IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received 1 May 2024, accepted 30 June 2024, date of publication 4 July 2024, date of current version 7 October 2024.

Digital Object Identifier 10.1109/ACCESS.2024.3423387

RESEARCH ARTICLE

q-Spherical Fuzzy Rough Einstein Geometric Aggregation Operator for Image Understanding and Interpretations

AHMAD BIN AZIM[1], ASAD ALI[1], ABDUL SAMAD KHAN[2], SUMBAL ALI[1], FUAD A. AWWAD[3], AND EMAD A. A. ISMAIL[3]

[1]Department of Mathematics and Statistics, Hazara University, Mansehra, Khyber Pakhtunkhwa 21300, Pakistan
[2]Research Centre for Computational Science, School of Mathematics and Statistics, Northwestern Polytechnical University, Xi'an 710129, China
[3]Department of Quantitative Analysis, College of Business Administration, King Saud University, Riyadh 11587, Saudi Arabia

Corresponding author: Asad Ali (asad_maths@hu.edu.pk)

This work was supported by King Saud University, Riyadh, Saudi Arabia, through the Researchers Supporting Project RSPD2024R1060.

ABSTRACT The combination of q-spherical fuzzy sets and rough sets has emerged as a useful paradigm for fuzzy mathematics and decision-making. The hybrid structure of q-spherical fuzzy sets and rough sets has shown to be useful in the fields of fuzzy mathematics and decision-making. The primary goal of this research is to present Einstein's operational principles for q-spherical rough numbers (q-SFRNs). The fundamental goal of this research is to develop geometric aggregation operators (AOs), such as q-spherical fuzzy rough Einstein weighted geometric (q-SFREWG) and q-spherical fuzzy rough Einstein ordered weighted geometric (q-SFREOWG) operators. We will look at the idempotency, boundedness, and other theorems linked with the suggested AOs. Recognizing the importance of multi-criteria decision-making (MCDM) in dealing with real-world difficulties, it is important to note that traditional MCDM procedures sometimes provide contradicting outcomes. Using the proposed AOs, this study presents a robust MCDM approach designed to address picture understanding and interpretation issues inside the q-SFRS framework. In addition, a complete comparative study is carried out to assess the suggested method's efficacy and value in comparison to existing procedures. The findings from these comparison investigations show that our developed technique outperforms current approaches. The study emphasizes the expanded capabilities of the suggested technique in resolving the complexities of picture perception and interpretation within the q-SFRS environment, bringing a potential addition to the field of decision-making and fuzzy mathematics.

INDEX TERMS q-spherical fuzzy rough sets, Einstein operators, MCDM involving image understanding and interpretations.

I. INTRODUCTION

Zadeh [1] introduced the concept of fuzzy sets, which inspired subsequent extensions such as interval-valued fuzzy sets (IVFSs) [2], Atanassov's intuitionistic fuzzy sets (IFSs) [3], and interval-valued intuitionistic fuzzy sets (IVIFSs) [4] as alternative structures within the fuzzy set framework. However, Zadeh's study [5] and other studies [6], [7] have demonstrated that Atanassov's intuitionistic fuzzy sets (IFS) and IVFSs are mathematically equal. IFS is closely linked to the representation of intuitionistic fuzzy information, necessitating a thorough understanding of both intuitive fuzzy sets and aggregation methods [8]. The concept of IFSs has evolved, allowing non-membership and membership functions to take interval values, leading to the development of interval-valued intuitionistic fuzzy sets (IVIFSs) [9]. IVIFS offers a more complicated representation of uncertainty and imprecision in fuzzy set theory. The invention of intuition-

The associate editor coordinating the review of this manuscript and approving it for publication was Muhammad Sharif.

istic fuzzy numbers (IFNs) was a significant step towards determining decision outcomes. To elaborate, Xu [10] presented other aggregation operators, including IFWA, IFOWA, and IFHA, and detailed their unique properties. Notably, Xu and Yager [11] introduced three new geometric aggregation operators for IFNs: IFWG, IFOGA, and IFHG. The application of these mathematical principles to a wide range of topics and situations is intriguing. Zeng and Su [12] examined the development of an intuitionistic fuzzy ordered weighted distance (IFOWD) operator and its use in group decision-making for system selection. Their finding throws light on this vital location, paving the way for future discoveries. The IFOWD operator shows fascinating applications throughout a wide spectrum. When decision-makers express preference values like $\zeta_A = 0.6$ and $\xi_A = 0.7$, the total of both values exceeds 1. This violates the criterion of an intuitionistic fuzzy set. Yager's Pythagorean fuzzy sets [13] solve these problems by ensuring that $\left((\zeta_A)^2 + (\xi_A)^2\right) \leq 1$. Pythagorean fuzzy sets are recognized to manage uncertainty better than intuitionistic fuzzy sets (IFSs), making Pythagorean fuzzy set theory a more popular and exciting research topic. Yager and Abbasov [14] developed many aggregation approaches to address Multiple Criteria Decision Making (MCDM) issues in the Pythagorean fuzzy framework. The neutrosophic set [15] is a remarkable extension of conventional fuzzy sets that leads to neutrosophic cubic sets [16]. According to the existing literature, major research efforts have been focused on the study of neutrosophic sets (NSs), neutrosophic cubic sets (NCSs), and the accompanying aggregation operators. Alia et al. [17] investigate the notion of NCSs and their use in pattern recognition. Furthermore, Je [18] developed operations and aggregation methods specifically for NCSs. Ajay et al. [19] used geometric Bonferroni mean operators for multicriteria decision-making (MCDM) using neutrosophic cubic sets (NCSs). Coung and Kreinovich [20] introduced the concept of picture fuzzy sets, with the restriction that the sum of all memberships lies inside the interval [0,1]. In more recent work, Atta et al. [21] used the notion of neutrosophic sets (NSs) in an upgraded picture steganography system that is based on modification direction. Gundogdu and Kahraman [22] developed the notion of spherical fuzzy sets (SFS) and its associated theory, describing it as a unique extension of fuzzy set theory. This paradigm is differentiated by its triple membership structure, which contains membership, non-membership, and hesitation functions. They look at the positive, neutral, and negative membership functions with total squares equal to or less than one. In dealing with uncertainty, imprecision, and vagueness, the SFS model surpasses Pythagorean fuzzy sets. A thorough examination of the most recent literature demonstrates a growing preference for investigations into SFS. Ashraf and Abdullah [23] developed a set of aggregation strategies, particularly for a spherical fuzzy framework. Ashraf et al. [24] took a unique approach by providing a grey technique (GRA) based on the groundbreaking notion of spherical linguistic fuzzy Choquet integrals. Furthermore, Jin et al. [25] developed and utilized logarithmic operators designed for spherical fuzzy sets (SFSs) in decision support systems. Rafiq et al. [26] introduced a cosine similarity measure tailored particularly to the SFS model, intending to improve decision-making in scenarios including ambiguous and imprecise data. Furthermore, Ashraf et al. [27] proposed a group decision-making technique customized for the spherical fuzzy environment and used it to solve challenges in multi-criteria group decision-making (MCGM). Gundogdu et al. [28] updated the well-known VIKOR technique to include the spherical fuzzy set (SFS) model and used it in a Multi-Criteria Decision Making (MCDM) context within a spherical fuzzy setting. Acharjya and Rathi [29] proposed an integrated decision-making approach that combines fuzzy rough sets and genetic algorithm models. They evaluated its effectiveness in a relevant MCDM scenario about smart agriculture. Sharaff et al. [30], [31] investigated a fuzzy-based technique for text summarization extraction and proposed a document categorization strategy based on a fuzzy clustering algorithm. Gou et al. [32] developed exponential operating rules for interval fuzzy sets (IFSs) and innovative aggregation operators for the IFS framework. Gou et al. [32] created new exponential operating laws for the Pythagorean Fuzzy Set (PFS) model, as well as aggregation operators based on these laws, to improve the management of uncertainty, imprecision, and ambiguity in data. Furthermore, Borg et al. [33] developed decision-making projection models based on PFS exponential operational laws, whereas Haque et al. [34] expanded the notion of SFS by integrating exponential operational laws. Akram et al. [35] investigated spherical fuzzy graphs and presented results on symmetric difference, rejection, degree, and total degrees. In a different setting, Ashraf et al. [36] developed a unique integrated strategy by combining established Multi-Criteria Decision Making (MCDM) techniques such as the Strategy for Order of Preference by Similarity to Ideal Solution (TOPSIS) and Complex Proportional Assessment of Alternatives (COPRAS). Using this integrated method, they addressed a Multi-Criteria Group Decision-Making (MCGDM) difficulty in emergency response during the COVID-19 pandemic. Quek et al. [37] proposed fresh operational concepts for the T-SFS model, as well as two versions of Einstein aggregation operators tailored particularly to these models. They then used these operators to solve a multi-attribute, multi-perception decision-making issue with pollution levels in five major Chinese cities. Aydogdu and Gul [38] presented a new entropy measure designed particularly for the SFS model and conducted a performance comparison with current measures in the literature. Shishavan et al. [39] developed Jaccard, exponential, and square root cosine similarity metrics in a spherical fuzzy environment and used them to solve Multi-Criteria Decision Making (MCDM) problems in medical diagnostics and supplier selection. Ali et al. [40] paved the road for complicated T-SFSs, defining associated operational concepts and offering two novel aggregation operators designed specifically for this architecture. Similarly, Garg et al. [41] proposed power aggregation operators for the T-SFS model and developed an MCDM approach using these operators. In a similar context, Liu et al. [42] developed the notion of linguistic T-spherical fuzzy numbers, as well as a weighted aggregation operator and two unique Multi-Criteria Decision Making (MCDM) algorithms designed for this concept. Furthermore, Guleria and Bajaj [43] described the T-spherical fuzzy soft set model and its associated aggregation methods. Sharaf and Khalil [44] applied the widely used MCDM technique known as Tomada de Decisao Interactive e Multicriterio (TODIM) to the spherical fuzzy environment by adding SFS-based models into Multi-Criteria Decision Making (MCDM) procedures. This strategy allows decision-makers to communicate their levels of reluctance. Meanwhile, Mathew et al. [45] suggested a novel decision-making framework that combines the well-known Analytic Hierarchy Process (AHP) and the Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) techniques in a spherical fuzzy environment. Gundogdu and Kahraman [45] proposed the idea of Interval Valued Spherical Fuzzy Sets (IV-SFS) and provided key supporting concepts for this model, such as score and accuracy functions, arithmetic, and geometric mean operators. Sharaf and Khalil [44] used the widely used MCDM technique, Tomada de Decisao Interactive e Multicriteria (TODIM), to a spherical fuzzy environment by combining SFS-based models into Multi-Criteria Decision Making (MCDM) procedures. This method allows decision-makers to communicate their levels of reluctance. Meanwhile, Mathew et al. [45] developed a novel decision-making framework that combines the well-known Analytic Hierarchy Process (AHP) and Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) algorithms in a spherical fuzzy environment. Gundogdu and Kahraman [46] developed the notion of Interval Valued Spherical Fuzzy Sets (IV-SFS) and defined key model support concepts including score and accuracy functions, as well as arithmetic and geometric mean operators. They also created a Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) based on IV-SFS and used it in a Multi-Criteria Decision Making (MCDM) scenario involving the choosing of 3D printers. Barukab et al. [47] developed an updated TOPSIS-based solution for the SFS model that was specially designed to solve issues in Multi-Criteria Group Decision-Making (MCGDM). They developed a generalized distance measure for SFSs based on spherical fuzzy entropy, with undetermined weights for the criterion. In a similar vein, Farrokhizadeh et al. [48] used the original maximum deviation technique to determine criterion weights in the spherical fuzzy environment, using both single-valued and interval-valued SFS. Akram et al. [49] suggested four separate aggregation operators for the intricate SFS model and used them to apply the multi-criteria optimization and compromise solution (VIKOR) approach to the complex spherical fuzzy environment. Simultaneously, Ali et al. [50] created a TOPSIS technique based on the complicated SFS model, as well as two Bonferroni mean aggregation processes that were adapted to this model. In their efforts to remove uncertainty, Kahraman and his research team [51] proposed the innovative concept of a q-spherical fuzzy set (q-SFS). This novel concept has been tremendously effective in encouraging informed decision-making, and it is frequently viewed as an extension of the standard "q-spherical fuzzy set." Each aspect in the q-SFS framework is classified as either positive, neutral, or negative. It is important to follow the requirement $0 \leq (\zeta_{\mathcal{A}})^q + (\eta_A)^q + (\xi_A)^q \leq 1$. The q-spherical fuzzy set has a wide range of responses, including positive, negative, doubtful, and even abstention from replying. This condition ensures that the total q powers of ζ_A, η_A and ξ_A are not more than one. The q-SFS with the parameter q provides decision-makers with a greater variety of options, allowing them to specify their preferences for membership, non-membership, and the level of ambiguity.

The concept of rough sets (RS) was first introduced by Pawlak [52], [53] as a means of dealing with uncertainty. When examined from a mathematical perspective, this configuration demonstrates attributes that could be construed as vagueness and indeterminacy. Rough set theory (RST) is a modification of the traditional set theory, that uses the notion of connection to elucidate the operations of information systems. Researchers have acknowledged that the applicability of the equivalence relation in Pawlak's relational semantic theory is subject to notable constraints in a range of real-world situations, a point emphasized by multiple scholars. Consequently, several researchers have expanded upon Pawlak's rough set theory [54], [55]. IFSs, PyFSs, and q-ROFSs, while serving different purposes, are all confined to binary choices, such as positive or negative choices. Nevertheless, it is essential to recognize that individuals' viewpoints are never as straightforward as a binary affirmation or negation. Consider exploring the process of voting as an illustrative instance. There exist four distinct potential outcomes: casting a vote in favor (voting yes), casting a vote against (voting no), refraining from casting a vote (abstaining from voting), or not participating in the voting process altogether. The specific incident in question lacks a valid explanation within the existing recognized framework. It's worth mentioning that both the PFS and the SFS can be employed to address such issues, although they come with their inherent limitations. The utilization of the q-spherical fuzzy set has been identified as the optimal approach for resolving this problem. Additionally, it's crucial to ensure that the existing theories adeptly handle any possible concerns or challenges within their respective frameworks or contexts, as exemplified below. The extent of plagiarism is considerably high. Swift action is imperative to entirely rectify this matter. The theories of IFRS [56], PyFRS [57], and q-ROFRS [58], [59] are well-established in the field and have been acknowledged for their contributions.

However, it is essential to acknowledge that these theories encounter substantial limitations when striving to encompass all three potential grades within a given dataset: positive, neutral, and negative grades. The examination of voting can be approached through the utilization of a theoretical framework termed a picture fuzzy rough set (PFRS). Nevertheless, the existence of lower and upper approximations, represented as $\left(\underline{\zeta_A}+\underline{\eta_A}+\underline{\xi_A}\right) \in [0, 1]$ and $(\overline{\zeta_A}+\overline{\eta_A}+\overline{\xi_A}) \in [0, 1]$, imposes certain constraints. The level of plagiarism is excessively high. Immediate and decisive measures are necessary to completely address this issue. Nevertheless, it is crucial to acknowledge that when decision-makers are presented with information in the format of SFRS, involving both the lower and upper approximations like {(07,0.8,0.9), (0.9,0.8,0.7)} and so on, it is important to acknowledge that the combine values of the lower and upper approximations exceed the interval [0, 1]. This suggests that the values $(0 \not\leq 0.7^2 + 0.8^2 + 0.9^2 \not\leq 1)$ and $(0 \not\leq 0.9^2 + 0.8^2 + 0.7^2 \not\leq 1)$ are inappropriate with the SFRS framework, hence, this imparts a limitation on the extent to which the SFRS concept can be effectively applied. To remove this difficulty Azim et al. [60] defined the notions of q-SFRS in their research paper published in 2023. Now with information in the format of q-SFRS, involving both the lower and upper approximations like (07,0.8,0.9), (0.9,0.8,0.7) and so on, it is important to acknowledge that the combine values of the lower and upper approximations do not exceed the interval $[\mathbf{0}, \mathbf{1}]$. This fuzzy set combines the advantages inherent in both the RS and the q-SFS. This research introduces a practical approach to decision-making within the framework of q-spherical fuzzy rough sets, thereby expanding the existing knowledge in this field. Within q-SFRS, three distinct parameters involve lower and upper approximations. Our main objective in this study is to advance future research by devising novel aggregation operators alongside defuzzification methods. After a comprehensive analysis, it becomes clear that the concept of q-SFRSs holds substantial potential as an innovative idea, thereby paving the way for numerous opportunities in future research endeavors.

A. LITERATURE REVIEW

Image understanding and interpretation play pivotal roles in the realm of computer vision, finding applications across diverse domains such as medical diagnosis and autonomous driving. As this field continues to evolve, researchers have explored various approaches aimed at enhancing robots' ability to comprehend visual information. This literature review provides a comprehensive overview of key advancements, methodologies, and challenges encountered in image understanding and interpretation. Initially, early efforts in image understanding primarily relied on conventional computer vision algorithms. These methods, including handmade feature extraction, template matching, and rule-based systems, demonstrated effectiveness in certain contexts. However, their utility was limited when confronted with the intricate complexity and variability inherent in real-world images. The advent of deep learning marked a significant paradigm shift in image understanding. Convolutional Neural Networks (CNNs) emerged as the predominant framework, delivering remarkable achievements in tasks such as image classification, object recognition, and segmentation. Foundational architectures like AlexNet, VGG, and ResNet laid the groundwork for subsequent advancements in deep learning-based image interpretation. Fuzzy logic has emerged as a valuable tool for addressing uncertainties and ambiguities present in image data. Fuzzy sets and fuzzy rule-based systems have been instrumental in representing imprecise relationships within images. The application of fuzzy logic in image interpretation has shown promise in tackling complex scenarios where conventional methods fall short. Furthermore, rough set theory has been leveraged to handle uncertainty and granularity in image interpretation. By encoding image attributes as rough sets, researchers have achieved enhanced robustness in tasks such as image segmentation and pattern recognition. The flexibility of rough sets in accommodating imprecision aligns well with the inherent uncertainties inherent in visual data. Recent research has focused on integrating fuzzy logic with rough set theory, giving rise to fuzzy rough sets. This hybrid approach aims to leverage the strengths of both paradigms, resulting in a more comprehensive framework for picture analysis. Fuzzy rough sets offer an adaptable representation of uncertainty, enabling detailed analysis of image data. To address the limitations of traditional fuzzy rough sets, q-spherical fuzzy rough sets have emerged as a promising development. This methodology introduces a spherical fuzzy set framework, enhancing the capability to describe and reason about uncertainty in image data. q-spherical fuzzy rough sets serve as sophisticated tools for picture interpretation, particularly in challenging scenarios. The integration of multi-criteria decision-making techniques into image interpretation has gained traction, allowing for comprehensive assessments and comparisons of alternative approaches. Criteria such as accuracy, efficiency, robustness, and generalization play crucial roles in evaluating image interpretation algorithms. Understanding the strengths and limitations of these methodologies is essential for driving future advancements in image understanding and interpretation within the rapidly evolving field of computer vision.

B. GAP BEFORE ESTABLISHING OUR PROPOSED OPERATORS

The preceding gap in the domain of q-SFRsets may be briefly expressed as follows:

Traditional fuzzy sets helped control uncertainty, but they struggled to absorb complicated information and effectively express decision-makers' preferences. This constraint created a gap in decision-making processes, especially when dealing with complex and unexpected data that required more efficient treatment. The introduction of q-SFRsets addressed this gap by improving the ability of conventional fuzzy sets to handle complicated information. However, a critical necessity remained for sophisticated aggregation operators capable of negotiating the intricacies of q-SFRsets while accurately conveying decision-makers' preferences and uncertainties. Einstein's operations and operators, such as q-SFREWG and q-SFREOG, were designed to fill the current gap. These operators combine the advantages of q-SFR sets with Einstein aggregation techniques, allowing for more accurate and adaptive decision-making in complex and unpredictable circumstances. They provide powerful and comprehensive ways of gathering information and accurately reflecting decision-makers' preferences. In essence, before the advent of Einstein's operations and operators, there was a lack of sophisticated aggregation approaches capable of processing complicated information and properly representing decision-makers' preferences. Einstein's operators effectively bridge this gap by incorporating powerful and thorough aggregation methods into decision-making procedures.

C. MOTIVATION FOR THE PROPOSED OPERATORS

In the field of fuzzy mathematics and decision-making, the combination of orthopair q-spherical fuzzy sets and rough sets has emerged as a viable approach. The combination of these two paradigms provides a distinct viewpoint that has proven useful in resolving the complexity inherent in real-world decision-making circumstances. Significant advances in fuzzy mathematics have been made as a result of the use of this hybrid structure, notably in the contexts of uncertainty modeling and decision support systems. The motivation for this study derives from the practical applicability of q-spherical fuzzy sets in conjunction with rough sets. As academics continue to investigate novel techniques to improve the robustness and flexibility of decision-making processes, the hybrid framework under consideration provides a convenient scenario in which theoretical advances may be turned into practical solutions. Einstein's operational rules have been extensively recognized for their effectiveness in a variety of mathematical contexts. Extending these rules to q-spherical rough numbers (q-SFRNs) offers an attractive opportunity to expand the theoretical underpinnings of fuzzy mathematics. This study seeks to contribute to this evolution by proposing and investigating geometric aggregation operators (AOs) designed for q-spherical fuzzy rough sets, specifically q-spherical fuzzy rough Einstein weighted geometric (q-SFREWG) and q-spherical fuzzy rough Einstein ordered weighted geometric (q-SFREOWG) operators. The primary motivation for proposing these operators is their ability to handle basic issues in multi-criteria decision-making (MCDM). While MCDM approaches are essential for dealing with real-world difficulties, current methods frequently yield inconsistent results. The suggested method tries to provide a robust MCDM technique inside the q-SFRS framework, with a focus on image understanding and interpretation situations. Ultimately, the objective of this study is to connect theoretical advances in fuzzy mathematics with actual decision-making issues. This work aims to significantly improve image understanding and interpretation methodologies by developing and validating innovative aggregation operators and decision-making techniques, with potential applications in a variety of fields characterized by uncertainty and decision complexity.

The primary contributions of our research can be summarized as follows:

This study proposes two innovative geometric aggregation operators specifically tailored for q-spherical fuzzy rough sets, namely q-spherical fuzzy rough Einstein weighted geometric (q-SFREWG) and q-spherical fuzzy rough Einstein ordered weighted geometric (q-SFREOWG) operators. These operators extend Einstein's operational principles to q-SFRNs, offering robust and adaptive aggregation methods for multi-criteria decision-making (MCDM) tasks.

This research emphasizes the practical applicability of q-SFRNs and AOs in resolving complexities in image understanding and interpretation, particularly within the q-SFRS framework. By integrating these novel techniques into MCDM processes, we aim to provide a more reliable and consistent approach to decision-making, addressing the limitations of existing methodologies.

This research article conducts a thorough comparative analysis to evaluate the efficacy and value of our proposed method against existing procedures in the literature. Our findings demonstrate that the developed technique outperforms current approaches, highlighting its enhanced capabilities in addressing real-world challenges.

By extending Einstein's operational rules to q-SFRNs and developing tailored aggregation operators, our study contributes to the theoretical advancements in fuzzy mathematics and decision-making. This novel integration expands the theoretical underpinnings of fuzzy mathematics, paving the way for practical solutions in various domains characterized by uncertainty and decision complexity.

This research presents a significant advancement in the field of fuzzy mathematics and decision-making by introducing novel q-SFRNs and geometric aggregation operators tailored for image understanding and interpretation tasks. We believe that our contributions fill a critical gap in the existing literature and offer valuable insights for both theoretical advancements and practical applications.

The comprehensive structure of the paper is shown in Figure 1.

II. PRELIMINARIES

In this section, we will look at a variety of mathematical ideas, beginning with an in-depth review of FS, IFS, PFS, SPS, q-SFS, and RS.

Definition 1: In 1965, Zadeh [1] proposed the idea of a fuzzy set as an extension of the conventional crisp set. The formal definition of a fuzzy set can be represented mathematically as follows:

$$\mathcal{A} = \{\langle \mathfrak{x}, \zeta_{\mathcal{A}}(\mathfrak{x})\rangle : \mathfrak{x} \in \mathcal{X}\} \tag{1}$$

where $0 \leq \zeta_{\mathcal{A}}(\mathfrak{x}) \leq 1$.

Definition 2: In 1986, Atanassov [3] proposed the intuitionistic fuzzy set (IFS) as an extension of the fuzzy set. The

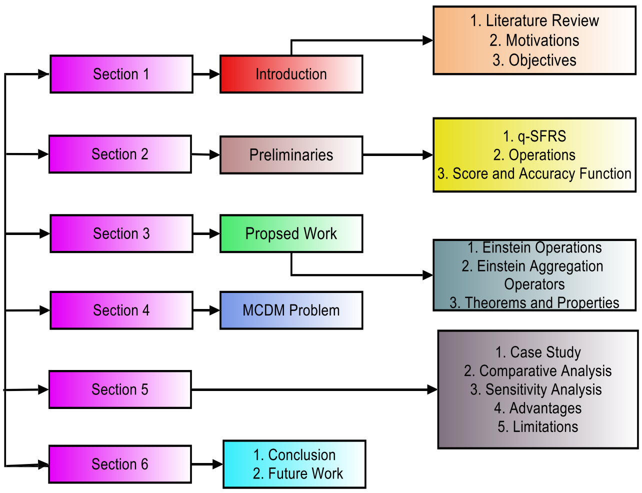

FIGURE 1. Structure of the research article.

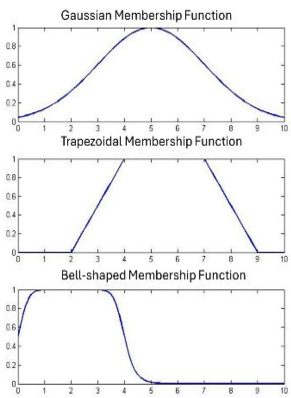

FIGURE 2. Some graphical representations of fuzzy spaces.

formal mathematical representation of an IFS is as follows:

$$\mathcal{A} = \{\langle \mathfrak{x}, \zeta_{\mathcal{A}}(\mathfrak{x}), \xi_{\mathcal{A}}(\mathfrak{x})\rangle : \mathfrak{x} \in \mathcal{X}\} \tag{2}$$

where $0 \leq \zeta_{\mathcal{A}}(\mathfrak{x}) + \xi_{\mathcal{A}}(\mathfrak{x}) \leq 1$.

Definition 3: [13] Let $\mathcal{X}$ be a non-empty finite set. A PyFS $\mathcal{A}$ over $\mathfrak{x} \in \mathcal{X}$ is defined as follows:

$$\mathcal{A} = \{\langle \mathfrak{x}, \zeta_{\mathcal{A}}(\mathfrak{x}), \xi_{\mathcal{A}}(\mathfrak{x})\rangle : \mathfrak{x} \in \mathcal{X}\} \tag{3}$$

where $\zeta_{\mathcal{A}}(\mathfrak{x})$ and $\xi_{\mathcal{A}}(\mathfrak{x})$ represent the MD and NMD of $\mathcal{A}$ respectively such that $\xi_{\mathcal{A}}(\mathfrak{x}), \eta_{\mathcal{A}}(\mathfrak{x}) \in [0, 1]$ and where $0 \leq (\zeta_{\mathcal{A}}(\mathfrak{x}))^2 + (\xi_{\mathcal{A}})^2 \leq 1$.

Definition 4: Building on the fundamental principles of FSs and IFSs, Cuong and his team [20] introduced the idea of a picture fuzzy set in 2014. Its definition can be expressed mathematically as follows:

$$\mathcal{A} = \{\langle \mathfrak{x}, \zeta_{\mathcal{A}}(\mathfrak{x}), \eta_{\mathcal{A}}(\mathfrak{x}), \xi_{\mathcal{A}}(\mathfrak{x})\rangle : \mathfrak{x} \in \mathcal{X}\} \tag{4}$$

where $0 \leq \zeta_{\mathcal{A}}(\mathfrak{x}) + \eta_{\mathcal{A}}(\mathfrak{x}) + \xi_{\mathcal{A}}(\mathfrak{x}) \leq 1$.

The following symbols represent the representation of the membership functions for a fuzzy set in this situation, which includes positive, neutral, and negative aspects: $\zeta_{\mathcal{A}}(\mathfrak{x})(\mathfrak{x})$: $X \rightarrow [0,1]$, $\eta_{\mathcal{A}}(\mathfrak{x})$: $\mathcal{X} \rightarrow [0,1]$ and $\xi_{\mathcal{A}}(\mathfrak{x})$: $\mathcal{X} \rightarrow [0,1]$ respectively.

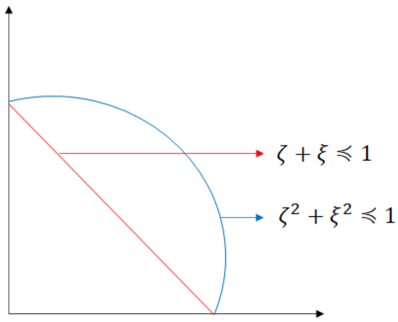

FIGURE 3. A comparison of the differences between Pythagorean and intuitionistic fuzzy spaces.

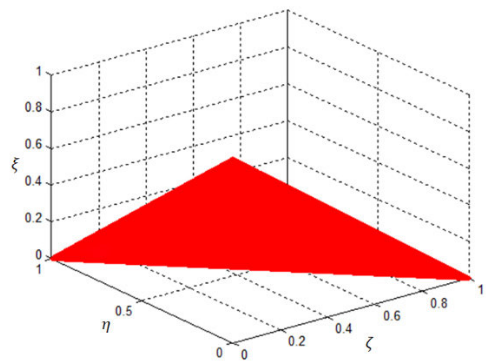

FIGURE 4. Picture membership grade space.

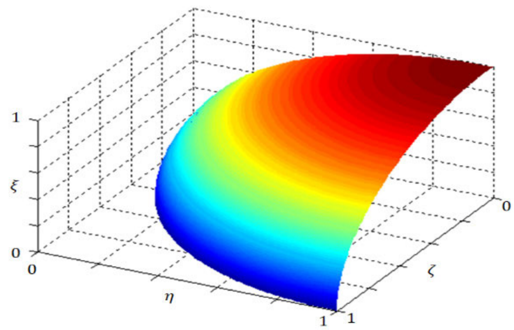

FIGURE 5. The condition $0 \leq (\zeta_{\mathcal{A}}(\mathfrak{x}))^2 + (\eta_{\mathcal{A}}(\mathfrak{x}))^2 + (\xi_{\mathcal{A}}(\mathfrak{x}))^2 \leq 1$ describes a spherical fuzzy set-in three-dimensional space.

Definition 5: Gündoğdu et al. [22] introduced the idea of a spherical fuzzy set in 2019, further advancing the picture fuzzy set framework. The concept can be expressed in the following way from a mathematical standpoint:

$$\mathcal{A} = \{\langle \mathfrak{x}, \zeta_{\mathcal{A}}(\mathfrak{x})(\mathfrak{x}), \eta_{\mathcal{A}}(\mathfrak{x}), \xi_{\mathcal{A}}(\mathfrak{x})\rangle : \mathfrak{x} \in \mathcal{X}\} \tag{5}$$

where $0 \leq (\zeta_{\mathcal{A}}(\mathfrak{x}))^2 + (\eta_{\mathcal{A}}(\mathfrak{x}))^2 + (\xi_{\mathcal{A}}(\mathfrak{x}))^2 \leq 1$.

Where the positive, neutral, and negative membership function for a fuzzy set is represented by $\zeta_{\mathcal{A}}(\mathfrak{x})$: $\mathcal{X} \rightarrow [0,1]$, $\eta_{\mathcal{A}}(\mathfrak{x})$: $\mathcal{X} \rightarrow [0,1]$ and $\xi_{\mathcal{A}}(\mathfrak{x})$: $\mathcal{X} \rightarrow [0,1]$ respectively.

Definition 6: The idea of a q-SFS was introduced by Kahraman et al. [51] in the year 2020, as an extension of the existing notion of a spherical fuzzy set. Mathematically, the

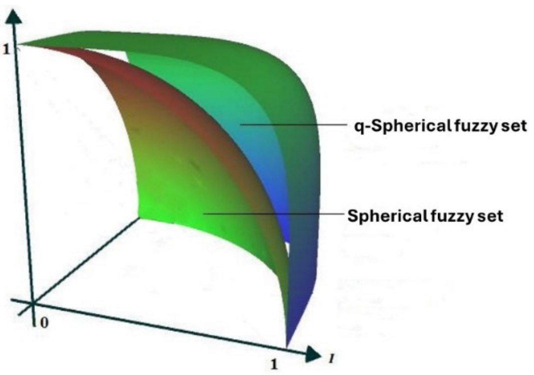

FIGURE 6. Graphical representation between spherical fuzzy set and q-spherical fuzzy set in three-dimensional space.

concept may be formally defined in the following manner.

$$\mathcal{A}=\{\langle\mathfrak{x},\zeta_{\mathcal{A}}(\mathfrak{x})(\mathfrak{x}),\eta_{\mathcal{A}}(\mathfrak{x}),\xi_{\mathcal{A}}(\mathfrak{x})\rangle:\mathfrak{x}\in\mathcal{X}\} \tag{6}$$

such that $0\leq(\zeta_{\mathcal{A}}(\mathfrak{x}))^q+(\eta_{\mathcal{A}}(\mathfrak{x}))^q+(\xi_{\mathcal{A}}(\mathfrak{x}))^q\leq 1$ for all $q\geq 1$. Where $\zeta_{\mathcal{A}}$:$\mathcal{X}\to$ [0,1], $\eta_{\mathcal{A}}$: $\mathcal{X}\to$ [0,1] and $\xi_{\mathcal{A}}$: $\mathcal{X}\to$ [0,1] correspond to the positive, neutral, and negative membership functions, respectively.

Definition 7: Pawlak [52] introduced the notion of RS in back 1982. The definition of rough set is as follows: The triplet $(\mathcal{G}_1,\mathcal{G}_2,\ \mathfrak{R})$ is referred to as an approximation space when considering an arbitrary binary relation $\mathfrak{R}$ on $\mathcal{G}_1\times\mathcal{G}_2$. The $\underline{\mathfrak{R}}(\mathcal{A})$ and $\overline{\mathfrak{R}}(\mathcal{A})$ are defined for sets $\mathcal{X}\subseteq\mathcal{G}_1$ and $\boldsymbol{\mathcal{A}}\subseteq\mathcal{G}_2$.

$$\begin{pmatrix}\underline{\mathfrak{R}}(\mathcal{A})=\{\mathfrak{x}\in\mathcal{G}_1:[\mathfrak{x}]_{\mathcal{A}}\subseteq\mathcal{G}\}\\ \overline{\mathfrak{R}}(\mathcal{A})=\left\{\mathfrak{x}\in\mathcal{G}_1:[\mathfrak{x}]_{\mathcal{A}}\bigcap\mathcal{G}\neq\phi\right\}\end{pmatrix} \tag{7}$$

where $[\mathfrak{x}]_{\mathcal{A}}$ represents the idea of indiscernibility.

The set $\left(\underline{\mathfrak{R}}(\mathcal{A}),\overline{\mathfrak{R}}(\mathcal{A})\right)$ is sometimes referred to as a rough set.

Definition 8: [60] A q-spherical fuzzy relation $\mathfrak{R}$ in is a q-spherical fuzzy subset of $\mathcal{G}_1\times\mathcal{G}_2$. and is given by

$$\begin{aligned}\mathfrak{R}&=\Big\{\langle(\mathfrak{w},x):\zeta_{\mathfrak{R}}(\mathfrak{w},x),\eta_{\mathfrak{R}}(\mathfrak{w},x),\xi_{\mathfrak{R}}(r,s)\rangle:\big((\zeta_{\mathfrak{R}}(\mathfrak{w},x))^q\\&\quad+(\eta_{\mathfrak{R}}(\mathfrak{w},x))^q+(\xi_{\mathfrak{R}}(\mathfrak{w},x))^q\big)\leq 1:\forall\,\mathfrak{w}\in\mathcal{G}_1,x\in\mathcal{G}_2\Big\},\end{aligned}$$

where $\zeta_R:\mathcal{X}\to[0,1]$, $\eta_R:\mathcal{X}\to[0,1]$ *and* $\xi_R:\mathcal{X}\to[0,1]$.

Definition 9: Azim et al. [60] introduced the concept of a q-spherical fuzzy rough set, which is defined as:

For a universal set $\mathcal{G}_1$ and $\mathcal{G}_2$ is a set of attributes. Let $\mathfrak{R}$ be a q-SF relation from $\mathcal{G}_1$to $\mathcal{G}_2$. Then the triplet $(\mathcal{G}_1,\mathcal{G}_2,\ \mathfrak{R})$ is called q-SF approximation space. Now for any element $\mathfrak{w}\in$ q $-$ SFRS, the lower and upper approximation space of $\mathfrak{w}$ w.r.t approximation space $(\mathcal{G}_1,\mathcal{G}_2,\ \mathfrak{R})$ are presented and given as:

$$\mathcal{A}=(\underline{\mathcal{A}},\overline{\mathcal{A}})=\left\{\mathfrak{w},\begin{pmatrix}\underline{\zeta}_{\mathcal{A}}(\mathfrak{w}),\underline{\zeta}_{\mathcal{A}}(\mathfrak{w}),\underline{\zeta}_{\mathcal{A}}(\mathfrak{w}),\\ \overline{\zeta}_{\mathcal{A}}(\mathfrak{w}),\overline{\zeta}_{\mathcal{A}}(\mathfrak{w}),\overline{\zeta}_{\mathcal{A}}(\mathfrak{w})\end{pmatrix}:\mathfrak{w}\in\mathcal{G}_1\right\} \tag{8}$$

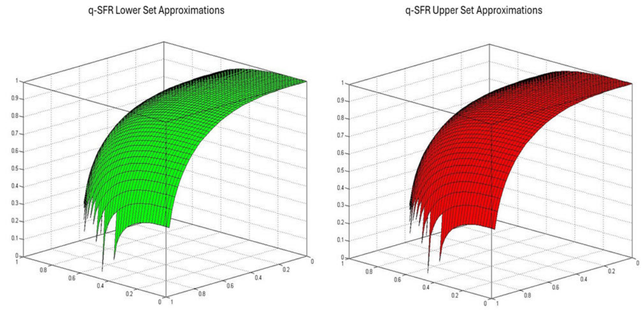

FIGURE 7. Graphical representation of q-spherical fuzzy rough set-in three-dimensional space.

where,

$$\begin{aligned}\underline{\zeta}_{\mathcal{A}}(\mathfrak{w})&=\bigwedge\nolimits_{x\in\mathcal{G}_2}\left\{\zeta_R(\mathfrak{w},x)\bigwedge\zeta_A(x)\right\},\\ \underline{\eta}_{\mathcal{A}}(\mathfrak{w})&=\bigvee\nolimits_{x\in\mathcal{G}_2}\left\{\eta_R(\mathfrak{w},x)\bigvee\eta_A(x)\right\},\\ \underline{\xi}_{\mathcal{A}}(\mathfrak{w})&=\bigvee\nolimits_{x\in\mathcal{G}_2}\left\{\xi_R(\mathfrak{w},x)\bigvee\xi_A(x)\right\},\\ \overline{\zeta}_{\mathcal{A}}(\mathfrak{w})&=\bigvee\nolimits_{x\in\mathcal{G}_2}\left\{\zeta_R(\mathfrak{w},x)\bigvee\zeta_A(x)\right\},\\ \overline{\eta}_{\mathcal{A}}(\mathfrak{w})&=\bigwedge\nolimits_{x\in U_2}\left\{\eta_R(\mathfrak{w},x)\bigwedge\eta_A(x)\right\},\\ \overline{\xi}_{\mathcal{A}}(\mathfrak{w})&=\bigwedge\nolimits_{x\in\mathcal{G}_2}\left\{\xi_R(\mathfrak{w},x)\bigwedge\xi_A(x)\right\},\end{aligned}$$

with the condition that $(0\leq\underline{\zeta}_{\mathcal{A}}{}^q(\mathfrak{w})+\underline{\eta}_{\mathcal{A}}{}^q(\mathfrak{w})+\underline{\xi}_{\mathcal{A}}{}^q(\mathfrak{w})\leq 1))$ and $\left(0\leq\overline{\zeta}_{\mathcal{A}}{}^q(\mathfrak{w})+\overline{\eta}_{\mathcal{A}}{}^q(\mathfrak{w})+\overline{\xi}_{\mathcal{A}}{}^q(\mathfrak{w})\leq 1\right)$.

The q-SFRS is defined as a pair of q-SFSs, where $\underline{\mathcal{A}}$ is distinct from $\overline{\mathcal{A}}$. To facilitate comprehension, we denote the given concept as $\mathcal{A}=(\underline{\mathcal{A}},\overline{\mathcal{A}})$, which is referred to as a q-spherical fuzzy rough number. The notation $\mathcal{A}_i$ represents the set that encompasses all q-SFR numbers.

Definition 10: The q-spherical fuzzy rough number $\mathcal{A}=(\underline{\mathcal{A}},\overline{\mathcal{A}})$ consists of two components:

Lower Approximation Space $\underline{\mathcal{A}}$: Represents the lower approximation space of $\mathfrak{w}$ in the q-spherical fuzzy rough set. It captures the lower bound of uncertainty associated with the element $\mathfrak{w}$.

Upper Approximation Space $\overline{\mathcal{A}}$: Denotes the upper approximation space of $\mathfrak{w}$ in the q-spherical fuzzy rough set. It captures the upper bound of uncertainty associated with the element $\mathfrak{w}$.

Definition 11: [60] Let $\mathcal{A}_1=(\underline{\zeta}_1,\underline{\eta}_1,\underline{\xi}_1,\overline{\zeta}_1,\overline{\eta}_1,\overline{\xi}_1)$, $\mathcal{A}_2=(\underline{\zeta}_2,\underline{\eta}_2,\underline{\xi}_2,\overline{\zeta}_2,\overline{\eta}_2,\overline{\xi}_2)$ and $\mathcal{A}=(\underline{\zeta},\underline{\eta},\underline{\xi},\overline{\zeta},\overline{\eta},\overline{\xi})$ be any three $q-$SFRNs, and $\omega>0$, then,

1. $\mathcal{A}_1\oplus\mathcal{A}_2$
$$=\left\langle\begin{matrix}\sqrt[q]{\underline{\zeta}_1^q+\underline{\zeta}_2^q-\underline{\zeta}_1^q\underline{\zeta}_2^q},\underline{\eta}_1^q\underline{\eta}_2^q, & \sqrt[q]{\left(1-\underline{\zeta}_2^q\underline{\xi}_1^q+1-\underline{\zeta}_1^q\underline{\xi}_2^q\right)-\underline{\xi}_1^q\underline{\xi}_2^q},\\ \sqrt[q]{\overline{\zeta}_1^q+\overline{\zeta}_2^q-\overline{\zeta}_1^q\overline{\zeta}_2^q},\overline{\eta}_1^q\overline{\eta}_2^q, & \sqrt[q]{\left(1-\overline{\zeta}_2^q\overline{\xi}_1^q+1-\overline{\zeta}_1^q\overline{\xi}_2^q\right)-\overline{\xi}_1^q\overline{\xi}_2^q}\end{matrix}\right\rangle,$$
2. $\mathcal{A}_1\otimes\mathcal{A}_2$
$$=\left\langle\begin{matrix}\underline{\zeta}_1^q\underline{\zeta}_2^q, & \sqrt[q]{\underline{\eta}_1^q+\underline{\eta}_2^q-\underline{\eta}_1^q\underline{\eta}_2^q}, & \sqrt[q]{\left(1-\underline{\eta}_2^q\underline{\xi}_1^q+1-\underline{\eta}_1^q\underline{\xi}_2^q\right)-\underline{\xi}_1^q\underline{\xi}_2^q},\\ \overline{\zeta}_1^q\overline{\zeta}_2^q, & \sqrt[q]{\overline{\eta}_1^q+\overline{\eta}_2^q-\overline{\eta}_1^q\overline{\eta}_2^q}, & \sqrt[q]{\left(1-\overline{\eta}_2^q\overline{\xi}_1^q+1-\overline{\eta}_1^q\overline{\xi}_2^q\right)-\overline{\xi}_1^q\overline{\xi}_2^q}\end{matrix}\right\rangle$$

3. $\mathcal{A}^{\omega}$
$$= \left\langle \begin{array}{l} \underline{\zeta}^{\omega}, \sqrt[q]{1-(1-\underline{\eta}^{q})^{\omega}}, \sqrt[q]{(1-\underline{\eta}^{q})^{\omega}-\left(1-\underline{\eta}^{q}-\underline{\xi}^{q}\right)^{\omega}}, \\ \overline{\zeta}^{\omega}, \sqrt[q]{1-(1-\overline{\eta}^{q})^{\omega}}, \sqrt[q]{(1-\overline{\eta}^{q})^{\omega}-\left(1-\overline{\eta}^{q}-\overline{\xi}^{q}\right)^{\omega}} \end{array} \right\rangle,$$
4. $\omega\mathcal{A}$
$$= \left\langle \begin{array}{l} \sqrt[q]{1-(1-\underline{\zeta}^{q})^{\omega}}, \quad \underline{\eta}^{\omega}, \sqrt[q]{\left(1-\underline{\zeta}^{q}-\underline{\xi}^{q}\right)^{\omega}}, \\ \sqrt[q]{1-(1-\overline{\zeta}^{q})^{\omega}}, \quad \overline{\eta}^{\omega}, \sqrt[q]{\left(1-\overline{\zeta}^{q}-\overline{\xi}^{q}\right)^{\omega}} \end{array} \right\rangle,$$
5. $\mathcal{A}_1 = \mathcal{A}_2$ if and only if $\underline{\zeta}_1 = \underline{\zeta}_2, \underline{\eta}_1 = \underline{\eta}_2$ $\underline{\xi}_1 = \underline{\xi}_2$ and $\overline{\zeta}_1 = \overline{\zeta}_2, \overline{\eta}_1 = \overline{\eta}_2$ $\overline{\xi}_1 = \overline{\xi}_2$.

Definition 12: [60] Let $\mathcal{A} = (\underline{\zeta}, \underline{\eta}, \underline{\xi}, \overline{\zeta}, \overline{\eta}, \overline{\xi})$ be a $q-$SFRN. Then the score value which is denoted as A_Q can be determined by the following function.

$$Sco(\mathcal{A}) = \frac{2+\left(\underline{\zeta}\right)^{q}+(\overline{\zeta})^{q}-\left(\underline{\eta}\right)^{q}-(\overline{\eta})^{q}-\left(\underline{\xi}\right)^{q}-(\overline{\xi})^{q}}{3} \tag{9}$$

where,

$$0 \leq Sco\,(\mathcal{A}) \leq 1.$$

Definition 13: [60] Let $\mathcal{A} = (\underline{\zeta}, \underline{\eta}, \underline{\xi}, \overline{\zeta}, \overline{\eta}, \overline{\xi})$ be a $q-$SFRN. The accuracy of $\mathcal{A}$ is calculated by using the formula mentioned in Equation No. 10.

$$Acc\,(\mathcal{A}) = \frac{\left(\underline{\zeta}\right)^{q}+\left(\underline{\zeta}\right)^{q}-\left(\underline{\xi}\right)^{q}-(\overline{\xi})^{q}}{2} \tag{10}$$

where $-1 \leq Acc\,(\mathcal{A}) \leq 1$.

Definition 14: [60] Let $\mathcal{A}_1 = (\underline{\zeta}_1, \underline{\eta}_1, \underline{\xi}_1, \overline{\zeta}_1, \overline{\eta}_1, \overline{\xi}_1)$ and $\mathcal{A}_2 = (\underline{\zeta}_2, \underline{\eta}_2, \underline{\xi}_2, \overline{\zeta}_2, \overline{\eta}_2, \overline{\xi}_2)$ are two $q-$SFRNs, then

1. If $Sco(\mathcal{A}_1) < Sco(\mathcal{A}_2)$ then $\mathcal{A}_1 < \mathcal{A}_2$,
2. If $Sco(\mathcal{A}_1) > Sco(\mathcal{A}_2)$ then $\mathcal{A}_1 > \mathcal{A}_2$,
3. If $Sco\,(\mathcal{A}_1) = Sco(\mathcal{A}_2)$ then
   - If $Acc(\mathcal{A}_1) < Acc(\mathcal{A}_2)$ then $\mathcal{A}_1 < \mathcal{A}_2$,
   - If $Acc(\mathcal{A}_1) > Acc(\mathcal{A}_2)$ then $\mathcal{A}_1 > \mathcal{A}_2$,
   - If $Acc(\mathcal{A}_1) = Acc(\mathcal{A}_2)$ then $\mathcal{A}_1 = \mathcal{A}_2$.

Definition 15: [60] Let $\mathcal{A}_1 = (\underline{\zeta}_1, \underline{\eta}_1, \underline{\xi}_1, \overline{\zeta}_1, \overline{\eta}_1, \overline{\xi}_1)$ and $\mathcal{A}_2 = (\underline{\zeta}_2, \underline{\eta}_2, \underline{\xi}_2, \overline{\zeta}_2, \overline{\eta}_2, \overline{\xi}_2)$ and $\mathcal{A} = (\underline{\zeta}, \underline{\eta}, \underline{\xi}, \overline{\zeta}, \overline{\eta}, \overline{\xi})$ be any three $q-$SFRNs, and ω, ω_1 and ω_2 are any positive integers then the following properties are held.

1. $\mathcal{A}_1 \oplus \mathcal{A}_2 = \mathcal{A}_2 \oplus \mathcal{A}_1$,
2. $\mathcal{A}_1 \otimes \mathcal{A}_2 = \mathcal{A}_2 \otimes \mathcal{A}_1$
3. $\omega\,(\mathcal{A}_1 \oplus \mathcal{A}_2) = \omega\mathcal{A}_1 \oplus \omega\mathcal{A}_2$,
4. $\omega_1\mathcal{A} \oplus \omega_2\mathcal{A} = (\omega_1+\omega_2)\,\mathcal{A}$,
5. $(\mathcal{A}_1 \otimes \mathcal{A}_2)^{\omega} = \mathcal{A}_1^{\omega} \otimes \mathcal{A}_2^{\omega}$,
6. $\mathcal{A}^{\omega_1} \otimes \mathcal{A}^{\omega_2} = \mathcal{A}^{\omega_1+\omega_2}$.

Definition 16: Using the t-norm T, t-conorm $\mathcal{S}$, Einstein's operations are as follows:

$$\mathrm{T}_{\underset{\sim}{E}}\,(\mathrm{X}\,,\,\mathrm{Y}) = \frac{\mathrm{X}+\mathrm{Y}}{1+(1-\mathrm{X})(1-\mathrm{Y})} \tag{11}$$

$$\mathcal{S}_{\underset{\sim}{E}}\,(\mathrm{X}, \mathrm{Y}) = \frac{\mathrm{X}+\mathrm{Y}}{1+\mathrm{XY}} \tag{12}$$

III. PROPOSED OPERATIONAL LAWS FOR q-SFRNs

In this section, we develop a set of operational laws using Equations (11) and (12) in the context of q-spherical fuzzy rough numbers. Using these established operational laws, we provide a diversified collection of Aggregation Operators (AOs) designed specifically for the integration of q-spherical fuzzy rough information. This technique greatly increases the flexibility and accuracy of aggregation procedures within the stated framework, resulting in higher decision-making efficacy in complex settings.

A. OPERATIONAL LAWS

Definition 17: Let $\mathcal{A}_1 = (\underline{\zeta}_1, \underline{\eta}_1, \underline{\xi}_1, \overline{\zeta}_1, \overline{\eta}_1, \overline{\xi}_1)$, $\mathcal{A}_2 = (\underline{\zeta}_2, \underline{\eta}_2, \underline{\xi}_2, \overline{\zeta}_2, \overline{\eta}_2, \overline{\xi}_2)$ and $\mathcal{A} = (\underline{\zeta}, \underline{\eta}, \underline{\xi}, \overline{\zeta}, \overline{\eta}, \overline{\xi})$ be any three q-SFRNs, where $\omega > 0$, the essential Einstein's operations for q-SFRNs are presented as follows:

(i). Addition Operation ($\mathcal{A}_1 \oplus_{\underset{\sim}{E}} \mathcal{A}_2$):

$$\mathcal{A}_1 \oplus_{\underset{\sim}{E}} \mathcal{A}_2 = \left[\left\langle \begin{array}{l} \sqrt[q]{\frac{\underline{\zeta}_1^q+\underline{\zeta}_2^q}{1+\underline{\zeta}_1^q\cdot\underline{\zeta}_2^q}}, \sqrt[q]{\frac{\underline{\eta}_1^q+\underline{\eta}_2^q}{1+\underline{\eta}_1^q\cdot\underline{\eta}_2^q}}, \sqrt[q]{\frac{\underline{\xi}_1^q\cdot\underline{\xi}_2^q}{1+\left(1-\underline{\xi}_1^q\right)\cdot\left(1-\underline{\xi}_2^q\right)}}, \\ \sqrt[q]{\frac{\overline{\zeta}_1^q+\overline{\zeta}_2^q}{1+\overline{\zeta}_1^q\cdot\overline{\zeta}_2^q}}, \sqrt[q]{\frac{\overline{\eta}_1^q+\overline{\eta}_2^q}{1+\overline{\eta}_1^q\cdot\overline{\eta}_2^q}}, \sqrt[q]{\frac{\overline{\xi}_1^q\cdot\overline{\xi}_2^q}{1+\left(1-\overline{\xi}_1^q\right)\cdot\left(1-\overline{\xi}_2^q\right)}} \end{array} \right\rangle\right]$$

(ii). Multiplication Operation ($\mathcal{A}_1 \otimes_{\underset{\sim}{E}} \mathcal{A}_2$):

$$\mathcal{A}_1 \otimes_{\underset{\sim}{E}} \mathcal{A}_2 = \left[\left\langle \begin{array}{l} \sqrt[q]{\frac{\underline{\zeta}_1^q\cdot\underline{\zeta}_2^q}{1+\left(1-\underline{\zeta}_1^q\right)\cdot\left(1-\underline{\zeta}_2^q\right)}}, \sqrt[q]{\frac{\underline{\eta}_1^q\cdot\underline{\eta}_2^q}{1+\left(1-\underline{\eta}_1^q\right)\cdot\left(1-\underline{\eta}_2^q\right)}}, \sqrt[q]{\frac{\underline{\xi}_1^q+\underline{\xi}_2^q}{1+\underline{\xi}_1^q\cdot\underline{\xi}_2^q}}, \\ \sqrt[q]{\frac{\overline{\zeta}_1^q\cdot\overline{\zeta}_2^q}{1+\left(1-\overline{\zeta}_1^q\right)\cdot\left(1-\overline{\zeta}_2^q\right)}}, \sqrt[q]{\frac{\overline{\eta}_1^q\cdot\overline{\eta}_2^q}{1+\left(1-\overline{\eta}_1^q\right)\cdot\left(1-\overline{\eta}_2^q\right)}}, \sqrt[q]{\frac{\overline{\xi}_1^q+\overline{\xi}_2^q}{1+\overline{\xi}_1^q\cdot\overline{\xi}_2^q}} \end{array} \right\rangle\right]$$

(iii). Scalar Multiplication Operation ($\omega_{\underset{\sim}{E}}\mathcal{A}$):

$$\omega_{\underset{\sim}{E}}\mathcal{A} = \left[\left\langle \begin{array}{l} \sqrt[q]{\frac{\left(1+\underline{\zeta}^q\right)^{\omega}-\left(1-\underline{\zeta}^q\right)^{\omega}}{\left(1+\underline{\zeta}^q\right)^{\omega}+\left(1-\underline{\zeta}^q\right)^{\omega}}}, \sqrt[q]{\frac{\left(1+\underline{\eta}^q\right)^{\omega}-\left(1-\underline{\eta}^q\right)^{\omega}}{\left(1+\underline{\eta}^q\right)^{\omega}+(1-\eta^q)^{\omega}}}, \sqrt[q]{\frac{2\left(\underline{\xi}^q\right)^{\omega}}{\left(2-\underline{\xi}^q\right)^{\omega}+\left(\underline{\xi}^q\right)^{\omega}}}, \\ \sqrt[q]{\frac{\left(1+\overline{\zeta}^q\right)^{\omega}-\left(1-\overline{\zeta}^q\right)^{\omega}}{\left(1+\overline{\zeta}^q\right)^{\omega}+\left(1-\overline{\zeta}^q\right)^{\omega}}}, \sqrt[q]{\frac{(1+\overline{\eta}^q)^{\omega}-(1-\overline{\eta}^q)^{\omega}}{(1+\overline{\eta}^q)^{\omega}+(1-\overline{\eta}^q)^{\omega}}}, \sqrt[q]{\frac{2\left(\overline{\xi}^q\right)^{\omega}}{\left(2-\overline{\xi}^q\right)^{\omega}+\left(\overline{\xi}^q\right)^{\omega}}} \end{array} \right\rangle\right]$$

(iv). Power Operation ($\mathcal{A}^{\underset{\sim}{E}^{\omega}}$):

$$\mathcal{A}^{\underset{\sim}{E}^{\omega}} = \left[\left\langle \begin{array}{l} \sqrt[q]{\frac{2\left(\underline{\zeta}^q\right)^{\omega}}{\left(2-\underline{\zeta}^q\right)^{\omega}+\left(\underline{\zeta}^q\right)^{\omega}}}, \sqrt[q]{\frac{2\left(\underline{\eta}^q\right)^{\omega}}{\left(2-\underline{\eta}^q\right)^{\omega}+\left(\underline{\eta}^q\right)^{\omega}}}, \sqrt[q]{\frac{\left(1+\underline{\xi}^q\right)^{\omega}-\left(1-\underline{\xi}^q\right)^{\omega}}{\left(1+\underline{\xi}^q\right)^{\omega}+\left(1-\underline{\xi}^q\right)^{\omega}}}, \\ \sqrt[q]{\frac{2\left(\overline{\zeta}^q\right)^{\omega}}{\left(2-\overline{\zeta}^q\right)^{\omega}+\left(\overline{\zeta}^q\right)^{\omega}}}, \sqrt[q]{\frac{2(\overline{\eta}^q)^{\omega}}{(2-\eta^q)^{\omega}+(\overline{\eta}^q)^{\omega}}}, \sqrt[q]{\frac{\left(1+\overline{\xi}^q\right)^{\omega}-\left(1-\overline{\xi}^q\right)^{\omega}}{\left(1+\overline{\xi}^q\right)^{\omega}+\left(1-\overline{\xi}^q\right)^{\omega}}} \end{array} \right\rangle\right].$$

These operational laws are essential for performing geometric operations and transformations involving q-spherical fuzzy rough numbers, providing a foundation for subsequent developments and applications in decision-making tasks.

Example 1: Let $\mathcal{A}_1 = (0.4, 0.1, 0.3, 0.2, 0.4, 0.3)$, $\mathcal{A}_2 = (0.5, 0.9, 0.5, 0.8, 0.6, 0.2)$ and $\mathcal{A} =$

(0.4, 0.6, 0.7, 0.5, 0.2, 0.8) be any three q-SFRNs if $\omega = 0.5$ and $q = 3$ then the operational laws defined in Definition 16 can be calculated as:

$$\mathcal{A}_1 \oplus_{\underset{\sim}{\mathrm{E}}} \mathcal{A}_2$$

$$= \left[\left\langle \begin{array}{c} \sqrt[q]{\frac{\underline{\zeta}_1^q+\underline{\zeta}_2^q}{1+\underline{\zeta}_1^q.\underline{\zeta}_2^q}}, \sqrt[q]{\frac{\underline{\eta}_1^q+\underline{\eta}_2^q}{1+\underline{\eta}_1^q.\underline{\eta}_2^q}}, \sqrt[q]{\frac{\underline{\xi}_1^q.\underline{\xi}_2^q}{1+\left(1-\underline{\xi}_1^q\right).\left(1-\underline{\xi}_2^q\right)}}, \\ \sqrt[q]{\frac{\overline{\zeta}_1^q+\overline{\zeta}_2^q}{1+\overline{\zeta}_1^q.\overline{\zeta}_2^q}}, \sqrt[q]{\frac{\overline{\eta}_1^q+\overline{\eta}_2^q}{1+\overline{\eta}_1^q.\overline{\eta}_2^q}}, \sqrt[q]{\frac{\overline{\xi}_1^q.\overline{\xi}_2^q}{1+\left(1-\overline{\xi}_1^q\right).\left(1-\overline{\xi}_2^q\right)}} \end{array} \right\rangle \right]$$

$$= \left[\left\langle \begin{array}{c} \sqrt[3]{\frac{0.4^3+0.5^3}{1+0.4^3.0.5^3}}, \sqrt[3]{\frac{0.1^3+0.9^3}{1+0.1^3.0.9^3}}, \sqrt[3]{\frac{0.3^3.0.5^3}{1+(1-0.3^3).(1-0.5^3)}}, \\ \sqrt[3]{\frac{0.2^3+0.8^3}{1+0.2^3.0.8^3}}, \sqrt[3]{\frac{0.4^3+0.6^3}{1+0.4^3.0.6^3}}, \sqrt[3]{\frac{0.3^3.0.2^3}{1+(1-0.3^3).(1-0.2^3)}} \end{array} \right\rangle \right]$$

$$= (0.5729, 0.9002, 0.1222, 0.8031, 0.6519, 0.0479)$$

$$\mathcal{A}_1 \otimes_{\underset{\sim}{\mathrm{E}}} \mathcal{A}_2$$

$$= \left[\left\langle \begin{array}{c} \sqrt[q]{\frac{\underline{\zeta}_1^q.\underline{\zeta}_2^q}{1+\left(1-\underline{\zeta}_1^q\right).\left(1-\underline{\zeta}_2^q\right)}}, \sqrt[q]{\frac{\underline{\eta}_1^q.\underline{\eta}_2^q}{1+\left(1-\underline{\eta}_1^q\right).\left(1-\underline{\eta}_2^q\right)}}, \sqrt[q]{\frac{\underline{\xi}_1^q+\underline{\xi}_2^q}{1+\underline{\xi}_1^q.\underline{\xi}_2^q}}, \\ \sqrt[q]{\frac{\overline{\zeta}_1^q.\overline{\zeta}_2^q}{1+\left(1-\overline{\zeta}_1^q\right).\left(1-\overline{\zeta}_2^q\right)}}, \sqrt[q]{\frac{\overline{\eta}_1^q.\overline{\eta}_2^q}{1+\left(1-\overline{\eta}_1^q\right).\left(1-\overline{\eta}_2^q\right)}}, \sqrt[q]{\frac{\overline{\xi}_1^q+\overline{\xi}_2^q}{1+\overline{\xi}_1^q.\overline{\xi}_2^q}} \end{array} \right\rangle \right]$$

$$= \left[\left\langle \begin{array}{c} \sqrt[3]{\frac{0.4^3.0.5^3}{1+\left(1-0.4^3\right).\left(1-0.5^3\right)}}, \sqrt[3]{\frac{0.1^3.0.9^3}{1+\left(1-0.1^3\right).\left(1-0.9^3\right)}}, \sqrt[3]{\frac{0.3^3+0.5^3}{1+0.3^3.0.5^3}}, \\ \sqrt[3]{\frac{0.2^3.0.8^3}{1+\left(1-0.2^3\right).\left(1-0.8^3\right)}}, \sqrt[3]{\frac{0.4^3.0.6^3}{1+\left(1-0.4^3\right).\left(1-0.6^3\right)}}, \sqrt[3]{\frac{0.3^3+0.2^3}{1+0.3^3.0.2^3}} \end{array} \right\rangle \right]$$

$$= (0.1638,\ 0.0831, 0.5332, 0.3553, 0.5064, 0.5104)$$

$$\omega_{\underset{\sim}{\mathrm{E}}} \mathcal{A}$$

$$= \left[\left\langle \begin{array}{c} \sqrt[q]{\frac{\left(1+\underline{\zeta}^q\right)^\omega-\left(1-\underline{\zeta}^q\right)^\omega}{\left(1+\underline{\zeta}^q\right)^\omega+\left(1-\underline{\zeta}^q\right)^\omega}}, \sqrt[q]{\frac{\left(1+\underline{\eta}^q\right)^\omega-\left(1-\underline{\eta}^q\right)^\omega}{\left(1+\underline{\eta}^q\right)^\omega+\left(1-\underline{\eta}^q\right)^\omega}}, \sqrt[q]{\frac{2\left(\underline{\xi}^q\right)^\omega}{\left(2-\underline{\xi}^q\right)^\omega+\left(\underline{\xi}^q\right)^\omega}}, \\ \sqrt[q]{\frac{\left(1+\overline{\zeta}^q\right)^\omega-\left(1-\overline{\zeta}^q\right)^\omega}{\left(1+\overline{\zeta}^q\right)^\omega+\left(1-\overline{\zeta}^q\right)^\omega}}, \sqrt[q]{\frac{\left(1+\overline{\eta}^q\right)^\omega-\left(1-\overline{\eta}^q\right)^\omega}{\left(1+\overline{\eta}^q\right)^\omega+\left(1-\overline{\eta}^q\right)^\omega}}, \sqrt[q]{\frac{2\left(\overline{\xi}^q\right)^\omega}{\left(2-\overline{\xi}^q\right)^\omega+\left(\overline{\xi}^q\right)^\omega}} \end{array} \right\rangle \right]$$

$$\omega_{\underset{\sim}{\mathrm{E}}} \mathcal{A}$$

$$= \left[\left\langle \begin{array}{c} \sqrt[3]{\frac{\left(1+0.4^3\right)^{0.5}-\left(1-0.4^3\right)^{0.5}}{\left(1+0.4^3\right)^{0.5}+\left(1-0.4^3\right)^{0.5}}}, \sqrt[3]{\frac{\left(1+0.6^3\right)^{0.5}-\left(1-0.6^3\right)^{0.5}}{\left(1+0.6^3\right)^{0.5}+\left(1-0.6^3\right)^{0.5}}}, \\ \sqrt[3]{\frac{2\left(0.7^3\right)^{0.5}}{\left(2-0.7^3\right)^{0.5}+\left(0.7^3\right)^{0.5}}}, \sqrt[3]{\frac{\left(1+0.5^3\right)^{0.5}-\left(1-0.5^3\right)^{0.5}}{\left(1+0.5^3\right)^{0.5}+\left(1-0.5^3\right)^{0.5}}}, \\ \sqrt[3]{\frac{\left(1+0.2^3\right)^{0.5}-\left(1-0.2^3\right)^{0.5}}{\left(1+0.2^3\right)^{0.5}+\left(1-0.2^3\right)^{0.5}}}, \sqrt[3]{\frac{2(0.8)^{0.5}}{\left(2-0.8^3\right)^{0.5}+\left(0.8^3\right)^{0.5}}} \end{array} \right\rangle \right]$$

$$= (0.3176, 0.4781, 0.8552, 0.3974, 0.1587, 0.9043)$$

$$\mathcal{A}^{\overset{\omega}{\underset{\sim}{\mathrm{E}}}}$$

$$= \left[\left\langle \begin{array}{c} , \sqrt[q]{\frac{2\left(\underline{\zeta}^q\right)^\omega}{\left(2-\underline{\zeta}^q\right)^\omega+\left(\underline{\zeta}^q\right)^\omega}}, \sqrt[q]{\frac{2\left(\underline{\eta}^q\right)^\omega}{\left(2-\underline{\eta}^q\right)^\omega+\left(\underline{\eta}^q\right)^\omega}}, \sqrt[q]{\frac{\left(1+\underline{\xi}^q\right)^\omega-\left(1-\underline{\xi}^q\right)^\omega}{\left(1+\underline{\xi}^q\right)^\omega+\left(1-\underline{\xi}^q\right)^\omega}} \\ \sqrt[q]{\frac{\sqrt[q]{2}(\overline{\zeta})^\omega}{\sqrt[q]{(2-\overline{\zeta})^\omega+(\overline{\zeta})^\omega}}}, \sqrt[q]{\frac{\sqrt[q]{2}(\overline{\eta}^q)^\omega}{\sqrt[q]{(2-\overline{\eta}^q)^\omega+(\overline{\eta}^q)^\omega}}}, \sqrt[q]{\frac{\left(1+\overline{\xi}^q\right)^\omega-\left(1-\overline{\xi}^q\right)^\omega}{\left(1+\overline{\xi}^q\right)^\omega+\left(1-\overline{\xi}^q\right)^\omega}} \end{array} \right\rangle \right]$$

$$\mathcal{A}^{\overset{\omega}{\underset{\sim}{\mathrm{E}}}}$$

$$= \left[\left\langle \begin{array}{c} \sqrt[3]{\frac{2\left(0.4^3\right)^{0.5}}{\left(2-0.4^3\right)^{0.5}+\left(0.4^3\right)^{0.5}}}, \sqrt[3]{\frac{2\left(0.6^3\right)^{0.5}}{\left(2-0.6^3\right)^{0.5}+\left(0.6^3\right)^{0.5}}}, \\ \sqrt[3]{\frac{\left(1+0.7^3\right)^{0.5}-\left(1-0.7^3\right)^{0.5}}{\left(1+0.7^3\right)^{0.5}+\left(1-0.7^3\right)^{0.5}}}, \sqrt[3]{\frac{2\left(0.5^3\right)^{0.5}}{\left(2-0.5^3\right)^{0.5}+\left(0.5^3\right)^{0.5}}}, \\ \sqrt[3]{\frac{2\left(0.2^3\right)^{0.5}}{\left(2-0.2^3\right)^{0.5}+\left(0.2^3\right)^{0.5}}}, \sqrt[3]{\frac{\left(1+0.8^3\right)^{0.5}-\left(1-0.8^3\right)^{0.5}}{\left(1+0.8^3\right)^{0.5}+\left(1-0.8^3\right)^{0.5}}} \end{array} \right\rangle \right]$$

$$= (0.6751, 0.8022, 0.5613, 0.7432, 0.4921, 0.6506)\,.$$

B. q-SFREWG OPERATORS

Definition 18: Assuming $\mathcal{A}_i = \left(\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i\right)$ $(i = 1, 2, \ldots, n)$ ne a collection of q-SFRNs, the q-spherical fuzzy rough Einstein geometric operator (q-SFREWG) operator is defined as a mapping $q - SFREWG : \mathcal{A}^n \longrightarrow \mathcal{A}$ characterized by

$$q - SFREWG\,(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$$

$$= \otimes_{\underset{\sim}{\mathrm{E}}}{}_{i=1}^{n} \mathcal{A}_i{}^{\omega_i}$$

$$= \left[\left\langle \begin{array}{c} \sqrt[q]{\frac{2\prod_{i=1}^n\left(\underline{\zeta}_i^q\right)^{\omega_i}}{\prod_{i=1}^n\left(2-\underline{\zeta}_i^q\right)^{\omega_i}+\prod_{i=1}^n\left(\underline{\zeta}_i^q\right)^{\omega_i}}}, \sqrt[q]{\frac{\prod_{i=1}^n\left(1+\underline{\eta}_i^q\right)^{\omega_i}-\prod_{i=1}^n\left(1-\underline{\eta}_i^q\right)^{\omega_i}}{\prod_{i=1}^n\left(1+\underline{\eta}_i^q\right)^{\omega_i}+\prod_{i=1}^n\left(1-\underline{\eta}_i^q\right)^{\omega_i}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^n\left(1+\underline{\xi}_i^q\right)^{\omega_i}-\prod_{i=1}^n\left(1-\underline{\xi}_i^q\right)^{\omega_i}}{\prod_{i=1}^n\left(1+\underline{\xi}_i^q\right)^{\omega_i}+\prod_{i=1}^n\left(1-\underline{\xi}_i^q\right)^{\omega_i}}}, \sqrt[q]{\frac{2\prod_{i=1}^n\left(\overline{\xi}_i^q\right)^{\omega_i}}{\prod_{i=1}^n\left(2-\overline{\xi}_i^q\right)^{\omega_i}+\prod_{i=1}^n\left(\overline{\xi}_i^q\right)^{\omega_i}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^n\left(1+\overline{\eta}_i^q\right)^{\omega_i}-\prod_{i=1}^n\left(1-\overline{\eta}_i^q\right)^{\omega_i}}{\prod_{i=1}^n\left(1+\overline{\eta}_i^q\right)^{\omega_i}+\prod_{i=1}^n\left(1-\overline{\eta}_i^q\right)^{\omega_i}}}, \sqrt[q]{\frac{\prod_{i=1}^n\left(1+\overline{\xi}_i^q\right)^{\omega_i}-\prod_{i=1}^n\left(1-\overline{\xi}_i^q\right)^{\omega_i}}{\prod_{i=1}^n\left(1+\overline{\xi}_i^q\right)^{\omega_i}+\prod_{i=1}^n\left(1-\overline{\xi}_i^q\right)^{\omega_i}}} \end{array} \right\rangle \right] \tag{13}$$

Hence $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ signifies the weight vector of $\mathcal{A}_i = \left(\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i\right)$ $(i = 1, 2, \ldots, n)$ adhering the conditions $\omega_i > 0$ and the constrain $\sum_{i=1}^n \omega_i = 1$.

Theorem 1: Assuming $\mathcal{A}_i = \left(\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i\right)$ $(i = 1, 2, \ldots, n)$ be a collection of $q - SFRNs$ and $(\omega_1, \omega_2, \ldots, \omega_n)^T$ signifies the weight vector adhering the condition $\omega_i > 0$ and the constrain $\sum_{i=1}^n \omega_i = 1$. Then the aggregated values obtained by q-SFREWG operator is also a q-SFRN and cab be expressed as:

$$q - SFREWG\,(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$$

$$= \otimes_{\underset{\sim}{\mathrm{E}}}{}_{i=1}^{n} \mathcal{A}_i{}^{\omega_i}$$

$$= \left[\left\langle \begin{array}{c} \sqrt[q]{\frac{2\prod_{i=1}^n\left(\underline{\zeta}_i^q\right)^{\omega_i}}{\prod_{i=1}^n\left(2-\underline{\zeta}_i^q\right)^{\omega_i}+\prod_{i=1}^n\left(\underline{\zeta}_i^q\right)^{\omega_i}}}, \sqrt[q]{\frac{\prod_{i=1}^n\left(1+\underline{\eta}_i^q\right)^{\omega_i}-\prod_{i=1}^n\left(1-\underline{\eta}_i^q\right)^{\omega_i}}{\prod_{i=1}^n\left(1+\underline{\eta}_i^q\right)^{\omega_i}+\prod_{i=1}^n\left(1-\underline{\eta}_i^q\right)^{\omega_i}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^n\left(1+\underline{\xi}_i^q\right)^{\omega_i}-\prod_{i=1}^n\left(1-\underline{\xi}_i^q\right)^{\omega_i}}{\prod_{i=1}^n\left(1+\underline{\xi}_i^q\right)^{\omega_i}+\prod_{i=1}^n\left(1-\underline{\xi}_i^q\right)^{\omega_i}}}, \sqrt[q]{\frac{2\prod_{i=1}^n\left(\overline{\xi}_i^q\right)^{\omega_i}}{\prod_{i=1}^n\left(2-\overline{\xi}_i^q\right)^{\omega_i}+\prod_{i=1}^n\left(\overline{\xi}_i^q\right)^{\omega_i}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^n\left(1+\overline{\eta}_i^q\right)^{\omega_i}-\prod_{i=1}^n\left(1-\overline{\eta}_i^q\right)^{\omega_i}}{\prod_{i=1}^n\left(1+\overline{\eta}_i^q\right)^{\omega_i}+\prod_{i=1}^n\left(1-\overline{\eta}_i^q\right)^{\omega_i}}}, \sqrt[q]{\frac{\prod_{i=1}^n\left(1+\overline{\xi}_i^q\right)^{\omega_i}-\prod_{i=1}^n\left(1-\overline{\xi}_i^q\right)^{\omega_i}}{\prod_{i=1}^n\left(1+\overline{\xi}_i^q\right)^{\omega_i}+\prod_{i=1}^n\left(1-\overline{\xi}_i^q\right)^{\omega_i}}} \end{array} \right\rangle \right] \tag{14}$$

Proof: This proof can be easily established using mathematical induction about the natural number n.

Step 1: For $n=2$, we have,

$$\mathcal{A}_1^{\omega_1}=\left[\begin{array}{c}\sqrt[q]{\frac{2\left(\underline{\zeta}_1^q\right)^{\omega_1}}{\left(2-\underline{\zeta}_1^q\right)^{\omega_1}+\left(\underline{\zeta}_1^q\right)^{\omega_1}}},\sqrt[q]{\frac{\left(1+\underline{\eta}_1^q\right)^{\omega_1}-\left(1-\underline{\eta}_1^q\right)^{\omega_1}}{\left(1+\underline{\eta}_1^q\right)^{\omega_1}+\left(1-\underline{\eta}_1^q\right)^{\omega_1}}},\\ \left\langle\sqrt[q]{\frac{\left(1+\underline{\xi}_1^q\right)^{\omega_1}-\left(1-\underline{\xi}_1^q\right)^{\omega_1}}{\left(1+\underline{\xi}_1^q\right)^{\omega_1}+\left(1-\underline{\xi}_1^q\right)^{\omega_1}}},\sqrt[q]{\frac{2\left(\overline{\zeta}_1^q\right)^{\omega_1}}{\left(2-\overline{\zeta}_1^q\right)^{\omega_1}+\left(\overline{\zeta}_1^q\right)^{\omega_1}}},\right\rangle\\ \sqrt[q]{\frac{(1+\overline{\eta}_1^q)^{\omega_1}-(1-\overline{\eta}_1^q)^{\omega_1}}{(1+\overline{\eta}_1^q)^{\omega_1}+(1-\overline{\eta}_1^q)^{\omega_1}}},\sqrt[q]{\frac{\left(1+\overline{\xi}_1^q\right)^{\omega_1}-\left(1-\overline{\xi}_1^q\right)^{\omega_1}}{\left(1+\overline{\xi}_1^q\right)^{\omega_1}+\left(1-\overline{\xi}_1^q\right)^{\omega_1}}}\end{array}\right]$$

$$\mathcal{A}_2^{\omega_2}=\left[\begin{array}{c}\sqrt[q]{\frac{2\left(\underline{\zeta}_2^q\right)^{\omega_2}}{\left(2-\underline{\zeta}_2^q\right)^{\omega_2}+\left(\underline{\zeta}_2^q\right)^{\omega_2}}},\sqrt[q]{\frac{\left(1+\underline{\eta}_2^q\right)^{\omega_2}-\left(1-\underline{\eta}_2^q\right)^{\omega_2}}{\left(1+\underline{\eta}_2^q\right)^{\omega_2}+\left(1-\underline{\eta}_2^q\right)^{\omega_2}}},\\ \left\langle\sqrt[q]{\frac{\left(1+\underline{\xi}_2^q\right)^{\omega_2}-\left(1-\underline{\xi}_2^q\right)^{\omega_2}}{\left(1+\underline{\xi}_2^q\right)^{\omega_2}+\left(1-\underline{\xi}_2^q\right)^{\omega_2}}},\sqrt[q]{\frac{2\left(\overline{\zeta}_2^q\right)^{\omega_2}}{\left(2-\overline{\zeta}_2^q\right)^{\omega_2}+\left(\overline{\zeta}_2^q\right)^{\omega_2}}},\right\rangle\\ \sqrt[q]{\frac{(1+\overline{\eta}_2^q)^{\omega_2}-(1-\overline{\eta}_2^q)^{\omega_2}}{(1+\overline{\eta}_2^q)^{\omega_2}+(1-\overline{\eta}_2^q)^{\omega_2}}},\sqrt[q]{\frac{\left(1+\overline{\xi}_2^q\right)^{\omega_2}-\left(1-\overline{\xi}_2^q\right)^{\omega_2}}{\left(1+\overline{\xi}_2^q\right)^{\omega_2}+\left(1-\overline{\xi}_2^q\right)^{\omega_2}}}\end{array}\right]$$

as shown in the equation at the bottom of the next page, where $\sum_{i=1}^{2}\omega_i=1$. Thus, the condition $n=2$ is true for Equation No. 14

Step 2: Assume that Equation (14) is valid for $n=k$, where k is any real number, given this supposition, Equation (13) can be represented as:

$$q-SFREWG\left(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_k\right)$$
$$=\otimes_{\underset{\sim}{\mathrm{E}}\,i=1}^{k}\mathcal{A}_i^{\omega_i}$$
$$=\left[\left\langle\begin{array}{c}\sqrt[q]{\frac{2\prod_{i=1}^{k}\left(\underline{\zeta}_i^q\right)^{\omega_i}}{\prod_{i=1}^{k}\left(2-\underline{\zeta}_i^q\right)^{\omega_i}+\prod_{i=1}^{n}\left(\underline{\zeta}_i^q\right)^{\omega_i}}},\sqrt[q]{\frac{\prod_{i=1}^{k}\left(1+\underline{\eta}_i^q\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\underline{\eta}_i^q\right)^{\omega_i}}{\prod_{i=1}^{k}\left(1+\underline{\eta}_i^q\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\underline{\eta}_i^q\right)^{\omega_i}}},\\ \sqrt[q]{\frac{\prod_{i=1}^{k}\left(1+\underline{\xi}_i^q\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\underline{\xi}_i^q\right)^{\omega_i}}{\prod_{i=1}^{k}\left(1+\underline{\xi}_i^q\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\underline{\xi}_i^q\right)^{\omega_i}}},\sqrt[q]{\frac{2\prod_{i=1}^{k}\left(\overline{\zeta}_i^q\right)^{\omega_i}}{\prod_{i=1}^{k}\left(2-\overline{\zeta}_i^q\right)^{\omega_i}+\prod_{i=1}^{n}\left(\overline{\zeta}_i^q\right)^{\omega_i}}},\\ \sqrt[q]{\frac{\prod_{i=1}^{k}\left(1+\overline{\eta}_i^q\right)^{\omega_i}-\prod_{i=1}^{k}\left(1-\overline{\eta}_i^q\right)^{\omega_i}}{\prod_{i=1}^{k}\left(1+\overline{\eta}_i^q\right)^{\omega_i}+\prod_{i=1}^{k}\left(1-\overline{\eta}_i^q\right)^{\omega_i}}},\sqrt[q]{\frac{\prod_{i=1}^{k}\left(1+\overline{\xi}_i^q\right)^{\omega_i}-\prod_{i=1}^{k}\left(1-\overline{\xi}_i^q\right)^{\omega_i}}{\prod_{i=1}^{k}\left(1+\overline{\xi}_i^q\right)^{\omega_i}+\prod_{i=1}^{k}\left(1-\overline{\xi}_i^q\right)^{\omega_i}}}\end{array}\right\rangle\right]$$

Step 3: Now for $n=k+1$, we are examining the following equations: As shown in the equation at the bottom of page 12.

Hence for $k=n+1$ Equation (14) holds. By combining observations from steps (1), (2), and (3), this result applies to all values of n inside the natural numbers.

Example 2: Consider four q-SFRNs $\mathcal{A}_1=(0.3,0.4,0.1,0.3,0.2,0.4)$, $\mathcal{A}_2=(0.5,0.5,0.9,0.2,0.8,0.6)$, $\mathcal{A}_3=(0.4,0.5,0.9,0.3,0.2,0.6)$ and $\mathcal{A}_4=(0.5,0.2,0.5,0.2,0.8,0.9)$ be any four q-SFRNs, if $\omega=(0.3,0.1,0.4,0.2)^T$ and $q=3$ then the q-SFREWG operator defined in Definition (14) can be calculated as:

$$\sqrt[q]{\frac{2\prod_{i=1}^{4}\left(\underline{\zeta}_i^q\right)^{\omega_i}}{\prod_{i=1}^{4}\left(2-\underline{\zeta}_i^q\right)^{\omega_i}+\prod_{i=1}^{4}\left(\underline{\zeta}_i^q\right)^{\omega_i}}}$$

$$=\sqrt[q]{\frac{2\left(\underline{\zeta}_i^q\right)^{\omega_1}\left(\underline{\zeta}_2^q\right)^{\omega_2}\left(\underline{\zeta}_3^q\right)^{\omega_3}\left(\underline{\zeta}_4^q\right)^{\omega_4}}{\left(2-\underline{\zeta}_i^q\right)^{\omega_1}\left(2-\underline{\zeta}_2^q\right)^{\omega_2}\left(2-\underline{\zeta}_3^q\right)^{\omega_3}\left(2-\underline{\zeta}_4^q\right)^{\omega_4}+\left(\underline{\zeta}_i^q\right)^{\omega_1}\left(\underline{\zeta}_2^q\right)^{\omega_2}\left(\underline{\zeta}_3^q\right)^{\omega_3}\left(\underline{\zeta}_4^q\right)^{\omega_4}}}$$

$$=\sqrt[3]{\frac{2(0.3^3)^{0.3}(0.5^3)^{0.1}(0.4^3)^{0.4}(0.5^3)^{0.2}}{(2-0.3^3)^{0.3}(2-0.5^3)^{0.1}(2-0.4^3)^{0.4}(2-0.5^3)^{0.2}+(0.3^3)^{0.3}(0.5^3)^{0.1}(0.4^3)^{0.4}(0.5^3)^{0.2}}}$$

$=(0.6385)$

$$\sqrt[q]{\frac{\prod_{i=1}^{4}\left(1+\underline{\eta}_i^q\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\underline{\eta}_i^q\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\underline{\eta}_i^q\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\underline{\eta}_i^q\right)^{\omega_i}}}$$

$$=\sqrt[q]{\frac{\left(1+\underline{\eta}_1^q\right)^{\omega_1}\left(1+\underline{\eta}_2^q\right)^{\omega_1}\left(1+\underline{\eta}_3^q\right)^{\omega_3}\left(1+\underline{\eta}_4^q\right)^{\omega_4}-\left(1-\underline{\eta}_1^q\right)^{\omega_1}\left(1-\underline{\eta}_2^q\right)^{\omega_2}\left(1-\underline{\eta}_3^q\right)^{\omega_3}\left(1-\underline{\eta}_4^q\right)^{\omega_4}}{\left(1+\underline{\eta}_1^q\right)^{\omega_1}\left(1+\underline{\eta}_2^q\right)^{\omega_2}\left(1+\underline{\eta}_3^q\right)^{\omega_3}\left(1+\underline{\zeta}_4^q\right)^{\omega_4}+\left(1-\underline{\eta}_1^q\right)^{\omega_1}\left(1-\underline{\eta}_2^q\right)^{\omega_2}\left(1-\underline{\eta}_3^q\right)^{\omega_3}\left(1-\underline{\eta}_4^q\right)^{\omega_4}}}$$

$$=\sqrt[3]{\frac{(1+0.4^3)^{0.3}(1+0.5^3)^{0.1}(1+0.5^3)^{0.4}(1+0.2^3)^{0.2}-(1-0.4^3)^{0.3}(1-0.5^3)^{0.1}(1-0.5^3)^{0.4}(1-0.2^3)^{0.2}}{(1+0.4^3)^{0.3}(1+0.5^3)^{0.1}(1+0.5^3)^{0.4}(1+0.2^3)^{0.2}+(1-0.4^3)^{0.3}(1-0.5^3)^{0.1}(1-0.5^3)^{0.4}(1-0.2^3)^{0.2}}}$$

$=(0.3916)$

$$\sqrt[q]{\frac{\prod_{i=1}^{4}\left(1+\underline{\xi}_i^q\right)^{\omega_i}-\prod_{i=1}^{4}\left(1-\underline{\xi}_i^q\right)^{\omega_i}}{\prod_{i=1}^{4}\left(1+\underline{\xi}_i^q\right)^{\omega_i}+\prod_{i=1}^{4}\left(1-\underline{\xi}_i^q\right)^{\omega_i}}}$$

$$=\sqrt[q]{\frac{\left(1+\underline{\xi}_1^q\right)^{\omega_1}\left(1+\underline{\xi}_2^q\right)^{\omega_1}\left(1+\underline{\xi}_3^q\right)^{\omega_3}\left(1+\underline{\xi}_4^q\right)^{\omega_4}-\left(1-\underline{\xi}_1^q\right)^{\omega_1}\left(1-\underline{\xi}_2^q\right)^{\omega_2}\left(1-\underline{\xi}_3^q\right)^{\omega_3}\left(1-\underline{\xi}_4^q\right)^{\omega_4}}{\left(1+\underline{\xi}_1^q\right)^{\omega_1}\left(1+\underline{\xi}_2^q\right)^{\omega_2}\left(1+\underline{\xi}_3^q\right)^{\omega_3}\left(1+\underline{\xi}_4^q\right)^{\omega_4}+\left(1-\underline{\xi}_1^q\right)^{\omega_1}\left(1-\underline{\xi}_2^q\right)^{\omega_2}\left(1-\underline{\xi}_3^q\right)^{\omega_3}\left(1-\underline{\xi}_4^q\right)^{\omega_4}}}$$

$$=\sqrt[3]{\frac{(1+0.1^3)^{0.3}(1+0.9^3)^{0.1}(1+0.9^3)^{0.4}(1+0.5^3)^{0.2}-(1-0.1^3)^{0.3}(1-0.9^3)^{0.1}(1-0.9^3)^{0.4}(1-0.5^3)^{0.2}}{(1+0.1^3)^{0.3}(1+0.9^3)^{0.1}(1+0.9^3)^{0.4}(1+0.5^3)^{0.2}+(1-0.1^3)^{0.3}(1-0.9^3)^{0.1}(1-0.9^3)^{0.4}(1-0.5^3)^{0.2}}}$$

$=(0.6820)$

$$\sqrt[q]{\frac{2\prod_{i=1}^{4}\left(\overline{\zeta}_i^q\right)^{\omega_i}}{\prod_{i=1}^{4}\left(2-\overline{\zeta}_i^q\right)^{\omega_i}+\prod_{i=1}^{k}\left(\overline{\zeta}_i^q\right)^{\omega_i}}}$$

$$= \sqrt[q]{\frac{2\left(\overline{\zeta}_1^q\right)^{\omega_1}\left(\overline{\zeta}_2^q\right)^{\omega_2}\left(\overline{\zeta}_3^q\right)^{\omega_3}\left(\overline{\zeta}_4^q\right)^{\omega_4}}{\left(2-\overline{\zeta}_1^q\right)^{\omega_1}\left(2-\overline{\zeta}_2^q\right)^{\omega_2}\left(2-\overline{\zeta}_3^q\right)^{\omega_3}\left(2-\overline{\zeta}_4^q\right)^{\omega_4} +\left(\overline{\zeta}_1^q\right)^{\omega_1}\left(\overline{\zeta}_2^q\right)^{\omega_2}\left(\overline{\zeta}_3^q\right)^{\omega_3}\left(\overline{\zeta}_4^q\right)^{\omega_4}}}$$

$$= \sqrt[3]{\frac{2\left(0.3^3\right)^{0.3}\left(0.2^3\right)^{0.1}\left(0.3^3\right)^{0.4}\left(0.3^3\right)^{0.2}}{\left(2-0.3^3\right)^{0.3}\left(2-0.2^3\right)^{0.1}\left(2-0.3^3\right)^{0.4}\left(2-0.3^3\right)^{0.2} +\left(0.3^3\right)^{0.3}\left(0.2^3\right)^{0.1}\left(0.3^3\right)^{0.4}\left(0.3^3\right)^{0.2}}}$$

$$= (\,0.5366)$$

$$\sqrt[q]{\frac{\prod_{i=1}^4\left(1+\overline{\eta}_i^q\right)^{\omega_i}-\prod_{i=1}^n\left(1-\overline{\eta}_i^q\right)^{\omega_i}}{\prod_{i=1}^4\left(1+\overline{\eta}_i^q\right)^{\omega_i}+\prod_{i=1}^n\left(1-\overline{\eta}_i^q\right)^{\omega_i}}}$$

$$= \sqrt[q]{\frac{\left(1+\overline{\eta}_1^q\right)^{\omega_1}\left(1+\overline{\eta}_2^q\right)^{\omega_2}\left(1+\overline{\eta}_3^q\right)^{\omega_3}\left(1+\overline{\eta}_4^q\right)^{\omega_4} -\left(1-\overline{\eta}_1^q\right)^{\omega_1}\left(1-\overline{\eta}_2^q\right)^{\omega_2}\left(1-\overline{\eta}_3^q\right)^{\omega_3}\left(1-\overline{\eta}_4^q\right)^{\omega_4}}{\left(1+\overline{\eta}_1^q\right)^{\omega_1}\left(1+\overline{\eta}_2^q\right)^{\omega_2}\left(1+\overline{\eta}_3^q\right)^{\omega_3}\left(1+\overline{\eta}_4^q\right)^{\omega_4} +\left(1-\overline{\eta}_1^q\right)^{\omega_1}\left(1-\overline{\eta}_2^q\right)^{\omega_2}\left(1-\overline{\eta}_3^q\right)^{\omega_3}\left(1-\overline{\eta}_4^q\right)^{\omega_4}}}$$

$$= \sqrt[3]{\frac{\left(1+0.2^3\right)^{0.3}\left(1+0.8^3\right)^{0.1}\left(1+0.2^3\right)^{0.4}\left(1+0.8^3\right)^{0.2} -\left(1-0.2^3\right)^{0.3}\left(1-0.8^3\right)^{0.1}\left(1-0.2^3\right)^{0.4}\left(1-0.8^3\right)^{0.2}}{\left(1+0.2^3\right)^{0.3}\left(1+0.8^3\right)^{0.1}\left(1+0.2^3\right)^{0.4}\left(1+0.8^3\right)^{0.2} +\left(1-0.2^3\right)^{0.3}\left(1-0.8^3\right)^{0.1}\left(1-0.2^3\right)^{0.4}\left(1-0.8^3\right)^{0.2}}}$$

$$= (0.6535)$$

$$\sqrt[q]{\frac{\prod_{i=1}^4\left(1+\overline{\xi}_i^q\right)^{\omega_i}-\prod_{i=1}^n\left(1-\overline{\xi}_i^q\right)^{\omega_i}}{\prod_{i=1}^4\left(1+\overline{\xi}_i^q\right)^{\omega_i}+\prod_{i=1}^n\left(1-\overline{\xi}_i^q\right)^{\omega_i}}}$$

$$\mathcal{A}_1{}^{\omega_1}\otimes_{\underset{\sim}{E}}\mathcal{A}_2{}^{\omega_2} = \left[\left\langle \begin{array}{c} \sqrt[q]{\frac{2\left(\underline{\zeta}_1^q\right)^{\omega_1}}{\left(2-\underline{\zeta}_1^q\right)^{\omega_1}+\left(\underline{\zeta}_1^q\right)^{\omega_1}}}, \sqrt[q]{\frac{\left(1+\underline{\eta}_1^q\right)^{\omega_1}-\left(1-\underline{\eta}_1^q\right)^{\omega_1}}{\left(1+\underline{\eta}_1^q\right)^{\omega_1}+\left(1-\underline{\eta}_1^q\right)^{\omega_1}}}, \\ \sqrt[q]{\frac{\left(1+\underline{\xi}_1^q\right)^{\omega_1}-\left(1-\underline{\xi}_1^q\right)^{\omega_1}}{\left(1+\underline{\xi}_1^q\right)^{\omega_1}+\left(1-\underline{\xi}_1^q\right)^{\omega_1}}}, \sqrt[q]{\frac{2\left(\overline{\zeta}_1^q\right)^{\omega_1}}{\left(2-\overline{\zeta}_1^q\right)^{\omega_1}+\left(\overline{\zeta}_1^q\right)^{\omega_1}}}, \\ \sqrt[q]{\frac{\left(1+\overline{\eta}_1^q\right)^{\omega_1}-\left(1-\overline{\eta}_1^q\right)^{\omega_1}}{\left(1+\overline{\eta}_1^q\right)^{\omega_1}+\left(1-\overline{\eta}_1^q\right)^{\omega_1}}}, \sqrt[q]{\frac{\left(1+\overline{\xi}_1^q\right)^{\omega_1}-\left(1-\overline{\xi}_1^q\right)^{\omega_1}}{\left(1+\overline{\xi}_1^q\right)^{\omega_1}+\left(1-\overline{\xi}_1^q\right)^{\omega_1}}} \end{array} \right\rangle\right]$$

$$\otimes_{\underset{\sim}{E}} \left[\left\langle \begin{array}{c} \sqrt[q]{\frac{2\left(\underline{\zeta}_2^q\right)^{\omega_2}}{\left(2-\underline{\zeta}_2^q\right)^{\omega_2}+\left(\underline{\zeta}_2^q\right)^{\omega_2}}}, \sqrt[q]{\frac{\left(1+\underline{\eta}_2^q\right)^{\omega_2}-\left(1-\underline{\eta}_2^q\right)^{\omega_2}}{\left(1+\underline{\eta}_2^q\right)^{\omega_2}+\left(1-\underline{\eta}_2^q\right)^{\omega_2}}}, \\ \sqrt[q]{\frac{\left(1+\underline{\xi}_2^q\right)^{\omega_2}-\left(1-\underline{\xi}_2^q\right)^{\omega_2}}{\left(1+\underline{\xi}_2^q\right)^{\omega_2}+\left(1-\underline{\xi}_2^q\right)^{\omega_2}}}, \sqrt[q]{\frac{2\left(\overline{\zeta}_2^q\right)^{\omega_2}}{\left(2-\overline{\zeta}_2^q\right)^{\omega_2}+\left(\overline{\zeta}_2^q\right)^{\omega_2}}}, \\ \sqrt[q]{\frac{\left(1+\overline{\eta}_2^q\right)^{\omega_2}-\left(1-\overline{\eta}_2^q\right)^{\omega_2}}{\left(1+\overline{\eta}_2^q\right)^{\omega_2}+\left(1-\overline{\eta}_2^q\right)^{\omega_2}}}, \sqrt[q]{\frac{\left(1+\overline{\xi}_2^q\right)^{\omega_2}-\left(1-\overline{\xi}_2^q\right)^{\omega_2}}{\left(1+\overline{\xi}_2^q\right)^{\omega_2}+\left(1-\overline{\xi}_2^q\right)^{\omega_2}}} \end{array} \right\rangle\right]$$

$$= \left[\left\langle \begin{array}{c} \sqrt[q]{\frac{2\left(\underline{\zeta}_1^q\right)^{\omega_1}\left(\underline{\zeta}_2^q\right)^{\omega_2}}{\left(2-\underline{\zeta}_1^q\right)^{\omega_1}\left(2-\underline{\zeta}_2^q\right)^{\omega_2}+\left(\underline{\zeta}_1^q\right)^{\omega_1}\left(\underline{\zeta}_2^q\right)^{\omega_2}}}, \sqrt[q]{\frac{\left(1+\underline{\eta}_1^q\right)^{\omega_1}\left(1+\underline{\eta}_2^q\right)^{\omega_2}-\left(1-\underline{\eta}_1^q\right)^{\omega_1}\left(1-\underline{\eta}_2^q\right)^{\omega_2}}{\left(1+\underline{\eta}_1^q\right)^{\omega_1}\left(1+\underline{\eta}_2^q\right)^{\omega_2}+\left(1-\underline{\eta}_1^q\right)^{\omega_1}\left(1-\underline{\eta}_2^q\right)^{\omega_2}}}, \\ \sqrt[q]{\frac{\left(1+\underline{\xi}_1^q\right)^{\omega_1}\left(1+\underline{\xi}_2^q\right)^{\omega_2}-\left(1+\underline{\xi}_1^q\right)^{\omega_1}\left(1+\underline{\xi}_2^q\right)^{\omega_2}}{\left(1+\underline{\xi}_1^q\right)^{\omega_1}\left(1+\underline{\xi}_2^q\right)^{\omega_2}+\left(1+\underline{\xi}_1^q\right)^{\omega_1}\left(1+\underline{\xi}_2^q\right)^{\omega_2}}}, \sqrt[q]{\frac{2\left(\overline{\zeta}_1^q\right)^{\omega_1}\left(\overline{\zeta}_2^q\right)^{\omega_2}}{\left(2-\overline{\zeta}_1^q\right)^{\omega_1}\left(2-\overline{\zeta}_2^q\right)^{\omega_2}+\left(\overline{\zeta}_1^q\right)^{\omega_1}\left(\overline{\zeta}_2^q\right)^{\omega_2}}}, \\ \sqrt[q]{\frac{\left(1+\overline{\eta}_1^q\right)^{\omega_1}\left(1+\overline{\eta}_2^q\right)^{\omega_2}-\left(1-\overline{\eta}_1^q\right)^{\omega_1}\left(1-\overline{\eta}_2^q\right)^{\omega_2}}{\left(1+\overline{\eta}_1^q\right)^{\omega_1}\left(1+\overline{\eta}_2^q\right)^{\omega_2}+\left(1-\overline{\eta}_1^q\right)^{\omega_1}\left(1-\overline{\eta}_2^q\right)^{\omega_2}}}, \sqrt[q]{\frac{\left(1+\overline{\xi}_1^q\right)^{\omega_1}\left(1+\overline{\xi}_2^q\right)^{\omega_2}-\left(1+\overline{\xi}_1^q\right)^{\omega_1}\left(1+\overline{\xi}_2^q\right)^{\omega_2}}{\left(1+\overline{\xi}_1^q\right)^{\omega_1}\left(1+\overline{\xi}_2^q\right)^{\omega_2}+\left(1+\overline{\xi}_1^q\right)^{\omega_1}\left(1+\overline{\xi}_2^q\right)^{\omega_2}}} \end{array} \right\rangle\right]$$

$$= \left[\left\langle \begin{array}{c} \sqrt[q]{\frac{2\prod_{i=1}^2\left(\underline{\zeta}_i^q\right)^{\omega_i}}{\prod_{i=1}^2\left(2-\underline{\zeta}_i^q\right)^{\omega_i}+\prod_{i=1}^2\left(\underline{\zeta}_i^q\right)^{\omega_i}}}, \sqrt[q]{\frac{\prod_{i=1}^2\left(1+\underline{\eta}_i^q\right)^{\omega_i}-\prod_{i=1}^2\left(1-\underline{\eta}_i^q\right)^{\omega_i}}{\prod_{i=1}^2\left(1+\underline{\eta}_i^q\right)^{\omega_i}+\prod_{i=1}^2\left(1-\underline{\eta}_i^q\right)^{\omega_i}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^2\left(1+\underline{\xi}_i^q\right)^{\omega_i}-\prod_{i=1}^2\left(1-\underline{\xi}_i^q\right)^{\omega_i}}{\prod_{i=1}^2\left(1+\underline{\xi}_i^q\right)^{\omega_i}+\prod_{i=1}^2\left(1-\underline{\xi}_i^q\right)^{\omega_i}}}, \sqrt[q]{\frac{2\prod_{i=1}^2\left(\overline{\zeta}_i^q\right)^{\omega_i}}{\prod_{i=1}^2\left(2-\overline{\zeta}_i^q\right)^{\omega_i}+\prod_{i=1}^2\left(\overline{\zeta}_i^q\right)^{\omega_i}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^2\left(1+\overline{\eta}_i^q\right)^{\omega_i}-\prod_{i=1}^2\left(1-\overline{\eta}_i^q\right)^{\omega_i}}{\prod_{i=1}^2\left(1+\overline{\eta}_i^q\right)^{\omega_i}+\prod_{i=1}^2\left(1-\overline{\eta}_i^q\right)^{\omega_i}}}, \sqrt[q]{\frac{\prod_{i=1}^2\left(1+\overline{\xi}_i^q\right)^{\omega_i}-\prod_{i=1}^2\left(1-\overline{\xi}_i^q\right)^{\omega_i}}{\prod_{i=1}^2\left(1+\overline{\xi}_i^q\right)^{\omega_i}+\prod_{i=1}^2\left(1-\overline{\xi}_i^q\right)^{\omega_i}}} \end{array} \right\rangle\right]$$

$$= \sqrt[q]{\frac{\left(1+\overline{\xi}_1^q\right)^{\omega_1}\left(1+\overline{\xi}_2^q\right)^{\omega_2}\left(1+\overline{\xi}_3^q\right)^{\omega_3}\left(1+\overline{\xi}_4^q\right)^{\omega_4} - \left(1-\overline{\xi}_1^q\right)^{\omega_1}\left(1-\overline{\xi}_2^q\right)^{\omega_2}\left(1-\overline{\xi}_3^q\right)^{\omega_3}\left(1-\overline{\xi}_4^q\right)^{\omega_4}}{\left(1+\overline{\xi}_1^q\right)^{\omega_1}\left(1+\overline{\xi}_2^q\right)^{\omega_2}\left(1+\overline{\xi}_3^q\right)^{\omega_3}\left(1+\overline{\xi}_4^q\right)^{\omega_4} + \left(1-\overline{\xi}_1^q\right)^{\omega_1}\left(1-\overline{\xi}_2^q\right)^{\omega_2}\left(1-\overline{\xi}_3^q\right)^{\omega_3}\left(1-\overline{\xi}_4^q\right)^{\omega_4}}}$$

$$= \sqrt[3]{\frac{(1+0.4^3)^{0.3}(1+0.6^3)^{0.1}(1+0.6^3)^{0.4}(1+0.9^3)^{0.2} - (1-0.4^3)^{0.3}(1-0.6^3)^{0.1}(1-0.6^3)^{0.4}(1-0.9^3)^{0.2}}{(1+0.4^3)^{0.3}(1+0.6^3)^{0.1}(1+0.6^3)^{0.4}(1+0.9^3)^{0.2} + (1-0.4^3)^{0.3}(1-0.6^3)^{0.1}(1-0.6^3)^{0.4}(1-0.9^3)^{0.2}}}$$

$$= (0.7528)$$

Hence

$$\left[\left\langle \begin{array}{c} \sqrt[q]{\frac{2\prod_{i=1}^4\left(\underline{\zeta}_i^q\right)^{\omega_i}}{\prod_{i=1}^4\left(2-\underline{\zeta}_i^q\right)^{\omega_i}+\prod_{i=1}^4\left(\underline{\zeta}_i^q\right)^{\omega_i}}}, \sqrt[q]{\frac{\prod_{i=1}^4\left(1+\underline{\eta}_i^q\right)^{\omega_i}-\prod_{i=1}^4\left(1-\underline{\eta}_i^q\right)^{\omega_i}}{\prod_{i=1}^4\left(1+\underline{\eta}_i^q\right)^{\omega_i}+\prod_{i=1}^4\left(1-\underline{\eta}_i^q\right)^{\omega_i}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^4\left(1+\underline{\xi}_i^q\right)^{\omega_i}-\prod_{i=1}^4\left(1-\underline{\xi}_i^q\right)^{\omega_i}}{\prod_{i=1}^4\left(1+\underline{\xi}_i^q\right)^{\omega_i}+\prod_{i=1}^4\left(1-\underline{\xi}_i^q\right)^{\omega_i}}}, \sqrt[q]{\frac{2\prod_{i=1}^4\left(\overline{\zeta}_i^q\right)^{\omega_i}}{\prod_{i=1}^4\left(2-\overline{\zeta}_i^q\right)^{\omega_i}+\prod_{i=1}^4\left(\overline{\zeta}_i^q\right)^{\omega_i}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^4\left(1+\overline{\eta}_i^q\right)^{\omega_i}-\prod_{i=1}^4\left(1-\overline{\eta}_i^q\right)^{\omega_i}}{\prod_{i=1}^4\left(1+\overline{\eta}_i^q\right)^{\omega_i}+\prod_{i=1}^4\left(1-\overline{\eta}_i^q\right)^{\omega_i}}}, \sqrt[q]{\frac{\prod_{i=1}^4\left(1+\overline{\xi}_i^q\right)^{\omega_i}-\prod_{i=1}^4\left(1-\overline{\xi}_i^q\right)^{\omega_i}}{\prod_{i=1}^4\left(1+\overline{\xi}_i^q\right)^{\omega_i}+\prod_{i=1}^4\left(1-\overline{\xi}_i^q\right)^{\omega_i}}}, \end{array} \right\rangle\right]$$

$$= (0.6385, 0.3916, 0.6820, 0.5366, 6535, 0.7528\ ).$$

Theorem 2 (Idempotency): Assuming $\mathcal{A}_i = \left(\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i\right)$ $(i = 1, 2, \ldots, n)$ be a collection of $q - SFRNs$ and $(\omega_1, \omega_2, \ldots, \omega_n)^T$ signifies the weight vector adhering to the condition $\omega_i > 0$ and the constrain $\sum_{i=1}^n \omega_i = 1$. $\mathcal{A}_i$ $(i = 1, 2, \ldots, n)$ are the same $\forall i$, then

$$q - SFREWG\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right) = \mathcal{A}$$

Proof : From Theorem 1, we have $q - SFREWG : \mathcal{A}^n \longrightarrow \mathcal{A}$ characterized by

$$q - SFREWG\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right) = \otimes_{\underset{\sim}{\mathrm{E}}}{}_{i=1}^{n} \mathcal{A}_i^{\omega_i}$$

$$= \left[\left\langle \begin{array}{c} \sqrt[q]{\frac{2\prod_{i=1}^n\left(\underline{\zeta}_i^q\right)^{\omega_i}}{\prod_{i=1}^n\left(2-\underline{\zeta}_i^q\right)^{\omega_i}+\prod_{i=1}^n\left(\underline{\zeta}_i^q\right)^{\omega_i}}}, \sqrt[q]{\frac{\prod_{i=1}^n\left(1+\underline{\eta}_i^q\right)^{\omega_i}-\prod_{i=1}^n\left(1-\underline{\eta}_i^q\right)^{\omega_i}}{\prod_{i=1}^n\left(1+\underline{\eta}_i^q\right)^{\omega_i}+\prod_{i=1}^n\left(1-\underline{\eta}_i^q\right)^{\omega_i}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^n\left(1+\underline{\xi}_i^q\right)^{\omega_i}-\prod_{i=1}^n\left(1-\underline{\xi}_i^q\right)^{\omega_i}}{\prod_{i=1}^n\left(1+\underline{\xi}_i^q\right)^{\omega_i}+\prod_{i=1}^n\left(1-\underline{\xi}_i^q\right)^{\omega_i}}}, \sqrt[q]{\frac{2\prod_{i=1}^n\left(\overline{\xi}_i^q\right)^{\omega_i}}{\prod_{i=1}^n\left(2-\overline{\xi}_i^q\right)^{\omega_i}+\prod_{i=1}^n\left(\overline{\xi}_i^q\right)^{\omega_i}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^n\left(1+\overline{\eta}_i^q\right)^{\omega_i}-\prod_{i=1}^n\left(1-\overline{\eta}_i^q\right)^{\omega_i}}{\prod_{i=1}^n\left(1+\overline{\eta}_i^q\right)^{\omega_i}+\prod_{i=1}^n\left(1-\overline{\eta}_i^q\right)^{\omega_i}}}, \sqrt[q]{\frac{\prod_{i=1}^n\left(1+\overline{\xi}_i^q\right)^{\omega_i}-\prod_{i=1}^n\left(1-\overline{\xi}_i^q\right)^{\omega_i}}{\prod_{i=1}^n\left(1+\overline{\xi}_i^q\right)^{\omega_i}+\prod_{i=1}^n\left(1-\overline{\xi}_i^q\right)^{\omega_i}}} \end{array} \right\rangle\right]$$

Since $\mathcal{A}_i$ $(i = 1, 2, \ldots, n)$ are the same ∀i, so, as shown in the equation at the bottom of the next page.

Theorem 3 (Boundness): Assuming

$$q - SFREWG\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k, \mathcal{A}_{k+1}\right) = \otimes_{\underset{\sim}{\mathrm{E}}}{}_{i=1}^{k} \mathcal{A}_i^{\omega_i} \otimes_{\underset{\sim}{\mathrm{E}}} \mathcal{A}_{k+1}{}^{\omega_{k+1}} = q - SFREWG\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right) = \otimes_{\underset{\sim}{\mathrm{E}}}{}_{i=1}^{k+1} \mathcal{A}_i^{\omega_i}$$

$$= \left[\left\langle \begin{array}{c} \sqrt[q]{\frac{2\prod_{i=1}^k\left(\underline{\zeta}_i^q\right)^{\omega_i}}{\prod_{i=1}^k\left(2-\underline{\zeta}_i^q\right)^{\omega_i}+\prod_{i=1}^n\left(\underline{\zeta}_i^q\right)^{\omega_i}}}, \sqrt[q]{\frac{\prod_{i=1}^k\left(1+\underline{\eta}_i^q\right)^{\omega_i}-\prod_{i=1}^n\left(1-\underline{\eta}_i^q\right)^{\omega_i}}{\prod_{i=1}^k\left(1+\underline{\eta}_i^q\right)^{\omega_i}+\prod_{i=1}^n\left(1-\underline{\eta}_i^q\right)^{\omega_i}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^k\left(1+\underline{\xi}_i^q\right)^{\omega_i}-\prod_{i=1}^n\left(1-\underline{\xi}_i^q\right)^{\omega_i}}{\prod_{i=1}^k\left(1+\underline{\xi}_i^q\right)^{\omega_i}+\prod_{i=1}^n\left(1-\underline{\xi}_i^q\right)^{\omega_i}}}, \sqrt[q]{\frac{2\prod_{i=1}^k\left(\overline{\zeta}_i^q\right)^{\omega_i}}{\prod_{i=1}^k\left(2-\overline{\zeta}_i^q\right)^{\omega_i}+\prod_{i=1}^n\left(\overline{\zeta}_i^q\right)^{\omega_i}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^k\left(1+\overline{\eta}_i^q\right)^{\omega_i}-\prod_{i=1}^k\left(1-\overline{\eta}_i^q\right)^{\omega_i}}{\prod_{i=1}^k\left(1+\overline{\eta}_i^q\right)^{\omega_i}+\prod_{i=1}^k\left(1-\overline{\eta}_i^q\right)^{\omega_i}}}, \sqrt[q]{\frac{\prod_{i=1}^k\left(1+\overline{\xi}_i^q\right)^{\omega_i}-\prod_{i=1}^k\left(1-\overline{\xi}_i^q\right)^{\omega_i}}{\prod_{i=1}^k\left(1+\overline{\xi}_i^q\right)^{\omega_i}+\prod_{i=1}^k\left(1-\overline{\xi}_i^q\right)^{\omega_i}}} \end{array} \right\rangle\right]$$

$$\otimes_{\underset{\sim}{\mathrm{E}}} \left[\left\langle \begin{array}{c} \sqrt[q]{\frac{2\left(\underline{\zeta}_{k+1}^q\right)^{\omega_{k+1}}}{\left(2-\underline{\zeta}_{k+1}^q\right)^{\omega_{k+1}}+\left(\underline{\zeta}_{k+1}^q\right)^{\omega_{k+1}}}}, \sqrt[q]{\frac{\left(1+\underline{\eta}_{k+1}^q\right)^{\omega_{k+1}}-\left(1-\underline{\eta}_{k+1}^q\right)^{\omega_{k+1}}}{\left(1+\underline{\eta}_{k+1}^q\right)^{\omega_{k+1}}+\left(1-\underline{\eta}_{k+1}^q\right)^{\omega_{k+1}}}}, \\ \sqrt[q]{\frac{\left(1+\underline{\xi}_{k+1}^q\right)^{\omega_{k+1}}-\left(1-\underline{\xi}_{k+1}^q\right)^{\omega_{k+1}}}{\left(1+\underline{\xi}_{k+1}^q\right)^{\omega_{k+1}}+\left(1-\underline{\xi}_{k+1}^q\right)^{\omega_{k+1}}}}, \sqrt[q]{\frac{2\prod_{i=1}^k\left(\overline{\zeta}_{k+1}^q\right)^{\omega_{k+1}}}{\left(2-\overline{\zeta}_{k+1}^q\right)^{\omega_{k+1}}+\left(\overline{\zeta}_{k+1}^q\right)^{\omega_{k+1}}}}, \\ \sqrt[q]{\frac{\left(1+\overline{\eta}_{k+1}^q\right)^{\omega_{k+1}}-\left(1-\overline{\eta}_{k+1}^q\right)^{\omega_{k+1}}}{\left(1+\overline{\eta}_{k+1}^q\right)^{\omega_{k+1}}+\left(1-\overline{\eta}_{k+1}^q\right)^{\omega_{k+1}}}}, \sqrt[q]{\frac{\left(1+\overline{\xi}_{k+1}^q\right)^{\omega_{k+1}}-\left(1-\overline{\xi}_{k+1}^q\right)^{\omega_{k+1}}}{\left(1+\overline{\xi}_{k+1}^q\right)^{\omega_{k+1}}+\left(1-\overline{\xi}_{k+1}^q\right)^{\omega_{k+1}}}} \end{array} \right\rangle\right]$$

$$= \left[\left\langle \begin{array}{c} \sqrt[q]{\frac{2\prod_{i=1}^{k+1}\left(\underline{\zeta}_i^q\right)^{\omega_i}}{\prod_{i=1}^{k+1}\left(2-\underline{\zeta}_i^q\right)^{\omega_i}+\prod_{i=1}^{k+1}\left(\underline{\zeta}_i^q\right)^{\omega_i}}}, \sqrt[q]{\frac{\prod_{i=1}^{k+1}\left(1+\underline{\eta}_i^q\right)^{\omega_i}-\prod_{i=1}^{k+1}\left(1-\underline{\eta}_i^q\right)^{\omega_i}}{\prod_{i=1}^{k+1}\left(1+\underline{\eta}_i^q\right)^{\omega_i}+\prod_{i=1}^{k+1}\left(1-\underline{\eta}_i^q\right)^{\omega_i}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^{k+1}\left(1+\underline{\xi}_i^q\right)^{\omega_i}-\prod_{i=1}^{k+1}\left(1-\underline{\xi}_i^q\right)^{\omega_i}}{\prod_{i=1}^{k+1}\left(1+\underline{\xi}_i^q\right)^{\omega_i}+\prod_{i=1}^{k+1}\left(1-\underline{\xi}_i^q\right)^{\omega_i}}}, \sqrt[q]{\frac{2\prod_{i=1}^{k+1}\left(\overline{\zeta}_i^q\right)^{\omega_i}}{\prod_{i=1}^{k+1}\left(2-\overline{\zeta}_i^q\right)^{\omega_i}+\prod_{i=1}^{k+1}\left(\overline{\zeta}_i^q\right)^{\omega_i}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^{k+1}\left(1+\overline{\eta}_i^q\right)^{\omega_i}-\prod_{i=1}^{k+1}\left(1-\overline{\eta}_i^q\right)^{\omega_i}}{\prod_{i=1}^{k+1}\left(1+\overline{\eta}_i^q\right)^{\omega_i}+\prod_{i=1}^{k+1}\left(1-\overline{\eta}_i^q\right)^{\omega_i}}}, \sqrt[q]{\frac{\prod_{i=1}^{k+1}\left(1+\overline{\xi}_i^q\right)^{\omega_i}-\prod_{i=1}^{k+1}\left(1-\overline{\xi}_i^q\right)^{\omega_i}}{\prod_{i=1}^{k+1}\left(1+\overline{\xi}_i^q\right)^{\omega_i}+\prod_{i=1}^{k+1}\left(1-\overline{\xi}_i^q\right)^{\omega_i}}} \end{array} \right\rangle\right].$$

$\mathcal{A}_i = \left(\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i\right)$ $(i = 1, 2, \ldots, n)$ be a collection of $q - SFRNs$and$(\omega_1, \omega_2, \ldots, \omega_n)^T$ signifies the weight vector adhering to the condition $\omega_i > 0$ and the constrain $\sum_{i=1}^{n} \omega_i = 1$.Let$\mathcal{A}^-\left(\min\underline{\zeta}_i, \min\underline{\eta}_i, \max\underline{\xi}_i, \min\overline{\zeta}_i, \min\overline{\eta}_i, \max\overline{\xi}_i\right)$ and $\mathcal{A}^+ = \left(\max\underline{\zeta}_i, \max\underline{\eta}_i, \min\underline{\xi}_i, \max\overline{\zeta}_i, \max\overline{\eta}_i, \min\overline{\xi}_i\right)$ Then $\mathcal{A}^- \leq q - SFREWG(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \mathcal{A}^+$

Proof: From Theorem 1, we have

$$q - SFREWG(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \otimes_{\underset{\sim}{E}\, i=1}^{n} \mathcal{A}_i^{\omega_i}$$

$$= \left[\left\langle \begin{array}{c} \sqrt[q]{\frac{2\prod_{i=1}^{n}\left(\underline{\zeta}_i^q\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\underline{\zeta}_i^q\right)^{\omega_i}+\prod_{i=1}^{n}\left(\underline{\zeta}_i^q\right)^{\omega_i}}}, \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\underline{\eta}_i^q\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\underline{\eta}_i^q\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\underline{\eta}_i^q\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\underline{\eta}_i^q\right)^{\omega_i}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\underline{\xi}_i^q\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\underline{\xi}_i^q\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\underline{\xi}_i^q\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\underline{\xi}_i^q\right)^{\omega_i}}}, \sqrt[q]{\frac{2\prod_{i=1}^{n}\left(\overline{\xi}_i^q\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\overline{\xi}_i^q\right)^{\omega_i}+\prod_{i=1}^{n}\left(\overline{\xi}_i^q\right)^{\omega_i}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\overline{\eta}_i^q\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\overline{\eta}_i^q\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\overline{\eta}_i^q\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\overline{\eta}_i^q\right)^{\omega_i}}}, \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\overline{\xi}_i^q\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\overline{\xi}_i^q\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\overline{\xi}_i^q\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\overline{\xi}_i^q\right)^{\omega_i}}} \end{array} \right\rangle\right]$$

For lower and upper memberships:

We have

$$\min_i \underline{\zeta}_i \leq \underline{\zeta}_i \leq \max_i \underline{\zeta}_i$$

$$\Longrightarrow \min_i \underline{\zeta}_i{}^q \leq \underline{\zeta}_i{}^q \leq \max_i \underline{\zeta}_i{}^q$$

$$\Longrightarrow 2\min_i\left(\underline{\zeta}_i{}^q\right)^{\omega_i} \leq 2\prod_{j=1}^{n}\left(\underline{\zeta}_i{}^q\right)^{\omega_i} \leq 2\max_i\left(\underline{\zeta}_i{}^q\right)^{\omega_i}$$

$$\Longrightarrow \frac{2\min_i\left(\underline{\zeta}_i{}^q\right)^{\omega_i}}{\left(2-\min_i\underline{\zeta}_i{}^q\right)^{\omega_i}+\min_i\left(\underline{\zeta}_i{}^q\right)^{\omega_i}} \leq \frac{2\prod_{i=1}^{n}\left(\underline{\zeta}_i{}^q\right)^{\omega_i}}{\left(2-\underline{\zeta}_i{}^q\right)^{\omega_i}+\prod_{i=1}^{n}\left(\underline{\zeta}_i{}^q\right)^{\omega_i}} \leq \frac{2\max_i\left(\underline{\zeta}_i{}^q\right)^{\omega_i}}{\left(2-\max_i\underline{\zeta}_i{}^q\right)^{\omega_i}+\max_i\left(\underline{\zeta}_i{}^q\right)^{\omega_i}}$$

$$\Longrightarrow \sqrt[q]{\frac{2\min_i\left(\underline{\zeta}_i{}^q\right)^{\omega_i}}{\left(2-\min_i\underline{\zeta}_i{}^q\right)^{\omega_i}+\min_i\left(\underline{\zeta}_i{}^q\right)^{\omega_i}}}$$

$$q - SFREWG(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$$

$$= \left[\left\langle \begin{array}{c} \sqrt[q]{\frac{2\prod_{i=1}^{n}\left(\underline{\zeta}_i^q\right)^{\sum_{i=1}^{n}\omega_i=1}}{\prod_{i=1}^{n}\left(2-\underline{\zeta}_i^q\right)^{\sum_{i=1}^{n}\omega_i=1}+\prod_{i=1}^{n}\left(\underline{\zeta}_i^q\right)^{\sum_{i=1}^{n}\omega_i=1}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\underline{\eta}_i^q\right)^{\sum_{i=1}^{n}\omega_i=1}-\prod_{i=1}^{n}\left(1-\underline{\eta}_i^q\right)^{\sum_{i=1}^{n}\omega_i=1}}{\prod_{i=1}^{n}\left(1+\underline{\eta}_i^q\right)^{\sum_{i=1}^{n}\omega_i=1}+\prod_{i=1}^{n}\left(1-\underline{\eta}_i^q\right)^{\sum_{i=1}^{n}\omega_i=1}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\underline{\xi}_i^q\right)^{\sum_{i=1}^{n}\omega_i=1}-\prod_{i=1}^{n}\left(1-\underline{\xi}_i^q\right)^{\sum_{i=1}^{n}\omega_i=1}}{\prod_{i=1}^{n}\left(1+\underline{\xi}_i^q\right)^{\sum_{i=1}^{n}\omega_i=1}+\prod_{i=1}^{n}\left(1-\underline{\xi}_i^q\right)^{\sum_{i=1}^{n}\omega_i=1}}}, \\ \sqrt[q]{\frac{2\prod_{i=1}^{n}\left(\overline{\zeta}_i^q\right)^{\sum_{i=1}^{n}\omega_i=1}}{\prod_{i=1}^{n}\left(2-\overline{\zeta}_i^q\right)^{\sum_{i=1}^{n}\omega_i=1}+\prod_{i=1}^{n}\left(\overline{\zeta}_i^q\right)^{\sum_{i=1}^{n}\omega_i=1}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^{n}(1+\overline{\eta}_i^q)^{\sum_{i=1}^{n}\omega_i=1}-\prod_{i=1}^{n}(1-\overline{\eta}_i^q)^{\sum_{i=1}^{n}\omega_i=1}}{\prod_{i=1}^{n}(1+\overline{\eta}_i^q)^{\sum_{i=1}^{n}\omega_i=1}+\prod_{i=1}^{n}(1-\overline{\eta}_i^q)^{\sum_{i=1}^{n}\omega_i=1}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\overline{\xi}_i^q\right)^{\sum_{i=1}^{n}\omega_i=1}-\prod_{i=1}^{n}\left(1-\overline{\xi}_i^q\right)^{\sum_{i=1}^{n}\omega_i=1}}{\prod_{i=1}^{n}\left(1+\overline{\xi}_i^q\right)^{\sum_{i=1}^{n}\omega_i=1}+\prod_{i=1}^{n}\left(1-\overline{\xi}_i^q\right)^{\sum_{i=1}^{n}\omega_i=1}}} \end{array} \right\rangle\right]$$

$$= \left[\left\langle \begin{array}{c} \sqrt[q]{\frac{2\left(\underline{\zeta}_i^q\right)}{\left(2-\underline{\zeta}_i^q\right)+\left(\underline{\zeta}_i^q\right)}}, \sqrt[q]{\frac{\left(1+\underline{\eta}_i^q\right)-\left(1-\underline{\eta}_i^q\right)}{\left(1+\underline{\eta}_i^q\right)+\left(1-\underline{\eta}_i^q\right)}}, \sqrt[q]{\frac{\left(1+\underline{\xi}_i^q\right)-\left(1-\underline{\xi}_i^q\right)}{\left(1+\underline{\xi}_i^q\right)+\left(1-\underline{\xi}_i^q\right)}} \\ \sqrt[q]{\frac{2\left(\overline{\zeta}_i^q\right)}{\left(2-\overline{\zeta}_i^q\right)+\left(\overline{\zeta}_i^q\right)}}, \sqrt[q]{\frac{(1+\overline{\eta}_i^q)-(1-\overline{\eta}_i^q)}{(1+\overline{\eta}_i^q)+(1-\overline{\eta}_i^q)}}, \sqrt[q]{\frac{\left(1+\overline{\xi}_i^q\right)-\left(1-\overline{\xi}_i^q\right)}{\left(1+\overline{\xi}_i^q\right)+\left(1-\overline{\xi}_i^q\right)}} \end{array} \right\rangle\right]$$

$$= (\underline{\zeta}, \underline{\eta}, \underline{\xi}, \overline{\zeta}, \overline{\eta}, \overline{\xi}) = \mathcal{A}.$$

$$\leq \sqrt[q]{\frac{2\prod_{i=1}^{n}\left(\underline{\zeta}_i^{\,q}\right)^{\omega_i}}{\left(2-\underline{\zeta}_i^{\,q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(\underline{\zeta}_i^{\,q}\right)^{\omega_i}}}$$

$$\leq \sqrt[q]{\frac{2\max_i\left(\underline{\zeta}_i^{\,q}\right)^{\omega_i}}{\left(2-\max_i\underline{\zeta}_i^{\,q}\right)^{\omega_i}+\max_i\left(\underline{\zeta}_i^{\,q}\right)^{\omega_i}}}$$

Also, we have

$$\min_i\overline{\zeta}_i \leq \overline{\zeta}_i \leq \max_i\overline{\zeta}_i$$

$$\Longrightarrow \min_i\overline{\zeta}_i^{\,q} \leq \overline{\zeta}_i^{\,q} \leq \max_i\overline{\zeta}_i^{\,q}$$

$$\Longrightarrow 2\min_i\left(\overline{\zeta}_i^{\,q}\right)^{\omega_i} \leq 2\prod_{j=1}^{n}\left(\overline{\zeta}_i^{\,q}\right)^{\omega_i}$$

$$\leq 2\max_i\left(\overline{\zeta}_i^{\,q}\right)^{\omega_i}$$

$$\Longrightarrow \frac{2\min_i\left(\overline{\zeta}_i^{\,q}\right)^{\omega_i}}{\left(2-\min_i\overline{\zeta}_i^{\,q}\right)^{\omega_i}+\min_i\left(\overline{\zeta}_i^{\,q}\right)^{\omega_i}}$$

$$\leq \frac{2\prod_{i=1}^{n}\left(\overline{\zeta}_i^{\,q}\right)^{\omega_i}}{\left(2-\overline{\zeta}_i^{\,q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(\overline{\zeta}_i^{\,q}\right)^{\omega_i}}$$

$$\leq \frac{2\max_i\left(\overline{\zeta}_i^{\,q}\right)^{\omega_i}}{\left(2-\max_i\overline{\zeta}_i^{\,q}\right)^{\omega_i}+\max_i\left(\overline{\zeta}_i^{\,q}\right)^{\omega_i}}$$

$$\Longrightarrow \sqrt[q]{\frac{2\min_i\left(\overline{\zeta}_i^{\,q}\right)^{\omega_i}}{\left(2-\min_i\overline{\zeta}_i^{\,q}\right)^{\omega_i}+\min_i\left(\overline{\zeta}_i^{\,q}\right)^{\omega_i}}}$$

$$\leq \sqrt[q]{\frac{2\prod_{i=1}^{n}\left(\overline{\zeta}_i^{\,q}\right)^{\omega_i}}{\left(2-\overline{\zeta}_i^{\,q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(\overline{\zeta}_i^{\,q}\right)^{\omega_i}}}$$

$$\leq \sqrt[q]{\frac{2\max_i\left(\overline{\zeta}_i^{\,q}\right)^{\omega_i}}{\left(2-\max_i\overline{\zeta}_i^{\,q}\right)^{\omega_i}+\max_i\left(\overline{\zeta}_i^{\,q}\right)^{\omega_i}}}$$

For lower and upper neutral memberships:
We have

$$\max_i\underline{\eta}_i$$

$$\geq \underline{\eta}_i \geq \min_i\underline{\eta}_i$$

$$\max_i\underline{\eta}_i^{\,q}$$

$$\geq \underline{\eta}_i^{\,q} \geq \min_i\underline{\eta}_i^{\,q}$$

$$\Longrightarrow \left(1+\max_i\underline{\eta}_i^{\,q}\right)^{\omega_i} \geq \prod_{i=1}^{n}\left(1+\underline{\eta}_i^{\,q}\right)^{\omega_i}$$

$$\geq \left(1+\min_i\underline{\eta}_i^{\,q}\right)^{\omega_i}$$

$$\Longrightarrow \prod_{i=1}^{n}\left(1+\max_i\underline{\eta}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\max_i\underline{\eta}_i^{\,q}\right)^{\omega_i}$$

$$\geq \prod_{i=1}^{n}\left(1+\underline{\eta}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\underline{\eta}_i^{\,q}\right)^{\omega_i}$$

$$\geq \prod_{i=1}^{n}\left(1+\min_i\underline{\eta}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\min_i\underline{\eta}_i^{\,q}\right)^{\omega_i}$$

$$\Longrightarrow \frac{\prod_{i=1}^{n}\left(1+\max_i\underline{\eta}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\max_i\underline{\eta}_i^{\,q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\max_i\underline{\eta}_i^{\,q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\max_i\underline{\eta}_i^{\,q}\right)^{\omega_i}}$$

$$\geq \frac{\prod_{i=1}^{n}\left(1+\underline{\eta}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\underline{\eta}_i^{\,q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\underline{\eta}_i^{\,q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\underline{\eta}_i^{\,q}\right)^{\omega_i}}$$

$$\geq \frac{\prod_{i=1}^{n}\left(1+\min_i\underline{\eta}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\min_i\underline{\eta}_i^{\,q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\min_i\underline{\eta}_i^{\,q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\min_i\underline{\eta}_i^{\,q}\right)^{\omega_i}}$$

$$\Longrightarrow \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\max_i\underline{\eta}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\max_i\underline{\eta}_i^{\,q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\max_i\underline{\eta}_i^{\,q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\max_i\underline{\eta}_i^{\,q}\right)^{\omega_i}}}$$

$$\geq \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\underline{\eta}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\underline{\eta}_i^{\,q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\underline{\eta}_i^{\,q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\underline{\eta}_i^{\,q}\right)^{\omega_i}}}$$

$$\geq \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\min_i\underline{\eta}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\min_i\underline{\eta}_i^{\,q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\min_i\underline{\eta}_i^{\,q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\min_i\underline{\eta}_i^{\,q}\right)^{\omega_i}}}$$

And since

$$\max_i\overline{\eta}_i$$

$$\geq \overline{\eta}_i \geq \min_i\overline{\eta}_i$$

$$\Longrightarrow \max_i\overline{\eta}_i^{\,q} \geq \overline{\eta}_i^{\,q} \geq \min_i\overline{\eta}_i^{\,q}$$

$$\Longrightarrow \left(1+\max_i\overline{\eta}_i^{\,q}\right)^{\omega_i} \geq \prod_{i=1}^{n}\left(1+\overline{\eta}_i^{\,q}\right)^{\omega_i}$$

$$\geq \left(1+\min_i\overline{\eta}_i^{\,q}\right)^{\omega_i}$$

$$\Longrightarrow \prod_{i=1}^{n}\left(1+\max_i\overline{\eta}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\max_i\overline{\eta}_i^{\,q}\right)^{\omega_i}$$

$$\geq \prod_{i=1}^{n}\left(1+\overline{\eta}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\overline{\eta}_i^{\,q}\right)^{\omega_i}$$

$$\geq \prod_{i=1}^{n}\left(1+\min_i\overline{\eta}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\min_i\overline{\eta}_i^{\,q}\right)^{\omega_i}$$

$$\Longrightarrow \frac{\prod_{i=1}^{n}\left(1+\max_i \overline{\eta}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\max_i \overline{\eta}_i^{\,q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\max_i \overline{\eta}_i^{\,q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\max_i \overline{\eta}_i^{\,q}\right)^{\omega_i}}$$

$$\geq \frac{\prod_{i=1}^{n}\left(1+\overline{\eta}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\overline{\eta}_i^{\,q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\overline{\eta}_i^{\,q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\overline{\eta}_i^{\,q}\right)^{\omega_i}}$$

$$\geq \frac{\prod_{i=1}^{n}\left(1+\min_i \overline{\eta}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\min_i \overline{\eta}_i^{\,q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\min_i \overline{\eta}_i^{\,q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\min_i \overline{\eta}_i^{\,q}\right)^{\omega_i}}$$

$$\Longrightarrow \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\max_i \overline{\eta}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\max_i \overline{\eta}_i^{\,q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\max_i \overline{\eta}_i^{\,q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\max_i \overline{\eta}_i^{\,q}\right)^{\omega_i}}}$$

$$\geq \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\overline{\eta}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\overline{\eta}_i^{\,q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\overline{\eta}_i^{\,q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\overline{\eta}_i^{\,q}\right)^{\omega_i}}}$$

$$\geq \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\min_i \overline{\eta}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\min_i \overline{\eta}_i^{\,q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\min_i \overline{\eta}_i^{\,q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\min_i \overline{\eta}_i^{\,q}\right)^{\omega_i}}}$$

For lower and upper non memberships:
We have

$$\max_i \underline{\xi}_i \geq \underline{\xi}_i \geq \min_i \underline{\xi}_i$$

$$\max_i \underline{\xi}_i^{\,q} \geq \underline{\xi}_i^{\,q} \geq \min_i \underline{\xi}_i^{\,q}$$

$$\Longrightarrow \left(1+\max_i \underline{\xi}_i^{\,q}\right)^{\omega_i} \geq \prod_{i=1}^{n}\left(1+\underline{\xi}_i^{\,q}\right)^{\omega_i} \geq \left(1+\min_i \underline{\xi}_i^{\,q}\right)^{\omega_i}$$

$$\Longrightarrow \prod_{i=1}^{n}\left(1+\max_i \underline{\xi}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\max_i \underline{\xi}_i^{\,q}\right)^{\omega_i} \geq \prod_{i=1}^{n}\left(1+\underline{\xi}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\underline{\xi}_i^{\,q}\right)^{\omega_i} \geq \prod_{i=1}^{n}\left(1+\min_i \underline{\xi}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\min_i \underline{\xi}_i^{\,q}\right)^{\omega_i}$$

$$\Longrightarrow \frac{\prod_{i=1}^{n}\left(1+\max_i \underline{\xi}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\max_i \underline{\xi}_i^{\,q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\max_i \underline{\xi}_i^{\,q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\max_i \underline{\xi}_i^{\,q}\right)^{\omega_i}}$$

$$\geq \frac{\prod_{i=1}^{n}\left(1+\underline{\xi}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\underline{\xi}_i^{\,q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\underline{\xi}_i^{\,q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\underline{\xi}_i^{\,q}\right)^{\omega_i}}$$

$$\geq \frac{\prod_{i=1}^{n}\left(1+\min_i \underline{\xi}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\min_i \underline{\xi}_i^{\,q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\min_i \underline{\xi}_i^{\,q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\min_i \underline{\xi}_i^{\,q}\right)^{\omega_i}}$$

$$\Longrightarrow \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\max_i \underline{\xi}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\max_i \underline{\xi}_i^{\,q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\max_i \underline{\xi}_i^{\,q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\max_i \underline{\xi}_i^{\,q}\right)^{\omega_i}}}$$

$$\geq \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\underline{\xi}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\underline{\xi}_i^{\,q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\underline{\xi}_i^{\,q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\underline{\xi}_i^{\,q}\right)^{\omega_i}}}$$

$$\geq \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\min_i \underline{\xi}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\min_i \underline{\xi}_i^{\,q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\min_i \underline{\xi}_i^{\,q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\min_i \underline{\xi}_i^{\,q}\right)^{\omega_i}}}$$

And since

$$\max_i \overline{\xi}_i \geq \overline{\xi}_i \geq \min_i \overline{\xi}_i$$

$$\Longrightarrow \max_i \overline{\xi}_i^{\,q} \geq \overline{\xi}_i^{\,q} \geq \min_i \overline{\xi}_i^{\,q}$$

$$\Longrightarrow \left(1+\max_i \overline{\xi}_i^{\,q}\right)^{\omega_i} \geq \prod_{i=1}^{n}\left(1+\overline{\xi}_i^{\,q}\right)^{\omega_i} \geq \left(1+\min_i \overline{\xi}_i^{\,q}\right)^{\omega_i}$$

$$\Longrightarrow \prod_{i=1}^{n}\left(1+\max_i \overline{\xi}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\max_i \overline{\xi}_i^{\,q}\right)^{\omega_i} \geq \prod_{i=1}^{n}\left(1+\overline{\xi}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\overline{\xi}_i^{\,q}\right)^{\omega_i} \geq \prod_{i=1}^{n}\left(1+\min_i \overline{\xi}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\min_i \overline{\xi}_i^{\,q}\right)^{\omega_i}$$

$$\Longrightarrow \frac{\prod_{i=1}^{n}\left(1+\max_i \overline{\xi}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\max_i \overline{\xi}_i^{\,q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\max_i \overline{\xi}_i^{\,q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\max_i \overline{\xi}_i^{\,q}\right)^{\omega_i}}$$

$$\geq \frac{\prod_{i=1}^{n}\left(1+\overline{\xi}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\overline{\xi}_i^{\,q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\overline{\xi}_i^{\,q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\overline{\xi}_i^{\,q}\right)^{\omega_i}}$$

$$\geq \frac{\prod_{i=1}^{n}\left(1+\min_i \overline{\xi}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\min_i \overline{\xi}_i^{\,q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\min_i \overline{\xi}_i^{\,q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\min_i \overline{\xi}_i^{\,q}\right)^{\omega_i}}$$

$$\Longrightarrow \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\max_i \overline{\xi}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\max_i \overline{\xi}_i^{\,q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\max_i \overline{\xi}_i^{\,q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\max_i \overline{\xi}_i^{\,q}\right)^{\omega_i}}}$$

$$\geq \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\overline{\xi}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\overline{\xi}_i^{\,q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\overline{\xi}_i^{\,q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\overline{\xi}_i^{\,q}\right)^{\omega_i}}}$$

$$\geq \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\min_i\overline{\xi}_i^{\,q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\min_i\overline{\xi}_i^{\,q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\min_i\overline{\xi}_i^{\,q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\min_i\overline{\xi}_i^{\,q}\right)^{\omega_i}}}.$$

Hence $\mathcal{A}^- \leq q-SFREWG(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n) \leq \mathcal{A}^+$.

Example 3: Consider four q-SFRNs $\mathcal{A}_1=(0.3, 0.4, 0.1, 0.3, 0.2, 0.4)$, $\mathcal{A}_2 = (0.5, 0.5, 0.9, 0.2, 0.8, 0.6)$, $\mathcal{A}_3 = (0.4, 0.5, 0.9, 0.3, 0.2, 0.6)$ and $\mathcal{A}_4=(0.5, 0.2, 0.5, 0.2, 0.8, 0.9)$ be any four q-SFRNs, if $\omega=(0.3, 0.1, 0.4, 0.2)^T$ and $q=3$ then the q-SFREWG operator values solved in Example (2) can be write as:

$$\left[\left\langle \begin{array}{c} \sqrt[q]{\frac{2\prod_{i=1}^{4}\left(\underline{\zeta}_i^{q}\right)^{\omega_i}}{\prod_{i=1}^{4}\left(2-\underline{\zeta}_i^{q}\right)^{\omega_i}+\prod_{i=1}^{4}\left(\underline{\zeta}_i^{q}\right)^{\omega_i}}},\ \sqrt[q]{\frac{\prod_{i=1}^{4}\left(1+\underline{\eta}_i^{q}\right)^{\omega_i}-\prod_{i=1}^{4}\left(1-\underline{\eta}_i^{q}\right)^{\omega_i}}{\prod_{i=1}^{4}\left(1+\underline{\eta}_i^{q}\right)^{\omega_i}+\prod_{i=1}^{4}\left(1-\underline{\eta}_i^{q}\right)^{\omega_i}}},\\ \sqrt[q]{\frac{\prod_{i=1}^{4}\left(1+\underline{\xi}_i^{q}\right)^{\omega_i}-\prod_{i=1}^{4}\left(1-\underline{\xi}_i^{q}\right)^{\omega_i}}{\prod_{i=1}^{4}\left(1+\underline{\xi}_i^{q}\right)^{\omega_i}+\prod_{i=1}^{4}\left(1-\underline{\xi}_i^{q}\right)^{\omega_i}}},\ \sqrt[q]{\frac{2\prod_{i=1}^{4}\left(\overline{\zeta}_i^{q}\right)^{\omega_i}}{\prod_{i=1}^{4}\left(2-\overline{\zeta}_i^{q}\right)^{\omega_i}+\prod_{i=1}^{4}\left(\overline{\zeta}_i^{q}\right)^{\omega_i}}},\\ \sqrt[q]{\frac{\prod_{i=1}^{4}\left(1+\overline{\eta}_i^{q}\right)^{\omega_i}-\prod_{i=1}^{4}\left(1-\overline{\eta}_i^{q}\right)^{\omega_i}}{\prod_{i=1}^{4}\left(1+\overline{\eta}_i^{q}\right)^{\omega_i}+\prod_{i=1}^{4}\left(1-\overline{\eta}_i^{q}\right)^{\omega_i}}},\ \sqrt[q]{\frac{\prod_{i=1}^{4}\left(1+\overline{\xi}_i^{q}\right)^{\omega_i}-\prod_{i=1}^{4}\left(1-\overline{\xi}_i^{q}\right)^{\omega_i}}{\prod_{i=1}^{4}\left(1+\overline{\xi}_i^{q}\right)^{\omega_i}+\prod_{i=1}^{4}\left(1-\overline{\xi}_i^{q}\right)^{\omega_i}}} \end{array}\right\rangle\right]$$

$= (0.6385, 0.3916, 0.6820, 0.5366, 0.6535, 0.7528)$

$\mathcal{A}^-$

$= \left(\min\underline{\zeta}_i,\ \max\underline{\eta}_i,\ \max\underline{\xi}_i,\ \min\overline{\zeta}_i,\ \max\overline{\eta}_i,\ \max\overline{\xi}_i\right)$

$= (0.3, 0.5, 0.9, 0.2, 0.8, 0.9)$

$\mathcal{A}^+$

$= \left(\max\underline{\zeta}_i,\ \min\underline{\eta}_i,\ \min\underline{\xi}_i,\ \max\overline{\zeta}_i,\ \min\overline{\eta}_i,\ \min\overline{\xi}_i\right)$

$= (0.5, 0.2, 0.1, 0.3, 0.2, 0.4)$

Since $\mathcal{A}^- \leq q-SFREWG(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n) \leq \mathcal{A}^+$ holds, it confirms the boundness property as stated in Theorem 3.

Theorem 4 (Monotonicity): Assuming $\mathcal{A}_i = \left(\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i\right)$ $(i=1,2,\ldots,n)$ and $\mathcal{A}_i^* = \left(\underline{\zeta}_i^*, \underline{\eta}_i^*, \underline{\xi}_i^*, \overline{\zeta}_i^*, \overline{\eta}_i^*, \overline{\xi}_i^*\right)$ $(i=1,2,\ldots,n)$ be a collection of two $q-SFRNs$ such that $\mathcal{A}_i \leq \mathcal{A}_i^*$ for all i, then $q-SFREWG(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n) \leq$

$$q-SFREWG\left(\mathcal{A}_1^*,\mathcal{A}_2^*,\ldots,\mathcal{A}_n^*\right).$$

Proof: From Theorem 1, we have

$$q-SFREWG(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n) = \otimes_{\underset{\sim}{\mathrm{E}}\,i=1}^{n}\mathcal{A}_i^{\omega_i}$$

$$= \left[\left\langle \begin{array}{c} \sqrt[q]{\frac{2\prod_{i=1}^{n}\left(\underline{\zeta}_i^{q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\underline{\zeta}_i^{q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(\underline{\zeta}_i^{q}\right)^{\omega_i}}},\ \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\underline{\eta}_i^{q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\underline{\eta}_i^{q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\underline{\eta}_i^{q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\underline{\eta}_i^{q}\right)^{\omega_i}}},\\ \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\underline{\xi}_i^{q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\underline{\xi}_i^{q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\underline{\xi}_i^{q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\underline{\xi}_i^{q}\right)^{\omega_i}}},\ \sqrt[q]{\frac{\prod_{i=1}^{n}\left(\overline{\zeta}_i^{q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\overline{\zeta}_i^{q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(\overline{\zeta}_i^{q}\right)^{\omega_i}}},\\ \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\overline{\eta}_i^{q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\overline{\eta}_i^{q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\overline{\eta}_i^{q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\overline{\eta}_i^{q}\right)^{\omega_i}}},\ \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\overline{\xi}_i^{q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\overline{\xi}_i^{q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\overline{\xi}_i^{q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\overline{\xi}_i^{q}\right)^{\omega_i}}} \end{array}\right\rangle\right]$$

and

$$q-SFREWG\left(\mathcal{A}_1^*,\mathcal{A}_2^*,\ldots,\mathcal{A}_n^*\right) = \otimes_{\underset{\sim}{\mathrm{E}}\,i=1}^{n}\mathcal{A}_i^{*\omega_i}$$

$$= \left[\left\langle \begin{array}{c} \sqrt[q]{\frac{2\prod_{i=1}^{n}\left(\underline{\zeta}_i^{*q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\underline{\zeta}_i^{*q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(\underline{\zeta}_i^{*q}\right)^{\omega_i}}},\ \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\underline{\eta}_i^{*q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\underline{\eta}_i^{*q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\underline{\eta}_i^{*q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\underline{\eta}_i^{*q}\right)^{\omega_i}}},\\ \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\underline{\xi}_i^{*q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\underline{\xi}_i^{*q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\underline{\xi}_i^{*q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\underline{\xi}_i^{*q}\right)^{\omega_i}}},\ \sqrt[q]{\frac{\prod_{i=1}^{n}\left(\overline{\zeta}_i^{*q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\overline{\zeta}_i^{*q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(\overline{\zeta}_i^{*q}\right)^{\omega_i}}},\\ \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\overline{\eta}_i^{*q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\overline{\eta}_i^{*q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\overline{\eta}_i^{*q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\overline{\eta}_i^{*q}\right)^{\omega_i}}},\ \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\overline{\xi}_i^{*q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\overline{\xi}_i^{*q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\overline{\xi}_i^{*q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\overline{\xi}_i^{*q}\right)^{\omega_i}}} \end{array}\right\rangle\right]$$

For the lower and upper memberships:

As, $\underline{\zeta}_i \leq \underline{\zeta}_i^* \Longrightarrow \underline{\zeta}_i^q \leq \underline{\zeta}_i^{*q} \Longrightarrow \left(\underline{\zeta}_i^q\right)^{\omega_i} \leq \left(\underline{\zeta}_i^{*q}\right)^{\omega_i}$

$$\Longrightarrow 2\prod_{i=1}^{n}\left(\underline{\zeta}_i^q\right)^{\omega_i} \leq 2\prod_{i=1}^{n}\left(\underline{\zeta}_i^{*q}\right)^{\omega_i}$$

$$\sqrt[q]{\frac{2\prod_{i=1}^{n}\left(\underline{\zeta}_i^q\right)^{\omega_i}}{\prod_{i=1}^{n}(2-\underline{\zeta}_i^q)^{\omega_i}+\prod_{i=1}^{n}\left(\underline{\zeta}_i^q\right)^{\omega_i}}}$$

$$\leq \sqrt[q]{\frac{2\prod_{i=1}^{n}\left(\underline{\zeta}_i^{*q}\right)^{\omega_i}}{\prod_{i=1}^{n}(2-\underline{\zeta}_i^{*q})^{\omega_i}+\prod_{i=1}^{n}\left(\underline{\zeta}_i^{*q}\right)^{\omega_i}}}$$

And, $\overline{\zeta}_i \leq \overline{\zeta}_i^* \Longrightarrow \overline{\zeta}_i^q \leq \overline{\zeta}_i^{*q} \Longrightarrow \left(\overline{\zeta}_i^q\right)^{\omega_i} \leq \left(\overline{\zeta}_i^{*q}\right)^{\omega_i}$

$$\Longrightarrow 2\prod_{i=1}^{n}\left(\overline{\zeta}_i^q\right)^{\omega_i} \leq 2\prod_{i=1}^{n}\left(\overline{\zeta}_i^{*q}\right)^{\omega_i}$$

$$\sqrt[q]{\frac{2\prod_{i=1}^{n}\left(\overline{\zeta}_i^q\right)^{\omega_i}}{\prod_{i=1}^{n}(2-\overline{\zeta}_i^q)^{\omega_i}+\prod_{i=1}^{n}\left(\overline{\zeta}_i^q\right)^{\omega_i}}}$$

$$\leq \sqrt[q]{\frac{2\prod_{i=1}^{n}\left(\overline{\zeta}_i^{*q}\right)^{\omega_i}}{\prod_{i=1}^{n}(2-\overline{\zeta}_i^{*q})^{\omega_i}+\prod_{i=1}^{n}\left(\overline{\zeta}_i^{*q}\right)^{\omega_i}}}.$$

For the lower and upper neutral memberships:

As, $\underline{\eta}_i \geq \underline{\eta}_i^* \Longrightarrow \underline{\eta}_i^q \geq \underline{\eta}_i^{*q} \Longrightarrow 1+\underline{\eta}_i^q \geq 1+\underline{\eta}_i^{*q}$

$$\Longrightarrow \left(1+\underline{\eta}_i^q\right)^{\omega_i} \geq \left(1+\underline{\eta}_i^{*q}\right)^{\omega_i}$$

$$\Longrightarrow \prod_{i=1}^{n}\left(1+\underline{\eta}_i^q\right)^{\omega_i} \geq \prod_{i=1}^{n}\left(1+\underline{\eta}_i^{*q}\right)^{\omega_i}$$

$$\sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\underline{\eta}_i^q\right)^{\omega_i}-\prod_{j=1}^{n}\left(1-\underline{\eta}_i^q\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\underline{\eta}_i^q\right)^{\omega_i}+\prod_{j=1}^{n}\left(1-\underline{\eta}_i^q\right)^{\omega_i}}}$$

$$\geq \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\underline{\eta}_i^{*q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\underline{\eta}_i^{*q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\underline{\eta}_i^{*q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\underline{\eta}_i^{*q}\right)^{\omega_i}}}$$

and $\overline{\eta}_i \geq \overline{\eta}_i^*$ for all i then $\overline{\eta}_i^q \geq \overline{\eta}_i^{*q} \Longrightarrow 1+\overline{\eta}_i^q \geq 1+\overline{\eta}_i^{*q}$

$$\Longrightarrow (1+\overline{\eta}_i^q)^{\omega_i} \geq (1+\overline{\eta}_i^{*q})^{\omega_i}$$

$$\Longrightarrow \prod_{i=1}^{n}(1+\overline{\eta}_i^q)^{\omega_i} \geq \prod_{i=1}^{n}(1+\overline{\eta}_i^{*q})^{\omega_i}$$

$$\sqrt[q]{\frac{\prod_{i=1}^{n}(1+\overline{\eta}_i^q)^{\omega_i}-\prod_{i=1}^{n}(1-\overline{\eta}_i^q)^{\omega_i}}{\prod_{i=1}^{n}(1+\overline{\eta}_i^q)^{\omega_i}+\prod_{i=1}^{n}(1-\overline{\eta}_i^q)^{\omega_i}}}$$

$$\geq \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\overline{\eta}_i^{*q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\overline{\eta}_i^{*q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\overline{\eta}_i^{*q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\overline{\eta}_i^{*q}\right)^{\omega_i}}}$$

For the lower and upper non-memberships:

As, $\underline{\xi}_i \geq \underline{\xi}_i^* \Longrightarrow \underline{\xi}_i^q \geq \underline{\xi}_i^{*q} \Longrightarrow 1+\underline{\xi}_i^q \geq 1+\underline{\xi}_i^{*q}$

$$\Longrightarrow \left(1+\underline{\xi}_i^q\right)^{\omega_i} \geq \left(1+\underline{\xi}_i^{*q}\right)^{\omega_i}$$

$$\Longrightarrow \prod_{i=1}^{n}\left(1+\underline{\xi}_i^q\right)^{\omega_i} \geq \prod_{i=1}^{n}\left(1+\underline{\xi}_i^{*q}\right)^{\omega_i}$$

$$\sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\underline{\xi}_i^q\right)^{\omega_i}-\prod_{j=1}^{n}\left(1-\underline{\xi}_i^q\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\underline{\xi}_i^q\right)^{\omega_i}+\prod_{j=1}^{n}\left(1-\underline{\xi}_i^q\right)^{\omega_i}}}$$

$$\geq \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\underline{\xi}_i^{*q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\underline{\xi}_i^{*q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\underline{\xi}_i^{*q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\underline{\xi}_i^{*q}\right)^{\omega_i}}}$$

and $\overline{\xi}_i \geq \overline{\xi}_i^* \Longrightarrow \overline{\xi}_i^q \geq \overline{\xi}_i^{*q} \Longrightarrow 1+\overline{\xi}_i^q \geq 1+\overline{\xi}_i^{*q}$

$$\Longrightarrow \left(1+\overline{\xi}_i\right)^{\omega_i} \geq \left(1+\overline{\xi}_i^*\right)^{\omega_i}$$

$$\Longrightarrow \prod_{i=1}^{n}\left(1+\overline{\xi}_i^q\right)^{\omega_i} \geq \prod_{i=1}^{n}\left(1+\overline{\xi}_i^{*q}\right)^{\omega_i}$$

$$\sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\overline{\xi}_i^q\right)^{\omega_i}-\prod_{i=1}^{n}\left(1+\overline{\xi}_i^q\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\overline{\xi}_i^q\right)^{\omega_i}+\prod_{i=1}^{n}\left(1+\overline{\xi}_i^q\right)^{\omega_i}}}$$

$$\geq \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\overline{\xi}_i^{*q}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\overline{\xi}_i^{*q}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\overline{\xi}_i^{*q}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\overline{\xi}_i^{*q}\right)^{\omega_i}}}$$

C. SOME SPECIFIC CASES REGARDING q-SFREWG OPERATOR

From Theorem 1, we have

$$q-SFREWG\left(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n\right)=\otimes_{\underset{\sim}{\mathrm{E}}\,i=1}^{n}\mathcal{A}_i^{\omega_i}$$

$$=\left[\left\langle \begin{array}{l} \sqrt[q]{\frac{2\prod_{i=1}^{n}\left(\underline{\zeta}_i^q\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\underline{\zeta}_i^q\right)^{\omega_i}+\prod_{i=1}^{n}\left(\underline{\zeta}_i^q\right)^{\omega_i}}}, \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\underline{\eta}_i^q\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\underline{\eta}_i^q\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\underline{\eta}_i^q\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\underline{\eta}_i^q\right)^{\omega_i}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\underline{\xi}_i^q\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\underline{\xi}_i^q\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\underline{\xi}_i^q\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\underline{\xi}_i^q\right)^{\omega_i}}}, \sqrt[q]{\frac{2\prod_{i=1}^{n}\left(\overline{\zeta}_i^q\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\overline{\zeta}_i^q\right)^{\omega_i}+\prod_{i=1}^{n}\left(\overline{\zeta}_i^q\right)^{\omega_i}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\overline{\eta}_i^q\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\overline{\eta}_i^q\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\overline{\zeta}_i^q\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\overline{\eta}_i^q\right)^{\omega_i}}}, \sqrt[q]{\frac{2\prod_{i=1}^{n}\left(\overline{\xi}_i^q\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\overline{\xi}_i^q\right)^{\omega_i}+\prod_{i=1}^{n}\left(\overline{\xi}_i^q\right)^{\omega_i}}} \end{array} \right\rangle\right]$$

We are facing the following cases:

Case 1: If $\underline{\zeta}_i=\underline{\eta}_i=\underline{\xi}_i=0$ *and* $q=2$ then

$$q-SFREWG\left(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n\right)=\otimes_{\underset{\sim}{\mathrm{E}}\,i=1}^{n}\mathcal{A}_i^{\omega_i}$$

$$=\left[\left\langle \begin{array}{l} \sqrt{\frac{2\prod_{i=1}^{n}\left(\overline{\zeta}_i^2\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\overline{\zeta}_i^2\right)^{\omega_i}+\prod_{i=1}^{n}\left(\overline{\zeta}_i^2\right)^{\omega_i}}}, \\ \sqrt{\frac{\prod_{i=1}^{n}\left(1+\overline{\eta}_i^2\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\overline{\eta}_i^2\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\overline{\eta}_i^2\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\overline{\eta}_i^2\right)^{\omega_i}}}, \\ \frac{\prod_{i=1}^{n}\left(1+\overline{\xi}_i^2\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\overline{\xi}_i^2\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\overline{\xi}_i^2\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\overline{\xi}_i^2\right)^{\omega_i}} \end{array} \right\rangle\right]$$

$$=SFEWG\left(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n\right)=\otimes_{\underset{\sim}{\mathrm{E}}\,i=1}^{n}\mathcal{A}_i^{\omega_i}$$

(The spherical fuzzy rough Einstein weighted geometric operator).

Case 2: If $\underline{\xi}_i=\overline{\xi}_i=0$ *and* $q=2$ then

$$q-SFREWG\left(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n\right)=\otimes_{\underset{\sim}{\mathrm{E}}\,i=1}^{n}\mathcal{A}_i^{\omega_i}$$

$$=\left[\left\langle \begin{array}{l} \sqrt{\frac{2\prod_{i=1}^{n}\left(\underline{\zeta}_i^2\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\underline{\zeta}_i^2\right)^{\omega_i}+\prod_{i=1}^{n}\left(\underline{\zeta}_i^2\right)^{\omega_i}}}, \sqrt{\frac{\prod_{i=1}^{n}\left(1+\underline{\eta}_i^2\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\underline{\eta}_i^2\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\underline{\eta}_i^2\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\underline{\eta}_i^2\right)^{\omega_i}}}, \\ \sqrt{\frac{2\prod_{i=1}^{n}\left(\overline{\zeta}_i^2\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\overline{\zeta}_i^2\right)^{\omega_i}+\prod_{i=1}^{n}\left(\overline{\zeta}_i^2\right)^{\omega_i}}}, \sqrt{\frac{\prod_{i=1}^{n}\left(1+\overline{\eta}_i^2\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\overline{\eta}_i^2\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\overline{\eta}_i^2\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\overline{\eta}_i^2\right)^{\omega_i}}} \end{array} \right\rangle\right]$$

$$=PyFREWG\left(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n\right)=\otimes_{\underset{\sim}{\mathrm{E}}\,i=1}^{n}\left(\omega_i\mathcal{A}_i\right)$$

(The Pythagorean fuzzy rough Einstein weighted geometric operator).

Case 3: If $q=1$ then

$$q-SFREWG\left(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n\right)=\otimes_{\underset{\sim}{\mathrm{E}}\,i=1}^{n}\mathcal{A}_i^{\omega_i}$$

$$=\left[\left\langle \begin{array}{l} \frac{2\prod_{i=1}^{n}\left(\underline{\zeta}_i\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\underline{\zeta}_i\right)^{\omega_i}+\prod_{i=1}^{n}\left(\underline{\zeta}_i\right)^{\omega_i}}, \frac{\prod_{i=1}^{n}\left(1+\underline{\eta}_i\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\underline{\eta}_i\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\underline{\eta}_i\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\underline{\eta}_i\right)^{\omega_i}}, \\ \frac{\prod_{i=1}^{n}\left(1+\underline{\xi}_i\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\underline{\xi}_i\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\underline{\xi}_i\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\underline{\xi}_i\right)^{\omega_i}}, \frac{2\prod_{i=1}^{n}\left(\overline{\zeta}_i\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\overline{\zeta}_i\right)^{\omega_i}+\prod_{i=1}^{n}\left(\overline{\zeta}_i\right)^{\omega_i}}, \\ \frac{\prod_{i=1}^{n}\left(1+\overline{\eta}_i\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\overline{\eta}_i\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\overline{\eta}_i\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\overline{\eta}_i\right)^{\omega_i}}, \frac{\prod_{i=1}^{n}\left(1+\overline{\xi}_i\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\overline{\xi}_i\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\overline{\xi}_i\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\overline{\xi}_i\right)^{\omega_i}} \end{array} \right\rangle\right]$$

$$=PFREWG\left(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n\right)=\otimes_{\underset{\sim}{\mathrm{E}}\,i=1}^{n}\mathcal{A}_i^{\omega_i}$$

(The picture fuzzy rough Einstein weighted geometric operator).

Case 4: If $\underline{\xi}_i=\overline{\xi}_i=0$ *and* $q=1$ then

$$q-SFREWG\left(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n\right)=\otimes_{\underset{\sim}{\mathrm{E}}\,i=1}^{n}\mathcal{A}_i^{\omega_i}$$

$$=\left[\left\langle \begin{array}{l} \frac{2\prod_{i=1}^{n}\left(\underline{\zeta}_i\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\underline{\zeta}_i\right)^{\omega_i}+\prod_{i=1}^{n}\left(\underline{\zeta}_i\right)^{\omega_i}}, \frac{\prod_{i=1}^{n}\left(1+\underline{\eta}_i\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\underline{\eta}_i\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\underline{\eta}_i\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\underline{\eta}_i\right)^{\omega_i}}, \\ \frac{2\prod_{i=1}^{n}\left(\overline{\zeta}_i\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\overline{\zeta}_i\right)^{\omega_i}+\prod_{i=1}^{n}\left(\overline{\zeta}_i\right)^{\omega_i}}, \frac{\prod_{i=1}^{n}\left(1+\overline{\eta}_i\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\overline{\eta}_i\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\overline{\eta}_i\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\overline{\eta}_i\right)^{\omega_i}} \end{array} \right\rangle\right]$$

$$=IFREWG\left(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n\right)=\otimes_{\underset{\sim}{\mathrm{E}}\,i=1}^{n}\mathcal{A}_i^{\omega_i}$$

(The intuitionistic fuzzy rough Einstein weighted geometric operator).

D. q-SFREOWG OPERATOR

Definition 19: Assuming $\mathcal{A}_i=\left(\underline{\zeta}_i,\underline{\eta}_i,\underline{\xi}_i,\overline{\zeta}_i,\overline{\eta}_i,\overline{\xi}_i\right)$ $(i=1,2,\ldots,n)$ be a collection of q-SFRNs, the q-spherical fuzzy rough Einstein-ordered geometric operator (q-SFREOWG) the operator is defined as a mapping $q-SFREOWG:\mathcal{A}^n\longrightarrow\mathcal{A}$ associated with the weight vector $\left(\omega_1,\omega_2,\ldots,\omega_n\right)^T$ adhering the condition $\omega_i>0$ and the constrain $\sum_{i=1}^{n}\omega_i=1$.

$$q-SFREOWG\left(\mathcal{A}_{\delta(1)},\mathcal{A}_{\delta(2)},\ldots,\mathcal{A}_{\delta(n)}\right)$$
$$=\mathcal{A}_{\delta(1)}\otimes\mathcal{A}_{\delta(2)},\ldots,\otimes\mathcal{A}_{\delta(n)}=\otimes_{i=1}^{n}\left(\mathcal{A}_{\delta(i)}\right)^{\omega_i}$$

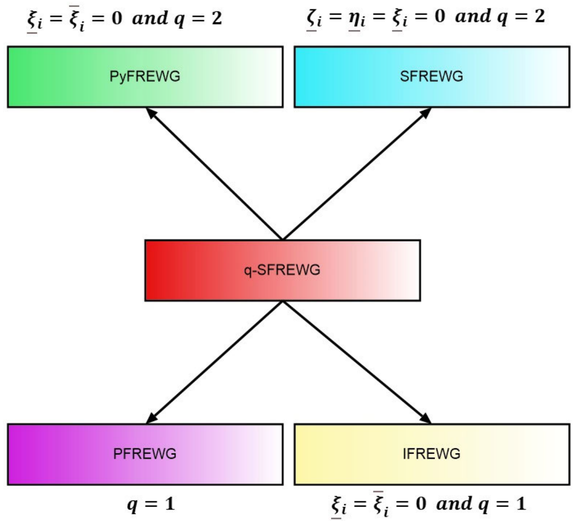

FIGURE 8. Specific cases regarding q-SFREWG operator.

where $\delta(1), \delta(2), \ldots, \delta(n)$ is a permutation of $(1, 2, 3, .., n)$ such that $\mathcal{A}_{\delta(1)} \leq \mathcal{A}_{\delta(i-1)}$ for all $i = 1, 2, 3, .., n.$, as shown in the equation at the bottom of the next page.

Theorem 5: Assuming $\mathcal{A}_i = \left(\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i\right)$ $(i = 1, 2, \ldots, n)$ be a collection of q-SFRNs and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector adhering the condition $\omega_i > 0$ and the constrain $\sum_{i=1}^{n} \omega_i = 1$. If it meets the requirements, it is known as $q - SFREOWG$ operator. As shown in the equation at the bottom of the next page.

Proof: The proof is same as Theorem 1.

Theorem 6 (Idempotency): Assuming $\mathcal{A}_i = \left(\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i\right)$ $(i = 1, 2, \ldots, n)$ be a collection of $q - SFRNs$ and $(\omega_1, \omega_2, \ldots, \omega_n)^T$ signifies the weight vector adhering to the condition $\omega_i > 0$ and the constrain $\sum_{i=1}^{n} \omega_i = 1$. $\mathcal{A}_i$ $(i = 1, 2, \ldots, n)$ are the same $\forall i$, then

$$q - SFREOWG\left(\mathcal{A}_{\delta(1)}, \mathcal{A}_{\delta(2)}, \ldots, \mathcal{A}_{\delta(n)}\right) = \mathcal{A}$$

Proof : The proof is the same as Theorem 2.

Theorem 7 (Boundness): Assuming $\mathcal{A}_i = \left(\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i\right)$ $(i = 1, 2, \ldots, n)$ be a collection of $q - SFRNs$ and $(\omega_1, \omega_2, \ldots, \omega_n)^T$ signifies the weight vector adhering to the condition $\omega_i > 0$ and the constrain $\sum_{i=1}^{n} \omega_i = 1$.

Let

$$\mathcal{A}^{-} = \left(\min\underline{\zeta}_i,\ \max\underline{\eta}_i,\ \max\underline{\xi}_i,\ \min\overline{\zeta}_i,\ \max\overline{\eta}_i,\ \max\overline{\xi}_i\right)$$

and

$$\mathcal{A}^{+} = \left(\max\underline{\zeta}_i,\ \min\underline{\eta}_i,\ \min\underline{\xi}_i,\ \max\overline{\zeta}_i, \min\overline{\eta}_i,\ \min\overline{\xi}_i\right)$$

Then

$$\mathcal{A}^{-} \leq q - SFREOWG\left(\mathcal{A}_{\delta(1)}, \mathcal{A}_{\delta(2)}, \ldots, \mathcal{A}_{\delta(n)}\right) \leq \mathcal{A}^{+}$$

Proof : The proof is the same as Theorem 3.

Theorem 8 (Monotonicity): Assuming

$$\mathcal{A}_i = \left(\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i\right) (i = 1, 2, \ldots, n)$$

and

$$\mathcal{A}_i^{*} = \left(\underline{\zeta}_i^{*}, \underline{\eta}_i^{*}, \underline{\xi}_i^{*}, \overline{\zeta}_i^{*}, \overline{\eta}_i^{*}, \overline{\xi}_i^{*}\right) (i = 1, 2, \ldots, n)$$

be a collection of two $q - SFRNs$ such that $\mathcal{A}_i \leq \mathcal{A}_i^{*}$ for all i, then

$$q - SFREOWG\left(\mathcal{A}_{\delta(1)}, \mathcal{A}_{\delta(2)}, \ldots, \mathcal{A}_{\delta(n)}\right) \leq q - SFREOWG\left(\mathcal{A}^{*}_{\delta(1)}, \mathcal{A}^{*}_{\delta(2)}, \ldots, \mathcal{A}^{*}_{\delta(n)}\right).$$

Proof: The proof is the same as Theorem 4.

E. SOME SPECIFIC CASES REGARDING q-SFREOWG OPERATOR

From Theorem 5, we have, as shown in the equation at the bottom of the next page.

We are facing the following cases:

Case 1: If $\underline{\zeta}_{\delta(i)} = \underline{\eta}_{\delta(i)} = \underline{\xi}_{\delta(i)} = 0$ *and* $q = 2$ then

$$q - SFREOWG\left(\mathcal{A}_{\delta(1)}, \mathcal{A}_{\delta(2)}, \ldots, \mathcal{A}_{\delta(n)}\right) = \otimes_{\underset{\sim}{\mathrm{E}}\,i=1}^{n}\left(\mathcal{A}_{\delta(i)}\right)^{\omega_i}$$

$$= \left[\left\langle \begin{array}{l} \sqrt{\dfrac{2\prod_{i=1}^{n}\left(\overline{\zeta}_{\delta(i)}^{2}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\overline{\zeta}_{\delta(i)}^{2}\right)^{\omega_i}+\prod_{i=1}^{n}\left(\overline{\zeta}_{\delta(i)}^{2}\right)^{\omega_i}}}, \\ \sqrt{\dfrac{\prod_{i=1}^{n}\left(1+\overline{\eta}_{\delta(i)}^{2}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\overline{\eta}_{\delta(i)}^{2}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\overline{\eta}_{\delta(i)}^{2}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\overline{\eta}_{\delta(i)}^{2}\right)^{\omega_i}}}, \\ \sqrt{\dfrac{\prod_{i=1}^{n}\left(1+\overline{\xi}_{\delta(i)}^{2}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\overline{\xi}_{\delta(i)}^{2}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\overline{\xi}_{\delta(i)}^{2}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\overline{\xi}_{\delta(i)}^{2}\right)^{\omega_i}}} \end{array}\right\rangle\right]$$

$$= SFREOWG\left(\mathcal{A}_{\delta(1)}, \mathcal{A}_{\delta(2)}, \ldots, \mathcal{A}_{\delta(n)}\right) = \otimes_{\underset{\sim}{\mathrm{E}}\,i=1}^{n}\left(\mathcal{A}_{\delta(i)}\right)^{\omega_i}$$

(The spherical fuzzy rough Einstein ordered weighted geometric operator).

Case 2: If $\underline{\xi}_{\delta(i)} = \overline{\xi}_{\delta(i)} = 0$ *and* $q = 2$ then, as shown in the equation at the bottom of the next page.

(The Pythagorean fuzzy rough Einstein ordered weighted geometric operator).

Case 3: If $q = 1$ then, as shown in the equation at the bottom of the next page.

(The picture fuzzy rough Einstein ordered weighted geometric operator).

Case 4: If $\underline{\xi}_{\delta(i)} = \overline{\xi}_{\delta(i)} = 0$ *and* $q = 1$ then

$$q - SFREOWG\left(\mathcal{A}_{\delta(1)}, \mathcal{A}_{\delta(2)}, \ldots, \mathcal{A}_{\delta(n)}\right)$$

$$= \otimes_{\underset{\sim}{\mathrm{E}}\,i=1}^{n}\left(\mathcal{A}_{\delta(i)}\right)^{\omega_i}$$

$$= \left[\left\langle \begin{array}{l} \dfrac{2\prod_{i=1}^{n}\left(\underline{\zeta}_{\delta(i)}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\underline{\zeta}_{\delta(i)}\right)^{\omega_i}+\prod_{i=1}^{n}\left(\underline{\zeta}_{\delta(i)}\right)^{\omega_i}}, \\ \dfrac{\prod_{i=1}^{n}\left(1+\underline{\eta}_{\delta(i)}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\underline{\eta}_{\delta(i)}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\underline{\eta}_{\delta(i)}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\underline{\eta}_{\delta(i)}\right)^{\omega_i}}, \\ \dfrac{\prod_{i=1}^{n}\left(1+\underline{\xi}_{\delta(i)}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\underline{\xi}_{\delta(i)}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\underline{\xi}_{\delta(i)}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\underline{\xi}_{\delta(i)}\right)^{\omega_i}} \end{array}\right\rangle\right]$$

$$= IFREOWG\left(\mathcal{A}_{\delta(1)}, \mathcal{A}_{\delta(2)}, \ldots, \mathcal{A}_{\delta(n)}\right) = \otimes_{\underset{\sim}{\mathrm{E}}\,i=1}^{n}\left(\mathcal{A}_{\delta(i)}\right)^{\omega_i}$$

(The intuitionistic fuzzy rough Einstein ordered weighted geometric operator).

Example 4: Consider four q-SFRNs $\mathcal{A}_1 = (0.3, 0.4, 0.1, 0.3, 0.2, 0.4)$, $\mathcal{A}_2 = (0.5, 0.5, 0.9, 0.2, 0.8, 0.6)$, $\mathcal{A}_3 = (0.4, 0.5, 0.9, 0.3, 0.2, 0.6)$ and $\mathcal{A}_4 = (0.5, 0.2, 0.5, 0.3, 0.8, 0.9)$ be any four q-SFRNs, if $\omega = (0.3, 0.1, 0.4, 0.2)^T$ and $q = 3$ then the q-SFREOWG operator defined in Definition (13) can be calculated as:

For this purpose first, we calculate the score values of $\mathcal{A}_i \, (i = 1, 2, 3, 4)$

$$Sco(\mathcal{A}_1) = \frac{2 + (0.3)^3 + (0.3)^3 - (0.4)^3 - (0.2)^3 - (0.1)^3 - (0.4)^3 -}{3} = 0.6390$$

$Sco(\mathcal{A}_2)$

$$q - SFREOWG(\mathcal{A}_{\delta(1)}, \mathcal{A}_{\delta(2)}, \ldots, \mathcal{A}_{\delta(n)}) = \left[\left\langle \begin{array}{l} \sqrt[q]{\frac{2\prod_{i=1}^n (\underline{\zeta}^q_{\delta(i)})^{\omega_i}}{\prod_{i=1}^n (2-\underline{\zeta}^q_{\delta(i)})^{\omega_i} + \prod_{i=1}^n (\underline{\zeta}^q_{\delta(i)})^{\omega_i}}}, \sqrt[q]{\frac{\prod_{i=1}^n (1+\underline{\eta}^q_{\delta(i)})^{\omega_i} - \prod_{i=1}^n (1-\underline{\eta}^q_{\delta(i)})^{\omega_i}}{\prod_{i=1}^n (1+\underline{\eta}^q_{\delta(i)})^{\omega_i} + \prod_{i=1}^n (1-\underline{\eta}^q_{\delta(i)})^{\omega_i}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^n (1+\underline{\xi}^q_{\delta(i)})^{\omega_i} - \prod_{i=1}^n (1-\underline{\xi}^q_{\delta(i)})^{\omega_i}}{\prod_{i=1}^n (1+\underline{\xi}^q_{\delta(i)})^{\omega_i} + \prod_{i=1}^n (1-\underline{\xi}^q_{\delta(i)})^{\omega_i}}}, \sqrt[q]{\frac{2\prod_{i=1}^n (\overline{\zeta}^q_{\delta(i)})^{\omega_i}}{\prod_{i=1}^n (2-\overline{\zeta}^q_{\delta(i)})^{\omega_i} + \prod_{i=1}^n (\overline{\zeta}^q_{\delta(i)})^{\omega_i}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^n (1+\overline{\eta}^q_{\delta(i)})^{\omega_i} - \prod_{i=1}^n (1-\overline{\eta}^q_{\delta(i)})^{\omega_i}}{\prod_{i=1}^n (1+\overline{\eta}^q_{\delta(i)})^{\omega_i} + \prod_{i=1}^n (1-\overline{\eta}^q_{\delta(i)})^{\omega_i}}}, \sqrt[q]{\frac{\prod_{i=1}^n (1+\overline{\xi}^q_{\delta(i)})^{\omega_i} - \prod_{i=1}^n (1-\overline{\xi}^q_{\delta(i)})^{\omega_i}}{\prod_{i=1}^n (1+\overline{\xi}^q_{\delta(i)})^{\omega_i} + \prod_{i=1}^n (1-\overline{\xi}^q_{\delta(i)})^{\omega_i}}} \end{array} \right\rangle\right]$$

$$q - SFREOWG(\mathcal{A}_{\delta(1)}, \mathcal{A}_{\delta(2)}, \ldots, \mathcal{A}_{\delta(n)}) = \left[\left\langle \begin{array}{l} \sqrt[q]{\frac{2\prod_{i=1}^n (\underline{\zeta}^q_{\delta(i)})^{\omega_i}}{\prod_{i=1}^n (2-\underline{\zeta}^q_{\delta(i)})^{\omega_i} + \prod_{i=1}^n (\underline{\zeta}^q_{\delta(i)})^{\omega_i}}}, \sqrt[q]{\frac{\prod_{i=1}^n (1+\underline{\eta}^q_{\delta(i)})^{\omega_i} - \prod_{i=1}^n (1-\underline{\eta}^q_{\delta(i)})^{\omega_i}}{\prod_{i=1}^n (1+\underline{\eta}^q_{\delta(i)})^{\omega_i} + \prod_{i=1}^n (1-\underline{\eta}^q_{\delta(i)})^{\omega_i}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^n (1+\underline{\xi}^q_{\delta(i)})^{\omega_i} - \prod_{i=1}^n (1-\underline{\xi}^q_{\delta(i)})^{\omega_i}}{\prod_{i=1}^n (1+\underline{\xi}^q_{\delta(i)})^{\omega_i} + \prod_{i=1}^n (1-\underline{\xi}^q_{\delta(i)})^{\omega_i}}}, \sqrt[q]{\frac{2\prod_{i=1}^n (\overline{\zeta}^q_{\delta(i)})^{\omega_i}}{\prod_{i=1}^n (2-\overline{\zeta}^q_{\delta(i)})^{\omega_i} + \prod_{i=1}^n (\overline{\zeta}^q_{\delta(i)})^{\omega_i}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^n (1+\overline{\eta}^q_{\delta(i)})^{\omega_i} - \prod_{i=1}^n (1-\overline{\eta}^q_{\delta(i)})^{\omega_i}}{\prod_{i=1}^n (1+\overline{\eta}^q_{\delta(i)})^{\omega_i} + \prod_{i=1}^n (1-\overline{\eta}^q_{\delta(i)})^{\omega_i}}}, \sqrt[q]{\frac{\prod_{i=1}^n (1+\overline{\xi}^q_{\delta(i)})^{\omega_i} - \prod_{i=1}^n (1-\overline{\xi}^q_{\delta(i)})^{\omega_i}}{\prod_{i=1}^n (1+\overline{\xi}^q_{\delta(i)})^{\omega_i} + \prod_{i=1}^n (1-\overline{\xi}^q_{\delta(i)})^{\omega_i}}} \end{array} \right\rangle\right]$$

$$q - SFREOWG(\mathcal{A}_{\delta(1)}, \mathcal{A}_{\delta(2)}, \ldots, \mathcal{A}_{\delta(n)}) = \left[\left\langle \begin{array}{l} \sqrt[q]{\frac{2\prod_{i=1}^n (\underline{\zeta}^q_{\delta(i)})^{\omega_i}}{\prod_{i=1}^n (2-\underline{\zeta}^q_{\delta(i)})^{\omega_i} + \prod_{i=1}^n (\underline{\zeta}^q_{\delta(i)})^{\omega_i}}}, \sqrt[q]{\frac{\prod_{i=1}^n (1+\underline{\eta}^q_{\delta(i)})^{\omega_i} - \prod_{i=1}^n (1-\underline{\eta}^q_{\delta(i)})^{\omega_i}}{\prod_{i=1}^n (1+\underline{\eta}^q_{\delta(i)})^{\omega_i} + \prod_{i=1}^n (1-\underline{\eta}^q_{\delta(i)})^{\omega_i}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^n (1+\underline{\xi}^q_{\delta(i)})^{\omega_i} - \prod_{i=1}^n (1-\underline{\xi}^q_{\delta(i)})^{\omega_i}}{\prod_{i=1}^n (1+\underline{\xi}^q_{\delta(i)})^{\omega_i} + \prod_{i=1}^n (1-\underline{\xi}^q_{\delta(i)})^{\omega_i}}}, \sqrt[q]{\frac{2\prod_{i=1}^n (\overline{\zeta}^q_{\delta(i)})^{\omega_i}}{\prod_{i=1}^n (2-\overline{\zeta}^q_{\delta(i)})^{\omega_i} + \prod_{i=1}^n (\overline{\zeta}^q_{\delta(i)})^{\omega_i}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^n (1+\overline{\eta}^q_{\delta(i)})^{\omega_i} - \prod_{i=1}^n (1-\overline{\eta}^q_{\delta(i)})^{\omega_i}}{\prod_{i=1}^n (1+\overline{\eta}^q_{\delta(i)})^{\omega_i} + \prod_{i=1}^n (1-\overline{\eta}^q_{\delta(i)})^{\omega_i}}}, \sqrt[q]{\frac{\prod_{i=1}^n (1+\overline{\xi}^q_{\delta(i)})^{\omega_i} - \prod_{i=1}^n (1-\overline{\xi}^q_{\delta(i)})^{\omega_i}}{\prod_{i=1}^n (1+\overline{\xi}^q_{\delta(i)})^{\omega_i} + \prod_{i=1}^n (1-\overline{\xi}^q_{\delta(i)})^{\omega_i}}} \end{array} \right\rangle\right]$$

$$q - SFREOWG(\mathcal{A}_{\delta(1)}, \mathcal{A}_{\delta(2)}, \ldots, \mathcal{A}_{\delta(n)}) = \otimes_{\underset{\sim}{\mathrm{E}}\, i=1}^{\ n} (\omega_i \mathcal{A}_{\delta(i)})$$

$$= \left[\left\langle \begin{array}{l} \sqrt{\frac{2\prod_{i=1}^n (\underline{\zeta}^2_{\delta(i)})^{\omega_i}}{\prod_{i=1}^n (2-\underline{\zeta}^2_{\delta(i)})^{\omega_i} + \prod_{i=1}^n (\underline{\zeta}^2_{\delta(i)})^{\omega_i}}}, \sqrt{\frac{\prod_{i=1}^n (1+\underline{\eta}^2_{\delta(i)})^{\omega_i} - \prod_{i=1}^n (1-\underline{\eta}^2_{\delta(i)})^{\omega_i}}{\prod_{i=1}^n (1+\underline{\eta}^2_{\delta(i)})^{\omega_i} + \prod_{i=1}^n (1-\underline{\eta}^2_{\delta(i)})^{\omega_i}}}, \\ \sqrt{\frac{2\prod_{i=1}^n (\overline{\zeta}^2_{\delta(i)})^{\omega_i}}{\prod_{i=1}^n (2-\overline{\zeta}^2_{\delta(i)})^{\omega_i} + \prod_{i=1}^n (\overline{\zeta}^2_{\delta(i)})^{\omega_i}}}, \sqrt{\frac{\prod_{i=1}^n (1+\overline{\eta}^2_{\delta(i)})^{\omega_i} - \prod_{i=1}^n (1-\overline{\eta}^2_{\delta(i)})^{\omega_i}}{\prod_{i=1}^n (1+\overline{\eta}^2_{\delta(i)})^{\omega_i} + \prod_{i=1}^n (1-\overline{\eta}^2_{\delta(i)})^{\omega_i}}} \end{array} \right\rangle\right]$$

$$= PyFREOWG(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \otimes_{\underset{\sim}{\mathrm{E}}\, i=1}^{\ n} (\mathcal{A}_{\delta(i)})^{\omega_i}$$

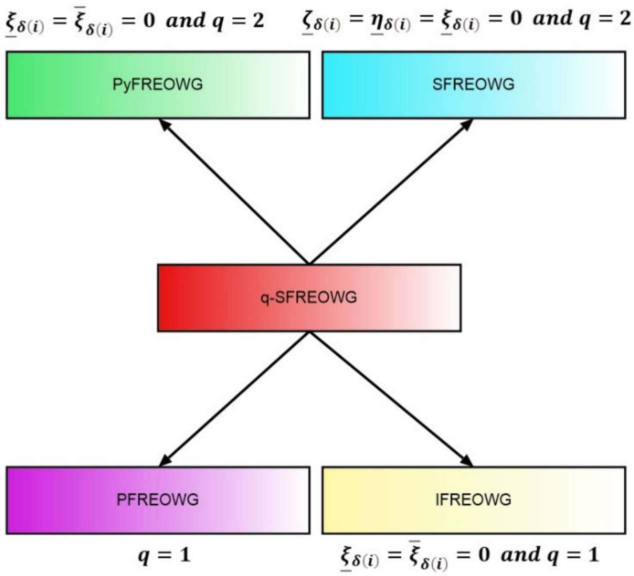

FIGURE 9. **Specific cases regarding q-SFREOWG operator.**

$$=\frac{2+\ (0.5)^3+(0.2)^3-(0.5)^3-(0.8)^3-(0.9)^3-(0.6)^3}{3}$$
$$=0.1837$$

$Sco\left(\mathcal{A}_3\right)$

$$=\frac{2+(0.4)^3+(0.3)^3-(0.5)^3-(0.2)^3-(0.9)^3-(0.6)^3}{3}$$
$$=0.3377$$

$Sco\left(\mathcal{A}_4\right)$

$$=\frac{2+(0.5)^3+(0.3)^3-(0.2)^3-(0.8)^3-(0.5)^3-(0.9)^3}{3}$$
$$=0.2593$$

Since $\quad Sco\left(\mathcal{A}_1\right)>\ Sco\left(\mathcal{A}_2\right)>\ Sco\left(\mathcal{A}_4\right)>\ Sco\left(\mathcal{A}_3\right)$

$$\left(\mathcal{A}_1\right)>\ \left(\mathcal{A}_3\right)>\ \left(\mathcal{A}_4\right)>\left(\mathcal{A}_2\right)$$

Hence

$$\begin{aligned}\mathcal{A}_{\delta(1)}&=\mathcal{A}^{\circ}{}_1=(0.3,0.4,0.1,0.3,0.2,0.4)\\ \mathcal{A}_{\delta(2)}&=\mathcal{A}^{\circ}{}_2=(0.4,0.5,0.9,0.3,0.2,0.6)\\ \mathcal{A}_{\delta(3)}&=\mathcal{A}^{\circ}{}_3=(0.5,0.2,0.5,0.3,0.8,0.9)\\ \mathcal{A}_{\delta(4)}&=\mathcal{A}^{\circ}{}_4=(0.5,0.5,0.9,0.2,0.8,0.6)\end{aligned}$$

For $\mathcal{A}_{\delta(i)}(i=1,2,3,4)$, the q-SFREOWG operator defined in Definition (19) can be calculated as:

$$\sqrt[q]{\frac{2\prod_{i=1}^{4}\left(\underline{\zeta}_{\delta(i)}^{q}\right)^{\omega_i}}{\prod_{i=1}^{4}\left(2-\underline{\zeta}_{\delta(i)}^{q}\right)^{\omega_i}+\prod_{i=1}^{4}\left(\underline{\zeta}_{\delta(i)}^{q}\right)^{\omega_i}}}$$

$$=\sqrt[q]{\frac{2\left(\underline{\zeta}_{\delta(1)}^{q}\right)^{\omega_1}\left(\underline{\zeta}_{\delta(2)}^{q}\right)^{\omega_2}\left(\underline{\zeta}_{\delta(3)}^{q}\right)^{\omega_3}\left(\underline{\zeta}_{\delta(4)}^{q}\right)^{\omega_4}}{\begin{array}{c}\left(2-\underline{\zeta}_{\delta(1)}^{q}\right)^{\omega_1}\left(2-\underline{\zeta}_{\delta(2)}^{q}\right)^{\omega_2}\left(2-\underline{\zeta}_{\delta(3)}^{q}\right)^{\omega_3}\left(2-\underline{\zeta}_{\delta(4)}^{q}\right)^{\omega_4}\\+\left(\underline{\zeta}_{\delta(1)}^{q}\right)^{\omega_1}\left(\underline{\zeta}_{\delta(2)}^{q}\right)^{\omega_2}\left(\underline{\zeta}_{\delta(3)}^{q}\right)^{\omega_3}\left(\underline{\zeta}_{\delta(4)}^{q}\right)^{\omega_4}\end{array}}}$$

$$=\sqrt[3]{\frac{2\left(0.3^3\right)^{0.3}\left(0.4^3\right)^{0.1}\left(0.5^3\right)^{0.4}\left(0.5^3\right)^{0.2}}{\begin{array}{c}\left(2-0.3^3\right)^{0.3}\left(2-0.4^3\right)^{0.1}\left(2-0.5^3\right)^{0.4}\left(2-0.5^3\right)^{0.2}\\+\left(0.3^3\right)^{0.3}\left(0.4^3\right)^{0.1}\left(0.5^3\right)^{0.4}\left(0.5^3\right)^{0.2}\end{array}}}$$
$$=0.6435$$

$$\sqrt[q]{\frac{\prod_{i=1}^{4}\left(1+\underline{\eta}_{\delta(i)}^{q}\right)^{\omega_i}-\prod_{i=1}^{4}\left(1-\underline{\eta}_{\delta(i)}^{q}\right)^{\omega_i}}{\prod_{i=1}^{4}\left(1+\underline{\eta}_{\delta(i)}^{q}\right)^{\omega_i}+\prod_{i=1}^{4}\left(1-\underline{\eta}_{\delta(i)}^{q}\right)^{\omega_i}}}$$

$$=\sqrt[q]{\frac{\begin{array}{c}\left(1+\underline{\eta}_{\delta(1)}^{q}\right)^{\omega_1}\left(1+\underline{\eta}_{\delta(2)}^{q}\right)^{\omega_2}\left(1+\underline{\eta}_{\delta(3)}^{q}\right)^{\omega_3}\left(1+\underline{\eta}_{\delta(4)}^{q}\right)^{\omega_4}\\-\left(1-\underline{\eta}_{\delta(1)}^{q}\right)^{\omega_1}\left(1-\underline{\eta}_{\delta(2)}^{q}\right)^{\omega_2}\left(1-\underline{\eta}_{\delta(3)}^{q}\right)^{\omega_3}\left(1-\underline{\eta}_{\delta(4)}^{q}\right)^{\omega_4}\end{array}}{\begin{array}{c}\left(1+\underline{\eta}_{\delta(1)}^{q}\right)^{\omega_1}\left(1+\underline{\eta}_{\delta(2)}^{q}\right)^{\omega_2}\left(1+\underline{\eta}_{\delta(3)}^{q}\right)^{\omega_3}\left(1+\underline{\eta}_{\delta(4)}^{q}\right)^{\omega_4}\\+\left(1-\underline{\eta}_{\delta(1)}^{q}\right)^{\omega_1}\left(1-\underline{\eta}_{\delta(2)}^{q}\right)^{\omega_2}\left(1-\underline{\eta}_{\delta(3)}^{q}\right)^{\omega_3}\left(1-\underline{\eta}_{\delta(4)}^{q}\right)^{\omega_4}\end{array}}}$$

$$=\sqrt[3]{\frac{\begin{array}{c}\left(1+0.4^3\right)^{0.3}\left(1+0.5^3\right)^{0.1}\left(1+0.2^3\right)^{0.4}\left(1+0.5^3\right)^{0.2}\\-\left(1-0.4^3\right)^{0.3}\left(1-0.5^3\right)^{0.1}\left(1-0.2^3\right)^{0.4}\left(1-0.5^3\right)^{0.2}\end{array}}{\begin{array}{c}\left(1+0.4^3\right)^{0.3}\left(1+0.5^3\right)^{0.1}\left(1+0.2^3\right)^{0.4}\left(1+0.5^3\right)^{0.2}\\+\left(1-0.4^3\right)^{0.3}\left(1-0.5^3\right)^{0.1}\left(1-0.2^3\right)^{0.4}\left(1-0.5^3\right)^{0.2}\end{array}}}$$
$$=0.3916$$

$$\sqrt[q]{\frac{\prod_{i=1}^{4}\left(1+\underline{\xi}_{\delta(i)}^{q}\right)^{\omega_i}-\prod_{i=1}^{4}\left(1-\underline{\xi}_{\delta(i)}^{q}\right)^{\omega_i}}{\prod_{i=1}^{4}\left(1+\underline{\xi}_{\delta(i)}^{q}\right)^{\omega_i}+\prod_{i=1}^{4}\left(1-\underline{\xi}_{\delta(i)}^{q}\right)^{\omega_i}}}$$

$$q-SFREOWG\left(\mathcal{A}_{\delta(1)},\mathcal{A}_{\delta(2)},\ldots,\mathcal{A}_{\delta(n)}\right)$$
$$=\left[\left\langle\begin{array}{c}\frac{2\prod_{i=1}^{n}\left(\underline{\zeta}_{\delta(i)}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\underline{\zeta}_{\delta(i)}\right)^{\omega_i}+\prod_{i=1}^{n}\left(\underline{\zeta}_{\delta(i)}\right)^{\omega_i}},\frac{\prod_{i=1}^{n}\left(1+\underline{\eta}_{\delta(i)}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\underline{\eta}_{\delta(i)}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\underline{\eta}_{\delta(i)}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\underline{\eta}_{\delta(i)}\right)^{\omega_i}},\\ \frac{\prod_{i=1}^{n}\left(1+\underline{\xi}_{\delta(i)}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\underline{\xi}_{\delta(i)}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\underline{\xi}_{\delta(i)}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\underline{\xi}_{\delta(i)}\right)^{\omega_i}},\frac{2\prod_{i=1}^{n}\left(\overline{\zeta}_{\delta(i)}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\overline{\zeta}_{\delta(i)}\right)^{\omega_i}+\prod_{i=1}^{n}\left(\overline{\zeta}_{\delta(i)}\right)^{\omega_i}},\\ \frac{\prod_{i=1}^{n}\left(1+\overline{\eta}_{\delta(i)}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\overline{\eta}_{\delta(i)}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\overline{\eta}_{\delta(i)}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\overline{\eta}_{\delta(i)}\right)^{\omega_i}},\frac{\prod_{i=1}^{n}\left(1+\overline{\xi}_{\delta(i)}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\overline{\xi}_{\delta(i)}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\overline{\xi}_{\delta(i)}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\overline{\xi}_{\delta(i)}\right)^{\omega_i}}\end{array}\right\rangle\right]$$
$$=PFREOWG\left(\mathcal{A}_{\delta(1)},\mathcal{A}_{\delta(2)},\ldots,\mathcal{A}_{\delta(n)}\right)=\otimes_{\underset{\sim}{E}\,i=1}^{n}\left(\mathcal{A}_{\delta(i)}\right)^{\omega_i}$$

$$= \sqrt[q]{\frac{\left(1+\underline{\xi}_{\delta(1)}^{q}\right)^{\omega_1}\left(1+\underline{\xi}_{\delta(2)}^{q}\right)^{\omega_2}\left(1+\underline{\xi}_{\delta(3)}^{q}\right)^{\omega_3}\left(1+\underline{\xi}_{\delta(4)}^{q}\right)^{\omega_4} -\left(1-\underline{\xi}_{\delta(1)}^{q}\right)^{\omega_1}\left(1-\underline{\xi}_{\delta(2)}^{q}\right)^{\omega_2}\left(1-\underline{\xi}_{\delta(3)}^{q}\right)^{\omega_3}\left(1-\underline{\xi}_{\delta(4)}^{q}\right)^{\omega_4}}{\left(1+\underline{\xi}_{\delta(1)}^{q}\right)^{\omega_1}\left(1+\underline{\xi}_{\delta(2)}^{q}\right)^{\omega_2}\left(1+\underline{\xi}_{\delta(3)}^{q}\right)^{\omega_3}\left(1+\underline{\xi}_{\delta(4)}^{q}\right)^{\omega_4} +\left(1-\underline{\xi}_{\delta(1)}^{q}\right)^{\omega_1}\left(1-\underline{\xi}_{\delta(2)}^{q}\right)^{\omega_2}\left(1-\underline{\xi}_{\delta(3)}^{q}\right)^{\omega_3}\left(1-\underline{\xi}_{\delta(4)}^{q}\right)^{\omega_4}}}$$

$$= \sqrt[3]{\frac{(1+0.1^3)^{0.3}(1+0.9^3)^{0.1}(1+0.5^3)^{0.4}(1+0.9^3)^{0.2} -(1-0.1^3)^{0.3}(1-0.9^3)^{0.1}(1-0.5^3)^{0.4}(1-0.9^3)^{0.2}}{(1+0.1^3)^{0.3}(1+0.9^3)^{0.1}(1+0.5^3)^{0.4}(1+0.9^3)^{0.2} +(1-0.1^3)^{0.3}(1-0.9^3)^{0.1}(1-0.5^3)^{0.4}(1-0.9^3)^{0.2}}}$$

$$= 0.6820$$

$$\sqrt[q]{\frac{2\prod_{i=1}^{4}\left(\overline{\xi}_{\delta(i)}^{q}\right)^{\omega_i}}{\prod_{i=1}^{4}\left(2-\overline{\xi}_{\delta(i)}^{q}\right)^{\omega_i}+\prod_{i=1}^{4}\left(\overline{\xi}_{\delta(i)}^{q}\right)^{\omega_i}}}$$

$$= \frac{\sqrt[q]{2}\left(\overline{\xi}_{\delta(1)}^{q}\right)^{\omega_1}\left(\overline{\xi}_{\delta(2)}^{q}\right)^{\omega_2}\left(\overline{\xi}_{\delta(3)}^{q}\right)^{\omega_3}\left(\overline{\xi}_{\delta(4)}^{q}\right)^{\omega_4}}{\sqrt[q]{\left(2-\overline{\xi}_{\delta(1)}^{q}\right)^{\omega_1}\left(2-\overline{\xi}_{\delta(2)}^{q}\right)^{\omega_2}\left(2-\overline{\xi}_{\delta(3)}^{q}\right)^{\omega_3}\left(2-\overline{\xi}_{\delta(4)}^{q}\right)^{\omega_4} +\left(\overline{\xi}_{\delta(1)}^{q}\right)^{\omega_1}\left(\overline{\xi}_{\delta(2)}^{q}\right)^{\omega_2}\left(\overline{\xi}_{\delta(3)}^{q}\right)^{\omega_3}\left(\overline{\xi}_{\delta(4)}^{q}\right)^{\omega_4}}}$$

$$= \sqrt[3]{\frac{2(0.3^3)^{0.3}(0.3^3)^{0.1}(0.3^3)^{0.4}(0.2^3)^{0.2}}{(2-0.3^3)^{0.3}(2-0.3^3)^{0.1}(2-0.3^3)^{0.4}(2-0.2^3)^{0.2} +(0.3^3)^{0.3}(0.3^3)^{0.1}(0.3^3)^{0.4}(0.2^3)^{0.2}}}$$

$$= 0.5588$$

$$\sqrt[q]{\frac{\prod_{i=1}^{4}\left(1+\overline{\eta}_{\delta(i)}^{q}\right)^{\omega_i}-\prod_{i=1}^{4}\left(1-\overline{\eta}_{\delta(i)}^{q}\right)^{\omega_i}}{\prod_{i=1}^{4}\left(1+\overline{\eta}_{\delta(i)}^{q}\right)^{\omega_i}+\prod_{i=1}^{4}\left(1-\overline{\eta}_{\delta(i)}^{q}\right)^{\omega_i}}}$$

$$= \sqrt[q]{\frac{\left(1+\overline{\eta}_{\delta(1)}^{q}\right)^{\omega_1}\left(1+\overline{\eta}_{\delta(2)}^{q}\right)^{\omega_2}\left(1+\overline{\eta}_{\delta(3)}^{q}\right)^{\omega_3}\left(1+\overline{\eta}_{\delta(4)}^{q}\right)^{\omega_4} -\left(1-\overline{\eta}_{\delta(1)}^{q}\right)^{\omega_1}\left(1-\overline{\eta}_{\delta(2)}^{q}\right)^{\omega_2}\left(1-\overline{\eta}_{\delta(3)}^{q}\right)^{\omega_3}\left(1-\overline{\eta}_{\delta(4)}^{q}\right)^{\omega_4}}{\left(1+\overline{\eta}_{\delta(1)}^{q}\right)^{\omega_1}\left(1+\overline{\eta}_{\delta(2)}^{q}\right)^{\omega_2}\left(1+\overline{\eta}_{\delta(3)}^{q}\right)^{\omega_3}\left(1+\overline{\eta}_{\delta(4)}^{q}\right)^{\omega_4} +\left(1-\overline{\eta}_{\delta(1)}^{q}\right)^{\omega_1}\left(1-\overline{\eta}_{\delta(2)}^{q}\right)^{\omega_2}\left(1-\overline{\eta}_{\delta(3)}^{q}\right)^{\omega_3}\left(1-\overline{\eta}_{\delta(4)}^{q}\right)^{\omega_4}}}.$$

$$= \sqrt[3]{\frac{(1+0.2^3)^{0.3}(1+0.2^3)^{0.1}(1+0.8^3)^{0.4}(1+0.8^3)^{0.2} -(1-0.2^3)^{0.3}(1-0.2^3)^{0.1}(1-0.8^3)^{0.4}(1-0.8^3)^{0.2}}{(1+0.2^3)^{0.3}(1+0.2^3)^{0.1}(1+0.8^3)^{0.4}(1+0.8^3)^{0.2} +(1-0.2^3)^{0.3}(1-0.2^3)^{0.1}(1-0.8^3)^{0.4}(1-0.8^3)^{0.2}}}$$

$$= 0.6908$$

$$\sqrt[q]{\frac{\prod_{i=1}^{4}\left(1+\overline{\xi}_{\delta(i)}^{q}\right)^{\omega_i}-\prod_{i=1}^{4}\left(1-\overline{\xi}_{\delta(i)}^{q}\right)^{\omega_i}}{\prod_{i=1}^{4}\left(1+\overline{\xi}_{\delta(i)}^{q}\right)^{\omega_i}+\prod_{i=1}^{4}\left(1-\overline{\xi}_{\delta(i)}^{q}\right)^{\omega_i}}}$$

$$= \sqrt[q]{\frac{\left(1+\overline{\xi}_{\delta(1)}^{q}\right)^{\omega_1}\left(1+\overline{\xi}_{\delta(2)}^{q}\right)^{\omega_2}\left(1+\overline{\xi}_{\delta(3)}^{q}\right)^{\omega_3}\left(1+\overline{\xi}_{\delta(4)}^{q}\right)^{\omega_4} -\left(1-\overline{\xi}_{\delta(1)}^{q}\right)^{\omega_1}\left(1-\overline{\xi}_{\delta(2)}^{q}\right)^{\omega_2}\left(1-\overline{\xi}_{\delta(3)}^{q}\right)^{\omega_3}\left(1-\overline{\xi}_{\delta(4)}^{q}\right)^{\omega_4}}{\left(1+\overline{\xi}_{\delta(1)}^{q}\right)^{\omega_1}\left(1+\overline{\xi}_{\delta(2)}^{q}\right)^{\omega_2}\left(1+\overline{\xi}_{\delta(3)}^{q}\right)^{\omega_3}\left(1+\overline{\xi}_{\delta(4)}^{q}\right)^{\omega_4} +\left(1-\overline{\xi}_{\delta(1)}^{q}\right)^{\omega_1}\left(1-\overline{\xi}_{\delta(2)}^{q}\right)^{\omega_2}\left(1-\overline{\xi}_{\delta(3)}^{q}\right)^{\omega_3}\left(1-\overline{\xi}_{\delta(4)}^{q}\right)^{\omega_4}}}.$$

$$= \sqrt[3]{\frac{(1+0.4^3)^{0.3}(1+0.6^3)^{0.1}(1+0.9^3)^{0.4}(1+0.6^3)^{0.2} -(1-0.4^3)^{0.3}(1-0.6^3)^{0.1}(1-0.9^3)^{0.4}(1-0.6^3)^{0.2}}{(1+0.4^3)^{0.3}(1+0.6^3)^{0.1}(1+0.9^3)^{0.4}(1+0.6^3)^{0.2} +(1-0.4^3)^{0.3}(1-0.6^3)^{0.1}(1-0.9^3)^{0.4}(1-0.6^3)^{0.2}}}$$

$$= 0.7528$$

Hence, as shown in the equation at the bottom of the next page.

IV. APPLICATIONS OF THE PROPOSED OPERATORS

This section focuses on solving multi-attribute decision-making (MADM) problems using the previously described operators and q-SFR numbers. An example is presented to demonstrate the effectiveness and use of these operators in real-world scenarios.

Consider the following sets: $\mathcal{V} = \mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \ldots, \mathcal{V}_{\mathrm{m}}$ for the m alternatives, $\mathcal{J} = \{\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \ldots, \mathcal{J}_{\mathrm{n}}\}$ for n criteria, and $\mathcal{D} = \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, .., \mathcal{D}_{\mathrm{k}}$ for k experts. Consider the corresponding weight vector for alternatives as $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ represent the weighted vector for experts $\mathcal{D} = \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, .., \{\mathcal{D}_{\mathrm{k}}\}$. Both weight vectors meet the identical requirements and are in the closed interval [0,1], with their sum equal to one.

Let $\mathcal{A}_{ij} = \left(\underline{\zeta}_{ij}, \underline{\eta}_{ij}, \underline{\xi}_{ij}, \overline{\zeta}_{ij}, \overline{\eta}_{ij}, \overline{\xi}_{ij}\right)$ $(i = 1, 2, \ldots, n)$ and $(j = 1, 2, \ldots, m)$, where $(\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i)$ and $(\overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i)$ represents lower set approximation and upper aet approximation, subject to the constraint $(0 \leq \underline{\zeta}_{ij}{}^q + \underline{\eta}_{ij}{}^q + \underline{\xi}_{ij}{}^q \leq 1)$ and $(0 \leq \overline{\zeta}_{ij}{}^q + \overline{\eta}_{ij}{}^q + \overline{\xi}_{ij}{}^q \leq 1)$. Following is the procedure to solve a MCDM problem.

Step 1: Construct $\mathcal{D}^{(k)} = \left[\left(\mathcal{A}_{ij}^{(k)}\right)\right]_{m\times n}$ $(k = 1, 2, 3, \ldots, d)$ for decision.

Step 2: If the criteria have two types, such as benefit criteria and cost criteria, the $\mathcal{D}^{(k)} = \left[\left(\mathcal{A}_{ij}^{(k)}\right)\right]_{m\times n}$ $(k = 1, 2, 3, \ldots, d)$ van be converted into the normalized decision matrices $\mathcal{R}^{(s)}$=$(s = 1, 2, 3, \ldots, t)$ where

$$r^{(s)} = \begin{cases} \mathcal{A}_{ij}^{(k)} & \text{for benefit type of criteria} \\ \left[\left(\mathcal{A}_{ij}^{(k)}\right)\right]^C & \text{for cost type of citeria} \end{cases}$$

where $\left[\left(\mathcal{A}_{ij}^{(k)}\right)\right]^C$ is a complement of $\mathcal{A}_{ij}^{(k)}$.

Step 3: Utilize the proposed operators to aggregate $\mathcal{R}^{(k)} = \left[\left(\mathcal{A}_{ij}^{(k)}\right)\right]_{m\times n}$ into $\mathcal{R} = \left[\mathcal{A}_{ij}\right]_{m\times n}$

Step 4: Utilize the $\overline{\mathcal{A}_{ij}} = \sigma\overline{\lambda_i}\mathcal{A}_{ij}$.

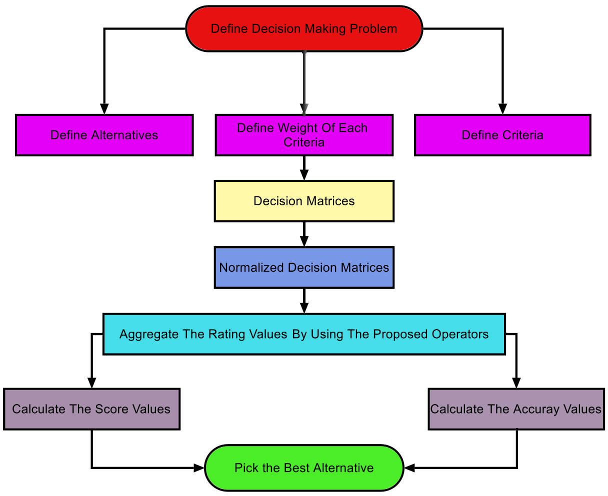

FIGURE 10. **Flow chart of the proposed model.**

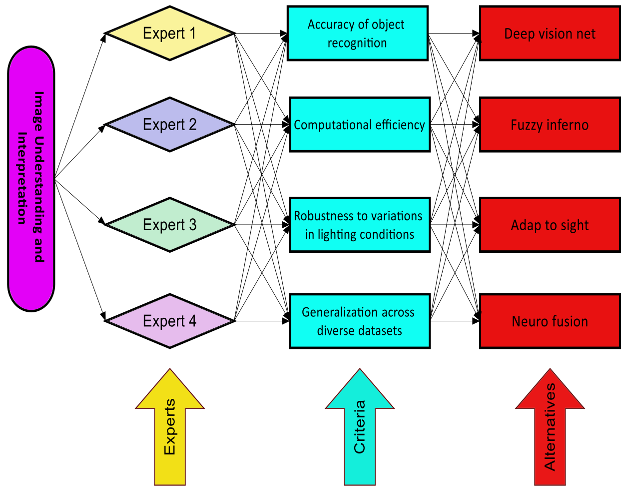

FIGURE 11. **Image understanding and interpretation.**

Step 5: Utilize the proposed operator to derive the overall preferences values**.**

Step 6: Calculate the scores of all values**.**

Step 7: Select the alternative which has the highest score value**.**

The flow chart of the proposed model is shown in Figure 10.

A. NUMERICAL EXAMPLE

To elucidate and illustrate the suggested technique, we offer an example in this section. In this scenario, consider an image recognition test where the goal is to identify objects in a complex scene. The four alternative solutions $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$ and $\mathcal{V}_4$ represent different image algorithms. The four criteria $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$ and $\mathcal{J}_4$ are defined as: $\mathcal{J}_1$ = Accuracy of object recognition, $\mathcal{J}_2$ = Computational efficiency, $\mathcal{J}_3$ = Robustness to variations in lighting conditions and $\mathcal{J}_4$ = Generalization across diverse datasets. The alternatives are $\mathcal{V}_1$ = Deep vision net, $\mathcal{V}_2$ = Fuzzy inferno, $\mathcal{V}_3$ = Adap to sight and $\mathcal{V}_4$ = neuro fusion. A group of four experts assigns weights to these criteria $\omega = (0.3, 0.1, 0.4, 0.2)^T$. The weight vector presents the importance of each criterion as determined by a group of experts. Now let's create a decision matrix $\mathcal{D}^{(\hbar)} = \left(\alpha_{ij}^{(\hbar)}\right)$ $(\hbar = 1, 2, 3, 4)$ for each automatic car. Tables 1-4 provide the decision matrix for evaluating image understanding and interpretations. The goal is to rank these cars and select the most suitable automatic car. Figure 11 illustrates a decision tree used for the image recognition test selection process.

Step 1: Construct the decision matrices (Tables 1-4).

Step 2: Construct the normalized decision matrices (Table 5-8). Since lower computational is better. As a result, a computational efficiency is regarded as advantageous, converting computational efficiency into a cost type criterion.

Step 3: Utilize the q-SFREOWA operator as shown in Table 9, where $\omega = (0.3, 0.1, 0.4, 0.2)^T$.

Step 4. Utilize $\overline{\mathcal{A}_{ij}} = \sigma \overline{\lambda}_i \mathcal{A}_{ij}$, where

$$\overline{\lambda} = (0.4122, 0.1222, 0.2315, 0.2341)^T.$$

$$\overline{\mathcal{A}_{11}} = (0.5246, 0.9826, 0.2478, 0.3456, 0.5496, 0.4528);$$

$$\overline{\mathcal{A}_{12}} = (0.2453, 0.2459, 0.9467, 0.6582, 0.4529, 03452)$$

$$\overline{\mathcal{A}_{13}} = (0.4652, 0.8956, 0.5298, 0.9746, 0.3246, 0.8576);$$

$$\overline{\mathcal{A}_{14}} = (0.2459, 0.7459, 0.3127, 0.4598, 0.2456, 0.4196)$$

$$\overline{\mathcal{A}_{21}} = (0.2546, 0.2498, 0.4278, 0.2457, 0.2546, 0.4527);$$

$$\overline{\mathcal{A}_{22}} = (0.5278, 0.2546, 0.2546, 0.2546, 0.5897, 0.2456)$$

$$\left[\left\langle \begin{array}{c} \sqrt[q]{\frac{2\prod_{i=1}^{4}\left(\underline{\zeta}_{\delta(i)}\right)^{\omega_i}}{\prod_{i=1}^{4}\left(2-\underline{\zeta}_{\delta(i)}\right)^{\omega_i}+\prod_{i=1}^{4}\left(\underline{\zeta}_{\delta(i)}\right)^{\omega_i}}}, \sqrt[q]{\frac{\prod_{i=1}^{4}\left(1+\underline{\eta}_{\delta(i)}\right)^{\omega_i}-\prod_{i=1}^{4}\left(1-\underline{\eta}_{\delta(i)}\right)^{\omega_i}}{\prod_{i=1}^{4}\left(1+\underline{\eta}_{\delta(i)}\right)^{\omega_i}+\prod_{i=1}^{4}\left(1-\underline{\eta}_{\delta(i)}\right)^{\omega_i}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^{4}\left(1+\underline{\xi}_{\delta(i)}\right)^{\omega_i}-\prod_{i=1}^{4}\left(1-\underline{\xi}_{\delta(i)}\right)^{\omega_i}}{\prod_{i=1}^{4}\left(1+\underline{\xi}_{\delta(i)}\right)^{\omega_i}+\prod_{i=1}^{4}\left(1-\underline{\xi}_{\delta(i)}\right)^{\omega_i}}}, \sqrt[q]{\frac{2\prod_{i=1}^{4}\left(\overline{\zeta}_{\delta(i)}\right)^{\omega_i}}{\prod_{i=1}^{4}(2-\overline{\zeta}_{\delta(i)})^{\omega_i}+\prod_{i=1}^{4}\left(\overline{\zeta}_{\delta(i)}\right)^{\omega_i}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^{4}(1+\overline{\eta}_{\delta(i)})^{\omega_i}-\prod_{i=1}^{4}(1-\overline{\eta}_{\delta(i)})^{\omega_i}}{\prod_{i=1}^{4}(1+\overline{\eta}_{\delta(i)})^{\omega_i}+\prod_{i=1}^{4}(1-\overline{\eta}_{\delta(i)})^{\omega_i}}}, \sqrt[q]{\frac{\prod_{i=1}^{4}(1+\overline{\xi}_{\delta(i)})^{\omega_i}-\prod_{i=1}^{4}(1-\overline{\xi}_{\delta(i)})^{\omega_i}}{\prod_{i=1}^{4}(1+\overline{\xi}_{\delta(i)})^{\omega_i}+\prod_{i=1}^{4}(1-\overline{\xi}_{\delta(i)})^{\omega_i}}} \end{array}\right\rangle\right]$$
$$= (0.6435, 0.3916, \ 0.6820, 0.5588, 0.6908, 0.7528).$$

TABLE 1. Decision matrix $\mathcal{D}^1$.

Alternatives	$\mathcal{J}_1$	$\mathcal{J}_2$	$\mathcal{J}_3$	$\mathcal{J}_4$
$\mathcal{V}_1$	(0.96,0.53,0.55, 0.25,0.45,0.76)	(0.19,0.65,0.85, 0.48,0.08,0.47)	(0.95,0.16,0.95, 0.65,0.16,0.65)	(0.96,0.75,0.29, 0.22,0.73,0.48)
$\mathcal{V}_2$	(0.66,0.89,0.72, 0.45,0.85,0.45)	(0.75,0.06,0.25, 0.26,0.06,0.65)	(0.95,0.45,0.35, 0.19,0.45,0.55)	(0.95,0.66,0,85, 0.34,0.54,0.25)
$\mathcal{V}_3$	(0.25,0.26,0.53, 0.45,0.95,0.26)	(0.68,0.68,0.78, 0.98,0.28,0.24)	(0.19,0.28,0.23, 0.21,0.25,0.18)	(0.95,0.46,0.89, 0.46,0.35,0.26)
$\mathcal{V}_4$	(0.65,0.26,0.95, 0.25,0.85,0.85)	(0.38,0.95,0.74, 0.65,0.75,0.36)	(0.66,0.83,0.16, 0.99,0.93,0.86)	(0.26,0.36,0.25, 0.65,0.25,0.74)

TABLE 2. Decision matrix $\mathcal{D}^2$.

Alternatives	$\mathcal{J}_1$	$\mathcal{J}_2$	$\mathcal{J}_3$	$\mathcal{J}_4$
$\mathcal{V}_1$	(0.47,0.83,0.24, 0.24,0.25,0.47)	(0.74,0.17,0.69, 0.96,0.94,0.36)	(0.74,0.83,0.25, 0.65,0.39,0.87)	(0.96,0.36,0.25, 0.25,0.52,0.96)
$\mathcal{V}_2$	(0.74,0.32,0.24, 0.79,0.62,0.39)	(0.36,0.36,0.25, 0.24,0.69,0.47)	(0.54,0.61,0.74, 0.74,0.86,0.36)	(0.47,0.65,0.35, 0.24,0.25,0.54)
$\mathcal{V}_3$	(0.25,0.62,0.47, 0.24,0.35,0.74)	(0.36,0.73,0.25, 0.36,0.74,0.47)	(0.98,0.37,0.42, 0.36,0.86,0.47)	(0.36,0.23,0.74, 0.38,0.86,0.27)
$\mathcal{V}_4$	(0.24,0.45,0.47, 0.96,0.57,0.45)	(0.45,0.86,0.74, 0.36,0.26,0.36)	(0.36,0.65,0.47, 0.74,0.55,0.36)	(0.57,0.67,0.38, 0.36,0.63,0.64)

TABLE 3. Decision matrix $\mathcal{D}^3$.

Alternatives	$\mathcal{J}_1$	$\mathcal{J}_2$	$\mathcal{J}_3$	$\mathcal{J}_4$
$\mathcal{V}_1$	(0.24,0.34,0.75 0.25,0.27,0.25)	(0.58,0.10,0.24, 0.24,0.08,0.25)	(0.27,0.89,0.25, 0.23,0.56,0.74)	(0.85,0.78,0.45, 0.69,0.62,0.56)
$\mathcal{V}_2$	(0.96,0.87,0.36, 0.85,0.34,0.78)	(0.36,0.29,0.52, 0.25,0.78,0.69)	(0.36,0.74,0.49, 0.74,0.89,0.29)	(0.79,0.56,0.92, 0.37,0.25,0.26)
$\mathcal{V}_3$	(0.41,0.27,0.85, 0.36,0.36,0.47)	(0.27,0.69,0.78, 0.79,0.85,0.65)	(0.14,0.63,0.26, 0.35,0.65,0.69)	(0.25,0.65,0.42, 0.56,0.47,0.25)
$\mathcal{V}_4$	(0.75,0.30,0.25, 0.47,0.09,0.24)	(0.79,0.69,0.76, 0.25,0.27,0.25)	(0.45,0.56,0.76, 0.89,0.58,0.52)	(0.85,0.30,0.56, 0.24,0.85,0.24)

TABLE 4. Decision matrix $\mathcal{D}^4$.

Alternatives	$\mathcal{J}_1$	$\mathcal{J}_2$	$\mathcal{J}_3$	$\mathcal{J}_4$
$\mathcal{V}_1$	(0.45,0.91,0.27, 0.36,0.23,0.25)	(0.28,0.37,0.25, 0.27,0.52,0.91)	(0.52,0.38,0.55, 0.25,0.53,0.37)	(0.27,0.31,0.25, 0.25,0.53,0.47)
$\mathcal{V}_2$	(0.41,0.25,0.89, 0.25,0.72,0.25)	(0.29,0.36,0.24, 0.99,0.57,0.74)	(0.72,0.36,0.56, 0.69,0.54,0.36)	(0.47,0.36,0.36, 0.63,0.54,0.37)
$\mathcal{V}_3$	(0.34,0.35,0.27, 0.4,0.51,0.27)	(0.29,0.34,0.42, 0.27,0.57,0.82)	(0.24,0.36,0.53, 0.36,0.54,0.34)	(0.74,0.39,0.85, 0.24,0.51,0.46)
$\mathcal{V}_4$	(0.64,0.36,0.25, 0.25,0.56,0.67)	(0.95,0.36,0.28, 0.25,0.57,0.74)	(0.27,0.36,0.52, 0.33,0.24,0.35)	(0.52,0.34,0.25, 0.58,0.55,0.28)

$$\overline{\mathcal{A}_{23}} = (0.4528, 0.2456, 05246, 0.9864, 0.6589, 0.2458);$$

$$\overline{\mathcal{A}_{24}} = (0.2475, 0.2458, 0.2549, 0.4726, 0.2548, 0.2549)$$

$$\overline{\mathcal{A}_{31}} = (0.5296, 0.2754, 0.8569, 0.5963, 0.4725, 0.8549);$$

$$\overline{\mathcal{A}_{32}} = (0.5134, 0.2546, 0.7849, 0.5467, 0.2579, 0.6498)$$

$$\overline{\mathcal{A}_{33}} = (0.5497, 0.2549, 0.6587, 0.2413, 0.9846, 0.57492);$$

$$\overline{\mathcal{A}_{34}} = (0.2549, 0.2549, 0.2549, 0.2589, 0.5421, 0.2542)$$

$$\overline{\mathcal{A}_{41}} = (0.2548, 0.2519, 0.4528, 0.3289, 0.9957, 0.5279);$$

$$\overline{\mathcal{A}_{42}} = (0.2471, 0.2587, 0.2573, 0.9735, 0.2549, 0.2589)$$

$$\overline{\mathcal{A}_{43}} = (0.5427, 0.5467, 0.5482, 0.2548, 0.2879, 0.6985);$$

$$\overline{\mathcal{A}_{44}} = (0.7413, 0.5821, 0.6428, 0.9856, 0.2546, 0.5496)$$

Step 5. Utilize the q-SFREWG operator to derive the overall preferences values.

Step 6. Utilize the q-SFREWG operator to derive the overall preferences values.

Calculate the overall preference values $\mathfrak{S}_i(i = 1, 2, 3, 4)$ for the alternative $\mathcal{V}_i(i = 1, 2, 3, 4)$ using the given data and the $q - SFREWA$ operator as shown below:

$$\mathfrak{S}_1 = \begin{pmatrix} 0.2573, 0.2791, 0.2781, \\ 0.5263, 0.9685, 0.8428 \end{pmatrix},$$

TABLE 5. Normalized decision matrix $\mathcal{R}^{(1)}$.

Alternatives	$\mathcal{J}_1$	$\mathcal{J}_2$
$\mathcal{V}_1$	(0.96,0.53,0.55, 0.25,0.45,0.76)	(0.85,0.65,0.19, 0.47,0.08,0.48)
$\mathcal{V}_2$	(0.66,0.89,0.72, 0.45,0.85,0.45)	(0.25,0.06,0.75, 0.65,0.06,0.26)
$\mathcal{V}_3$	(0.25,0.26,0.53, 0.45,0.95,0.26)	(0.78,0.68,0.68, 0.24,0.28,0.98)
$\mathcal{V}_4$	(0.65,0.26,0.95, 0.25,0.85,0.85)	(0.74,0.95,0.38, 0.36,0.75,0.65)

Alternatives	$\mathcal{J}_3$	$\mathcal{J}_4$
$\mathcal{V}_1$	(0.95,0.16,0.95, 0.65,0.16,0.65)	(0.96,0.75,0.29, 0.22,0.73,0.48)
$\mathcal{V}_2$	(0.95,0.45,0.35, 0.19,0.45,0.55)	(0.95,0.66,0,85, 0.34,0.54,0.25)
$\mathcal{V}_3$	(0.19,0.28,0.23, 0.21,0.25,0.18)	(0.95,0.46,0.89, 0.46,0.35,0.26)
$\mathcal{V}_4$	(0.66,0.83,0.16, 0.99,0.93,0.86)	(0.26,0.36,0.25, 0.65,0.25,0.74)

TABLE 6. Normalized decision matrix $\mathcal{R}^{(2)}$.

Alternatives	$\mathcal{J}_1$	$\mathcal{J}_2$
$\mathcal{V}_1$	(0.47,0.83,0.24, 0.24,0.25,0.47)	(0.69,0.17,0.74, 0.36,0.94,0.96)
$\mathcal{V}_2$	(0.74,0.32,0.24, 0.79,0.62,0.39)	(0.25,0.36,0.36, 0.47,0.69,0.24)
$\mathcal{V}_3$	(0.25,0.62,0.47, 0.24,0.35,0.74)	(0.25,0.73,0.36, 0.47,0.74,0.36)
$\mathcal{V}_4$	(0.24,0.45,0.47, 0.96,0.57,0.45)	(0.74,0.86,0.45, 0.36,0.26,0.36)

Alternatives	$\mathcal{J}_3$	$\mathcal{J}_4$
$\mathcal{V}_1$	(0.74,0.83,0.25, 0.65,0.39,0.87)	(0.96,0.36,0.25, 0.25,0.52,0.96)
$\mathcal{V}_2$	(0.54,0.61,0.74, 0.74,0.86,0.36)	(0.47,0.65,0.35, 0.24,0.25,0.54)
$\mathcal{V}_3$	(0.98,0.37,0.42, 0.36,0.86,0.47)	(0.36,0.23,0.74, 0.38,0.86,0.27)
$\mathcal{V}_4$	(0.36,0.65,0.47, 0.74,0.55,0.36)	(0.57,0.67,0.38, 0.36,0.63,0.64)

$$\mathfrak{S}_2 = \begin{pmatrix} 0.8576, 0.9489, 0.5249, \\ 0.2968, 0.4528, 0.8762 \end{pmatrix},$$

$$\mathfrak{S}_3 = \begin{pmatrix} 0.6975, 0.2375, 0.2897, \\ 0.2574, 0.3687, 0.5143 \end{pmatrix} \text{ and}$$

$$\mathfrak{S}_4 = \begin{pmatrix} 0.8864, 0.8296, 0.6374, \\ 0.2867, 0.2386, 0.5983 \end{pmatrix}.$$

By using Equation (9) we get $Sco(\mathfrak{S}_1) = 0.2042$, $Sco(\mathfrak{S}_2) = 0.2975$, $Sco(\mathfrak{S}_3) = 0.7108andSco(\mathfrak{S}_4) = 0.5541$. Using the score values, we can establish the ranking order of the available alternatives as follows:

$\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$. Hence fuzzy inferno is the best alternative. For these assessed alternatives, Table 11 provides a succinct illustration of the score's values and the ensuing ranking order utilizing the $q-SFREWG$ and $q-SFROWG$ operators.

The graphical representation of score values is shown in Figure 12.

This study also investigates the applicability of these operators in cases where decision-makers seek to adjust their choice aggregation approaches to their own preferences. Table 11 shows the results when various operators are employed, demonstrating how decision-makers may improve their decisions by considering both assigned values and expert opinions at the same time. The previous

TABLE 7. Normalized decision matrix $\mathcal{R}^{(3)}$.

Alternatives	$\mathcal{J}_1$	$\mathcal{J}_2$
$\mathcal{V}_1$	(0.24,0.34,0.75 0.25,0.27,0.25)	(0.24,0.10,0.58, 0.25,0.08,0.24)
$\mathcal{V}_2$	(0.96,0.87,0.36, 0.85,0.34,0.78)	(0.52,0.29,0.36, 0.69,0.78,0.25)
$\mathcal{V}_3$	(0.41,0.27,0.85, 0.36,0.36,0.47)	(0.78,0.69,0.27, 0.65,0.85,0.79)
$\mathcal{V}_4$	(0.75,0.30,0.25, 0.47,0.09,0.24)	(0.76,0.69,0.79, 0.25,0.27,0.25)

Alternatives	$\mathcal{J}_3$	$\mathcal{J}_4$
$\mathcal{V}_1$	(0.27,0.89,0.25, 0.23,0.56,0.74)	(0.85,0.78,0.45, 0.69,0.62,0.56)
$\mathcal{V}_2$	(0.36,0.74,0.49, 0.74,0.89,0.29)	(0.79,0.56,0.92, 0.37,0.25,0.26)
$\mathcal{V}_3$	(0.14,0.63,0.26, 0.35,0.65,0.69)	(0.25,0.65,0.42, 0.56,0.47,0.25)
$\mathcal{V}_4$	(0.45,0.56,0.76, 0.89,0.58,0.52)	(0.85,0.30,0.56, 0.24,0.85,0.24)

TABLE 8. Normalized decision matrix $\mathcal{R}^{(4)}$.

Alternatives	$\mathcal{J}_1$	$\mathcal{J}_2$
$\mathcal{V}_1$	(0.45,0.91,0.27, 0.36,0.23,0.25)	(0.25,0.37,0.28, 0.91,0.52,0.27)
$\mathcal{V}_2$	(0.41,0.25,0.89, 0.25,0.72,0.25)	(0.24,0.36,0.29, 0.74,0.57,0.99)
$\mathcal{V}_3$	(0.34,0.35,0.27, 0.4,0.51,0.27)	(0.42,0.34,0.29, 0.82,0.57,0.27)
$\mathcal{V}_4$	(0.64,0.36,0.25, 0.25,0.56,0.67)	(0.28,0.36,0.95, 0.74,0.57,0.25)

Alternatives	$\mathcal{J}_3$	$\mathcal{J}_4$
$\mathcal{V}_1$	(0.52,0.38,0.55, 0.25,0.53,0.37)	(0.27,0.31,0.25, 0.25,0.53,0.47)
$\mathcal{V}_2$	(0.72,0.36,0.56, 0.69,0.54,0.36)	(0.47,0.36,0.36, 0.63,0.54,0.37)
$\mathcal{V}_3$	(0.24,0.36,0.53, 0.36,0.54,0.34)	(0.74,0.39,0.85, 0.24,0.51,0.46)
$\mathcal{V}_4$	(0.27,0.36,0.52, 0.33,0.24,0.35)	(0.52,0.34,0.25, 0.58,0.55,0.28)

TABLE 9. Collective normalized decision matrix $\mathcal{R}$.

$\mathcal{V}_i$	$\mathcal{J}_1$	$\mathcal{J}_2$
$\mathcal{V}_1$	(0.5482,0.7469,0.2513, 0.9846,0.5428,0.2571)	(0.8543,0.2549,05486, 0.8572,0.5897,0.2469)
$\mathcal{V}_2$	(0.2749,0.5486,0.9543, 0.2876,0.9728,0.2749)	(0.9857,0.6857,0.5847, 0.2641,0.9852,0.3419)
$\mathcal{V}_3$	(0.2794,0.2486,0.2749, 0.5746,0.8543,0.2549)	(0.9574,0.6842,0.9845, 0.9514,0.6945,0.2749)
$\mathcal{V}_4$	(0.2749,0.5967,0.2489, 0.3891,0.9516,0.3574)	(0.9845,0.7463,0.7486, 0.2587,0.7412,0.2583)

$\mathcal{V}_i$	$\mathcal{J}_3$	$\mathcal{J}_4$
$\mathcal{V}_1$	(0.2549,0.2549,0.2573, 0.2574,0.3674,0.9823)	(0.5486,0.2586,0.5233, 0.1125,0.5246,0.9985)
$\mathcal{V}_2$	(0.1382,0.8346,0.8596, 0.2589,0.1452,0.2871)	(0.5874,0.5286,0.5876, 0.5286,0.2583,0.6589)
$\mathcal{V}_3$	(0.5249,0.5584,0.6422, 0.5482,0.9924,0.2489)	(0.2543,0.5876,0.2573, 0.3374,0.8467,0.2587)
$\mathcal{V}_4$	(0.2587,0.8576,0.5263, 0.2549,0.5233,0.5466)	(0.2569,0.5687,0.7413, 0.2394,0.5821,0.6791)

TABLE 10. Overall preferences values.

$\mathcal{V}_i$	$\mathcal{J}_1$	$\mathcal{J}_2$
$\mathcal{V}_1$	(0.5428,0.5682,0.2459, 0.6474,0.6872,0.6527)	(0.5864,0.5279,0.5236, 0.7419,0.5289,0.3652)
$\mathcal{V}_2$	(0.4529,0.9864,0.7439, 0.3567,0.2574,0.2568)	(0.2789,0.5268,0.1963, 0.2831,0.2891,0.5219)
$\mathcal{V}_3$	(0.5289,0.1374,0.4526, 0.2875,0.5637,0.8573)	(0.2537,0.2851,0.6537, 0.5894,0.2564,0.6589)
$\mathcal{V}_4$	(0.2563,0.6286,0.8976, 0.2542,0.7584,0.5486)	(0.3527,0.5791,0.864, 0.5284,0.5263,0.8546)

$\mathcal{V}_i$	$\mathcal{J}_3$	$\mathcal{J}_4$
$\mathcal{V}_1$	(0.5289,0.5469,0.8576, 0.5274,0.2538,0.5897)	(0.5493,0.2583,0.5496, 0.2763,0.2567,0.5927)
$\mathcal{V}_2$	(0.2576,0.9672,0.5823, 0.589,0.2563,0.8569)	(0.2583,0.7916,0.5637, 0.2895,0.2795,0.6253)
$\mathcal{V}_3$	(0.5286,0.8396,0.5796, 0.2314,0.5943,0.5826)	(0.8527,0.2763,0.9671, 0.2563,0.8196,0.5864)
$\mathcal{V}_4$	(0.5219,0.5623,0.5391, 0.2567,0.5897,0.3589)	(0.2578,0.2857,0.6729, 0.2587,0.5273,0.8543)

discussion shows that the proposed aggregation operators offer decision-makers a more adaptive framework for selecting viable choices. Furthermore, as compared to conventional aggregation methods, these operators offer more flexibility. This shows that the proposed operators can handle a broader range of decision-making scenarios while also providing better flexibility and relevance in several settings. By providing a more adaptive and inclusive framework, these aggregation operators enable decision-makers to make informed decisions that are in line with their needs and preferences. Furthermore, the generalizability of these operators assures their usefulness across a wide range of decision domains, hence improving the overall robustness and reliability of the decision-making process.

TABLE 11. Alternatives scores and sequence of ranking.

Operators	Score values			
	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$
$q-SFREWG$	0.3967	0.4527	0.9867	0.5896
$q-SFROWG$	0.2042	0.2975	0.7108	0.5541

Operators	Ranking
$q-SFREWG$	$\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$
$q-SFROWG$	$\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$

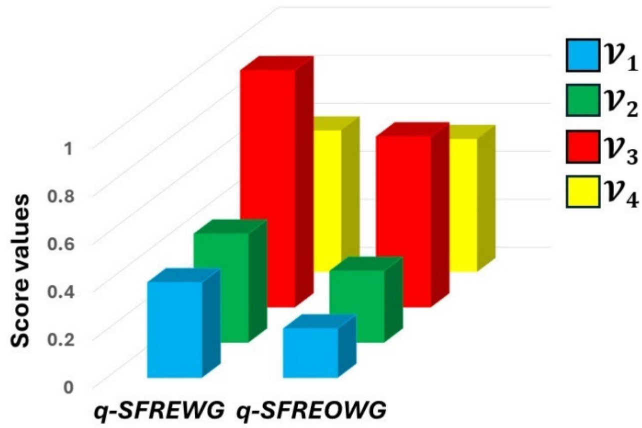

FIGURE 12. Graphical representation of score values of q-SFREWG and q-SFROWG.

B. EFFECT OF q ON RANKING ORDER AND SCORE VALUES

To fulfill the constraint requirement $(0 \le \underline{\zeta}_{\mathcal{A}}^{q}(\mathfrak{w}) + \underline{\eta}_{\mathcal{A}}^{q}(\mathfrak{w}) + \underline{\xi}_{\mathcal{A}}^{q}(\mathfrak{w}) \le 1)$ and $(0 \le \overline{\zeta}_{\mathcal{A}}^{q}(\mathfrak{w}) + \overline{\eta}_{\mathcal{A}}^{q}(\mathfrak{w}) + \overline{\xi}_{\mathcal{A}}^{q}(\mathfrak{w}) \le 1)$, and then by examining the attribute values, a decision-maker can determine which integer parameter, q, is the smallest. For example, while evaluating an alternative, if the attribute values are (0.8,0.7,0.9,0.9,0.8,0.7), one should choose q as 3 or q as 4, as both configurations meet the criterion. However, we employed several values of q of the novel approach to solve the case to fully evaluate the effect of parameter q on the experimental results. Table 12 presents the results of these modifications and indicates that $\mathcal{V}_3$ is at the top, followed by $\mathcal{V}_4$, $\mathcal{V}_2$, and finally, $\mathcal{V}_1$. Notable is the relevance of the best alternative and the unchanging ranking. Table 12 illustrates this point. Specifically, when q equals 1. The alternatives and ratings offered do not adhere to the requirements of either 1 (i.e., under the PFRS environment $\left(0 \le \underline{\zeta}_{\mathcal{A}}(\mathfrak{w}) + \underline{\eta}_{\mathcal{A}}(\mathfrak{w}) + \underline{\xi}_{\mathcal{A}}(\mathfrak{w}) \le 1\right)$ and $\left(0 \le \overline{\zeta}_{\mathcal{A}}(\mathfrak{w}) + \overline{\eta}_{\mathcal{A}}(\mathfrak{w}) + \overline{\xi}_{\mathcal{A}}(\mathfrak{w}) \le 1\right)$) or 2 (i.e., under SFRS environment $(0 \le \underline{\zeta}_{\mathcal{A}}^{2}(\mathfrak{w}) + \underline{\eta}_{\mathcal{A}}^{2}(\mathfrak{w}) + \underline{\xi}_{\mathcal{A}}^{2}(\mathfrak{w}) \le 1)$ and $(0 \le \overline{\zeta}_{\mathcal{A}}^{2}(\mathfrak{w}) + \overline{\eta}_{\mathcal{A}}^{2}(\mathfrak{w}) + \overline{\xi}_{\mathcal{A}}^{2}(\mathfrak{w}) \le 1)$.

TABLE 12. Sorting alternatives according to their respective parameter q values.

Parameter q	Ranking order	Best alternative
$q = 1$	$\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$	$\mathcal{V}_3$
$q = 2$	$\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$	$\mathcal{V}_3$
$q = 3$	$\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$	$\mathcal{V}_3$
$q = 4$	$\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$	$\mathcal{V}_3$
$q = 5$	$\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$	$\mathcal{V}_3$
$q = 6$	$\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$	$\mathcal{V}_3$
$q = 7$	$\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$	$\mathcal{V}_3$
$q = 8$	$\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$	$\mathcal{V}_3$
$q = 9$	$\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$	$\mathcal{V}_3$
$q = 10$	$\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$	$\mathcal{V}_3$
$q = 11$	$\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$	$\mathcal{V}_3$

Table. 12 shows how, for a range of q-parameter values, the ranking order of the alternatives stays consistent. This consistent ranking provides decision-makers with a robust framework to evaluate test alternatives within a given collection of finite alternatives. This gives decision-makers a secure and adaptable environment, facilitating careful examination and well-informed choices based on the specified parameters.

The q-SFREWG and q-SFROWG operators are applied within the framework of multi-criteria decision-making (MCDM) to address challenges in image understanding. In the context of MCDM for image understanding, decision-makers are faced with the task of analyzing and interpreting images based on multiple criteria or features extracted from the images. These criteria may include color, texture, shape, spatial relationships, and statistical properties, among others. The proposed q-SFREWG and q-SFROWG operators serve as aggregation functions that enable the integration and analysis of these multiple criteria. Specifically, the q-SFREWG operator combines the q-spherical fuzzy rough sets with Einstein-weighted aggregation, while the q-SFROWG operator combines these sets with ordered weighted aggregation. Decision-makers assign weights to each criterion based on their relative importance in the image understanding task. The q-SFREWG and q-SFROWG operators allow for the incorporation of these weights into the decision-making process. Once the criteria are weighted, the q-SFREWG and q-SFROWG operators aggregate the information from different criteria to obtain a comprehensive representation of the image. This aggregation process takes into account the relationships between criteria and the degree of uncertainty or imprecision associated with each criterion. The aggregated information obtained from the q-SFREWG and q-SFROWG operators serves as the basis for making informed decisions in image understanding tasks. Decision-makers can use the aggregated results to perform tasks such as object detection, classification, segmentation, and scene understanding with greater accuracy and reliability. By integrating q-spherical fuzzy rough sets with Einstein weighted aggregation and ordered weighted aggregation, the q-SFREWG and q-SFROWG operators offer a robust framework for handling uncertainty, imprecision, and vagueness in multi-criteria decision-making for image understanding. These operators provide decision-makers with the tools necessary to effectively analyze and interpret images based on diverse criteria, ultimately improving the quality of decision outcomes in image understanding applications.

C. TEST OF VALIDITY

To illustrate the adaptability of the proposed technique in various settings, we utilize the evaluation protocols developed by Wang and Trianaphyllou [61] in the following ways:

Step 1. Replacing the rating values of less-than-ideal alternatives with those of inferior quality shouldn't affect the identification of the best alternative, preserving the selection that is rated highest, and assuming stable relative weights for the criterion.

Step 2. Transitivity should be followed in the procedure.

Step 3. When using the same decision-making process for a given problem that has been broken into smaller ones, the initial ranking of the alternatives should be preserved.

Test of validity utilizing criteria 1.

The alternatives ranked by using our suggested method are $\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$. Based on test criteria 1, we replaced the non-optimal alternative $\mathcal{V}_1$ with the lowest alternative $\mathcal{V}_2^*$ to evaluate the stability of the suggested method. (0.25,0.87,0.29,0.64,0.85,0.74), (0.65,0.85,0.25,0.64,0.78,0.96), (0.25,0.83,0.28,0.73, 0.83,0.67) and (0.58,0.87,0.38,0.91,0.67,0.28) were used as the rating values of $\mathfrak{F}_1^*$. The aggregated score values for the alternatives were as follows after we used our suggested methodology: $Sco(\mathfrak{F}_1^*) = 0.2390$, $Sco(\mathfrak{F}_2) = 0.4342$, $Sco(\mathfrak{F}_3) = 0.9486$, $Sco(\mathfrak{F}_4) = 0.6879$, and as a result, the ranking order is $\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1^*$ with the best alternative remaing the same as the suggested approach. Thus, the findings consistently support test criteria 1.

Test of validity employing criteria 2 and 3.

The fragmented decision-making subcases are regarded as $\{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3\}$, $\{\mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4\}$ and $\{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_4\}$ to assess the validity based on criteria 2 and 3. They rank in the following sequence via the procedures mentioned: $\mathcal{V}_3 > \mathcal{V}_2 > \mathcal{V}_1$, $\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2$ and $\mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$. After combining all the findings, the overall ranking appears as $\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$, which is exactly in line with the outcomes of the initial decision-making process. As a result, our suggested strategy meets requirements 2 and 3.

V. MANAGERIAL IMPLICATIONS

The advent of the innovative q-SFR Einstein operators bears significant managerial implications. It plays a crucial role in aiding managers and decision-makers in shaping strategic choices and attaining robust, reliable results. This framework demonstrates impressive versatility across a range of industries and proves to be effective in diverse decision-making situations. Managers from various sectors can efficiently harness the potential of the image understanding and interpretation model. For example, it demonstrates its worth in the process of image understanding and interpretation selection by assisting in the evaluation of various factors to identify the most advantageous image understanding and interpretation technology. Moreover, it can aid in the selection of maintenance strategies, allowing managers to choose the most suitable maintenance approach for their equipment or systems. The assessment of robots in industrial settings is another field where the model can be utilized, aiding managers in evaluating the effectiveness and appropriateness of various robotic solutions. Additionally, it can be employed in the process of selecting material handling equipment, assisting managers in making informed decisions about the most optimal and productive equipment that suits their unique needs. Nevertheless, it is crucial to acknowledge that the decision-making process within this framework is dependent on the preferences of experts and individuals who are engaged in it. The model offers a methodical and organized method for decision-making, but the ultimate decisions and rankings will ultimately be based on the judgments and preferences of the decision-makers. Therefore, it becomes imperative to involve experts and stakeholders to ensure the precision and significance of the findings. To ensure the credibility and resilience of the obtained results, two pivotal analyses are conducted.

*Comparative Analysis:*The analysis serves as a valuable tool for decision-makers in assessing and comparing rankings and outcomes among different alternatives, each evaluated based on distinct criteria, IT enables a deeper understanding of trad0offs and facilitates well-informed decision-making by highlighting the strengths and weaknesses of each alternative.

Sensitivity Analysis: By doing so, it offers crucial insights into the stability and sensitivity of the outcomes. Decision-makers can thus evaluate how various factors influence their choices, enhancing their ability to make adaptive decisions in dynamic environments. By incorporating this analysis into the decision-making process, managers can enhance the reliability and confidence in their strategic decisions. The q-SFR Einstein operators, in conjunction with comparative and sensitivity analysis, provide a comprehensive framework that equips managers across diverse industries and applications with the tools needed to make informed and resilient decisions.

A. COMPARATIVE ANALYSIS

Table 13 presents a comparative evaluation of rankings achieved through the utilization of the q-SFR Einstein geometric operators in contrast to four other methods for Multi-Criteria Decision Making (MCDM). A review of recent research. The works from [62] to [70] add to the field of fuzzy aggregation operators and decision-making approaches. Wang and Liu provide intuitionistic fuzzy geometric aggregation operators based on Einstein's operations [62]. Arora and Garg examine robust aggregation operators for multi-criteria decision-making in an intuitionistic fuzzy soft set environment, emphasizing the significance of robustness in decision-making processes [63]. Rahman et al. present Pythagorean fuzzy Einstein weighted geometric aggregation procedures, which are particularly useful for multiple attribute group decision-making [64]. Zulqarnain and colleagues investigate a variety of applications, including green supply chain management [65], multi-attribute group decision-making [66], and mathematical issues in engineering, such as MAGDM [67]. Riaz et al. describe q-Rung orthopair fuzzy geometric aggregation operators for water loss management [68]. Furthermore, Chinram et al. propose geometric aggregation operations for multi-criteria decision-making using q-rung orthopair fuzzy soft information [69]. Finally, Zulqarnain et al. provide Einstein geometric aggregation procedures for Q-rung orthopair fuzzy soft sets, with a particular emphasis on their use in multi-criteria decision-making settings [70].

TABLE 13. Comparative analysis of rankings across differing methodologies.

Aggregation Operators	References	Score Values	Ranking
IFWG	[62]	(0.4612,0.4992,0.5572,0.5542)	$\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$
IFOWG	[62]	(0.4712,0.5050,0.5545,0.5426)	$\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$
IFSWG	[63]	(0.4754,0.5116,0.5681,0.5582)	$\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$
IFEWG	[62]	(0.4429,0.5007,0.5594,0.5561)	$\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$
PFEWG	[64]	(0.1632,0.2119,0.2525,0.2511)	$\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$
PFSWG	[65]	(0.4430,0.5147,0.5734,0.5537)	$\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$
PFSEWG	[66]	(0.4751,0.5234,0.5750,0.5497)	$\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$
PFSEOWG	[67]	(0.3674,0.3798,0.4975,0.4138)	$\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$
q-ROFWG	[68]	(0.2987,0.3502,0.4068,0.3805)	$\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$
q-ROFSWG	[69]	(0.3492,0.4048,0.4648,0.4337)	$\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$
q-ROFEOWG	[69]	(0.3600,0.4132,0.4677,0.4342)	$\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$
q-ROFSEWG	[70]	(0.4158,0.4307,0.4942,0.4862)	$\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$
q-ROFSEOWG	[70]	(0.4251,0.4467,0.5138,0.4568)	$\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$
q-SFREWG	[This Paper]	(0.3967,0.4527 ,0.9867,0.5896)	$\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$
q-SFREOWG	[This Paper]	(0.2042,0.2975,0.7108,0.5541)	$\mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_1$

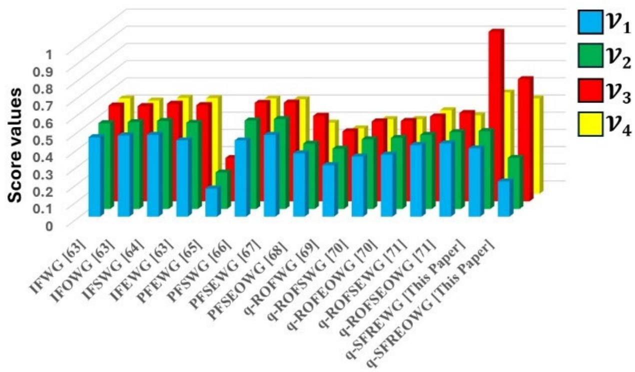

FIGURE 13. Visual representation of the rankings acquired using different methodologies.

Figure 13 presents a visual representation of the rankings acquired using different methodologies.

Based on the propositions, calculations, and applications discussed above, the following comparative remarks and advantages of employing the notion of q-spherical fuzzy rough sets emerge:

1. Traditional fuzzy sets and intuitionistic fuzzy sets exhibit limitations as they may fail to capture complete information specifications in certain scenarios. The

TABLE 14. The pros and cons of both the proposed and existing operators.

Operators	Approximations Set	Parameter q	Year
IFWG [62]	×	×	2011
IFOWG [62]	×	×	2011
IFSWG [63]	×	×	2018
IFEWG [62]	×	×	2011
PFEWG [64]	×	×	2017
PFSWG [65]	×	×	2021
PFSEWG [66]	×	×	2022
PFSEOWG [67]	×	×	2022
q-ROFWG [68]	×	✓	2020
q-ROFSWG [69]	×	✓	2021
q-ROFEOWG [69]	×	✓	2021
q-ROFSEWG [70]	×	✓	2022
q-ROFSEOWG [70]	×	✓	2022
q-SFREWG	✓	✓	2024
q-SFREOWG	✓	✓	2024

conditions of membership degrees and non-membership degrees may not always be satisfied, restricting decision-makers from expressing opinions freely.

2. To address these limitations, Yager proposed Pythagorean fuzzy sets, extending the representation to $\zeta^2 + \xi^2 \leq 1.$, enabling a wider range of applications.
3. In contexts involving uncertain information, such as voting systems, the introduction of "degree of refusal" necessitates the utilization of picture fuzzy sets. However, this approach presents its limitations in accommodating decision-maker flexibility.
4. Spherical fuzzy numbers offer a solution, capable of representing diverse information sets without exceeding the bounds of unity. This flexibility empowers decision-makers to allocate membership values according to their preferences.
5. The utilization of q-spherical fuzzy rough sets and associated algorithms, as demonstrated in selection processes, provides a generalized framework for impactful applications.
6. The proposed aggregation operators effectively handle imprecise information with a degree of refusal, offering superior reliability compared to existing approaches, as delineated in Table 13 and Table 14.
7. The applicability of q-spherical fuzzy rough sets extends to various domains, including stock investment analysis, airline service quality evaluation, investment banking authority selection, and electronic learning factor assessment, indicating their broad utility and relevance.
8. By leveraging the advantages of q-spherical fuzzy rough sets, decision-makers can navigate complex decision landscapes with greater confidence and precision.

Table 14 represents the pros and cons of the proposed operators and existing operators along with their year of publications.

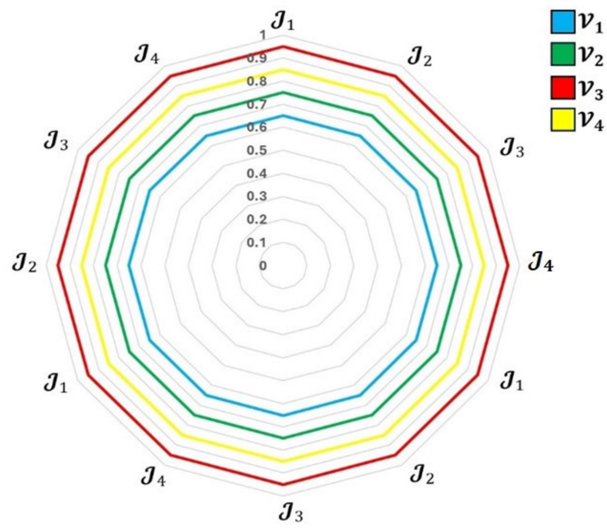

FIGURE 14. Alternative classification considering variations in criteria weights.

B. SENSITIVITY ANALYSIS

In this work, the developed model is verified using two separate sensitivity analyses that focus on changes in criteria and decision-making weights and examine their impact on final rankings. In the first research, a temporal sensitivity analysis is undertaken to explore the effect of varying the weights of reference criteria with high, equal, and low priority on the overall ranking. The model is then run separately for each criterion, assigning reference weights one at a time. Figure 14 displays the results from twenty distinct settings. In all circumstances, alternative $\mathcal{V}_4$ consistently ranks first, while alternative $\mathcal{V}_2$ regularly ranks last. Notably, despite considerable variations in the criterion weights, the model output is quite insensitive.

The second research focuses on adjusting the weights assigned to decision-makers, resulting in four possible situations with different weight distributions. Figure 15 displays the final rankings for these scenarios. In all scenarios, alternative $\mathcal{V}_3$ is consistently the greatest alternative, whereas alternative $\mathcal{V}_2$ is continuously the least loved. Although the relative rank of the alternatives differs according to the decision-maker weights utilized, the proposed approach is robust and consistent over a wide range of decision-weighting scenarios.

C. ADVANTAGES

The proposed technique has various benefits:

1. The addition of parameters q to the aggregation operators gives decision-makers a great deal of freedom. This versatility allows them to tailor the settings to the individual needs and preferences of the decision-making scenarios. The decision process's versatility allows for varying degrees of membership and non-membership, making it appropriate for a broad range of real-world scenarios.

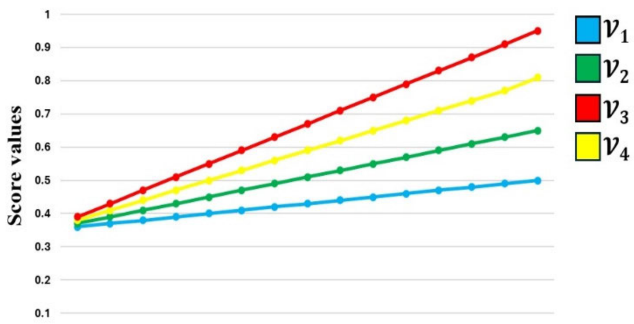

FIGURE 15. Alternative rankings in response to adjustments in decision-makers weights.

2. The parametric character of the suggested operators enables decision-makers to fine-tune the impact of membership and non-membership degrees. This degree of control enables decision-makers to accurately tailor the aggregation process to their preferences and the unique aspects of the situation at hand.
3. The symmetry of the suggested aggregation operators with respect to the parameter ensures that the ranking orders of alternatives stay generally consistent across parameter values. This stability is critical in decision-making because it prevents the outcomes from being impacted by the decision-makers' pessimism or optimism.

D. LIMITATIONS

Every research endeavor has limits, and the approach described in this study is not immune to such constraints. Following is a discussion of these limitations:

1. The applicability of the proposed technique may be limited to specific domains or decision contexts. Understanding these limitations is critical to determining the optimal use of the recommended strategy.
2. As with any research approach, the proposed method relies on certain assumptions and simplifications to facilitate analysis. It is important to recognize that these assumptions may not align perfectly with real-world scenarios, potentially limiting the broad or practical applicability of the results.
3. The accomplishment of the suggested framework is established through a case study including four alternatives and four criteria. It is critical to identify that the pattern may be expanded to integrate more possibilities and abilities in future efforts.
4. For several values of the parameter q, alternative ranking orders are calculated. It is important to note that more investigations might be conducted to investigate the hierarchical order for other values of these considerations.

E. IMPACTS

This research contributes to the field by proposing novel aggregation operators, namely q-spherical fuzzy rough Einstein weighted geometric (q-SFREWG) and q-spherical fuzzy rough Einstein ordered weighted geometric (q-SFREOWG) operators, specifically tailored for q-SFRS. These operators address the inherent complexities of multi-criteria decision-making (MCDM) scenarios by effectively integrating the orthopair q-spherical fuzzy sets with rough set theory. The impacts of our proposed operators extend beyond theoretical advancements; they offer practical solutions for real-world decision-making problems characterized by uncertainty and complexity. By bridging the gap between theoretical developments and practical applications, our work facilitates informed decision-making processes across various domains, including image understanding, pattern recognition, and system selection.

VI. CONCLUSION AND RECOMMENDATIONS FOR FUTURE WORK

In this paper, we looked at aggregation operators, with a particular emphasis on proposing new q-SFREWG and q-SFREOWA for q-spherical fuzzy rough sets (q-SFRSs). Clearly stated operational laws are critical in decision-making processes, and Einstein operators excel in accommodating experts' preferences over time. Our objective was to use these operators to enhance decision-making, resulting in a smoother and more effective end. We presented Einstein sum and Einstein product for q-spherical fuzzy rough numbers (q-SFRNs) and thoroughly investigated their properties. This investigation led to the creation of q-SFREWG and q-SFREOWG operators based on these rules, creating a framework for incorporating decision-makers' preferences. Furthermore, we investigated the underlying linkages between these aggregation operators, resulting in a thorough grasp of their interconnections. To apply these newly described operators to real-world decision-making scenarios, we suggested a novel technique for multiple attribute group decision-making (MAGDM), which addresses group choice difficulties. To evaluate the efficiency of our suggested strategy, we used a practical example involving the selection of an image understanding and interpretations. We extensively studied the approach's practicality and performance, including a comparative study with existing approaches and a sensitivity analysis to confirm its efficacy. Looking ahead, we will use the framework established by the new multiple attribute assessment models to address fuzziness and ambiguity in a variety of decision-making parameters, such as design choices, building options, site selection, and decision-making problems in a soft set environment.

CONFLICT OF INTEREST

The authors confirm that they do not possess any discernible conflicting financial interests or personal relationships that could appear to impact the research detailed in this paper.

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AHMAD BIN AZIM was born in Karak, Pakistan, in April 1992. He received the Graduate degree in mathematics from Kohat University of Science and Technology, in 2015, and the M.Phil. degree from Hazara University, Mansehra, Pakistan, where he is currently pursuing the Ph.D. degree with the Department of Mathematics and Statistics. He is also a Young Research Scientist with the Department of Mathematics and Statistics, Hazara University, where he is also a Visiting Lecturer, specializing in mathematics and statistics. With a notable publication history, he has contributed to over ten publications in SCI journals. His research interests include q-spherical fuzzy rough sets, aggregation operators, decision-making processes, soft sets, rough sets, machine learning and optimization, the neural network in artificial intelligence, fuzzy differential equations, fuzzy integral equations, fuzzy soft sets, fuzzy automaton, fuzzy languages, interval-valued fuzzy sets, aggregation operators, fuzzy aggregation operators, and application decision support systems and clustering.

ASAD ALI received the M.Phil. degree in mathematics from Hazara University, Mansehra, Pakistan, in 2012, and the Ph.D. degree in pure mathematics from Beijing Institute of Technology, in 2016. He spent 12 years as an Assistant Professor in mathematics with the Department of Mathematics, Hazara University. His research interests include fuzzy algebra and group theory. He was awarded a merit scholarship for an M.Phil. and a Chinese Government Scholarship for the Ph.D. degree.

ABDUL SAMAD KHAN received the Ph.D. degree from Northwestern Polytechnical University, China. He is currently a Distinguished Applied Mathematics Researcher passionate about theoretical hydrodynamics, non-Newtonian fluid flows, nanofluid flows, and heat transfer. He has dedicated his career to advancing the understanding of theoretical hydrodynamics. With nine years of academic experience, he has been a driving force in national and international research projects. Previously, he has published work in reputable journals and his expertise is widely recognized in the academic community. He is also an Assistant Professor with Northwestern Polytechnical University, he continues to influence applied mathematics through his research and mentorship of emerging scholars. He remains dedicated to pushing the boundaries of knowledge in applied mathematics and contributing to the broader academic discourse.

SUMBAL ALI was born in Swabi, Khyber Pakhtunkhwa, Pakistan, in 1994. She received the B.S. degree (Hons.) in mathematics and the Master of Philosophy in pure mathematics from Hazara University, Mansehra, Pakistan, in 2016 and 2018, respectively, where she is currently pursuing the Ph.D. degree with the Department of Mathematics and Statistics. Her research interests include fuzzy algebra, soft sets, aggregation operators, and decision-making.

FUAD A. AWWAD received the bachelor's degree in quantitative methods from the College of Business Administration, Saudi Arabia, in 1986, the master's Diploma degree in computing and statistics from Wales University, U.K., in 1994, and the Ph.D. degree in applied statistics from Lancaster University, U.K., in 1999. He is currently an Associate Professor in applied statistics and the Chairperson of the Executive Master of Insurance with the College of Business Administration, King Saud University. Throughout his career, he was a Statistical Consultant in several governmental and non-governmental organizations, such as the King Abdulaziz Center for National Dialogue (1999–2002), Ministry of Education (2002–2004), and Ministry of Interior Affairs (2004–2005). King Abdulaziz and his companion's Foundation for Giftedness and Creativity (2006–2012) and National Assessment Center (2107–2019). In addition to consulting, he has been actively involved in academia, teaching some intermediate, and advanced statistical courses in the college as an Assistant Professor and an Associate Professor, from 1999 to 2022. Instructing advanced courses in data analysis using SPSS to private and governmental sectors (health ministry, education ministry, interior ministry, and Qiyas), between 2002 and 2019. He has experience in the following statistical software S-PLUS, SPSS, SAS, EGRET, Latent Gold, and MINTAB; and the following programming languages R and Python. His research interests include statistical techniques on decision-making, statistical modeling in general and in epidemiology in specific labor market skills requirements, and education mismatches.

EMAD A. A. ISMAIL received the Ph.D. degree in risk management and insurance from the Faculty of Commerce, Cairo University, in 2005. He was a Professor in insurance and actuarial statistics with Cairo University. He is currently an Associate Professor in statistics with King Saud University for more than 17 years. He was a member of the Founding Committee for the establishment of the Executive Master's Program in Insurance. Also the Founder and the Chair of the Quality and Accreditation Unit, College of Business Administration, King Saud University. He has published more than 30 articles in risk assessment, underwriting in risks, property risk rating, reinsurance, reserves, estimating the maximum potential loss, mathematics, quantitative analysis, and Bayesian analysis of risk data. He earned within the team many competitive research grants (seven projects) from the King Abdulaziz City for Science and Technology (KACST) and some government agencies and ministries to prepare national research projects. He was the Award Rector of King Saud University to get his website ranked the first for the best website for a Faculty Member of the Faculty of Business Administration, King Saud University. He received numerous awards from other institutions.