_e = 1.75$, where it reaches its peak before it begins to decrease right after $w_e = 5.0$. The reasoning behind these adjustments is that, on the one hand, the EW expenditures for both reman and new products can rise with an increase in EW length, reducing total profit (when $w_e > 1.75$) and the reman product profit (when $w_e > 3.75$). Also, ensure that the EW expenditure made a positive effect on both reman and new products during both phases of the potential market. In addition to the disparity between the reman and new products' failure rates ($\lambda_n \leq \lambda_r$ and $\lambda_{n_1} \leq \lambda_{r_1}$), the reman product's profit grows more quickly than the new product's during phase−II when $w_e = 4.5$. In contrast, when prices remain unchanged, an increase in the EW can result in an increased demand for products (a positive demand impact when $w_e \leq 1.75$).

Furthermore, when ($w_e < 0.75$) or ($0.75 \leq w_e < 1.75$): for both ranges, the positive demand impact on both reman and new caused by EW length outweighs the EW expenditure influenced on total profit (Π). Because of this, the OEM is presented with the opportunity to foster an increase in both reman and new product demands, which in turn leads to a substantial boost in the OEM's overall profits. Also, on the potential market, there is no interest in purchasing the reman product (i.e., the solid red curve stays constantly decreasing). Additionally, no switches are available to move from reman products to new ones as the EW length increases. As a result,

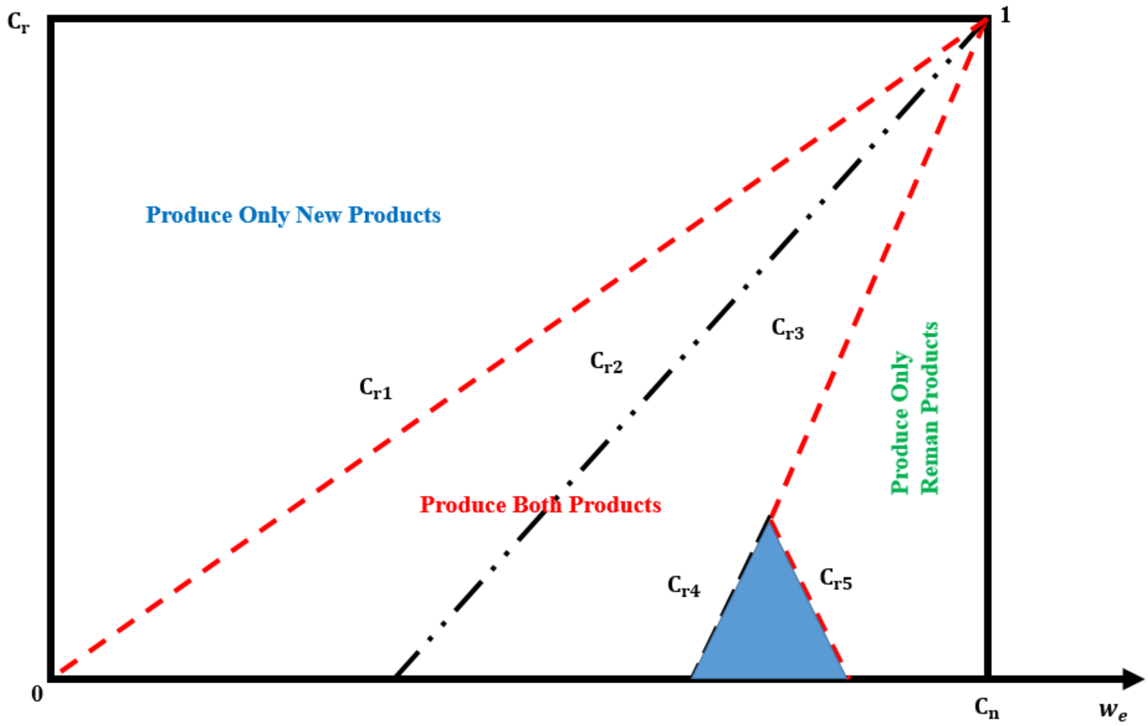


FIGURE 4. Mapping the most efficient production techniques in phase−*II*.

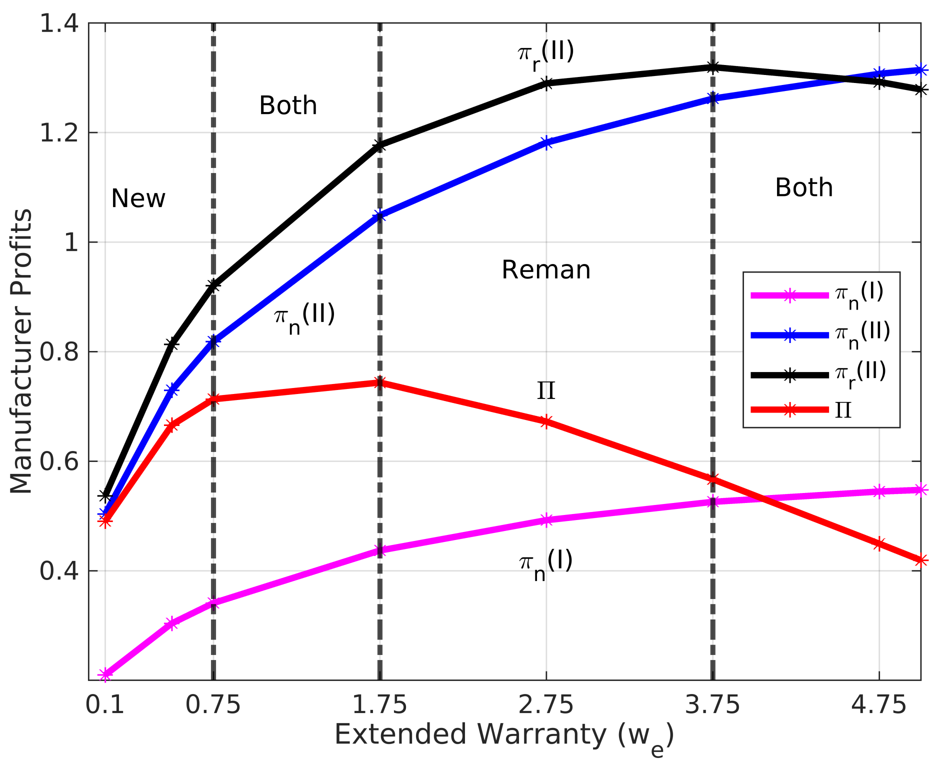


FIGURE 5. Impact of w_e on OEM profits.

there is a gradual reduction in the quantity of demand for the brand−new product during phase−I. Moreover, the profit for the reman product begins to fall at $w_e \geq 3.75$.

Next, it is interesting to investigate how the OEM's production strategy (B) impacts the customer responsiveness rates, product failure rate, and customer acceptance levels, are affected by the EW of the product.

B. SENSITIVITY TO THE EW OF THE REMAN AND NEW PRODUCTS (STRATEGY B)

This subsection analyzes the sensitivity to the EW of the reman and new products by varying w_e from 0.10 to 5.0 while retaining the values of all remaining parameters. Increasing the EW is beneficial to the pricing strategy adopted by the product OEM and results in an increase in demand for both reman and new products.

Based on the findings presented in Figure (6), it can be concluded that an increase in the length of an EW leads to a corresponding increase in the optimal price of both reman and new products. Additionally, Figure (6) indicates a varying impact on the optimal demand for new products and reman products. During phase−I, the demand for new products showed a consistent increase, while the demand for reman products during phase−II initially increased but eventually began to decrease when the value of w_e exceeded 1.75. These results have important implications for businesses operating in the (re)manufacturing industry, highlighting the need to carefully consider the length of EWs when deciding on optimal pricing strategies for their product offerings. In the broad sense, increasing the length of the EW resulted in a price increase for both the reman and new products. The present study indicates that an increase in the length of the EW results in a corresponding increase in the expenditure incurred by the OEM, both for reman and new products. This, in turn, forces the OEM to enhance its optimal prices. The findings suggest that the OEM may continue to charge a higher (increased) price if it offers superior customer service after the sale. The phase−II demand for the new product attains its maximum at $w_e = 1.75$. The explanation for these improvements is that an increase in the length of the EW can enhance the warranty expenditure for both reman and new products, leading to a decrease ($w_e > 1.75$) in the OEM's total profit (non−positive warranty expenditure effects). Furthermore, an investigation into how the EW influences the OEM's total profit has also additionally carried out. As the length of the EW increases, so does the total profit, and the optimal EW for the OEM is found to be ($w_e^* = 1.75$).

Because of the difference between the reman and new product failure rates ($\lambda_n \leq \lambda_r$ and $\lambda_{n_1} \leq \lambda_{r_1}$), the phase−$II$ reman product's EW expenditure increased faster than the new product's. However, EW length increases demand when the optimal price remains unchanged. When $w_e > 1.75$, the positive demand effect of boosting the length of the EW maximizes the EW expenditure effects that hurt the OEM's total profit for both reman and new products during both phases. Because of this, the OEM is able to encourage an increase in optimal demand for the reman and new products, which effectively leads to a significant boost in total profits. In the context of the manufacturing industry, it has been observed that (re)manufacturing products can be more profitable than producing new ones when the length of the EW falls between the range (0.10 to 1.75). Although the customers tend to value the EW on reman products more than on new ones ($\delta_{n_1} \leq \delta_{r_1}$), it has been noted that as the length of the EW increases, the customer's perception of the product's value also slightly increases. Therefore, it can be inferred that (re)manufacturing products can be a viable option for OEMs looking to optimize their profits, especially when the EW length falls within the aforementioned range. Since the new product is preferable ($\lambda_{n_1} \leq \lambda_{r_1}$), the reman product's EW expenditure rises faster than the new product's as the EW length keeps increasing. In phase−II, new product demand linearly increases when $w_e \leq 1.75$ and then starts to decrease. However, as the length of the EW increases, manufacturing the new product becomes more profitable. Simultaneously, the phase−I demand for the same new product experiences a gradual increase.

When $w_e > 2.75$, on the product market, there is a non-increasing demand for reman products (i.e., the red line starts to decrease significantly). In addition to the evidence presented in support, Table (6) demonstrates that the prices and demands increase as the length of EW increases, but the OEM profit drops after an optimal EW. In this specific scenario, the failure rate of the new product is significantly lower than that of the reman product. When the tenure of the EW increases, there is a changeover from reman to a new product (using the partial product returns to mitigate the possibility of increasing the reman product). Furthermore, the new product demand begins to rise and starts to decrease when $w_e > 1.75$ during phase−II.

In the meantime, an analysis of how sensitive failure rates are carried out. Examine the impacts of changing the product failure rate on optimal decisions using the OEM's optimal EW for both products. The findings are outlined in the following Table (7):

As the failure rate for both reman and new products increases, it stands to reason that the OEM's bottom line (profit) is decreasing. Further, unit production prices for reman and new products, as well as EW length, indicate similar patterns of findings. Therefore, the OEM does not extend the length of the product's EW or increase the production expenditure to reduce the negative effects on the OEM's total profit. In addition, the results from the table demonstrate that an improvement in the OEM's profit may be directly due to a lower product failure rate. As a result, the beneficial impact of the optimal profit grows more significant when the perceived rate of product failure tends to become smaller. The customer's responsiveness during the OEM's optimal EW is shown for both products in the following Table (8). If the expenditure per unit of production for either the reman or new product is held unchanged (c_r or c_n), then it is understandable that the OEM's optimal EW length will

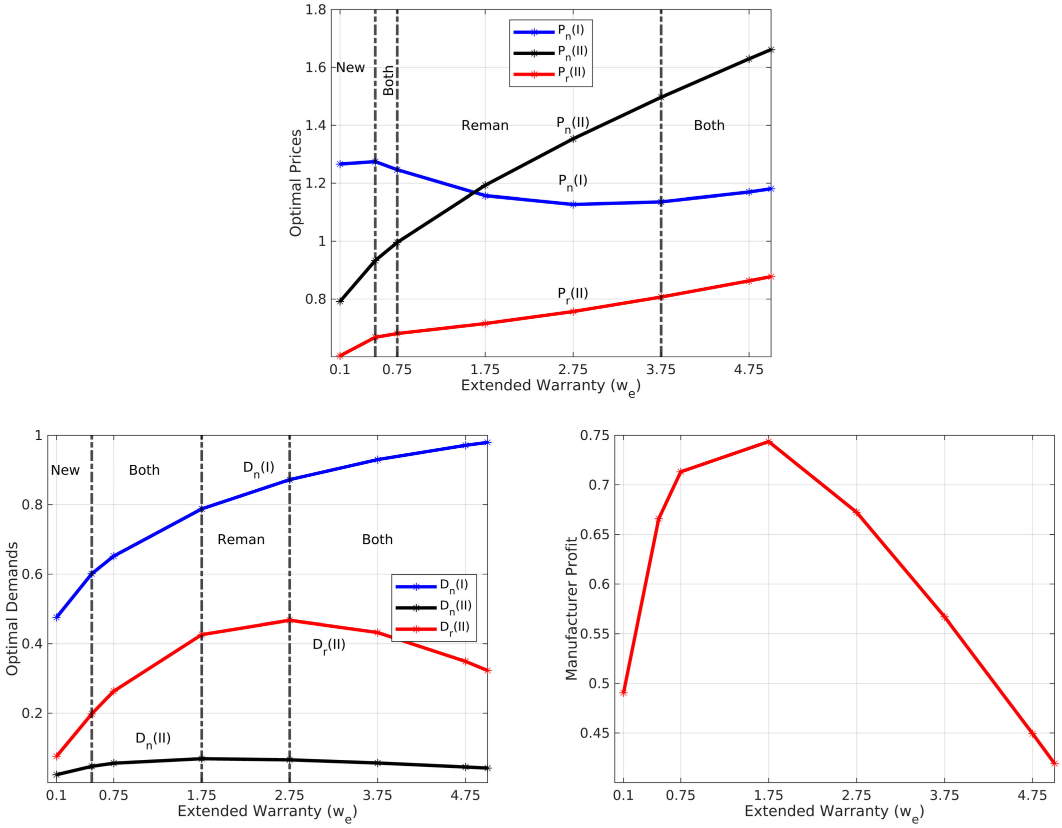


FIGURE 6. The implications of EW (w_e) on the optimal decisions.

TABLE 6. EW analysis on optimal decisions for the OEM.

	Extended Warranty (w_e)							
	0.10	0.25	0.50	**1.75**	2.75	3.75	4.75	5.0
$P_n(I)$	1.2660	1.2744	1.2464	1.1570	1.1267	1.1357	1.1694	1.1806
$P_n(II)$	0.7917	0.9329	0.9956	1.1927	1.3533	1.4967	1.6296	1.6616
$P_r(II)$	0.6037	0.6676	0.6807	0.7151	0.7567	0.8069	0.8628	0.8774
$D_n(I)$	0.4755	0.6011	0.6520	0.7880	0.8722	0.9300	0.9711	0.9795
$D_n(II)$	0.0228	0.0469	0.0559	0.0689	0.0656	0.0566	0.0450	0.0419
$D_r(II)$	0.0757	0.1980	0.2633	0.4261	0.4675	0.4323	0.3491	0.3230
Π	0.4906	0.6658	0.7132	**0.7437**	0.6724	0.5670	0.4494	0.4192

continue to increase. The identical conclusion occurs when analyzing the failure rate associated with a reman or new product when compared to the optimal EW length. If the expenditure for producing each unit does not change or if the rate of defective products increases, then the EW expenses will also increase. In order to counteract this adverse effect, the producer has significantly reduced the EW to decrease the expenditure spent on EW.

Furthermore, it was discovered that the unit manufacturing expenditure or the product failure rate for both the reman and new product increases, results in a significant decreases in the OEM's optimal EW length. Therefore, determining the right EW length for reman or new products is improved by customer response to the EW length. The rate of customer responsiveness improves across the board, benefiting both products. It has been discovered that not only does the optimal

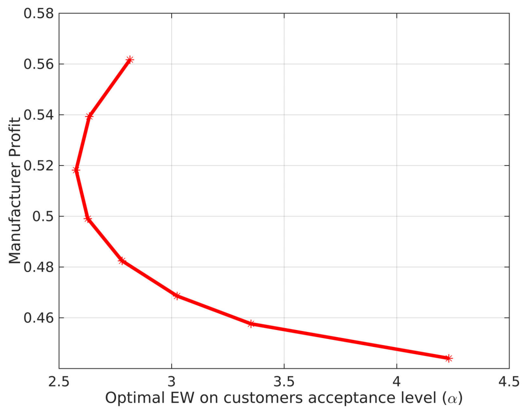


FIGURE 7. Significance of acceptance level on optimal EW length w_e.

profit enhance, but so does the sensitivity of the customer. As a result, the OEM pursues additional customers and generates a higher profit as a direct result of enhancing the customer responsiveness rate.

The importance of customer acceptance of reman products on the optimal EW is highlighted in Figure (7) and Table (9), and it proves that as α increases, so does the OEM's optimal profit. To highlight the trend more clearly, the value of optimal EW length is displayed in an integer with four decimal places. Hence, the results highlighted in Table (9) proved that when customers' acceptance (α) increases, the optimal EW length has decreased at $\alpha = 0.8$, then the optimal EW increases to a high of 2.6347 ($\alpha = 0.9$). Moreover, the aforementioned indicates that the level of acceptance granted by customers towards reman products is significantly inadequate. Hence, the OEM is not actively pursuing new customers. When the customer acceptability level of reman products is enough, very few customers will be willing to purchase either reman or new products. As a result, the acceptance of reman products has become sufficiently high, whichÂ led to a situation where the supply is unable to meet the demand, thus acting as a hurdle to the further evolution of the reman products in the market. It has been identified that an improvement in customer acceptance level of reman products leads to an increase in the OEM's overall profit. The most effective way to boost sales of refurbished (reman) products is to enhance the OEM's profit by increasing the value from the perspective of potential customers.

Next, it will be fascinating to investigate the impact of customer acceptance level rates on the OEM's optimal decisions.

C. THE IMPACT OF CUSTOMER ACCEPTANCE LEVEL (α) ON THE OEM OPTIMAL DECISIONS

This subsection analyzes the sensitivity of the customer's acceptance level on the OEM's optimal profits, pricing, and demands by varying α from 0.2 to 1 while retaining the values of all remaining parameters. The effect of α on the OEM's total profit for both phases is demonstrated in Figure (8).

Based on Figure (8), it can be observed that the most effective manufacturing strategy during Phase−II changes as the customer's acceptance level of the reman products increases. Initially, producing only new products is the most effective choice. However, as the acceptance level rises, it becomes more profitable to produce both reman and new products. Subsequently, the most efficient approach is to shift to producing only reman products. Finally, the optimal strategy reverts to producing both reman and new products. These findings suggest that companies should consider the customer's acceptance level when deciding on the manufacturing strategy for reman products. It occurs because the customers are more likely to purchase the reman products. The above results correlate with the findings of Ferguson and Toktay [79]. They highlight that with an increased value of α, the (re)manufacturing industry is a more viable option for implementation. In addition, when α is adequate ($\alpha = 0.36$ in this specific case), it is apparent that the profits generated by reman products exceed the aggregate profits realized during both planning phases. This is indicated by the solid black line, which lies above the blue solid curve line. The result occurs due to the fact that manufacturing enough reman products in phase−II necessitates a significant amount of customers purchasing the new products in phase−I at an affordable expenditure, which ultimately results in a profit gain substantially in phase−I, as demonstrated by a solid purple curve, which increases towards positive values. As a result, the overall profit will be less than the profit of phase−II production had focused only on reman products. Additionally, Figure (8) illustrates that the increase in the level of acceptance of reman products by customers had an adverse effect impact on the optimal overall profit that the OEM is expected to achieve when $\alpha > 0.7$ in this particular scenario. It brings more insights and repercussions about the research conducted by Ferguson and Toktay [79]. As the value of α increases, the perceived quality gap between reman and new products decreases. This phenomenon, in turn, drives a growing number of customers to consider purchasing reman products from an OEM that offers both options. The availability of such alternatives, coupled with the narrowing quality gap, can lead to an increase in the popularity of reman products among consumers. This trend has significant implications for OEMs seeking to expand their product offerings and differentiate themselves in a competitive marketplace.

If the OEM follows the optimal production and pricing strategy: more customers will opt to acquire the new products in phase−II, and fewer returns will be recommended in phase−I, as the higher price of reman products aims to discourage demand for reman products. When $\alpha > 0.8$, the new product's profit from phase−I increases, but the profit from reman products decreases slightly. It leads to a slight decrease in the total profit because the demand for reman products drops after $\alpha > 0.7$. This finding suggests that even though customers' favorable perspective regarding acquiring a reman product will beneficially impact their

TABLE 7. Sensitivity analysis for the failure rates while using optimal EW $w_e^* = 1.75$.

λ_n, λ_{n_1}	0.01, 0.03	0.05, 0.08	0.20, 0.25	0.23, 0.28	0.30, 0.35	0.32, 0.37
λ_r, λ_{r_1}	0.50, 0.55	0.60, 0.65	0.80, 0.85	0.83, 0.88	0.90, 0.95	0.92, 0.97
$P_n(I)$	1.5604	1.4666	1.2796	1.2553	1.2060	1.1936
$P_n(II)$	1.0853	1.1006	1.1533	1.1628	1.1849	1.1913
$P_r(II)$	0.8717	0.8367	0.7649	0.7556	0.7372	0.7326
$D_n(I)$	0.7757	0.7710	0.7501	0.7461	0.7366	0.7337
$D_n(II)$	0.0113	0.0274	0.0622	0.0657	0.0714	0.0724
$D_r(II)$	0.0315	0.0837	0.2312	0.2514	0.2897	0.2984
Π	0.9940	0.9339	0.7763	0.7486	0.6829	0.6640

TABLE 8. Analysis of customer responsiveness rates.

δ_n, δ_{n_1}	0.01, 0.05	0.06, 0.10	0.11, 0.15	0.16, 0.20	0.21, 0.25
δ_r, δ_{r_1}	0.08, 0.13	0.13, 0.18	0.18, 0.23	0.23, 0.28	0.28, 0.33
$P_n(I)$	0.9238	1.0008	1.0778	1.1547	1.2317
$P_n(II)$	0.6997	0.7478	0.7959	0.8440	0.8920
$P_r(II)$	0.4748	0.5086	0.5424	0.5762	0.6100
$D_n(I)$	0.3734	0.4217	0.4699	0.5181	0.5663
$D_n(II)$	0.0090	0.0102	0.0114	0.0125	0.0137
$D_r(II)$	0.0251	0.0284	0.0316	0.0349	0.0381
Π	0.2327	0.2966	0.3683	0.4478	0.5350

TABLE 9. Significance of acceptance level on optimal EW length w_e^*.

α	w_e^*	Profit	Use all returns?
0.2	4.2310	0.4440	No
0.4	3.3524	0.4576	No
0.5	3.0244	0.4686	No
0.6	2.7802	0.4825	No
0.7	2.6277	0.4991	No
0.8	2.5760	0.5182	No
0.9	2.6347	0.5393	**Yes**
1	2.8150	0.5617	**Yes**

intentions to buy [80], the OEM should refrain from investing significantly in loyalty programs or advertising aimed at increasing the value of α once it has reached an adequate level. This is because the sale of reman products is not profitable, and could result in back−orders, which may lead to customer dissatisfaction. Therefore, it is advisable to focus on alternative strategies to increase revenue and customer satisfaction.

Figures (9) and (10) show how the pricing and demands influence as α rises.

Figure (9) demonstrates that in phase−II, for a small value of α ($\alpha = 0.5$ in this scenario), the price of the new products steadily increases and is similar to in phase−I. The optimal price of phase−II's new products is significantly higher than that of phase−I's new products as α rises to its mid−point value. When α is low, the OEM decides to (re)manufacture a significant number of the returned products to meet market demand. Maintaining an excessive number of reman products from the market, the OEM may determine the new product price in phase−II to decrease the phase−I new product price, as the reman products influence the demand for the new products. As a result, the OEM needs to produce larger quantities of the new products during phase−I so that they may boost their sales. On the other hand, the volume of the reman product that the OEM produces extends (grows) in both situations as α rises and as α remains mild to moderate. Therefore, phase−I requires more returned products. Furthermore, the OEM reduced the new product price in phase−I to entice additional customers.

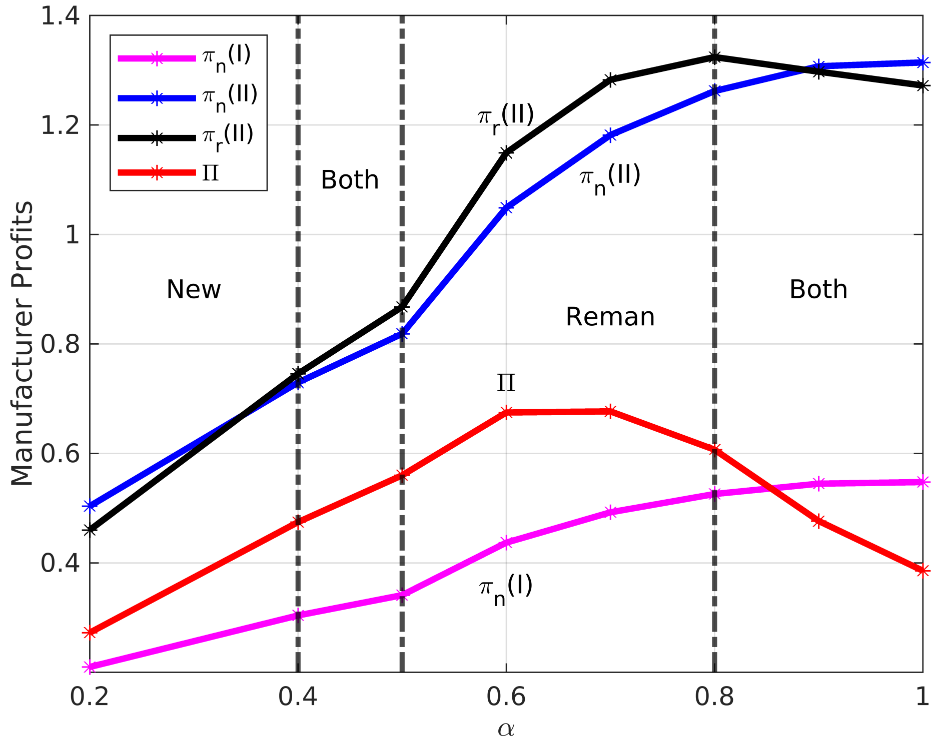


FIGURE 8. Impact of α on OEMs profits.

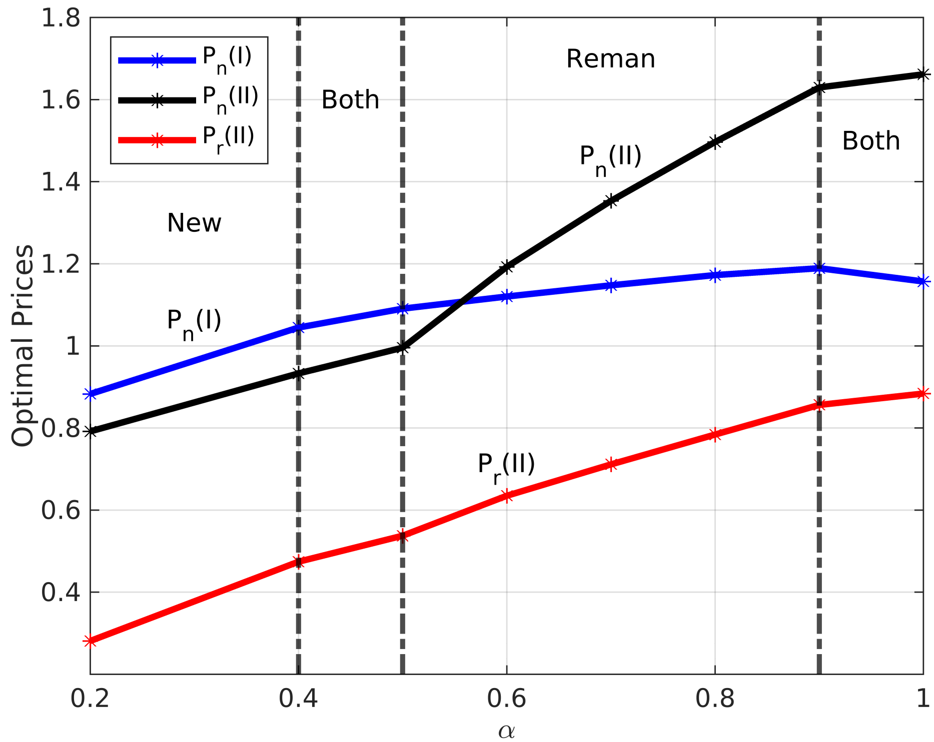


FIGURE 9. Impact of α on OEMs optimal prices.

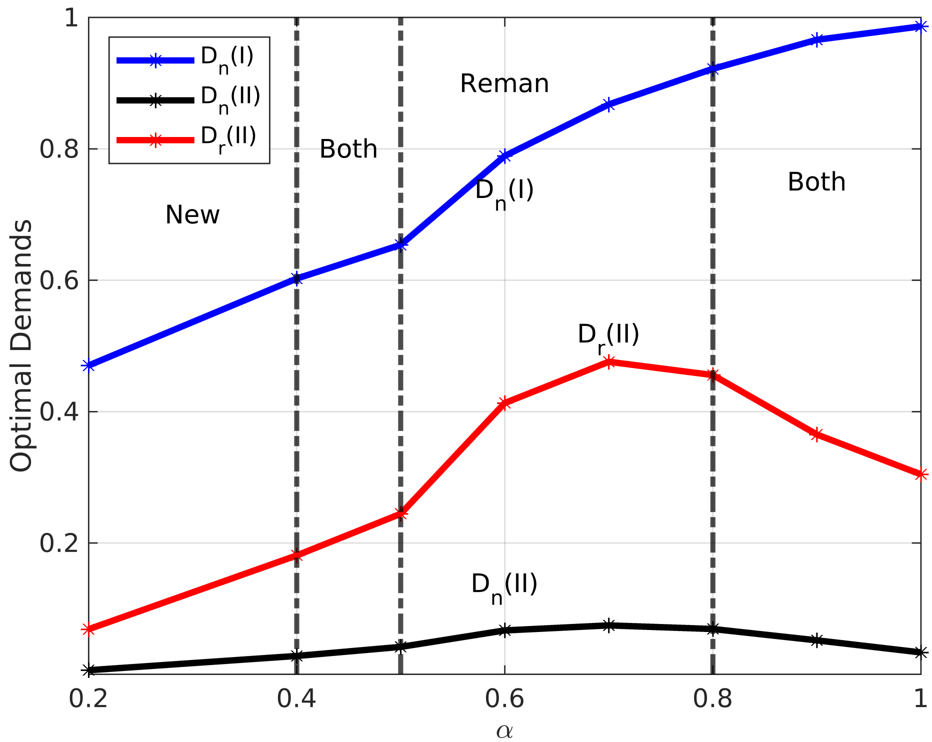


FIGURE 10. Impact of α on OEMs optimal demands.

D. MANAGERIAL IMPLICATIONS

The previous theoretical background leads us to the following managerial insights and industrial repercussions:

- **To what extent should the OEM (re)manufacture returned products, and when?** According to the model, the decision of OEMs to vend reman products is affected by several factors, such as the proportion of unit (re)manufacturing to manufacturing spending ($\frac{c_r}{c_n}$), the degree of customer acceptance of reman products, and the EW for both kinds of products. The optimal solution of our model offers valuable insights into the factors that influence the decision–making process of OEMs regarding the sale of reman products. The feasibility of selling reman products for OEMs is heavily influenced by a variety of factors. These factors must be taken into account when evaluating the potential commercial viability of such products. By increasing the length of the EW and enhancing the customer acceptance level, the expenditure of the unit (re)manufacturing to the unit manufacturing may decrease. This has the potential to result in a higher profit for the OEM from the sale of both reman and new products, compared to the sale of new products alone.
- **Does an OEM make an investment in the process of increasing the possibility that customers will accept the reman products?** Enhancing the customer's acceptance level of the reman products can entice additional prospective customers to make purchases. According to the above sensitivity study, however, a significant improvement in the reman products' response from customers may have a negative impact on the OEM's bottom line. To optimize cost efficiency, OEMs should refrain from investing in the improvement of customer acceptance levels of reman products when the current level already meets the necessary standards. Such investments may result in redundant expenses and a decrease in profitability. Therefore, OEMs must evaluate the existing acceptance level of reman products before considering any further investments in this regard.
- **How should the OEM set reman and new product prices when EW is offered?** Based on the results of the numerical studies and the sensitivity investigation, the OEM ought to enhance the pricing of both the newly manufactured and the reman products during phase–II when there is an increase in the EW length. Additionally, during phase–I of availability, the OEM implements a significant price increase for the new products.

VI. CONCLUSION

In this research, the authors explore the optimal pricing and production strategies of an EW for both reman and new products by adopting a two–phase mathematical framework strategy for a monopolistic OEM with the capability to manufacture the reman products by utilizing returned failure or faulty products during phase–I. During a Non–Renewing

EW, the demand function for both products is obtained using the customer's utility function. There is an explicit assumption that new products will always be priced more than their reman products. The research presented in this paper provides evidence that the market size (Q) for both reman and new has a convex relationship, which expresses the utility for purchasing decisions for the customer and the extent of the EW. In addition, numerical results were additionally carried out in order to bring attention to the significance of the EW towards reman products. It is possible to generate several production and price strategies, along with the conditions under which they exist. The impact of the customers' acceptance level of reman products and varying EW length on optimal manufacturing and price strategies are examined through the use of a sensitivity study.

Here, the conditions in which the product OEM's optimal decision to participate in (re)manufacturing during the phase−II is determined by the length of the EW (w_e) and the ratio of reman and new product expenditure in unit production ($\frac{c_r}{c_n}$). As a result, the (re)manufacturer will be more probably to (re)manufacture the product if the customer values the product significantly. It has been determined that an optimal EW for the OEM is influenced by the following factors such as the customer's willingness towards the purchase, the product's failure rate, and the level of customer acceptance for the reman product. Exploring the impact of the EW on the competitive landscape between reman and new products presents an intriguing field of study. An in−depth investigation into this area can shed light on the extent to which the EW influences the competition between these two product categories. Such an exploration can provide valuable insights into the dynamics of the market and can help businesses and academics better understand the factors that drive competition in this industry. OEMs might use different pricing structures to promote the reman products and entice more customers. The designed work aids the OEM and the consumer in helping to make decisions.

The numerical study also reveals the fascinating impact of the EW length for reman and new products on demand over the two phases. The EW length ($w_e = 1.75$) causes an increase in demand for both the new product manufactured in phase−I and the reman product manufactured in phase−II. Based on this trend, one can conclude that, including both reman and new products, an increase in EW has a positive influence when the EW length is low enough but a negative influence when the EW length is high enough ($w_e > 1.75$). To optimize sales, the OEM can determine the most appropriate pricing for their products. The goal is to ensure that, given a certain level of EW, customers are drawn to either the reman or the new products. Specifically, if the EW is low, the reman products will be more enticing to customers, whereas if the EW is high, the new products will be more appealing. By setting the optimal prices, the OEM can enhance the competitiveness of their products and improve their market position.

This paper's addresses the new perspectives into pricing and (re)manufacturing strategy, but there are many ways to extend the model in future studies. First, the framework only looks into how the non−renewable EW impacts an OEM's decision to participate in (re)manufacturing and how it impacts their pricing and production approaches. Standard warranty alternatives such as pro−rata, refund, and combination warranties may be part of the future task area to investigate and figure out the most effective (re)manufacturing warranty regulations. Further, the studies can expand their scope to consider how wholesalers and retailers factor into the process of setting prices and developing production methods. Second, in this paper, returns occur exclusively from products that failed in phase−I. It would be of considerable interest to undertake an investigation into the various influences that affect the optimal pricing, demand, and profit when returns stem from multiple sources, such as the recycling of End−of−Life (EOL) products. Furthermore, the model can be extended to incorporate a CLSC by encompassing several entities, including vendors, OEMs, merchants, dealers, third−party (re)manufacturers, and several others.

Data Availability Statement: Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing Interests: The authors declare that they have no competing interests.

Funding: This research did not receive any specific grant from funding agencies in the public, commercial, or not−for−profit sectors.

Author Contribution: The authors confirm their contribution to the paper as follows: P. Jalapathy and M. Mubashir Unnissa conceived of the presented idea. P. Jalapathy developed the idea and performed the computations. M. Mubashir Unnissa verified the analytical methods and encouraged P. Jalapathy to investigate and supervised the findings of this work. Both authors discussed the results and contributed to the final manuscript.

APPENDIX
PROOF OF THE PAPER

Proof of Lemma 1: This Lemma(1) intends to demonstrate that the Hessian Matrix (H_1) is symmetric and to establish whether or not the objective function is definite (7). The following is a list of the second−order Partial derivatives that can be derived from the Equations (4), (5) and (6):

$$\frac{\partial^2 \Pi}{\partial P_n^2(I)} = -2k_1; \quad \frac{\partial^2 \Pi}{\partial P_n(I)\partial P_n(II)} = 0;$$
$$\frac{\partial^2 \Pi}{\partial P_n(I)\partial P_r(II)} = 0$$
$$\frac{\partial^2 \Pi}{\partial P_n(II)\partial P_n(I)} = 0; \quad \frac{\partial^2 \Pi}{\partial P_n^2(II)} = \frac{-2Qk_1}{(1-\alpha)};$$
$$\frac{\partial^2 \Pi}{\partial P_n(II)\partial P_r(II)} = \frac{-Qk_2}{(1-\alpha)}$$

$$\frac{\partial^2 \Pi}{\partial P_r(II)\partial P_n(I)} = 0; \quad \frac{\partial^2 \Pi}{\partial P_r(II)\partial P_n(II)} = \frac{Qk_1}{(1-\alpha)};$$
$$\frac{\partial^2 \Pi}{\partial P_r^2(II)} = \frac{-2Qk_2}{\alpha(1-\alpha)}$$

According to the above partial derivatives of the second–order, the Hessian Matrix (H_1) of the objective function (7) is,

$$H_1 = \begin{bmatrix} -2k_1 & 0 & 0 \\ 0 & \frac{-2Qk_1}{(1-\alpha)} & \frac{-Qk_2}{(1-\alpha)} \\ 0 & \frac{Qk_1}{(1-\alpha)} & \frac{-2Qk_2}{\alpha(1-\alpha)} \end{bmatrix} \tag{A.1}$$

If the Hessian matrix (H_1) is symmetric matrix of size 3×3 and its three principal minors A_r, for $r = 1, 2, 3$ are given below: $|A_1| = -2k_1 < 0$, $|A_2| = \frac{4Qk_1^2}{(1-\alpha)} \geq 0$ and $|A_3| = -\frac{2Q^2k_1^2k_2}{(1-\alpha)^2}\left[\frac{4}{\alpha}+1\right] \leq 0$.

Therefore, the result of the matrix inferred that the optimization function (7) is concave, which indicates the maximization of total profits. For more information, one can refer to the works of Eiselt et al. [81] and Vandenberghe and Boyd [82].

Proof of Theorem 1: When $P_r(II) = 0$, according to the Equation (2) and (3), the OEM to determine $P_n(II) < \frac{(1-\alpha)}{k_1} + \frac{\left[\delta_n\sqrt{w} + \delta_{n_1}\sqrt{w_e}\right]}{k_1}$ to achieve the total profit from the product sales. Thus, the demand function for the reman and new products in phase–II ($D_n(II)$ and $D_r(II)$) provided by:

$$D_n(II) = Q\left[1 - \frac{k_1P_n(II) + [\delta_r - \delta_n]\sqrt{w} + \left[\delta_{r_1} - \delta_{n_1}\right]\sqrt{w_e}}{1-\alpha}\right] \tag{A.2}$$

$$D_r(II) = Q\left[\frac{\alpha k_1P_n(II) - (\alpha\delta_n - \delta_r)\sqrt{w} - \left[\alpha\delta_{n_1} - \delta_{r_1}\right]\sqrt{w_e}}{\alpha(1-\alpha)}\right] \tag{A.3}$$

If $P_r(II) = \frac{\alpha k_1P_n(II) - (\alpha\delta_n - \delta_r)\sqrt{w} - \left[\alpha\delta_{n_1} - \delta_{r_1}\right]\sqrt{w_e}}{k_2}$, the demand function for the reman and new products in phase–II ($D_n(II)$ and $D_r(II)$) provided by:

$$D_n(II) = Q\left[1 - k_1P_n(II) - \left[\delta_n\sqrt{w} + \delta_{n_1}\sqrt{w_e}\right]\right] \tag{A.4}$$
$$D_r(II) = 0 \tag{A.5}$$

When comparing these two scenarios, it is simple to observe that the OEM generates a higher profit in phase-II for the second scenario than for the first scenario. As a result, the demand for the new product has witnessed a significant increase. This can be attributed to various factors that have contributed to the increased market demand. In the second scenario, since $D_r(II) = 0$, the OEM is able to maximize profit in phase–I without taking into account phase–II. For this reason, the OEM should never settle for $P_r(II) = 0$. Thus, only the $P_r(II) > 0$ scenario needs to be taken into account.

When $P_r(II) > 0$ then $P_n(II) > 0$, because when $P_n(II) = 0$, the demand function for reman products in phase–II ($D_r(II)$) is non-positive according to Equation (3):

$$D_r(II) = -Q\left[\frac{(\delta_n - \delta_r)\sqrt{w} + k_2P_r(II) + (\delta_{n_1} - \delta_{r_1})\sqrt{w_e}}{(1-\alpha)}\right] \tag{A.6}$$

As a result, $P_n(j) > 0$ and $P_r(II) > 0$. Thus, the optimization Equation (7) is concave and the constraint Equation (8) is linear. In order to achieve the two–phase optimization model, one must solve the KKT conditions, which are as follows:

$$\frac{\partial \Pi}{\partial P_n(I)} + s_1B_1\frac{\partial D_n(I)}{\partial P_n(I)} = 0 \tag{A.7}$$
$$\frac{\partial \Pi}{\partial P_n(II)} - s_1N_r\frac{\partial D_r(II)}{\partial P_r(II)} = 0 \tag{A.8}$$
$$\frac{\partial \Pi}{\partial P_r(II)} - s_1N_r\frac{\partial D_r(II)}{\partial P_r(II)} = 0 \tag{A.9}$$
$$s_1\left[B_1D_n(I) - N_rD_r(II)\right] = 0 \tag{A.10}$$
$$B_1D_n(I) - N_rD_r(II) \geq 0 \tag{A.11}$$
$$P_n(I), P_n(II), P_r(II), s_1 \geq 0 \tag{A.12}$$

In two–phase KKT formulation, the value of the variable s_1 is utilized to clarify the impact of the boundedness. When $s_1 = 0$, then the limit of the constraint (A.10) is unbounded, and it has no influence on the objective function (7), as is evident from the Equations (A.7) to (A.12), and the total unit expenditure C_r is given by,

$$C_r \geq m_1 - \frac{\alpha E_3\left[QN_r + (1-\alpha)B_1\right]}{Qk_2N_r} \tag{A.13}$$

In addition, the limit of the constraint (A.10) is considered to be bounded if the value of $s_1 > 0$. Then the C_r value can be given by,

$$C_r < m_1 - \frac{\alpha E_3\left[QN_r + (1-\alpha)B_1\right]}{Qk_2N_r} \tag{A.14}$$

Using the demand function formulations in (2) and (3), the preceding scenarios can be split into two distinct cases.

Case 1: In this subsection (Only new products in phase–II), using $D_n(II) = Q(1-k_1P_n(II)+\delta_n\sqrt{w}+\delta_{n_1}\sqrt{w_e})$ and $D_r(II) = 0$ in Equations (A.7) to (A.12), the optimal price in both phase is,

$$P_n^*(I) = P_n^*(II) = p_1 = \frac{E_1}{2k_1} \tag{A.15}$$

Also, the optimal new products demand for both phases obtained by using (2) and (3),

$$D_n^*(I) = \frac{E_3}{2} \tag{A.16}$$

$$D_n^*(II) = d_1 = \frac{QE_3}{2} \tag{A.17}$$

By incorporating the Equations (A.15) to (A.17) and (10), the optimal total profit $\Pi^* = \pi_3$.

Case 2: In this subsection (Both reman and new products in phase−II), using $D_n(II)$ and $D_r(II)$, as shown at the bottom of the next page, in Equations (A.7) to (A.12), the reman and new products optimal prices in both phases are given below.

When $s_1 = 0$, $\left[C_r \geq m_1 - \frac{\alpha E_3 [QN_r + (1-\alpha)B_1]}{Qk_2N_r}\right]$, then

$$P_n^*(I) = P_n^*(II) = \frac{E_1}{2k_1} \tag{A.18}$$

$$P_r^*(II) = p_2 = \frac{E_2}{2k_2} \tag{A.19}$$

When $s_1 > 0$ $\left[C_r < m_1 - \frac{\alpha E_3 [QN_r + (1-\alpha)B_1]}{Qk_2N_r}\right]$, then the optimal prices for reman and new products ($P_n^*(I), P_n^*(II), P_r^*(II)$) and s_1^* in phase−II is,

$$P_n^*(I) = p_3 = \frac{E_1}{2k_1} + \frac{\alpha E_3 \left[(1-\alpha)B_1^2 + \theta QN_r\right]}{2k_1 \left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]} - \frac{\theta QN_r \left[\lambda_n w + \lambda_{n_1} w_e\right] E_4}{2k_1 \left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]} \tag{A.20}$$

$$P_n^*(II) = \frac{E_1}{2k_1} \tag{A.21}$$

$$P_r^*(II) = p_4 = \frac{E_2}{2k_2} + \frac{QN_rE_4}{2k_2 \left[QN_r^r + \alpha(1-\alpha)B_1^2\right]} - \frac{\alpha B_1 E_3 [(1-\alpha)B_1 + QN_r]}{2k_2 \left[QN_r^r + \alpha(1-\alpha)B_1^2\right]} \tag{A.22}$$

$$s_1^* = \frac{\alpha E_3 [(1-\alpha)B_1 + QN_r]}{QN_r^2 + \alpha(1-\alpha)B_1^2} - \frac{QN_rE_4}{QN_r^2 + \alpha(1-\alpha)B_1^2} \tag{A.23}$$

Obviously, $P_n^*(j) \geq 0$ and $P_r^*(II) \geq 0$. While $s_1^* \geq 0$ when:

$$C_r \leq m_1 - \frac{\alpha E_3 [QN_r + (1-\alpha)B_1]}{Qk_2N_r} \tag{A.24}$$

Also, the optimal reman and new products demand for both phases obtained by using (2) and (3).

When $s_1 = 0$ $\left[C_r \geq m_1 - \frac{\alpha E_3 [QN_r + (1-\alpha)B_1]}{Qk_2N_r}\right]$, then

$$D_n^*(I) = \frac{E_3}{2} \tag{A.25}$$

$$D_n^*(II) = d_2 = \frac{Q[E_3 - E_4]}{2(1-\alpha)} \tag{A.26}$$

$$D_r^*(II) = d_3 = \frac{Q[(E_4 - \alpha) - \alpha(E_3 - 1)]}{2\alpha(1-\alpha)} \tag{A.27}$$

Since all demand functions must be non-negative, require:

$$C_r < m_1 - \frac{E_3}{k_2} \tag{A.28}$$

$$C_r < \frac{\delta_r \sqrt{w} + \delta_{r_1}\sqrt{w_e}}{k_2} - \frac{\alpha\left[\delta_n \sqrt{w} + \delta_{n_1}\sqrt{w_e}\right]}{k_2} \tag{A.29}$$

Furthermore, the constraint Equation (8) should be satisfied, thus:

$$C_r > m_1 - \frac{\alpha}{k_2}\left[E_3 + \frac{(1-\alpha)B_1}{QN_r}\right] \tag{A.30}$$

By incorporating the Equations (A.18) to (A.19) and (7), the optimal total profit $\Pi^* = \pi_1$.

When $s_1 > 0$ $\left[C_r < m_1 - \frac{\alpha E_3 [QN_r + (1-\alpha)B_1]}{Qk_2N_r}\right]$, then the optimal demands in phase−II is,

$$D_n^*(I) = d_4 = \frac{E_3}{2} + \frac{QB_1E_4N_r}{2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]} - \frac{\alpha B_1 E_3((1-\alpha)B_1 + QN_r)}{2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]} \tag{A.31}$$

$$D_n^*(II) = d_5 = \frac{\alpha QB_1E_3 [B_1 + QN_r]}{2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]} - \frac{\alpha QB_1 [E_3N_r + B_1E_4]}{2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]} \tag{A.32}$$

$$D_r^*(II) = d_6 = \frac{QB_1 [E_3N_r + B_1E_4]}{2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]} - \frac{\alpha QB_1^2(\lambda_n w + \lambda_{n_1} w_e)E_3}{2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]} \tag{A.33}$$

Since all demand functions must be non−negative, therefore:

$$C_r > m_1 + \frac{E_3\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]}{k_1QB_1N_r} - \frac{\alpha E_3 [QN_r + (1-\alpha)B_1]}{k_1QN_r} \tag{A.34}$$

$$C_r < m_1 - \frac{E_3 [B_1 + (Q-1)N_r]}{k_2B_1} \tag{A.35}$$

$$C_r > m_1 + \frac{E_3\left[N_r - \alpha B_1 \left[\lambda_n w + \lambda_{n_1} w_e\right]\right]}{k_2B_1} \tag{A.36}$$

Taking into account the possible range of values for each parameter, when:

- Equation (A.28), Equation (A.30) are satisfied and

$$C_n \geq \frac{Nn}{k_2} - \frac{\alpha B_1}{k_2QN_r}$$

- Equation (A.29), Equation (A.30) are satisfied and

$$C_n \leq \frac{N_n}{k_1} - \frac{k_2m_1}{k_1\alpha} + \frac{(1-\alpha)B_1}{QN_r} + \alpha\left[(\delta_r - \alpha\delta_n)\sqrt{w} + (\delta_{r_1} - \alpha\delta_{n_1})\sqrt{w_e}\right]$$

- Equation (A.24), Equation (A.30) are satisfied and

$$C_n \geq \frac{N_n}{k_1} + \frac{\alpha[1-\alpha]B_1}{k_1QN_r[(\alpha-1)B_1 + (1-Q)N_r]},$$

the pricing method utilized in this scenario is the most effective strategy. The OEM's optimal prices for each scenario can be determined using the above solutions (refer

Equations: A.15 to A.36) and the optimal profit function from (7). By incorporating the Equations (A.20) to (A.23) and (7), the optimal total profit $\Pi^* = \pi_2$.

Proof of Lemma 2: From Table (4), only for strategies approach in N and B (in Areas 1 and 3) does the OEM offer new products during the phase−I and phase−II. The optimal pricing structure for Strategy N and B (in Area 2) remains unchanged for both phases. For Area 3 and Strategy B:

$$P_n^*(II) - P_n^*(I) = -\frac{\alpha E_3\left[(1-\alpha)B_1^2 + \theta Q N_r\right]}{2k_1\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]} + \frac{QN_rB_1E_4}{2k_1\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]}$$

In Area 3, for Plan Strategy B, the OEM using the product returns partially in Area 2 and using the all product returns in Area 3 (if $C_{r_4} < C_r < C_{r_2}$) and $Q > \frac{\alpha B_1}{N_r}$, then

$$\alpha E_3\left[(1-\alpha)B_1^2 + \theta Q N_r\right] - QN_rB_1E_4 \le 0.$$

Hence, $P_n^*(II) \ge P_n^*(I)$. Additionally, for return product Strategy N and B (in Areas 2 and 3), the new product price for the phase−II is unchanged and equivalent to $\frac{E_1}{2k_1}$. As a result, the only factors entail the warranty length, how the customer's acceptance influenced, and varying the EW to the new product. Thus, it is unaffected by reman product factors.

Proof of Lemma 3: While analyzing the EW, the OEM assumes that the standard warranty to neutral (or relaxed). To determine how the optimal solutions responds to enhance the customer's responsiveness δ_i, the first−order partial derivatives of the optimum prices $(P_i(j))$ and the demands $(D_i(j))$ are acquired with respect to δ_i.

For Area 1 (Strategy N):

$$\frac{\partial P_n^*(I)}{\partial \delta_{n_1}} = \frac{\partial P_n^*(II)}{\partial \delta_{n_1}} = \frac{\sqrt{w_e}}{2k_1} \ge 0 \quad \text{(A.37)}$$

$$\frac{\partial D_n^*(I)}{\partial \delta_{n_1}} = \frac{\sqrt{w_e}}{2} \ge 0; \quad \frac{\partial D_n^*(II)}{\partial \delta_{n_1}} = \frac{Q\sqrt{w_e}}{2} \ge 0 \quad \text{(A.38)}$$

$$\frac{\partial P_n^*(I)}{\partial \delta_{r_1}} = \frac{\partial P_n^*(II)}{\partial \delta_{r_1}} = \frac{\partial D_n^*(I)}{\partial \delta_{r_1}} = \frac{\partial D_n^*(II)}{\partial \delta_{r_1}} = 0 \quad \text{(A.39)}$$

For Area 2 (Strategy B):

$$\frac{\partial P_n^*(I)}{\partial \delta_{n_1}} = \frac{\partial P_n^*(II)}{\partial \delta_{n_1}} = \frac{\sqrt{w_e}}{2k_1} \ge 0; \quad \frac{\partial P_r^*(II)}{\partial \delta_{n_1}} = \frac{\sqrt{w_e}}{2k_2} \ge 0 \quad \text{(A.40)}$$

$$\frac{\partial D_n^*(I)}{\partial \delta_{n_1}} = \frac{\sqrt{w_e}}{2} \ge 0; \quad \frac{\partial D_n^*(II)}{\partial \delta_{n_1}} = \frac{Q\sqrt{w_e}}{2(1-\alpha)} \ge 0;$$

$$\frac{\partial D_r^*(II)}{\partial \delta_{n_1}} = \frac{-Q\sqrt{w_e}}{2\alpha(1-\alpha)} \le 0 \quad \text{(A.41)}$$

$$\frac{\partial P_n^*(I)}{\partial \delta_{r_1}} = \frac{\partial P_n^*(II)}{\partial \delta_{r_1}} = 0; \quad \frac{\partial P_r^*(II)}{\partial \delta_{r_1}} = \frac{\sqrt{w_e}}{2k_2} \ge 0 \quad \text{(A.42)}$$

$$\frac{\partial D_n^*(I)}{\partial \delta_{r_1}} = 0; \quad \frac{\partial D_n^*(II)}{\partial \delta_{r_1}} = \frac{-Q\sqrt{w_e}}{2(1-\alpha)} \le 0;$$

$$\frac{\partial D_r^*(II)}{\partial \delta_{r_1}} = \frac{Q\sqrt{w_e}}{2\alpha(1-\alpha)} \ge 0 \quad \text{(A.43)}$$

For Area 3 (Strategy B):

$$\frac{\partial P_n^*(I)}{\partial \delta_{n_1}} = \frac{\sqrt{w_e}}{2k_1} + \frac{\alpha\sqrt{w_e}\left[(1-\alpha)B_1^2 + Q\theta N_r\right]}{2k_1\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]} \ge 0 \quad \text{(A.44)}$$

$$\frac{\partial P_n^*(II)}{\partial \delta_{n_1}} = \frac{\sqrt{w_e}}{2k_1} \ge 0 \quad \text{(A.45)}$$

$$\frac{\partial P_r^*(II)}{\partial \delta_{n_1}} = -\frac{\alpha B_1\sqrt{w_e}\left[(1-\alpha)B_1 + QN_r\right]}{2k_2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]} \le 0 \quad \text{(A.46)}$$

$$\frac{\partial D_n^*(I)}{\partial \delta_{n_1}} = \frac{\sqrt{w_e}}{2} - \frac{\alpha B_1\sqrt{w_e}\left[(1-\alpha)B_1 + QN_r\right]}{2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]} \ge 0 \quad \text{(A.47)}$$

$$\frac{\partial D_n^*(II)}{\partial \delta_{n_1}} = \frac{\alpha QB_1\sqrt{w_e}\left[B_1 + (Q-1)N_r\right]}{2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]} \ge 0 \quad \text{(A.48)}$$

$$\frac{\partial D_r^*(II)}{\partial \delta_{n_1}} = \frac{QB_1\sqrt{w_e}\left[N_r - \alpha B_1(\lambda_n w + \lambda_{n_1}w_e)\right]}{2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]} \ge 0 \quad \text{(A.49)}$$

$$\frac{\partial P_n^*(I)}{\partial \delta_{r_1}} = \frac{-QB_1\left[\lambda_n w + \lambda_{n_1}w_e\right]\sqrt{w_e}}{2k_1\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]} \le 0; \quad \frac{\partial P_n^*(II)}{\partial \delta_{r_1}} = 0 \quad \text{(A.50)}$$

$$\frac{\partial P_r^*(II)}{\partial \delta_{r_1}} = \frac{\sqrt{w_e}}{2k_2} + \frac{QN_r\sqrt{w_e}}{2k_2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]} \ge 0 \quad \text{(A.51)}$$

$$\frac{\partial D_n^*(I)}{\partial \delta_{r_1}} = \frac{QB_1N_r\sqrt{w_e}}{2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]} \ge 0 \quad \text{(A.52)}$$

$$\frac{\partial D_n^*(II)}{\partial \delta_{r_1}} = -\frac{\alpha QB_1^2\sqrt{w_e}}{2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]} \le 0 \quad \text{(A.53)}$$

$$\frac{\partial D_r^*(II)}{\partial \delta_{r_1}} = \frac{QB_1^2\sqrt{w_e}}{2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]} \ge 0 \quad \text{(A.54)}$$

$$D_n(II) = Q\left[1 - \frac{k_1P_n(II) + (\delta_r - \delta_n)\sqrt{w} - k_2P_r(II) + (\delta_{r_1} - \delta_{n_1})\sqrt{w_e}}{(1-\alpha)}\right]$$

$$D_r(II) = Q\left[\frac{\alpha k_1P_n(II) + (\delta_r - \alpha\delta_n)\sqrt{w} - k_2P_r(II) + (\delta_{r_1}\sqrt{w} - \alpha\delta_{n_1})\sqrt{w_e}}{\alpha(1-\alpha)}\right]$$

For Area 4 (Strategy R):

$$\frac{\partial P_n^*(I)}{\partial \delta_{n_1}} = \frac{\partial P_n^*(II)}{\partial \delta_{n_1}} = \frac{\sqrt{w_e}}{2k_1} \geq 0 \tag{A.55}$$

$$\frac{\partial D_n^*(II)}{\partial \delta_{n_1}} = \frac{\sqrt{w_e}}{2} \geq 0 \tag{A.56}$$

$$\frac{\partial P_r^*(II)}{\partial \delta_{n_1}} = \frac{\partial D_n^*(I)}{\partial \delta_{n_1}} = \frac{\partial D_r^*(II)}{\partial \delta_{n_1}} = 0 \tag{A.57}$$

$$\frac{\partial P_n^*(I)}{\partial \delta_{r_1}} = \frac{\partial P_n^*(II)}{\partial \delta_{r_1}} = \frac{\partial D_n^*(I)}{\partial \delta_{r_1}} = \frac{\partial D_n^*(II)}{\partial \delta_{r_1}} = 0 \tag{A.58}$$

$$\frac{\partial P_r^*(II)}{\partial \delta_{r_1}} = \frac{\sqrt{w_e}}{2k_2} \geq 0; \quad \frac{\partial D_r^*(II)}{\partial \delta_{r_1}} = \frac{Q\sqrt{w_e}}{2\alpha} \geq 0 \tag{A.59}$$

For Area 5 (Strategy R):

$$\frac{\partial P_n^*(I)}{\partial \delta_{n_1}} = \frac{\sqrt{w_e}}{2k_1} + \frac{\alpha B_1^2\sqrt{w_e}}{2k_1\left[QN_r^2 + \alpha B_1^2\right]} \geq 0 \tag{A.60}$$

$$\frac{\partial P_n^*(II)}{\partial \delta_{n_1}} = \frac{\partial D_n^*(II)}{\partial \delta_{n_1}} = \frac{\partial P_n^*(II)}{\partial \delta_{r_1}} = \frac{\partial D_n^*(II)}{\partial \delta_{r_1}} = 0 \tag{A.61}$$

$$\frac{\partial P_r^*(II)}{\partial \delta_{n_1}} = -\frac{\alpha B_1 N_r\sqrt{w_e}}{2k_2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]} \leq 0 \tag{A.62}$$

$$\frac{\partial D_n^*(I)}{\partial \delta_{n_1}} = \frac{\sqrt{w_e}}{2} - \frac{\alpha B_1^2\sqrt{w_e}}{2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]} \geq 0 \tag{A.63}$$

$$\frac{\partial D_r^*(II)}{\partial \delta_{n_1}} = \frac{QB_1N_r\sqrt{w_e}}{2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]} \geq 0 \tag{A.64}$$

$$\frac{\partial P_n^*(I)}{\partial \delta_{r_1}} = -\frac{QB_1N_r\sqrt{w_e}}{2k_1\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]} \leq 0 \tag{A.65}$$

$$\frac{\partial P_r^*(II)}{\partial \delta_{r_1}} = \frac{\sqrt{w_e}}{2k_2} + \frac{QN_r^2\sqrt{w_e}}{2k_2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]} \geq 0 \tag{A.66}$$

$$\frac{\partial D_n^*(I)}{\partial \delta_{r_1}} = \frac{QB_1N_r\sqrt{w_e}}{2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]} \geq 0 \tag{A.67}$$

$$\frac{\partial D_r^*(II)}{\partial \delta_{r_1}} = \frac{QB_1^2(1-\alpha)\sqrt{w_e}}{2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]} \geq 0 \tag{A.68}$$

Equations (A.37) to (A.68) shows that the relationship between the two parameters (δ_{n_1} and δ_{r_1}). If the parameter value is greater than 0, it is positive; if it is lower than 0, it is negative; and if it is zero, the factors are independent.

Proof of Lemma 4: This Lemma(4) intends to demonstrate that the Hessian Matrix (H_2) definiteness of the objective function (10) is given:

$$\frac{\partial^2\Pi}{\partial P_n^2(I)} = -2k_1; \quad \frac{\partial^2\Pi}{\partial P_n(I)\partial P_n(II)} = 0$$
$$\frac{\partial^2\Pi}{\partial P_n(II)\partial P_n(I)} = 0; \quad \frac{\partial^2\Pi}{\partial P_n^2(II)} = -2Qk_1$$

Using the above second−order partial derivatives, the Hessian Matrix (H_2) of objective function (10) is

$$H_2 = \begin{bmatrix} -2k_1 & 0 \\ 0 & -2Qk_1 \end{bmatrix} \tag{A.69}$$

If the Hessian Matrix is 2 × 2 symmetric matrix and the alternative sign of its two leading principle minors A_4 and A_5 are as follows:

$$|A_4| = -2k_1 < 0, |A_5| = 4Qk_1^2 \geq 0.$$

It is observed that the Hessian Matrix is always non−positive semi−definite. Hence, it concludes that the objective function (10) is concave.

Proof of Lemma 5: This Lemma(5) intends to demonstrate that the Hessian Matrix (H_3) definiteness of the objective function (15) is given:

$$\frac{\partial^2\Pi}{\partial P_n^2(I)} = -2k_1; \quad \frac{\partial^2\Pi}{\partial P_n(I)\partial P_r(II)} = 0$$
$$\frac{\partial^2\Pi}{\partial P_r(II)\partial P_n(I)} = 0; \quad \frac{\partial^2\Pi}{\partial P_r^2(II)} = \frac{-2Qk_2}{\alpha}$$

Using the above second−order partial derivatives, the Hessian Matrix (H_3) of objective function (15) is

$$H_3 = \begin{bmatrix} -2k_1 & 0 \\ 0 & \dfrac{-2Qk_2}{\alpha} \end{bmatrix} \tag{A.70}$$

If the Hessian matrix is 2 × 2 symmetric matrix and the alternative sign of its two leading principle minors A_6 and A_7 are as follows:

$$|A_6| = -2k_1 < 0, |A_7| = \frac{4Qk_1k_2}{\alpha} \geq 0.$$

It is observed that the Hessian Matrix is always non−positive semi−definite. Hence, it concludes that the objective function (15) is concave.

Proof of Theorem 3: In accordance with the proofs for Lemma (5) and Theorem (1), the KKT conditions in this case when the OEM makes a decision to manufacture (produce) only reman products in phase−II are listed below:

$$\frac{\partial\Pi}{\partial P_n(I)} + s_1B_1\frac{\partial D_n(I)}{\partial P_n(I)} = 0 \tag{A.71}$$

$$\frac{\partial\Pi}{\partial P_r(II)} - s_1N_r\frac{\partial D_r(II)}{\partial P_r(II)} = 0 \tag{A.72}$$

$$s_1\left[B_1D_n(I) - N_rD_r(II)\right] = 0 \tag{A.73}$$

$$B_1D_n(I) - N_rD_r(II) \geq 0 \tag{A.74}$$

$$P_n(I), P_r(II), s_2 \geq 0 \tag{A.75}$$

In this section, only reman products in phase−II, using $D_r(II) = Q\left[1 - \frac{k_2P_r(II) - \delta_r\sqrt{w} - \delta_{r_1}\sqrt{w_e}}{\alpha}\right]$ and $D_n(II) = 0$ in Equations (A.71) to (A.75), the reman and new products optimal prices in both phase are given below:

When $s_2 = 0$, $\left[C_r \geq m_1 - \frac{\alpha E_3B_1}{Qk_2N_r}\right]$, then

$$P_n^*(I) = P_n^*(II) = \frac{E_1}{2k_1} \tag{A.76}$$

$$P_r^*(II) = \frac{E_2}{2k_2} \tag{A.77}$$

TABLE 10. Critical values for optimal prices p_t.

Critical Values	Expression
p_1	$\frac{E_1}{2k_1}$
p_2	$\frac{E_2}{2k_2}$
p_3	$\frac{E_1}{2k_1}+\frac{\alpha E_3\left[(1-\alpha)B_1^2+\theta QN_r\right]}{2k_1\left[QN_r^2+\alpha(1-\alpha)B_1^2\right]}-\frac{\theta QN_r\left[\lambda_n w+\lambda_{n_1}w_e\right]E_4}{2k_1\left[QN_r^2+\alpha(1-\alpha)B_1^2\right]}$
p_4	$\frac{E_2}{2k_2}+\frac{QN_rE_4}{2k_2\left[QN_r^r+\alpha(1-\alpha)B_1^2\right]}-\frac{\alpha B_1E_3\left[(1-\alpha)B_1+QN_r\right]}{2k_2\left[QN_r^r+\alpha(1-\alpha)B_1^2\right]}$
p_5	$\frac{E_1}{2k_1}+\frac{B_1\left[\alpha B_1E_3-QN_rE_4\right]}{2k_1\left[QN_r^2+\alpha B_1^2\right]}$
p_6	$\frac{E_2}{2k_2}+\frac{QE_4N_r^2}{2k_2\left[QN_r^2+\alpha(1-\alpha)B_1^2\right]}-\frac{\alpha B_1E_3N_r}{2k_2\left[QN_r^2+\alpha(1-\alpha)B_1^2\right]}$

TABLE 11. Critical values for s_t^*.

Critical Values	Expression
s_1^*	$\frac{\alpha E_3\left[(1-\alpha)B_1+QN_r\right]}{QN_r^2+\alpha(1-\alpha)B_1^2}-\frac{QN_rE_4}{QN_r^2+\alpha(1-\alpha)B_1^2}$
s_2^*	$\frac{(1-\alpha)\left[QN_rE_2-\alpha B_1E_3\right]}{\alpha(1-\alpha)B_1^2+QN_r^2}$

TABLE 12. Critical values for C_{n_t}.

Critical Values	Expression
C_{n_1}	$\frac{Nn}{k_2}-\frac{\alpha B_1}{k_2QN_r}$
C_{n_2}	$\frac{N_n}{k_1}+\frac{\alpha\left[1-\alpha\right]B_1}{k_1QN_r\left[(\alpha-1)B_1+(1-Q)N_r\right]}$
C_{n_3}	$\alpha\left[(\delta_r-\alpha\delta_n)\sqrt{w}+(\delta_{r_1}-\alpha\delta_{n_1})\sqrt{w_e}\right]+\frac{N_n}{k_1}-\frac{k_2m_1}{k_1\alpha}+\frac{(1-\alpha)B_1}{QN_r}$

When $s_2>0$, $\left[C_r<m_1-\frac{\alpha E_3B_1}{Qk_2N_r}\right]$, then

$$P_n^*(I)=p_5=\frac{E_1}{2k_1}+\frac{B_1\left[\alpha B_1E_3-QN_rE_4\right]}{2k_1\left[QN_r^2+\alpha B_1^2\right]} \quad \text{(A.78)}$$

$$P_r^*(II)=p_6=\frac{E_2}{2k_2}+\frac{QE_4N_r^2}{2k_2\left[QN_r^2+\alpha(1-\alpha)B_1^2\right]}-\frac{\alpha B_1E_3N_r}{2k_2\left[QN_r^2+\alpha(1-\alpha)B_1^2\right]} \quad \text{(A.79)}$$

$$s_2^*=\frac{(1-\alpha)\left[QN_rE_2-\alpha B_1E_3\right]}{\alpha(1-\alpha)B_1^2+QN_r^2} \quad \text{(A.80)}$$

Obviously, $P_n^*(j)\geq 0$ and $P_r^*(II)\geq 0$. While $s_2^*\geq 0$ when:

$$C_r\geq m_1-\frac{\alpha E_3B_1}{Qk_2N_r} \quad \text{(A.81)}$$

Also, the optimal reman and new products demand for both phases obtained by using (2) and (3).

TABLE 13. Critical values for optimal demands d_t.

Critical Values	Expression
d_1	$\frac{QE_3}{2}$
d_2	$\frac{Q[E_3 - E_4]}{2(1-\alpha)}$
d_3	$\frac{Q[(E_4-\alpha) - \alpha(E_3 - 1)]}{2\alpha(1-\alpha)}$
d_4	$\frac{E_3}{2} + \frac{QB_1E_4N_r}{2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]} - \frac{\alpha B_1 E_3((1-\alpha)B_1 + QN_r)}{2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]}$
d_5	$\frac{\alpha QB_1E_3[B_1 + QN_r]}{2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]} - \frac{\alpha QB_1[E_3N_r + B_1E_4]}{2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]}$
d_6	$\frac{QB_1[E_3N_r + B_1E_4]}{2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]} - \frac{\alpha QB_1^2(\lambda_n w + \lambda_{n_1} w_e)E_3}{2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]}$
d_7	$\frac{QE_4}{2\alpha}$
d_8	$\frac{E_3}{2} + \frac{B_1[QN_rE_4 - \alpha B_1E_3]}{2\left[QN_r^2 + \alpha B_1^2\right]}$
d_9	$\frac{QB_1[(1-\alpha)B_1E_4 + E_3N_r]}{2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]}$

When $s_2 = 0$, $\left[C_r \geq m_1 - \frac{\alpha E_3 B_1}{Qk_2N_r}\right]$, then

$$D_n^*(I) = D_n^*(II) = \frac{E_3}{2} \tag{A.82}$$

$$D_r^*(II) = d_7 = \frac{QE_4}{2\alpha} \tag{A.83}$$

Since all demand functions must be non−negative, which provides:

$$C_r > m_1 \tag{A.84}$$

By incorporating the Equations (A.76) to (A.77) and (15), the optimal total profit $\Pi^* = \pi_5$.

When $s_2 > 0$, $\left[C_r < m_1 - \frac{\alpha E_3 B_1}{Qk_2N_r}\right]$, then

$$D_n^*(I) = d_8 = \frac{E_3}{2} + \frac{B_1[QN_rE_4 - \alpha B_1E_3]}{2\left[QN_r^2 + \alpha B_1^2\right]} \tag{A.85}$$

$$D_r^*(II) = d_9 = \frac{QB_1[(1-\alpha)B_1E_4 + E_3N_r]}{2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]} \tag{A.86}$$

Since all demand functions must be non-negative, which provides:

$$C_r > m_1 + \frac{E_3\left[QN_r^2 + \alpha B_1^2\right]}{k_2QB_1N_r} - \frac{\alpha B_1E_3}{k_2QN_r} \tag{A.87}$$

$$C_r > m_1 + \frac{N_rE_3}{(1-\alpha)k_2B_1} \tag{A.88}$$

Furthermore, the constraint Equation (8) should be satisfied, thus:

$$\begin{aligned} C_r &\geq m_1 + \frac{\left[QN_r^2 + \alpha B_1^2\right]\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]}{k_2Q^2N_r^3B_1^2} \\ &\times \left[\frac{B_1E_3}{2} - \frac{\alpha B_1^2E_3}{2\left[QN_r^2 + \alpha B_1^2\right]} - \frac{QN_r^2B_1E_3}{2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]}\right] \end{aligned} \tag{A.89}$$

By incorporating the Equations (A.78) to (A.80) and (15), the optimal total profit $\Pi^* = \pi_4$.

Proof of Theorem 4: In Theorems 1 to 3, the optimal profits in each case refer by using $\pi_t(t = 1, 2 \ldots, 5)$. According to the theorems, π_3 stands for the optimal profit that may be generated by the OEM when producing only new products during phase−II. π_4, π_5 stands for the optimal profit that may be generated by the OEM when producing only reman products during phase−II. Finally, π_1, π_2 stands for the optimal profit that may be generated by the OEM when producing both new and reman products during phase−II.

Comparison between the production cases in phase−II of only manufacturing new products and manufacturing

TABLE 14. Critical values for C_{r_t}.

Critical Values	Expression
C_{r_1}	$m_1 - \frac{\alpha E_3 \left[QN_r + (1-\alpha)B_1\right]}{Qk_2 N_r}$
C_{r_2}	$\frac{\delta_r \sqrt{w} + \delta_{r_1}\sqrt{we}}{k_2} - \frac{\alpha \left[\delta_n \sqrt{w} + \delta_{n_1}\sqrt{we}\right]}{k_2}$
C_{r_3}	$m_1 - \frac{\alpha}{k_2}\left[E_3 + \frac{(1-\alpha)B_1}{QN_r}\right]$
C_{r_4}	$m_1 + \frac{E_3\left[N_r - \alpha B_1\left[\lambda_n w + \lambda_{n_1} we\right]\right]}{k_2 B_1}$
C_{r_5}	$m_1 + \frac{\left[QN_r^2 + \alpha B_1^2\right]\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]}{k_2 Q^2 N_r^3 B_1^2}\left[\frac{B_1 E_3}{2} - \frac{\alpha B_1^2 E_3}{2\left[QN_r^2 + \alpha B_1^2\right]} - \frac{QN_r^2 B_1 E_3}{2\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]}\right]$
Other Critical Values	
C_{r_6}	$m_1 - \frac{E_3}{k_2}$
C_{r_7}	$m_1 + \frac{E_3\left[QN_r^2 + \alpha(1-\alpha)B_1^2\right]}{k_1 QB_1 N_r} - \frac{\alpha E_3\left[QN_r + (1-\alpha)B_1\right]}{k_1 QN_r}$
C_{r_8}	$m_1 - \frac{E_3\left[B_1 + (Q-1)N_r\right]}{k_2 B_1}$
C_{r_9}	$m_1 - \frac{\alpha E_3 B_1}{Qk_2 N_r}$
$C_{r_{10}}$	m_1
$C_{r_{11}}$	$m_1 + \frac{E_3\left[QN_r^2 + \alpha B_1^2\right]}{k_2 QB_1 N_r} - \frac{\alpha B_1 E_3}{k_2 QN_r}$
$C_{r_{12}}$	$m_1 + \frac{N_r E_3}{(1-\alpha)k_2 B_1}$

both reman and new products:

$$\pi_3 - \pi_1 = \frac{(1+Q)E_3^2}{4k_1} - \frac{Q}{4(1-\alpha)} \times \left[\frac{E_1\left[E_3 - E_4\right]}{k_1} + \frac{E_2\left[E_4 - \alpha E_3\right]}{\alpha k_2}\right] - \frac{E_1 E_3}{4k_1} + \frac{C_n}{2}\left[E_3 + \frac{Q(E_3 - E_4)}{(1-\alpha)}\right] + \frac{QC_r\left[E_4 - \alpha E_3\right]}{2\alpha(1-\alpha)} \tag{A.90}$$

Taking into account the possible range of values for each parameter, if and only if $C_r = C_{r_1}$, then $\pi_3 - \pi_1 = 0$. Consequently, when $C_{r_3} < C_r < C_{r_1}$, the OEM ready to produce both reman and new products in phase$-II$.

According to Theorem (3), when $C_n > C_{n_1}$ and $C_{r_4} < C_r < C_{r_2}$ or when $C_n < C_{n_1}$ and $C_{r_5} < C_r < C_{r_2}$, the OEM optimal profit for this scenario is π_3, to acquire

$$\pi_3 \geq \pi_2 > \pi_1 \tag{A.91}$$

For that reason, in this scenario, it is more profitable to produce new products in phase$-II$. Additionally, When $C_n \geq C_{n_1}$ and $C_r < C_{r_5}$ or when $C_n < C_{n_1}$ and $C_r < C_{r_4}$, the OEM encourages producing reman and new products in phase$-II$.

Comparison between the production cases in phase$-II$ of only manufacturing reman products and manufacturing both reman and new products:

Taking into account the possible range of values for each parameter, when $C_{r_3} < C_r < C_{r_1}$:

$$\pi_1 - \pi_4 < 0 \tag{A.92}$$

$$\pi_1 - \pi_5 < 0 \tag{A.93}$$

When $C_n \geq C_{n_1}$ and $C_{r_5} < C_r < C_{r_3}$ or when $C_n < C_{n_1}$ and $C_r < min\left[C_{r_4}, C_{r_5}\right]$:

$$\pi_2 - \pi_4 < 0 \tag{A.94}$$

$$\pi_2 - \pi_5 < 0 \tag{A.95}$$

Thus, when $C_n > C_{n_1}$ and $C_{r_3} < C_r < C_{r_1}$ or when $C_n > C_{n_1}$ and $C_{r_5} < C_r < C_{r_3}$ or when $C_n \leq C_{n_1}$ and $C_r \leq C_{r_1}$, manufacturing both reman and new products in phase$-II$ has been shown to have a adverse impact on the profits of the OEM, so producing reman products in phase$-II$ is a viable choice.

Comparison between the production cases in phase$-II$ of only manufacturing new products and manufacturing only reman products:

Taking into account the possible range of values for each parameter, when $C_r > C_{r_1}$:

$$\pi_3 - \pi_4 > 0 \tag{A.96}$$

$$\pi_3 - \pi_5 > 0 \tag{A.97}$$

In this particular scenario, it is suggested that the OEM should consider focusing solely on the production of new

TABLE 15. Critical values for π_t.

Critical Values	Expression
π_1	$\frac{E_1E_3}{4k_1} + \frac{Q}{4(1-\alpha)}\left[\frac{E_1\,[E_3-E_4]}{k_1} + \frac{E_2\,[E_4-\alpha E_3]}{\alpha k_2}\right] - \frac{C_n}{2}\left[E_3 + \frac{Q(E_3-E_4)}{(1-\alpha)}\right] - \frac{QC_r\,[E_4-\alpha E_3]}{2\alpha(1-\alpha)}$
π_2	$\left[\frac{E_1}{2k_1} + \frac{\alpha E_3\left[(1-\alpha)B_1^2+\theta QN_r\right]}{2k_1\left[QN_r^2+\alpha(1-\alpha)B_1^2\right]} - \frac{\theta QN_r\left[\lambda_n w+\lambda_{n_1} w_e\right]E_4}{2k_1\left[QN_r^2+\alpha(1-\alpha)B_1^2\right]} - C_n\right] \times \left[\frac{E_3}{2} + \frac{QB_1E_4N_r}{2\left[QN_r^2+\alpha(1-\alpha)B_1^2\right]} - \frac{\alpha B_1E_3((1-\alpha)B_1+QN_r)}{2\left[QN_r^2+\alpha(1-\alpha)B_1^2\right]}\right] + \left[\frac{E_1}{2k_1} - C_n\right] \times \left[\frac{\alpha QB_1E_3\,[B_1+QN_r]}{2\left[QN_r^2+\alpha(1-\alpha)B_1^2\right]} - \frac{\alpha QB_1\,[E_3N_r+B_1E_4]}{2\left[QN_r^2+\alpha(1-\alpha)B_1^2\right]}\right] + \left[\frac{E_2}{2k_2} + \frac{QN_rE_4}{2k_2\left[QN_r^r+\alpha(1-\alpha)B_1^2\right]} - \frac{\alpha B_1E_3\,[(1-\alpha)B_1+QN_r]}{2k_2\left[QN_r^r+\alpha(1-\alpha)B_1^2\right]} - C_r\right] \times \left[\frac{QB_1\,[E_3N_r+B_1E_4]}{2\left[QN_r^2+\alpha(1-\alpha)B_1^2\right]} - \frac{\alpha QB_1^2(\lambda_n w+\lambda_{n_1} w_e)E_3}{2\left[QN_r^2+\alpha(1-\alpha)B_1^2\right]}\right]$
π_3	$\frac{(1+Q)E_3^2}{4k_1}$
π_4	$\frac{QN_r\,[N_rE_3+B_1E_4]}{4k_1\left[QN_r^2+\alpha B_1^2\right]^2}\left[\begin{array}{l}[E_1+E_3-2k_1C_n]\,\alpha B_1^2 - QB_1N_rE_4 \\ +\,QN_r^2\,[E_1-2k_1C_n]\end{array}\right] + \frac{QB_1\,[(1-\alpha)B_1E_4+N_rE_3]}{4k_2\left[QN_r^2+\alpha(1-\alpha)B_1^2\right]^2}\left[\begin{array}{l}\alpha(1-\alpha)\,[E_2-2k_2C_r]\,B_1^2 - \alpha B_1N_rE_3 \\ +\,QN_r^2\,[E_2+E_4-2k_2C_r]\end{array}\right]$
π_5	$\frac{E_3}{2}\left[\frac{E_1}{2k_1} - C_n\right] + \frac{QE_4}{2}\left[\frac{E_2}{2k_2} - C_r\right]$

products during phase$-II$ due to its potential to yield greater profitability. This approach is expected to be more viable for the OEM as compared to other alternatives.

Critical Values for Lemmas and Theorems.

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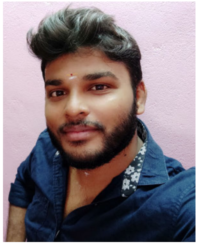

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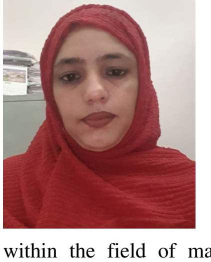

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