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RESEARCH ARTICLE

A Power Transformation for Non-Normal Processes Capability Estimation

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ABSTRACT The capability of a process is its ability to produce products that meet predefined requirements in industry and it is measured by capability indices. If data are not normally distributed, other techniques for capability estimation should be taken into consideration. One of frequently used methods are transformations to normal data distribution like Abbasi-Niaki, Box-Cox and Johnson transformation. In this paper, we propose modified power transformation method for non-normal process capability estimation. Proposed method is compared to Box-Cox, Johnson and Abbasi-Niaki transformation method using simulation studies under several theoretical non-normal distributions. Proposed modified transformation method finds optimal power to reduce data skewness in case of negatively and positively skewed data. In case the optimal power is not found, power to achieve minimal skewness is estimated. In case of negatively skewed data, algorithm transforms data to positively skewed. After applying each transformation, capability indices were estimated and then compared with theoretical indices by calculating relative bias. Proposed transformation method showed better performance than other methods in reducing skewness of negatively and positively skewed data. Study showed that performance of the proposed power transformation method was either better or comparable in estimating capability indices.

INDEX TERMS Box-Cox transformation, non-normal process, power transformation, process capability.

I. INTRODUCTION

Process capability analysis is part of quality control in the industry with the goal of supplying information on product design and process quality improvement for engineers and designers. The benefits from process capability control are to ensure that product is in its validated state and to ensure that the manufactured batches are conforming to the specifications and that there is no adverse trend which could lead to batch rejection. Having a good tool and statistical methods for estimating process capability is essential, as they have influence on business decisions and consumers.

Process capability is measured by capability indices, assuming six sigma approach ([1], [2]):

$$C_p = \frac{USL - LSL}{6s}, \tag{1}$$

and

$$C_{pk} = \min\left\{\frac{\bar{x} - LSL}{3s}, \frac{USL - \bar{x}}{3s}\right\}, \tag{2}$$

where USL and LSL represent upper and lower specification limit, $\bar{x}$ is the estimate of the mean value μ and s is the estimate of the process standard deviation σ. Indices C_p in C_{pk} can be reliably used to assess the ability of the process only in the case of normal data distribution. When the data do not follow the normal distribution, the calculation of the indices need to be adjusted.

Theoretically, three approaches are possible in the case of non-normal data distribution: calculation of process variability using percentile estimation, data transformation and adjusted index calculation.

In estimating percentiles, the goal is to estimate the 0.135 percentile ($x_{0.00135}$), the 99.865 percentile ($x_{0.99865}$) and the median ($x_{0.5}$), which are necessary for calculating C_p

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and C_{pk} indices:

$$C_p = \frac{USL - LSL}{x_{0.99865} - x_{0.00135}}, \tag{3}$$

and

$$C_{pk} = \min\left\{\frac{x_{0.5} - LSL}{x_{0.5} - x_{0.00135}}, \frac{USL - x_{0.5}}{x_{0.99865} - x_{0.5}}\right\}. \tag{4}$$

Author in [3] described two methods to estimate 0.135 percentile ($x_{0.00135}$), 99.865 percentile ($x_{0.99865}$) and median ($x_{0.5}$): Exact method and Clement's method [4]. Authors in [5] also discussed about Clement's method, although they assumed distribution of data is known. In case the data distribution is not known, the Chebycheff inequality is suggested as alternative way for calculating capability indices [5]:

$$p(\mid X - \mu \mid < k\sigma) \geq 1 - \frac{1}{k^2}, k > 1. \tag{5}$$

According to Chebycheff inequality, when we use $\bar{x}$ to estimate μ in Equation 5, at least $100(1 - \frac{1}{k^2})\%$ of the observed values will fall within $\mu \mp k\sigma$ regardless of the distribution. However, we should use intervals definition $\mu \mp k\sigma$ carefully, as they could exceed specification limits USL and LSL. For example, if we want 99.865% of the observed values to fall within $\mu \mp k\sigma$ according to Equation 3 and Equation 4, k should be larger than 28, which is far too conservative for the distributions usually associated with processes. Therefore, the Chebycheff inequality is not applicable in practice.

Using data transformation technique, the goal is to transform the data to normal distribution, which then allows us to estimate the indices according to Equation 1 and Equation 2. Authors in [6] suggest transforming data to normal by Johnson transformation based on Johnson curves [9] or Box-Cox transformation [8]. According to [1], approach based on Johnson curves should be regarded within considerable caution as not every data distribution can be described by Johnson curves and can yield unstable or inefficient curve parameters in some cases. Another power transformation method is outlined in [29], aiming to minimize skewness and kurtosis for the Jarque–Bera test. In the case data can't be transformed to normal and data distribution is unknown, Clement's method is proposed [6]. Burr method for percentile estimation is proposed by Ahmad et al. [10], who compared Burr based parameters, introduced by Burr [7], with Clement's and Box-Cox transformation methods. Another method for estimating percentiles without statistical tables and easy to use was given in [11]. Authors in [30] investigated non-normal process capability indices utilizing fuzzy information, developing a general type of process capability indices based on a proposed triangular fuzzy distance to address imprecise observations, fuzzy limits, and targets, with the proposed method performing effectively in real applications according to a simulation study. As the Box-Cox transformation is applicable only for positive data, Yeo and Johnson [13] introduced a new family of transformations that covers the entire real line. Rivera et al. [14] proposed transformation by using continuous monotonically increasing functions called logarithmic transformation, square root transformation, inverse transformation, inverse square root transformation, arcsin square root transformation and power transformation. A study in [14] showed that power transformation gives better results than other transformations. Somerville and Montgomery [15] used a square-root transformation to transform a skewed distribution into a normal distribution. The transformation method using the root transformation is proposed by Hosseinifard et al. [12], who showed that root transformation techniques have better performance than the Box-Cox transformation method.

Many authors went one step further and redefined new indices for non-normal data, as percentile estimation techniques did not provide accurate results for heavily skewed distributions: the weighted variance method [16], Gini's mean difference [17], Wright's index [18], flexible index by Johnson [19], index by Chen and Pearn [20], indices by Albing and Vännman [21], generalized new indices [28] and ranked probability score index [22]. In [31], authors delve into the utilization of gauge R&R studies as a tool for uncovering the sources of non-normality in data distributions. Among the methods which were already mentioned, other quantile transform approaches, distribution-free tolerance intervals approach and superstructure capability indices for process capability estimation can be found in [23] and [24].

In this paper, a modification of root transformation proposed by Hosseinifard et al. [12] is presented and compared with Box-Cox and Johnson transformation methods.

II. METHODS

Box-Cox transformation method identifies appropriate exponent λ for transforming into data which has the highest likelihood and takes the following form ([8], [26]):

$$f(Y) = \begin{cases} \dfrac{Y^\lambda - 1}{\lambda} & \text{if} \lambda \neq 0, \\ \text{Log(Y)} & \text{if} \lambda = 0, \end{cases} \tag{6}$$

where parameter λ is estimated by employing the maximum likelihood estimator (MLE) [26].

Johnson system of distribution uses Johnson curves (S_B, S_U, S_L) to transform the data ([9], [27]). Distributions (S_B, S_U, S_L) of Johnson curves have two shape (γ and η), one location (ϵ) and one scale (λ) real parameters. S_B cover bounded distributions as gamma, beta and other distributions. It is bounded on the lower end by ϵ, the upper end by $\epsilon + \lambda$ or both [27]. S_B covers bounded distributions, S_U is unbounded and S_L covers distributions which are bounded only on the lower side [27]. These three distributions are generated by transformations of the form [27]:

$$z = \gamma + \eta k_i(x; \lambda, \epsilon), i = 1, 2, 3, \tag{7}$$

where $k_i(x; \lambda, \epsilon)$ are chosen as follows [27]:

$$k_1(x; \lambda, \epsilon) = \ln\left(\frac{x-\epsilon}{\lambda+\epsilon-x}\right) \text{ in the case of } S_B \text{ distribution,} \tag{8}$$

$$k_2(x; \lambda, \epsilon) = \sinh^{-1}\left(\frac{x-\epsilon}{\lambda}\right) \text{ in the case of } S_U \text{ distribution,} \tag{9}$$

$$k_3(x; \lambda, \epsilon) = \ln\left(\frac{x-\epsilon}{\lambda}\right) \text{ in the case of } S_L \text{ distribution.} \tag{10}$$

In this study, parameters of Johnson system will be estimated by quantile estimation method described in [27].

Niaki and Abbasi [25] first mentioned root transformation technique for skewness reduction in multi-attribute process monitoring. One year later, Hosseinifard et al. [12] discussed same method for skewness reduction for univariate process parameters. The idea is based on knowing standardized central moments of the third and fourth order, named skewness and kurtosis respectively. The standardized central moment of the third order β_1 and fourth order β_2 of normally distributed random variable $X \sim N(\mu, \sigma)$ are given as follows:

$$\beta_1 = \frac{E[(X-\mu)^3]}{\sigma^3} = \frac{0}{\sigma^3} = 0, \tag{11}$$

and

$$\beta_2 = \frac{E[(X-\mu)^4]}{\sigma^4} = \frac{\frac{4!\sigma^4}{2^2 2!}}{\sigma^4} = 3. \tag{12}$$

In the root transformation method proposed by authors in [12] and [25], when sample Y is taken from non-normal data distribution, we search for the power r such that skewness of Y^r equals 0 as shown in Equation 11. To find power r, bisection method is used on predefined initial interval $(a_0, b_0) \in \mathbb{R}$ such that $f(a_0)f(b_0) < 0$, where function $f(a)$, $a \in \mathbb{R}$ is defined as skewness value of Y^a. After the initial interval and tolerance ϵ are determined, the well known bisection method is applied in [12] to find the root r such that $f(r) = 0$.

For positively skewed data, bisection method is trying to find a root r in the interval $(0, 1)$, however, when data distribution is heavily negatively skewed, algorithm proposed in [12] is no longer applicable. For that reason Abbasi et al. [26] proposed modified root transformation technique such that skewness of Y^r equals 0 also in case of heavily negative skewed data by using transformation $U = Y - min(Y)$ and then finding root r using bisection method such that

$$f(U^{\frac{1}{r}}) = 0. \tag{13}$$

Abbasi et al. [26] also proposed that in case skewness of Y^r is positive should be used as transformation function. Moreover, authors suggest that power should be found within interval $(0, 1)$.

A simulation study of modified root transformation method performed in [26] showed good performance in all right-skewed, left-skewed and logarithmic distributions. Moreover, authors in [26] showed that transformation method efficiently transforms non-normal data to normal data in almost all cases. However, Box-Cox method does slightly better than the procedure described in [25] for small sample sizes and in general has higher p-values of Shapiro-Wilk normality test.

A. PROPOSED POWER TRANSFORMATION METHOD

Applying described method by Abbasi et al. [26] on data simulated from different distributions revealed two types of data for which appropriate power can't be found:

1) Predefined initial interval $(a_0, b_0) \in \mathbb{R}$ such that $f(a_0)f(b_0) < 0$ (function $f(a)$, $a \in \mathbb{R}$ is defined as skewness value of Y^a) doesn't always exist. To demonstrate this type of data, sample was simulated from positively skewed Gamma distribution with sample size 30, and parameters 5 and 7. For simulated sample, function $f(a)$, $a \in \mathbb{R}$ always gives skewness larger than zero. Therefore, bisection method described in [25] can not be applied.
2) The best root for data transformation can be found outside interval $(0, 1)$. To demonstrate this type of data, sample was simulated from positively skewed Gumbel distribution with sample size 210, location parameter 5 and scale parameter 0.5.
   For simulated sample, function $f(a)$, $a \in \mathbb{R}$ equals zero within interval (-4,-2), which is outside (0,1) interval.

In this paper we propose modified power transformation method for both positively and negatively skewed data. In the case when data are positively skewed, we find initial interval (a, b) from bisection method within entire $\mathbb{R}$. In case such interval doesn't exist, interval with minimum skewness value is taken as initial interval for bisection method. If acceptable power can't be found by using bisection method based on skewness, kurtosis criteria is used for finding power r such that kurtosis value of Y^r equals 3 (according to Equation 12). Last, if appropriate power can't be found within initial interval, then initial interval should be redefined. Moreover, when searching for initial interval $(A, B) \in \mathbb{R}$, it should be considered that value $0 \notin (A, B)$, as it can lead algorithm to find optimal power which equals zero. When data are negatively skewed, we propose transformation to positively skewed data as follows:

$$Y = -Y + 2\,max(Y). \tag{14}$$

The linear transformation $Y = -Y + 2max(Y)$ is utilized to convert negatively skewed data into positively skewed data, as negative and positive skewed data can't be transformed to normal distribution by applying the same algorithm. The general approach to reverse the distribution of data involves applying a linear transformation of the form $Y = c - bX$, where c and b are constants. In this study, $X = Y$, $c =$

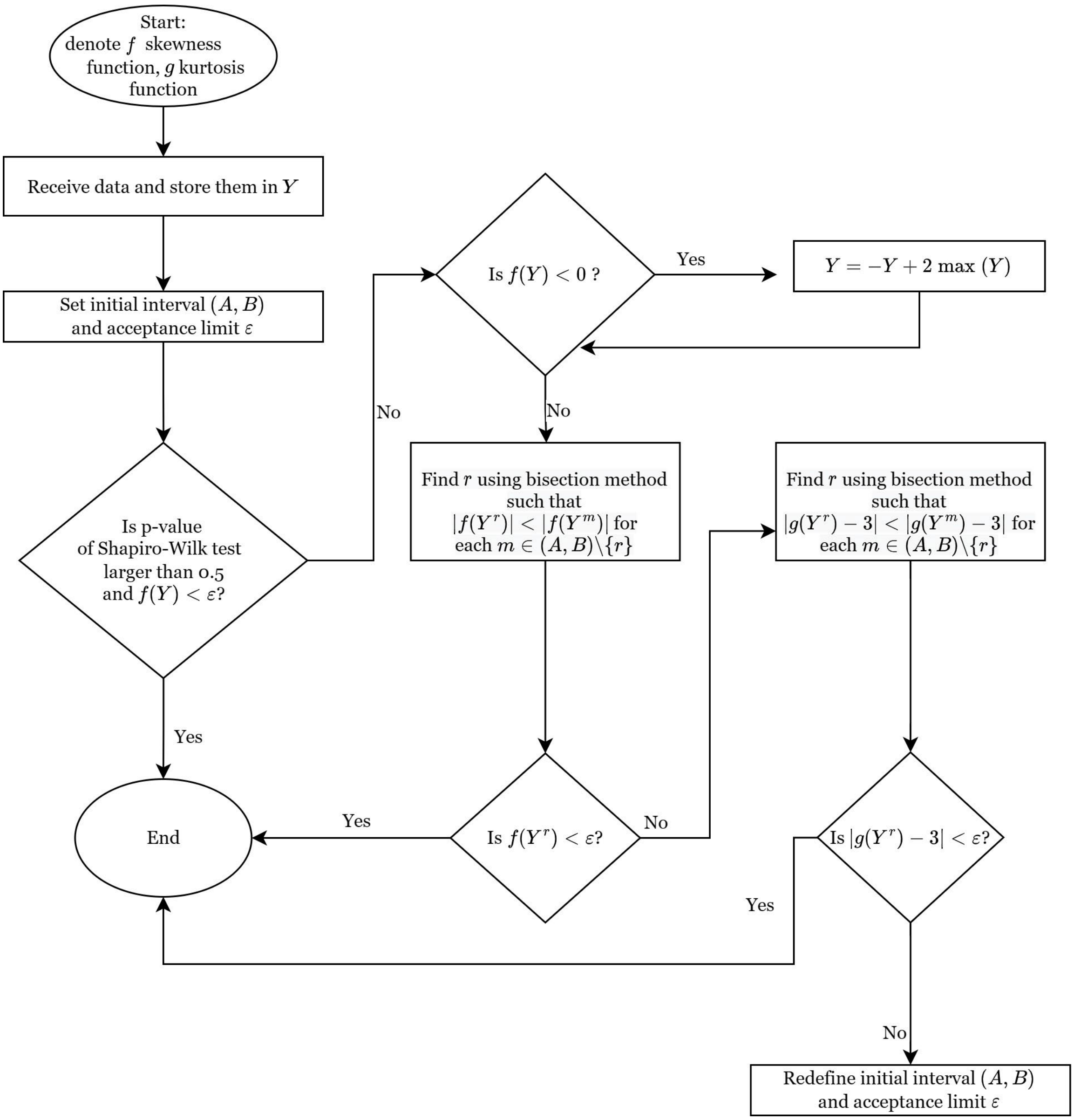

FIGURE 1. Flowchart of proposed method.

$2max(Y)$ and $b = 1$. The constant c can take any value that converts negative values $-Y$ to positive values to facilitate transformation methods. Thus, $c > max(Y)$. The authors' choice is $c = 2max(Y)$ but the transformation would function similarly if $c = 3max(Y)$. Notably, altering the constant c does not affect the capability indices since both the Upper Specification Limit (USL) and Lower Specification Limit (LSL) undergo the same transformation.

Flowchart in Figure 1 describes each step in proposed transformation method, along with pseudocode provided in Section V.

If we again take simulated data described in 1 and 2 and apply algorithm proposed in Figure 1, we get following results:

- For a sample simulated from positively skewed Gamma distribution with sample size 30, location parameter 5 and scale parameter 7, in Figure 2 can be seen that optimal power for transformation to normally distributed data is 0.525. As algorithm couldn't find power such that skewness of transformed data are below acceptance limit 0.05, kurtosis was used as criteria function to find optimal power. Kurtosis values based on different power transformations compared to optimal power transformation can be found in Figure 3. P-value of Shapiro-Wilk normality test before transformation was 0.01, and after transformation calculated Shapiro-Wilk p-value was 0.45.
- For sample simulated from positively skewed Gumbel distribution with sample size 210, location parameter 5 and scale parameter 0.5, the proposed transformation method described in Figure 1 gives optimal power -3.575. Skewness values compared to power values

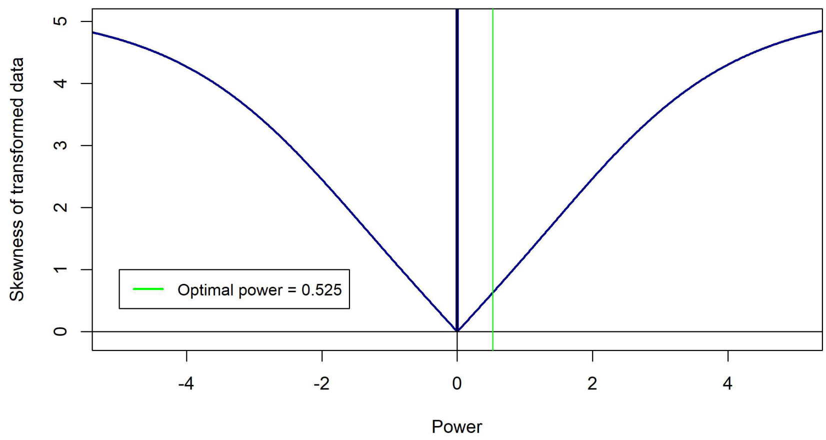

FIGURE 2. Skewness and optimal solution.

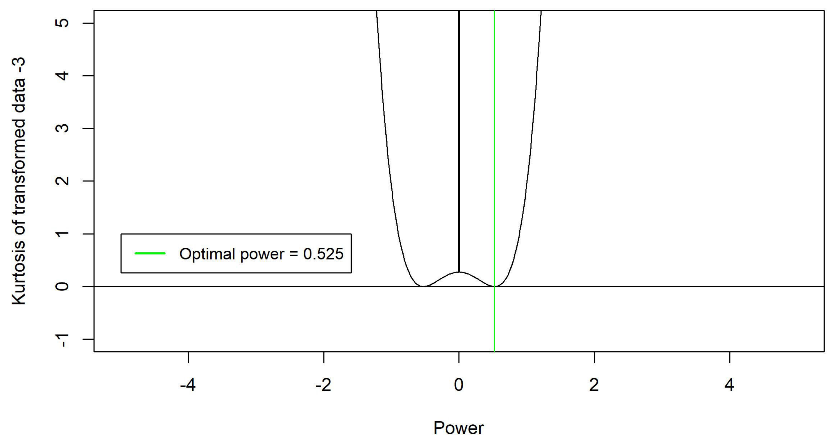

FIGURE 3. Kurtosis and optimal solution.

and optimal power are given in Figure 4. P-value of Shapiro-Wilk normality test before transformation was < 0.001, and after transformation calculated Shapiro-Wilk p-value was 0.1904.

III. RESULTS

In order to compare performance of proposed method to other transformation methods, data were simulated in statistical software R using functions for generating random samples from Beta and Gamma distribution, for which density functions are defined as follows:

- Beta(a, b) distribution

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, \tag{15}$$

  for $a > 0$, $b > 0$ and $0 \leq x \leq 1$,
- Gamma(a, b) distribution

$$f(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-xb}, \tag{16}$$

  for $a > 0$, $b > 0$ and $x \geq 0$.

To perform that evaluation, 32 different data distributions were chosen: Beta(1,2), Beta(1,4), Beta(1,7), Beta(2,2), Beta(2,1), Beta(2,4), Beta(2,5), Beta(2,7), Beta(4,1), Beta(4,2), Beta(4,4), Beta(4,5), Beta(4,7), Beta(5,2), Beta(5,4), Beta(5,7), Beta(7,1), Beta(7,2), Beta(7,4), Beta(7,5), Gamma(0.5,0.5), Gamma(0.5,2), Gamma(0.5,7), Gamma(1,0.5), Gamma(1,2), Gamma(1,7), Gamma(2,0.5), Gamma(2,2), Gamma(2,7), Gamma(5,0.5), Gamma(5,2) and Gamma(5,7). The goal of having 32 different data distribution is to ensure that wide enough range of skewness and kurtosis values were taken. For each distribution sample with size 10, 30 and 90 was generated 30 times to obtain representative sample. An overview of sample skewness, kurtosis and Shapiro-Wilk p-value before transformation is given in Figure 5. To reflect situation in practice, only non-negative simulated values were taken into account.

Results of simulation study are organized in two groups: transformation results and capability estimation. Statistical analysis of simulated data was performed in statistical software R using additional package jtrans for implementation of Johnson transformation.

A. TRANSFORMATION RESULTS

For each sample, proposed power transformation, Abbasi-Niaki, Box-Cox and Johnson transformation were conducted.

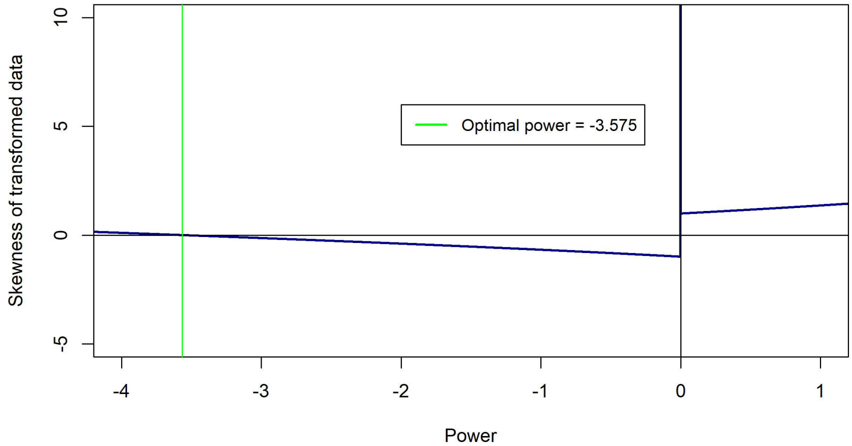

FIGURE 4. Skewness and optimal solution.

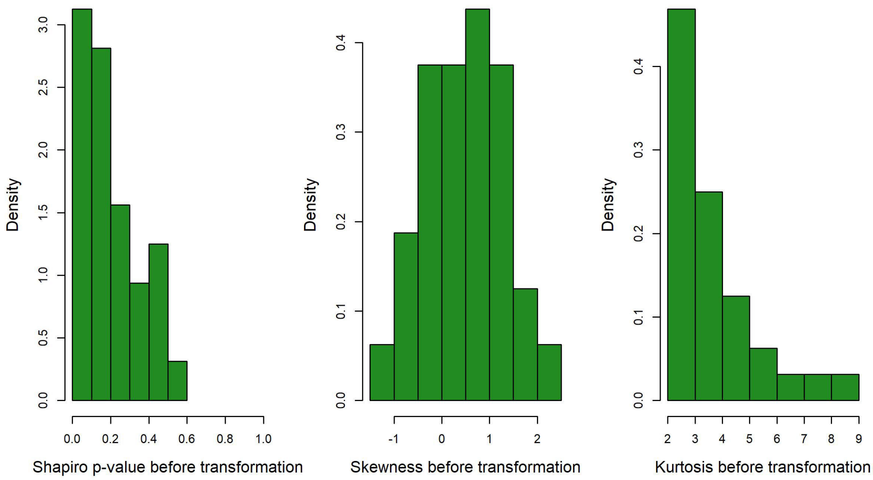

FIGURE 5. Simulated data.

Furthermore, for each sample and for each transformation method, sample standard deviation, sample mean, sample skewness, sample kurtosis and p-value of Shapiro-Wilk normality test were recorded before and after transformation. To obtain representative sample, average sample skewness, kurtosis and p-value of Shapiro-Wilk normality test were calculated for each distribution and sample size, which gave 96 average values. Difference between Shapiro-Wilk p-value before and after transformation was calculated and displayed in Table 4. Johnson transformation was the most successful compared to other methods as p-value of Shapiro-Wilk test before transformation was on average increased by 0.54. Proposed power transformation method increased p-value by 0.35 on average, which lead us to the conclusion that performance of the proposed power transformation method has better performance in terms of p-values increase than method by Abbasi-Niaki and Box-Cox transformation.

In terms of Shapiro-Wilk normality test, for data transformed by Johnson method null hypothesis on normality was rejected in 1.7 % of cases. Both, the proposed power transformation and Abbasi-Niaki transformation had better performance than Box-Cox transformation, as only for 7.2 % samples had null hypothesis of Shapiro-Wilk test rejected in the case of the proposed power transformation (Figure 6).

TABLE 1. Difference between Shapiro-Wilk p-values before and after transformation.

Transformation method	Average difference	Standard deviation of the difference
Johnson transformation	0.54	0.16
Proposed power transformation	0.35	0.19
Abbasi-Niaki transformation	0.31	0.21
Box-Cox transformation	0.29	0.19

For the proposed power transformation, 2.1 % of 2880 samples were transformed by using optimal kurtosis approach and 0.7 % of samples were transformed using minimum skewness approach as described in Figure 1. Remaining 97.2 % of samples where transformed using zero skewness approach. For Johnson transformation, 78 % of samples were transformed using S_B Johnson curve according to Equation 8. As the proposed transformation method has different approach in the case of negative skewness

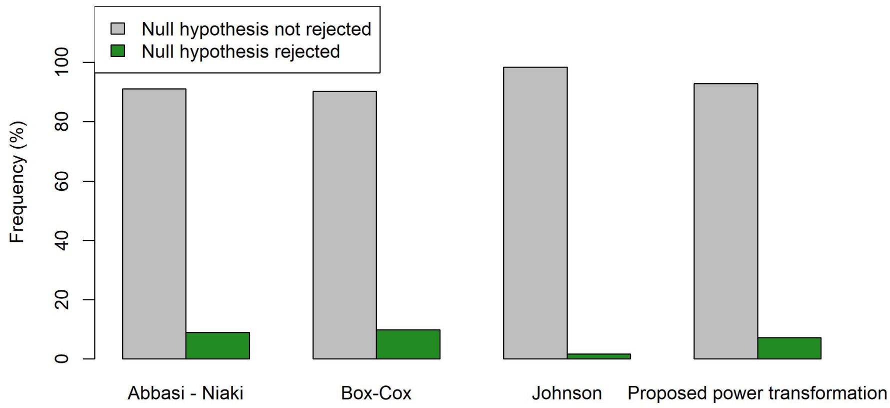

FIGURE 6. Normality test results.

compared to Abbasi-Niaki method, mean difference in p-value was compared for average sample with negative skewness: Abbasi-Niaki method on average increases p-value by 0.10 and proposed transformation method increases p-value by 0.19. For that reason, we can agree that proposed transformation method has better performance in the case of negatively skewed data. Overview of average sample skewness before and after transformation is provided in Figure 7. It can be seen that Box-Cox transformation method has the lowest performance in case of negatively skewed data and proposed power transformation has the lowest performance in case of data with skewness close to 0, but not lower compared to Box-Cox and Abbasi-Niaki method.

Median skewness and kurtosis value after transformation are also important indicators of transformation performance. According to Equation 11 and Equation 12, ideal skewness after transformation is 0 and ideal kurtosis after transformation is 3. In Figure 7 it is clear that proposed power transformation reduces skewness to zero better than any other method, as median value is closest to zero. Median kurtosis value is closest to 3 in the case of Johnson transformation and in case of other transformation methods, kurtosis values after transformation are comparable (Figure 8).

B. CAPABILITY ESTIMATION

Lower and upper specification limits in industry are usually defined by regulatory agencies and do not rely on theoretical data distribution of specific parameter. For that reason, for generated sample upper specification limit (USL) and lower specification limit (LSL) were chosen to reflect situation in practice. List of specification limits by distribution is given in Table 2.

For each sample 99.865, 50, and 0.135 percentiles were calculated using quantile function in R. Equation 3 and Equation 4 were then applied to calculate indices C_p and C_{pk} (target indices). After data transformation is performed and null hypothesis of normality test isn't rejected, transformed data were used to estimate process capability indices after transformation C'_p and C'_{pk} according to Equation 1 and Equation 2. Mean value and standard deviation were estimated from transformed sample, while USL and LSL were transformed in a same way as sample was transformed. For some samples transformation of USL and LSL couldn't be performed from mathematical reasons. For example, in case of Johnson transformation for 76.5 % of samples USL and LSL couldn't be transformed by Equation 8 as $x > \lambda+\epsilon$, which made most of samples fitted to S_B not successfully transformed. For Johnson transformation, authors in [27] used percentiles approach to estimate C_p and C_{pk}, which won't be discussed in this paper.

TABLE 2. Specification limits.

Distribution ID	Distribution	Lower specification limit (LSL)	Upper specification limit (USL)
1	Beta(1,2)	0.00020	1.44607
2	Beta(1,4)	0.00008	1.24463
3	Beta(1,7)	0.00006	1.02030
4	Beta(2,1)	0.00856	1.49899
5	Beta(2,2)	0.01016	1.47097
6	Beta(2,4)	0.00508	1.28923
7	Beta(2,5)	0.00304	1.27306
8	Beta(2,7)	0.00340	1.03644
9	Beta(4,1)	0.07540	1.49944
10	Beta(4,2)	0.05372	1.48546
11	Beta(4,4)	0.04493	1.41424
12	Beta(4,5)	0.03133	1.31159
13	Beta(4,7)	0.01880	1.25704
14	Beta(5,2)	0.09402	1.48726
15	Beta(5,4)	0.06199	1.42301
16	Beta(5,7)	0.04096	1.25247
17	Beta(7,1)	0.14638	1.49980
18	Beta(7,2)	0.12681	1.49449
19	Beta(7,4)	0.10275	1.43639
20	Beta(7,5)	0.09336	1.39298
21	Gamma(0.5,0.5)	0.00000	16.36701
22	Gamma(0.5,2)	0.00000	4.00712
23	Gamma(0.5,7)	0.00000	1.19715
24	Gamma(1,0.5)	0.00175	22.17356
25	Gamma(1,2)	0.00017	5.11075
26	Gamma(1,7)	0.00010	1.52708
27	Gamma(2,0.5)	0.05343	28.45444
28	Gamma(2,2)	0.01249	7.38301
29	Gamma(2,7)	0.00218	1.94618
30	Gamma(5,0.5)	0.58474	45.01437
31	Gamma(5,2)	0.17080	11.43345
32	Gamma(5,7)	0.05360	3.64273

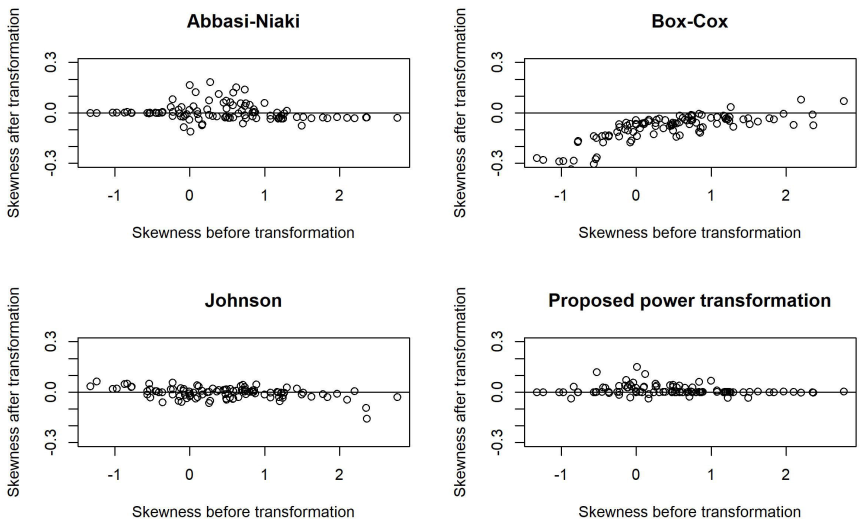

FIGURE 7. Skewness after transformation.

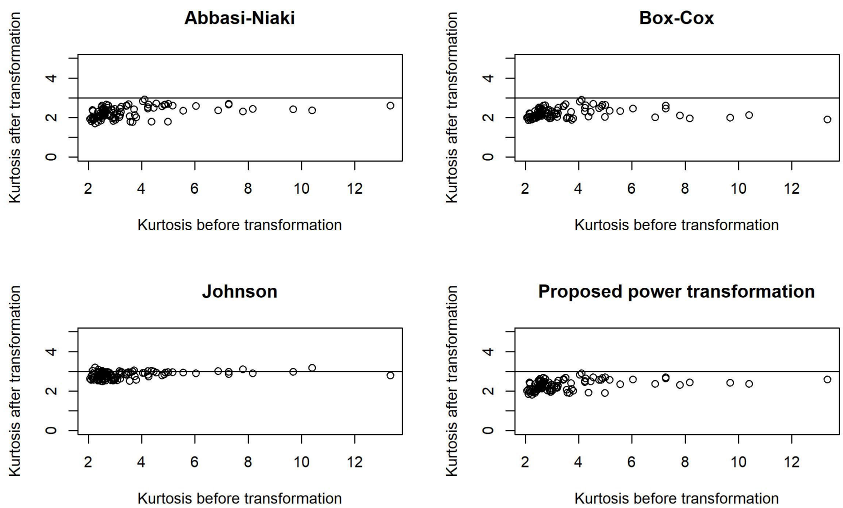

FIGURE 8. Kurtosis after transformation.

In order to identify the method which gives the most accurate capability estimation, differences $C_p - C'_p$ and $C_{pk} - C'_{pk}$ were observed. If calculated differences were bigger than 5, transformation was identified as unsuccessful. After samples for which transformation wasn't successful were excluded, it has been shown that proposed power transformation and Box-Cox method gave the most samples for which capability indices can be estimated according to Equation 3 and Equation 4 (23%, 67%, 84% and 87% of successful transformations for Johnson transformation, Abbasi-Niaki transformation, proposed power transformation and Box-Cox transformation, respectively).

In order to compare estimated capability indices, C^*_{pk} and C^*_p were calculated according to Equation 3 before data transformation was performed. Calculated differences are then interpreted as follows:

- Differences between target indices (C_p and C_{pk}) and estimated indices before data transformation (C^*_{pk} and C^*_p) are showing the risk of estimating capability of the process in case transformation is not preformed when data are not normally distributed. Difference was calculated for every data distribution, having in total 32 values. For extreme negatively skewed data there is a risk to underestimate C_{pk} index and overestimate the C_p index (Figure 9). Expected difference between estimated indices and target indices are decreasing as p-value of Shapiro-Wilk test is increasing (Figure 9).
- Differences between target indices (C_p and C_{pk}) and estimated indices after data transformation (C'_{pk} and C'_p) are showing how accurate process capability is estimated after data transformation is performed. Differences were calculated for every data distribution, having in total 32 values. In Figure 10 and Figure 11 it is clear that capability indices after transformation are closer to target indices compared to indices before data transformation.

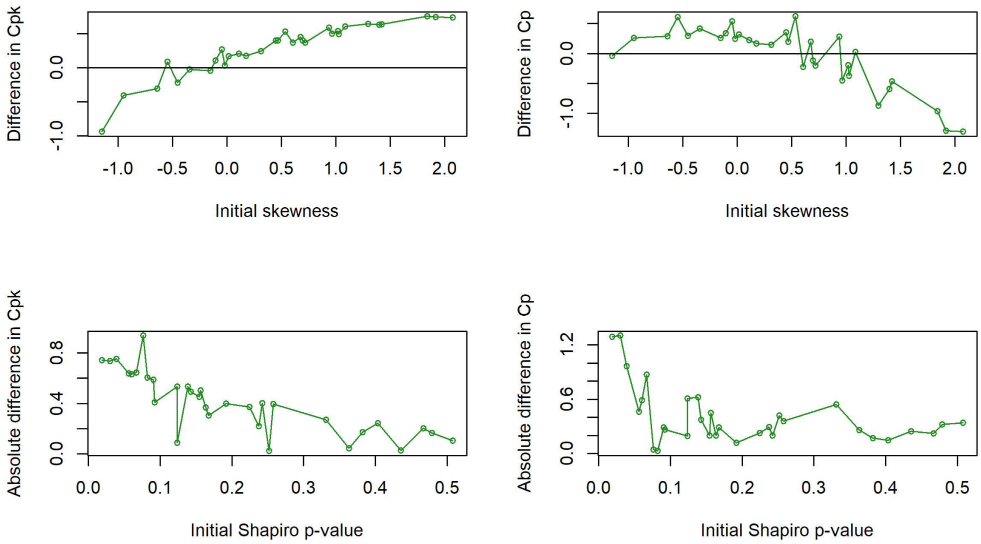

FIGURE 9. Difference between target indices and indices before transformation.

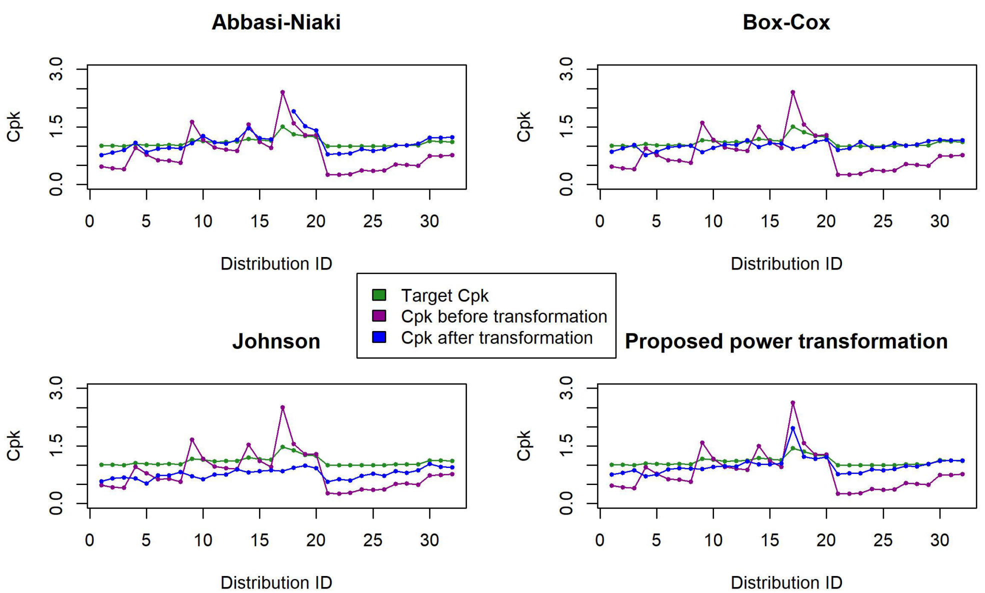

FIGURE 10. Difference between C_{pk} estimated true indices and indices after transformation.

- To identify which method performs best, differences between target capability indices and estimated capability indices after transformation were compared for each distribution separately by calculating relative bias (RB) according to [27]:

$$RB_{Cp_k} = \frac{1}{r}\sum_{i=1}^{r}\frac{C'_{pk} - C_{pk}}{C_{pk}}, \tag{17}$$

$$RB_{Cp} = \frac{1}{r}\sum_{i=1}^{r}\frac{C'_{p} - C_{p}}{C_{p}}, \tag{18}$$

where r denotes number of successful transformations for data generated from specific data distribution.
We define $\mu_X = \bar{RB}_{Cp}$, where $X \in$ {Abbasi-Niaki (A), Box-Cox (B), Johnson (J), Proposed power transformation (P)}. For each distribution, difference of relative bias for each method was tested by ANOVA. For example, in case of Beta(1,2) distribution hypothesis of statistical test for Cp index are defined as follows:

$$H_0 : \mu_A = \mu_B = \mu_J = \mu_P$$
H_1 : mean relative bias is different at least for one method.

In the case ANOVA revealed statistical difference between methods, Tukey HSD test was conducted to decide which method performs differently than others. Results are provided in Table 3. Transformation methods with comparable relative bias are grouped in parentheses, which are sorted by mean relative bias, starting with minimal value. For example, in the case of Beta(1,2) distribution p-value of ANOVA test for

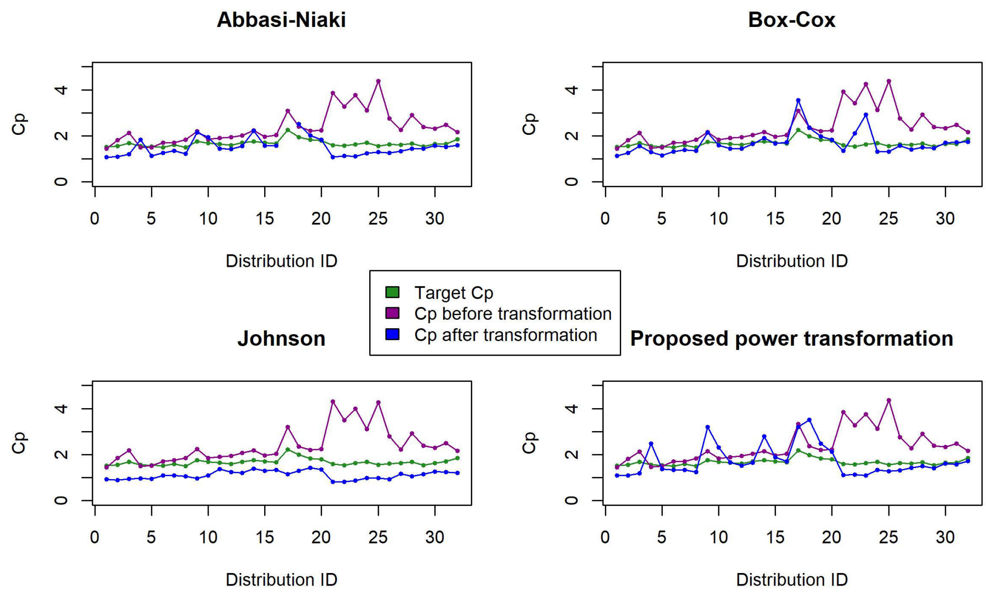

FIGURE 11. Difference between C_p estimated true indices and indices after transformation.

testing differences of C_{pk} mean relative bias defined in Equation 18 was < 0.05. Conducting Tukey HSD test revealed that Box-Cox method has minimal relative bias. Mean relative bias of Abbasi-Niaki and proposed power method were comparable whereas Johnson method had the biggest relative bias. Those results were then written in Table 3 as (B)(A-P)(J).

In almost all cases, Johnson method performed the worst. Proposed power transformation method performed better or equal to Box-Cox and Abbasi-Niaki in most cases. In other cases, proposed power transformation method was comparable to Box-Cox or Abbasi-Niaki method. In case of data generated from Beta(7,1) distribution, Abbasi-Niaki method was unsuccessful and couldn't provide estimate of C_p nor C_{pk}. Results of ANOVA analysis of relative bias of Cp in relation to skewness and kurtosis value is displayed in Figure 12. Proposed power transformation method showed significantly better or comparable performance in estimating C_p and C_{pk} in cases of positive skewness and kurtosis bigger than 3. In case of negative skewness and kurtosis lower than 3, proposed power transformation method showed better performance in estimating C_{pk} index.

TABLE 3. Results of ANOVA test of relative bias.

Distribution	RB_{Cp_k} p-value	Tukey HSD RB_{Cp_k}	RB_{Cp} p-value	Tukey HSD RB_{Cp}
Beta(1,2)	< 0.05	(B)(A-P)(J)	$\mathbf{> 0.05}$	(A-B-J-P)
Beta(1,4)	< 0.05	(B)(A-P)(J)	< 0.05	(B)(A-J-P)
Beta(1,7)	< 0.05	(B)(A-P)(J)	< 0.05	(B)(A-J-P)
Beta(2,1)	< 0.05	(A)(B-J-P)	< 0.05	(A-B-J)(P)
Beta(2,2)	< 0.05	(A-B)(P)(J)	< 0.05	(P)(A-B-J)
Beta(2,4)	< 0.05	(A-B)(P)(J)	< 0.05	(A-B-P)(J)
Beta(2,5)	< 0.05	(B)(A-P)(J)	< 0.05	(A-B-P)(J)
Beta(2,7)	< 0.05	(B)(A-J-P)	< 0.05	(A-B-P)(J)
Beta(4,1)	$\mathbf{> 0.05}$	(A-B-J-P)	< 0.05	(A-B)(J)(P)
Beta(4,2)	< 0.05	(A)(B-P)(J)	< 0.05	(B)(A)(J)(P)
Beta(4,4)	< 0.05	(A)(B-P)(J)	< 0.05	(P)(A-B-J)
Beta(4,5)	< 0.05	(A)(B-P)(J)	$\mathbf{> 0.05}$	(A-B-J-P)
Beta(4,7)	< 0.05	(A-B-P)(J)	< 0.05	(A-B-P)(J)
Beta(5,2)	< 0.05	(B-P-J)(A)	< 0.05	(A-B-J)(P)
Beta(5,4)	< 0.05	(A)(B-P)(J)	< 0.05	(A-B)(P)(J)
Beta(5,7)	< 0.05	(A-B)(P)(J)	< 0.05	(A-B-P)(J)
Beta(7,1)	< 0.05	(P)(B-J)(A failed)	< 0.05	(B-P)(J)(A failed)
Beta(7,2)	< 0.05	(P)(B-J)(A)	< 0.05	(A-B)(J)(P)
Beta(7,4)	< 0.05	(B-J-P)((A)	< 0.05	(A-B)(J)(P)
Beta(7,5)	< 0.05	(B-P)(A)(J)	< 0.05	(A-B)(P)(J)
Gamma(0.5,0.5)	< 0.05	(B)(A-P)(J)	< 0.05	(B)(A-J-P)
Gamma(0.5,2)	< 0.05	(B)(A-P)(J)	< 0.05	(A-J-P)(B)
Gamma(0.5,7)	< 0.05	(B)(A-P)(J)	< 0.05	(A-J-P)(B)
Gamma(1,0.5)	< 0.05	(A-B-P)(J)	< 0.05	(A-B-P)(J)
Gamma(1,2)	< 0.05	(B)(A-J-P)	< 0.05	(A-B-P)(J)
Gamma(1,7)	< 0.05	(B)(A-P)(J)	< 0.05	(B)(A-P)(J)
Gamma(2,0.5)	< 0.05	(A-B-P)(J)	< 0.05	(A-B-P)(J)
Gamma(2,2)	< 0.05	(A-B-P)(J)	< 0.05	(A-B-P)(J)
Gamma(2,7)	< 0.05	(P)(A-B)(J)	< 0.05	(A-B-P)(J)
Gamma(5,0.5)	< 0.05	(B-J-P)((A)	< 0.05	(A-B-P)(J)
Gamma(5,2)	< 0.05	(A-B-P)(J)	< 0.05	(A-B-P)(J)
Gamma(5,7)	< 0.05	(P)(A-B)(J)	< 0.05	(A-B-P)(J)

IV. EXAMPLE

The example for capability analysis was taken from [32]. The data shows an example for the production of a polymer, based on an actual scenario. A catalyst is required for the chemical reactions to occur to produce the polymer and contains a chemical that can create an impurity in the polymer. For this purpose, we consider the reaction time as main variable for capability analysis. Suppose the impurities need to be analyzed within the predefined limits of 80 and 100. The Shapiro-Wilk test (p-value 0.01774) and probability plot indicated that we cannot rely on standard capability analysis assuming normally distributed data. Figure 13 shows the data histogram along with the assumed specification limits. After applying all four algorithms—proposed power transformation, Abbasi-Niaki transformation, Box-Cox transformation, and Johnson

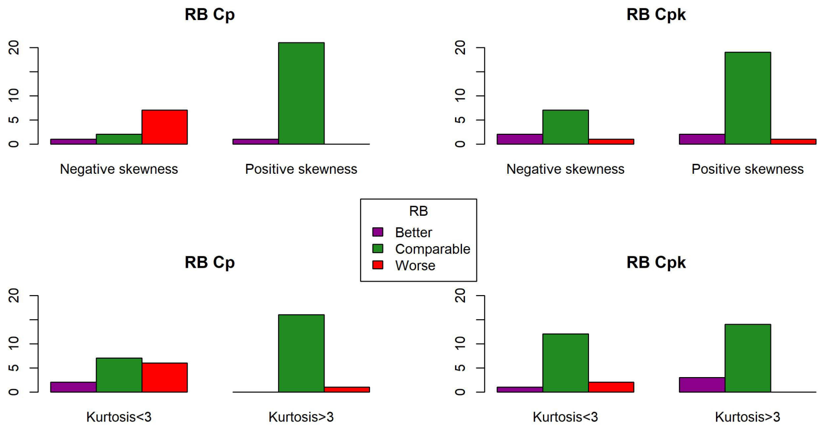

FIGURE 12. Performance of proposed power transformation compared to other methods.

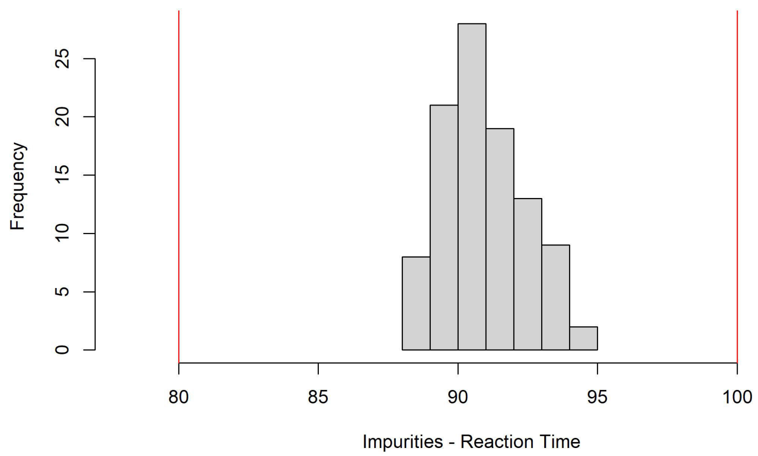

FIGURE 13. Impurities reaction time data.

TABLE 4. Shapiro-Wilk p-values after transformation.

Transformation method	P-value	Transformation details
Johnson transformation	0.79	S_B
Proposed power transformation	0.1401	skewness method
Abbasi-Niaki transformation	0.02378	log-transformation
Box-Cox transformation	0.03123	$\lambda = 0.01$

transformation—a Shapiro-Wilk test was performed on the transformed data, as shown in Table 4. Since only the proposed power transformation and Johnson transformation yielded successful results, capability indices were calculated for these two methods only, utilizing the transformed upper and lower specification limits. The C_p index after the power transformation is equal to 4.533, and C_{pk} is equal to 1.143. However, determining the Upper Specification Limit (USL) and Lower Specification Limit (LSL) after the Johnson transformation was not possible, as $x > \lambda + \epsilon$. Out of the four transformation methods considered, only the proposed power transformation method yielded usable results for capability estimation. The other methods—Abbasi-Niaki, Box-Cox, and Johnson transformations—did not provide reliable outcomes for this purpose.

V. CONCLUSION

In this paper, we introduce a modified power transformation method and compare it to three other transformation methods for non-normal distributed data. We conducted simulations on data from 32 different distributions and found that the Johnson and the proposed power transformations performed better than the Box-Cox and Abbasi-Niaki methods, resulting in larger reduction in skewness. However, in some cases, the Johnson transformation led to unrealistic results, making the Box-Cox and the proposed power transformation methods more practical.

The proposed algorithm offers the advantage of considering both skewness and kurtosis values for transformation, ensuring effectiveness even when skewness reduction is limited. The proposed transformation method demonstrated significant performance, particularly in cases of negatively skewed data. Although the analysis revealed that the capability index estimation using the proposed power transformation method is superior or comparable to the Box-Cox and Abbasi-Niaki methods, the Johnson transformation method performed the worst in this context. However, it's important to note that all transformation methods may yield unsuccessful results in the case of extreme skewness. Additionally,

Algorithm 1 Power Transformation Function

function=0 PowerTransform(A, B,ϵ, $data$, $iterations$, $flag$, $step$)
    if shapiro.test($data$) returns $p > 0.5 and |\text{skewness}(data)| < \epsilon$ **then** ▷ Data is already approximately normal
        return
    end if
    Initialize variables and data structures
    $a = A, b = B, r = 1$
    for a_k in sequence from A to B by 0.01 **do**
        $b_k \leftarrow a_k + step$
        if skewness($data^{a_k}$) $\cdot$ skewness($data^{b_k}$) $< 0 and\, a_k \cdot b_k > 0$ **then**
            Update $a = a_k$ and $b = b_k$
            Store parameters and output related to skewness
        end if
    end for
    if multiple solutions available **then**
        Select a and b based on minimum skewness
    end if
    if skewness is positive **then**
        Perform bisection on interval (a, b) until convergence ϵ for skewness
    else
        Invert data to positively skewed: $data \leftarrow -data + 2max(data)$ and recursively call PowerTransform ϵ, $data$, $iterations$, $flag$, $step$
    end if
    if power isn't found **then** PowerTransformKurtosis A, B, ϵ, $data$, $iterations$, $flag$, $step$
    end if
    return $result$
end function=0

when dealing with data that are already close to normally distributed, the proposed transformation may yield poorer results. Further analysis showed that the estimation of capability indices of the proposed power transformation method is better or comparable to the Box-Cox and Abbasi-Niaki methods, while the Johnson transformation method performed the worst in this regard. For future investigations, it is recommended to conduct a comparison of capability indices through the implementation of the proposed transformation technique with percentile estimation and adjusted index methodology.

When demonstrating the performance of each method using a real example, it became evident that only the proposed power transformation method was viable for capability estimation among the four transformation methods considered.

APPENDIX

See Algorithms 1 and 2.

Algorithm 2 Power Transformation Function With Kurtosis

function=0 PowerTransformKurtosis(A, B, ϵ, $data$, $iterations$, $flag$, $step$)
    Initialize variables and data structures
    $a = A, b = B, r = 1$
    for a_k in sequence from A to B by step **do**
        $b_k \leftarrow a_k + step$
        if (kurtosis($data^{a_k}$) $- 3) \cdot$ (kurtosis($data^{b_k}$) $- 3) < 0 and\, a_k \cdot b_k > 0$ **then**
            Update $a = a_k$ and $b = b_k$
            Store parameters and output related to kurtosis
        end if
    end for
    if skewness is positive **then**
        Perform bisection on interval (a, b) until convergence ϵ for kurtosis
    else
        Invert data to positively skewed: $data \leftarrow -data + 2max(data)$ and recursively call PowerTransformKurtosis A, B, ϵ, $data$, $iterations$, $flag$, $step$
    end if
    return $result$
end function=0

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