IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received 2 June 2024, accepted 21 June 2024, date of publication 28 June 2024, date of current version 10 July 2024.

Digital Object Identifier 10.1109/ACCESS.2024.3420711

RESEARCH ARTICLE

Frank-Based TOPSIS Methodology of Development and Operations Challenges Based on Intuitionistic Linguistic Aggregation Operators and Their Applications

ZEESHAN ALI[1], ZOHAIB AHMAD KHAN[2], TAPAN SENAPATI[3,4], AND SARBAST MOSLEM[5]

[1]Department of Information Mathematics, National Yunlin University of Science and Technology, Douliou, Yunlin 64002, Taiwan
[2]Department of Electrical Engineering, Riphah International University, Islamabad 44000, Pakistan
[3]School of Mathematics and Statistics, Southwest University, Beibei, Chongqing 400715, China
[4]Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Thandalam, Chennai 602105, India
[5]School of Architecture, Planning and Environmental Policy, University College Dublin, Belfield, Dublin 4, D04 V1W8 Ireland

Corresponding author: Sarbast Moslem (sarbast.moslem@ucd.ie)

This work was supported by the Digital Voting Hub for Sustainable Urban Transport System (VOTE-TRA) Project funded by the Science Foundation Ireland (SFI) under Grant 22/NCF/DR/11309.

ABSTRACT DevOps is a collection of principles, practices, and cultural philosophies aimed at enhancing communication and collaboration among information and technology operations groups and software development. The key objective of DevOps is to simplify the software development growth, from early development through operations, testing, and deployment, to distribute high-quality software products reliably, quickly, and feasibly. In this manuscript, we derive the frank operational laws based on IL information for FTN and FTCN. Additionally, we analyze the ILFWA operator, ILFOWA operator, ILFWG operator, and ILFOWG operator. Some fundamental laws for the above operators are also derived. Moreover, using the techniques, we evaluate the TOPSIS technique, which will help in the investigation of the best optimal. Further, we illustrate some practical examples based on initiated techniques for showing the supremacy and validity of the derived theory with the help of MADM methods by evaluating the problem of DevOps. Finally, to show the supremacy and validity of the presented techniques, we aim to compare the proposed ranking values with some existing techniques.

INDEX TERMS Decision-making problems, Frank averaging/geometric aggregation operators, intuitionistic linguistic sets.

I. INTRODUCTION

Development and operations [1] is a family of rules and performances aimed to enhance the collaboration and communications between information technology operations and software development. The key and major aim of the above procedure is to streamline the software delivery to achieve faster and massive flexible releases [2]. Further, these techniques are very flexible because of their structure, but the technique of decision-making problem is also very dominant and a lot of people have utilized it in many fields based on classical set theory, where the range of crisp set is {0, 1}, it is clear that we have just two option such as zero and one which is not enough for dealing with uncertain and unreliable information in genuine life problems because in the presence of crisp set we have no partial degree. Dealing with uncertainty in various genuine life situations is very complex, because of ambiguity and uncertainty, for instance, if simplifying objects based on their color, an object could be partially in the set of "red" if it is reddish but not entirely red. For this, the fuzzy set (FS) theory is very flexible and reliable because of its range, was initiated by Zadeh [3] in 1965. The truth grade $\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_{IL}}\left(\overline{\overline{\mathbb{x}}}\right)$ is a major part of the FS theory, but in various ways, we failed to cope with vague and conditional

The associate editor coordinating the review of this manuscript and approving it for publication was Yilun Shang.

information, because falsity, negative, or supporting against part is the valuable part of the many problems. For this, the intuitionistic FS (IFS) is the very dominant and flexible technique to cope with uncertain and unreliable information, because truth and falsity grades with a condition that the sum of the duplet will be restricted to the unit interval, was presented by Atanassov [4], [5] in 1983 and 1986. Zadeh [6], [7] initiated the novel concept of linguistic set (LS), where the idea of LS is involved in every field of life, for instance, if we talk about the temperature, then we all use the following linguistic terms, called could, very cold, normal, hot, and very hot.

A. LITERATURE REVIEW

The domain of FS is any finite universal set, but the range of FS is the unit interval. FS theory has a lot of applications, for example, decision support technique for fuzzy aggregation operators [8], fuzzy n-soft sets [9], fuzzy modified hybrid aggregation operators [10], hesitant fuzzy n-soft sets [11], analysis of hesitant fuzzy aggregation operators [12], multi-fuzzy n-soft sets [13], induced fuzzy modified aggregation operators [14], fuzzy superior mandelbrot sets [15], and (a, b)-fuzzy soft sets [16]. Further, IFS is massive attractive than FS, because of their structure, and due to this reason various scholars have utilized it in many fields, for instance, power information operators under the consideration of IFSs [17], novel similarity measures under the presence of IFSs [18], the hypervolume-based technique for evaluating the ranking values based on IFSs [19], analysis of TODIM technique based on Jenson-Shannon measures for IFSs [20], decision-making analysis for evaluating the ordered pairs based on IFSs [21], analysis of a novel theory, called circular IFSs [22], analysis of TOPSIS and Hamacher information operators for IFSs [23], intuitionistic multi fuzzy n-soft information [24], and analysis of COVID-19 based on MAIRCA technique for IFSs [25]. After the investigation of LS, many scholars have combined the LS with many different techniques, for example, fuzzy linguistic sets [26], linguistic IFSs [27], and intuitionistic linguistic sets [28].

The technique of TOPSIS [29] is also very important, which is computed based on positive and negative ideal solutions. Moreover, the fuzzy TOPSIS technique was developed by Kim et al. [30]. Furthermore, some applications have been developed by different scholars based on the TOPSIS technique, for instance, fuzzy TOPSIS technique and their applications [31], intuitionistic fuzzy TOPSIS techniques [32], and TOPSIS technique based on IFSs [33]. To compute any kind of operator is very awkward, where the triangular norms play a very beneficial role in the construction of any kind of operator, as proposed by Klement et al. [34] in 1997. Further, the modified version of the triangular norms was initiated by Frank [35], called Frank norms in 1979. Additionally, the Frank aggregation operators for intuitionistic fuzzy measures were derived by Iancu [36]. Zhang et al. [37] derived the Frank power operators for IFSs. In 2012, Xia et al. [38] evaluated the aggregation operators based on Archimedean norms for IFSs. Ecer and Haseli [39] proposed fuzzy ZE numbers. Moreover, a survey on the fuzzy TOPSIS technique was discussed by Salih et al. [40] in 2019. The above existing technique has a lot of advantages, but to aggregate the collection of information into a singleton set is very complex because up to date no one can derive any kind of operators based on intuitionistic linguistic sets, where the technique of FSs, IFSs, and LTSs are the part of the intuitionistic linguistic sets.

B. RESEARCH GAP/RESEARCH PROBLEMS

After all, according to all experts, the following problems are the major parts of the decision-making procedure, such as

1) The construction of new operational laws based on Frank norms is very awkward.
2) The derivation of aggregation operators based on Frank's operational laws is also complex.
3) The evaluation of the best optimal is also very complex among the collection of a finite number of alternatives.

The above three points play an important role in the environment of fuzzy set theory. Finding the solution to these problems are very awkward and challenging task for authors. Further, the technique of intuitionistic linguistic sets is very reliable because of their features, where the truth function, falsity function, and linguistic term are part of the intuitionistic linguistic sets. After the construction of the intuitionistic linguistic set, no one can derive the technique of Frank aggregation operators and the TOPSIS method based on intuitionistic linguistic values. Further, we have discussed the advantages and limitations of the TOPSIS technique, such as:

1. **Advantages:**
   - i) **Simple Concept:** The TOPSIS technique is a very simple method compared to other decision-making techniques which means that it is more accessible to a huge number of users.
   - ii) **Consideration of Multiple Criteria:** During an evaluation of the best alternative, the TOPSIS technique allows exports to select multiple criteria continuously.
   - iii) **Flexible:** The TOPSIS technique is more feasible and more accessible to cope with both qualitative and quantitative information.
   - iv) **Applicability:** The technique of the TOPSIS method can easily utilized in many fields, for instance, supplier selection, project selection, and service evaluations.
2. **Limitation**
   - i) **Subjectivity in Weight Assignment:** The evaluation of the weight vectors in the TOPSIS technique for each criterion depends on the expert's preferences or biases.
   - ii) **Sensitivity to Normalization:** The normalization of the criteria in the TOPSIS technique can affect the outcomes of the analysis.

iii) **Assumption of Independence:** The TOPSIS technique assumes that criteria are independent of each other, which does not hold in some genuine life problems or scenarios.

C. MOTIVATION AND MAJOR CONTRIBUTIONS

From the above observations, we noticed that the Frank operators and TOPSIS techniques are very dominant and flexible, but both are very difficult to define based on an intuitionistic linguistic set, where no one can propose it yet for such kind of ideas. Frank aggregation operators and TOPSIS techniques are valuable, and many techniques and operators are special cases of the above operators and TOPSIS techniques. Inspired by the above theory, our major contribution is listed below:

1. To evaluate the Frank operational laws based on IL variables and also derive their fundamental properties. These operational laws can help us in the construction of the aggregation operators.
2. To analyze the ILFWA operator, ILFOWA operator, ILFWG operator, ILFOWG operator, and discuss their basic properties.
3. To evaluate the TOPSIS technique based on the initiated operators to enhance the worth of the explored information.
4. To introduce the MADM methods based on initiated operators for evaluating the major tools and technology that are commonly utilized in DevOps workflows.
5. To select some existing techniques and try to compare their ranking results with our obtained ranking results to show the supremacy and validity of the presented techniques.

This manuscript is arranged in the following shape: In Section II, we discussed the Frank t-norm and Frank t-conorm based on a crisp set. Further, we reviewed the ILS and their operational laws based on fixed set $\overline{\overline{\mathbb{X}}}_{UI}$. In Section III, we evaluated the Frank operational laws based on IL variables and also derived their fundamental properties. In Section IV, we analyzed the ILFWA operator, ILFOWA operator, ILFWG operator, and ILFOWG operator, and discussed their basic properties. In Section V, we evaluated the TOPSIS technique based on the initiated operators to enhance the worth of the explored information. In Section VI, we introduced the MADM methods based on initiated operators for evaluating the major tools and technology that are commonly utilized in DevOps workflows. In Section VII, we selected some existing techniques and tried to compare their ranking results with our obtained ranking results to show the supremacy and validity of the presented techniques. Some concluding remarks are stated in Section VIII.

II. PRELIMINARIES

In this section, we discussed the Frank t-norm (FTN) and Frank t-conorm (FTCN) based on a crisp set. Further, we reviewed the ILS and its operational laws. The meanings of the symbols in this manuscript are listed in Table 1.

TABLE 1. Symbols and their meanings.

symbol	Meaning	symbol	Meaning
$\overline{\overline{\mathbb{X}}}_{UI}$	Universal set	$\overline{\overline{\mathbb{x}}}$	Element of the universal set
$\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_1}, \overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_2} \in [0,1]$	Any two positive values	$\overline{\mathcal{E}_1}, \pi \geq 1$	Any two scalers
$\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_z}$	Membership value	$\overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}}}_z}$	Non-membership value
$\mathbb{S}(\overline{\overline{\mathbb{N}}}_z)$	Score value	$\mathbb{H}(\overline{\overline{\mathbb{N}}}_z)$	Accuracy Value
$\mathbb{I}_{\mathcal{J}_z(\overline{\overline{\mathbb{x}}})}$	Linguistic number	$\mathfrak{P}$	Order of linguistic set

Definition 1 ([35]): Consider any two numbers $\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_1}, \overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_2} \in [0, 1]$ based on $\pi \geq 1$. Then

$$FTN\left(\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_1}, \overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_2}\right) = \log_\pi\left(1+\frac{\left(\pi^{\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_1}}-1\right)\left(\pi^{\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_2}}-1\right)}{\pi-1}\right) \quad (1)$$

$$FTCN\left(\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_1}, \overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_2}\right) = 1-\log_\pi\left(1+\frac{\left(\pi^{1-\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_1}}-1\right)\left(\pi^{1-\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_2}}-1\right)}{\pi-1}\right) \quad (2)$$

Describe the FTN and FTCN.

Definition 2 ([28]): The structure of ILS $\overline{\overline{\mathbb{N}}}_{IL}$ for a universal set $\overline{\overline{\mathbb{X}}}_{UI}$ is described in the shape:

$$\overline{\overline{\mathbb{N}}}_{IL} = \left\{\left(\mathbb{I}_{\mathcal{J}(\overline{\overline{\mathbb{x}}})}, \overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_{IL}}(\overline{\overline{\mathbb{x}}}), \overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}}}_{IL}}(\overline{\overline{\mathbb{x}}})\right) : \overline{\overline{\mathbb{x}}} \in \overline{\overline{\mathbb{X}}}_{UI}\right\} \quad (3)$$

The linguistic information is derived in the shape: $\mathbb{I}_{\mathcal{J}(\overline{\overline{\mathbb{x}}})} \in S = \left\{\mathbb{I}_{\mathcal{J}_z(\overline{\overline{\mathbb{x}}})} : z = 1, 2, \ldots, 2\mathfrak{P}\right\}$, where $\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_{IL}}(\overline{\overline{\mathbb{x}}})$ and $\overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}}}_{IL}}(\overline{\overline{\mathbb{x}}})$ describes the supporting and supporting against grades with a strong condition: $0 \leq \overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_{IL}}(\overline{\overline{\mathbb{x}}}) + \overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}}}_{IL}}(\overline{\overline{\mathbb{x}}}) \leq 1$. Furthermore, the information $\overline{\overline{R}}_{\overline{\overline{\mathbb{N}}}_{IL}}(\overline{\overline{\mathbb{x}}}) = 1 - \left(\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_{IL}}(\overline{\overline{\mathbb{x}}}) + \overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}}}_{IL}}(\overline{\overline{\mathbb{x}}})\right)$ uses as neutral information and the simple shape of ILNs is derived from the shape: $\overline{\overline{\mathbb{N}}}_z = \left(\mathbb{I}_{\mathcal{J}_z}, \overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_z}, \overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}}}_z}\right), z = 1, 2, \ldots, m$.

Definition 3 ([28]): Let $\overline{\overline{\mathbb{N}}}_z = \left(\mathbb{I}_{\mathcal{J}_z}, \overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_z}, \overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}}}_z}\right), z = 1, 2,$ be any two ILNs. Then

$$\overline{\overline{\mathbb{N}}}_1 \oplus \overline{\overline{\mathbb{N}}}_2 = \left(\begin{array}{c}\mathbb{I}_{\mathfrak{P}\left(\frac{\mathcal{J}_1}{\mathfrak{P}}+\frac{\mathcal{J}_2}{\mathfrak{P}}-\frac{\mathcal{J}_1}{\mathfrak{P}}\frac{\mathcal{J}_2}{\mathfrak{P}}\right)}, \\ \overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_1}+\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_2}-\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_1}\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_2}, \overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}}}_1}\overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}}}_2}\end{array}\right) \quad (4)$$

$$\overline{\overline{\mathbb{N}}}_1 \otimes \overline{\overline{\mathbb{N}}}_2 = \left(\mathbb{I}_{\mathfrak{P}\left(\frac{\mathcal{J}_1}{\mathfrak{P}} \frac{\mathcal{J}_2}{\mathfrak{P}}\right)}, \overline{\overline{\mathfrak{mm}}}_{\overline{\overline{\mathbb{N}}}_1} \overline{\overline{\mathfrak{mm}}}_{\overline{\overline{\mathbb{N}}}_2}, \overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_1} + \overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_2} - \overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_1} \overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_2} \right) \tag{5}$$

$$\overline{\overline{\mathcal{E}_1 \mathbb{N}_1}} = \left(\mathbb{I}_{\mathfrak{P}\left(1-\left(1-\frac{\mathcal{J}_1}{\mathfrak{P}}\right)^{\overline{\overline{\mathcal{E}}}_1}\right)}, 1-\left(1-\overline{\overline{\mathfrak{mm}}}_{\overline{\overline{\mathbb{N}}}_1}\right)^{\overline{\overline{\mathcal{E}}}_1}, \left(\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_1}\right)^{\overline{\overline{\mathcal{E}}}_1} \right) \tag{6}$$

$$\left(\overline{\overline{\mathbb{N}}}_1\right)^{\overline{\overline{\mathcal{E}}}_1} = \left(\mathbb{I}_{\mathfrak{P}\left(\left(\frac{\mathcal{J}_1}{\mathfrak{P}}\right)^{\overline{\overline{\mathcal{E}}}_1}\right)}, \left(\overline{\overline{\mathfrak{mm}}}_{\overline{\overline{\mathbb{N}}}_1}\right)^{\overline{\overline{\mathcal{E}}}_1}, 1-\left(1-\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_1}\right)^{\overline{\overline{\mathcal{E}}}_1} \right) \tag{7}$$

Definition 4 ([28]): Let $\overline{\overline{\mathbb{N}}}_z = \left(\mathbb{I}_{\mathcal{J}_z}, \overline{\overline{\mathfrak{mm}}}_{\overline{\overline{\mathbb{N}}}_z}, \overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_z}\right), z = 1,$ be any ILN. Then

$$\mathbb{S}\left(\overline{\overline{\mathbb{N}}}_z\right) = \frac{\mathcal{J}_z}{\mathfrak{P}} * \left(\overline{\overline{\mathfrak{mm}}}_{\overline{\overline{\mathbb{N}}}_z} - \overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_z}\right) \in [-1, 1] \tag{8}$$

$$\mathbb{H}\left(\overline{\overline{\mathbb{N}}}_z\right) = \frac{\mathcal{J}_z}{\mathfrak{P}} * \left(\overline{\overline{\mathfrak{mm}}}_{\overline{\overline{\mathbb{N}}}_z} - \overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_z}\right) \in [0, 1] \tag{9}$$

Describe the score function and accuracy function.

Definition 5 ([28]): Let $\overline{\overline{\mathbb{N}}}_z = \left(\mathbb{I}_{\mathcal{J}_z}, \overline{\overline{\mathfrak{mm}}}_{\overline{\overline{\mathbb{N}}}_z}, \overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_z}\right), z = 1,$ be any ILN. Then

1) When $\mathbb{S}\left(\overline{\overline{\mathbb{N}}}_1\right) > \mathbb{S}\left(\overline{\overline{\mathbb{N}}}_2\right) \Rightarrow \overline{\overline{\mathbb{N}}}_1 > \overline{\overline{\mathbb{N}}}_2$;
2) When $\mathbb{S}\left(\overline{\overline{\mathbb{N}}}_1\right) < \mathbb{S}\left(\overline{\overline{\mathbb{N}}}_2\right) \Rightarrow \overline{\overline{\mathbb{N}}}_1 < \overline{\overline{\mathbb{N}}}_2$;
3) When $\mathbb{S}\left(\overline{\overline{\mathbb{N}}}_1\right) = \mathbb{S}\left(\overline{\overline{\mathbb{N}}}_2\right)$, then
    i) When $\mathbb{H}\left(\overline{\overline{\mathbb{N}}}_1\right) > \mathbb{H}\left(\overline{\overline{\mathbb{N}}}_2\right) \Rightarrow \overline{\overline{\mathbb{N}}}_1 > \overline{\overline{\mathbb{N}}}_2$;
    ii) When $\mathbb{H}\left(\overline{\overline{\mathbb{N}}}_1\right) < \mathbb{H}\left(\overline{\overline{\mathbb{N}}}_2\right) \Rightarrow \overline{\overline{\mathbb{N}}}_1 < \overline{\overline{\mathbb{N}}}_2$.

III. FRANK OPERATIONAL LAWS FOR ILNS

This section introduces Frank's operational laws for ILNs and simplifies their valuable results. These techniques are the superior part of the algebraic operational laws.

Definition 6: Let $\overline{\overline{\mathbb{N}}}_z = \left(\mathbb{I}_{\mathcal{J}_z}, \overline{\overline{\mathfrak{mm}}}_{\overline{\overline{\mathbb{N}}}_z}, \overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_z}\right), z = 1, 2, \ldots, m,$ be any family of ILNs. Then

$$\overline{\overline{\mathbb{N}}}_1 \oplus \overline{\overline{\mathbb{N}}}_2 = \left(\begin{array}{l} \mathbb{I}_{\mathfrak{P}\left(1-\log_\pi\left(1+\frac{\left(\pi^{1-\frac{\mathcal{J}_1}{\mathfrak{P}}}-1\right)\left(\pi^{1-\frac{\mathcal{J}_2}{\mathfrak{P}}}-1\right)}{\pi-1}\right)\right)}, \\ 1-\log_\pi\left(1+\frac{\left(\pi^{1-\overline{\overline{\mathfrak{mm}}}_{\overline{\overline{\mathbb{N}}}_1}}-1\right)\left(\pi^{1-\overline{\overline{\mathfrak{mm}}}_{\overline{\overline{\mathbb{N}}}_2}}-1\right)}{\pi-1}\right), \\ \log_\pi\left(1+\frac{\left(\pi^{\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_1}}-1\right)\left(\pi^{\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_2}}-1\right)}{\pi-1}\right) \end{array} \right) \tag{10}$$

$$\overline{\overline{\mathbb{N}}}_1 \otimes \overline{\overline{\mathbb{N}}}_2 = \left(\begin{array}{l} \mathbb{I}_{\mathfrak{P}\left(\log_\pi\left(1+\frac{\left(\pi^{\frac{\mathcal{J}_1}{\mathfrak{P}}}-1\right)\left(\pi^{\frac{\mathcal{J}_2}{\mathfrak{P}}}-1\right)}{\pi-1}\right)\right)}, \\ \log_\pi\left(1+\frac{\left(\pi^{\overline{\overline{\mathfrak{mm}}}_{\overline{\overline{\mathbb{N}}}_1}}-1\right)\left(\pi^{\overline{\overline{\mathfrak{mm}}}_{\overline{\overline{\mathbb{N}}}_2}}-1\right)}{\pi-1}\right), \\ 1-\log_\pi\left(1+\frac{\left(\pi^{1-\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_1}}-1\right)\left(\pi^{1-\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_2}}-1\right)}{\pi-1}\right) \end{array} \right) \tag{11}$$

$$\overline{\overline{\mathcal{E}_1 \mathbb{N}_1}} = \left(\begin{array}{l} \mathbb{I}_{\mathfrak{P}\left(1-\log_\pi\left(1+\frac{\left(\pi^{1-\frac{\mathcal{J}_1}{\mathfrak{P}}}-1\right)^{\overline{\overline{\mathcal{E}}}_1}}{(\pi-1)^{\overline{\overline{\mathcal{E}}}_1-1}}\right)\right)}, \\ 1-\log_\pi\left(1+\frac{\left(\pi^{1-\overline{\overline{\mathfrak{mm}}}_{\overline{\overline{\mathbb{N}}}_1}}-1\right)^{\overline{\overline{\mathcal{E}}}_1}}{(\pi-1)^{\overline{\overline{\mathcal{E}}}_1-1}}\right), \\ \log_\pi\left(1+\frac{\left(\pi^{\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_1}}-1\right)^{\overline{\overline{\mathcal{E}}}_1}}{(\pi-1)^{\overline{\overline{\mathcal{E}}}_1-1}}\right) \end{array} \right) \tag{12}$$

$$\left(\overline{\overline{\mathbb{N}}}_1\right)^{\overline{\overline{\mathcal{E}}}_1} = \left(\begin{array}{l} \mathbb{I}_{\mathfrak{P}\left(\log_\pi\left(1+\frac{\left(\pi^{\frac{\mathcal{J}_1}{\mathfrak{P}}}-1\right)^{\overline{\overline{\mathcal{E}}}_1}}{(\pi-1)^{\overline{\overline{\mathcal{E}}}_1-1}}\right)\right)}, \log_\pi\left(1+\frac{\left(\pi^{\overline{\overline{\mathfrak{mm}}}_{\overline{\overline{\mathbb{N}}}_1}}-1\right)^{\overline{\overline{\mathcal{E}}}_1}}{(\pi-1)^{\overline{\overline{\mathcal{E}}}_1-1}}\right), \\ 1-\log_\pi\left(1+\frac{\left(\pi^{1-\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_1}}-1\right)^{\overline{\overline{\mathcal{E}}}_1}}{(\pi-1)^{\overline{\overline{\mathcal{E}}}_1-1}}\right) \end{array} \right) \tag{13}$$

Theorem 1: Let $\overline{\overline{\mathbb{N}}}_z = \left(\mathbb{I}_{\mathcal{J}_z}, \overline{\overline{\mathfrak{mm}}}_{\overline{\overline{\mathbb{N}}}_z}, \overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_z}\right), z = 1, 2, \ldots, m,$ be any family of ILNs. Then

1. $\overline{\overline{\mathbb{N}}}_1 \oplus \overline{\overline{\mathbb{N}}}_2 = \overline{\overline{\mathbb{N}}}_2 \oplus \overline{\overline{\mathbb{N}}}_1$.
2. $\overline{\overline{\mathbb{N}}}_1 \otimes \overline{\overline{\mathbb{N}}}_2 = \overline{\overline{\mathbb{N}}}_2 \otimes \overline{\overline{\mathbb{N}}}_1$.
3. $\overline{\overline{\mathcal{E}}}_1\left(\overline{\overline{\mathbb{N}}}_1 \oplus \overline{\overline{\mathbb{N}}}_2\right) = \overline{\overline{\mathcal{E}_1 \mathbb{N}_1}} \oplus \overline{\overline{\mathcal{E}_1 \mathbb{N}_2}}$.
4. $\left(\overline{\overline{\mathbb{N}}}_1 \otimes \overline{\overline{\mathbb{N}}}_2\right)^{\overline{\overline{\mathcal{E}}}_1} = \left(\overline{\overline{\mathbb{N}}}_1\right)^{\overline{\overline{\mathcal{E}}}_1} \otimes \left(\overline{\overline{\mathbb{N}}}_2\right)^{\overline{\overline{\mathcal{E}}}_1}$.
5. $\overline{\overline{\mathcal{E}_1 \mathbb{N}_1}} \oplus \overline{\overline{\mathcal{E}_2 \mathbb{N}_1}} = \left(\overline{\overline{\mathcal{E}}}_1 + \overline{\overline{\mathcal{E}}}_2\right) \overline{\overline{\mathbb{N}}}_1$.

6. $\left(\overline{\overline{\mathbb{N}_1}}\right)^{\overline{\overline{\mathcal{E}_1}}} \otimes \left(\overline{\overline{\mathbb{N}_1}}\right)^{\overline{\overline{\mathcal{E}_2}}} = \left(\overline{\overline{\mathbb{N}_1}}\right)^{\left(\overline{\overline{\mathcal{E}_1}}+\overline{\overline{\mathcal{E}_2}}\right)}$.

Proof:

1. Consider

$$\overline{\overline{\mathbb{N}_1}} \oplus \overline{\overline{\mathbb{N}_2}}$$
$$= \left(\begin{array}{l} \mathbb{I}\,\mathfrak{P}\left(1-\log_\pi\left(1+\frac{\left(\pi^{1-\frac{\overline{\overline{\mathcal{J}_1}}}{\mathfrak{P}}}-1\right)\left(\pi^{1-\frac{\overline{\overline{\mathcal{J}_2}}}{\mathfrak{P}}}-1\right)}{\pi-1}\right)\right), \\ 1-\log_\pi\left(1+\frac{\left(\pi^{1-\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}_1}}}}-1\right)\left(\pi^{1-\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}_2}}}}-1\right)}{\pi-1}\right), \\ \log_\pi\left(1+\frac{\left(\pi^{\overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}_1}}}}-1\right)\left(\pi^{\overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}_2}}}}-1\right)}{\pi-1}\right) \end{array} \right)$$
$$\left(\begin{array}{l} \mathbb{I}\,\mathfrak{P}\left(1-\log_\pi\left(1+\frac{\left(\pi^{1-\frac{\overline{\overline{\mathcal{J}_2}}}{\mathfrak{P}}}-1\right)\left(\pi^{1-\frac{\overline{\overline{\mathcal{J}_1}}}{\mathfrak{P}}}-1\right)}{\pi-1}\right)\right), \\ 1-\log_\pi\left(1+\frac{\left(\pi^{1-\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}_2}}}}-1\right)\left(\pi^{1-\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}_1}}}}-1\right)}{\pi-1}\right), \\ \log_\pi\left(1+\frac{\left(\pi^{\overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}_2}}}}-1\right)\left(\pi^{\overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}_1}}}}-1\right)}{\pi-1}\right) \end{array} \right)$$
$$= \overline{\overline{\mathbb{N}_2}} \oplus \overline{\overline{\mathbb{N}_1}}$$

2. Omitted.
3. Consider

$$\overline{\overline{\mathcal{E}_1}}\left(\overline{\overline{\mathbb{N}_1}} \oplus \overline{\overline{\mathbb{N}_2}}\right)$$
$$= \overline{\overline{\mathcal{E}_1}} \left(\begin{array}{l} \mathbb{I}\,\mathfrak{P}\left(1-\log_\pi\left(1+\frac{\left(\pi^{1-\frac{\overline{\overline{\mathcal{J}_1}}}{\mathfrak{P}}}-1\right)\left(\pi^{1-\frac{\overline{\overline{\mathcal{J}_2}}}{\mathfrak{P}}}-1\right)}{\pi-1}\right)\right), \\ 1-\log_\pi\left(1+\frac{\left(\pi^{1-\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}_1}}}}-1\right)\left(\pi^{1-\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}_2}}}}-1\right)}{\pi-1}\right), \\ \log_\pi\left(1+\frac{\left(\pi^{\overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}_1}}}}-1\right)\left(\pi^{\overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}_2}}}}-1\right)}{\pi-1}\right) \end{array} \right)$$
$$= \left(\begin{array}{l} \mathbb{I}\,\mathfrak{P}\left(1-\log_\pi\left(1+\frac{\left(\pi^{1-\frac{\overline{\overline{\mathcal{J}_1}}}{\mathfrak{P}}}-1\right)^{\overline{\overline{\mathcal{E}_1}}}\left(\pi^{1-\frac{\overline{\overline{\mathcal{J}_2}}}{\mathfrak{P}}}-1\right)^{\overline{\overline{\mathcal{E}_1}}}}{(\pi-1)^{\overline{\overline{\mathcal{E}_1}}-1}}\right)\right), \\ 1-\log_\pi\left(1+\frac{\left(\pi^{1-\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}_1}}}}-1\right)^{\overline{\overline{\mathcal{E}_1}}}\left(\pi^{1-\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}_2}}}}-1\right)^{\overline{\overline{\mathcal{E}_1}}}}{(\pi-1)^{\overline{\overline{\mathcal{E}_1}}-1}}\right), \\ \log_\pi\left(1+\frac{\left(\pi^{\overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}_1}}}}-1\right)^{\overline{\overline{\mathcal{E}_1}}}\left(\pi^{\overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}_2}}}}-1\right)^{\overline{\overline{\mathcal{E}_1}}}}{(\pi-1)^{\overline{\overline{\mathcal{E}_1}}-1}}\right) \end{array} \right)$$
$$= \left(\begin{array}{l} \mathbb{I}\,\mathfrak{P}\left(1-\log_\pi\left(1+\frac{\left(\pi^{1-\frac{\overline{\overline{\mathcal{J}_1}}}{\mathfrak{P}}}-1\right)^{\overline{\overline{\mathcal{E}_1}}}}{(\pi-1)^{\overline{\overline{\mathcal{E}_1}}-1}}\right)\right), \\ 1-\log_\pi\left(1+\frac{\left(\pi^{1-\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}_1}}}}-1\right)^{\overline{\overline{\mathcal{E}_1}}}}{(\pi-1)^{\overline{\overline{\mathcal{E}_1}}-1}}\right), \\ \log_\pi\left(1+\frac{\left(\pi^{\overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}_1}}}}-1\right)^{\overline{\overline{\mathcal{E}_1}}}}{(\pi-1)^{\overline{\overline{\mathcal{E}_1}}-1}}\right) \end{array} \right)$$
$$\oplus \left(\begin{array}{l} \mathbb{I}\,\mathfrak{P}\left(1-\log_\pi\left(1+\frac{\left(\pi^{1-\frac{\overline{\overline{\mathcal{J}_2}}}{\mathfrak{P}}}-1\right)^{\overline{\overline{\mathcal{E}_1}}}}{(\pi-1)^{\overline{\overline{\mathcal{E}_1}}-1}}\right)\right), \\ 1-\log_\pi\left(1+\frac{\left(\pi^{1-\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}_2}}}}-1\right)^{\overline{\overline{\mathcal{E}_1}}}}{(\pi-1)^{\overline{\overline{\mathcal{E}_1}}-1}}\right), \\ \log_\pi\left(1+\frac{\left(\pi^{\overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}_2}}}}-1\right)^{\overline{\overline{\mathcal{E}_1}}}}{(\pi-1)^{\overline{\overline{\mathcal{E}_1}}-1}}\right) \end{array} \right)$$
$$= \overline{\overline{\mathcal{E}_1\mathbb{N}_1}} \oplus \overline{\overline{\mathcal{E}_1\mathbb{N}_2}}$$

4. Omitted.
5. Consider

$$\overline{\overline{\mathcal{E}_1\mathbb{N}_1}} \oplus \overline{\overline{\mathcal{E}_2\mathbb{N}_1}}$$

$$= \begin{pmatrix} \mathbb{I}_{\mathfrak{P}\left(1-\log_\pi\left(1+\frac{\left(\pi^{1-\frac{\overline{\overline{\mathcal{J}_1}}}{\mathfrak{P}}}-1\right)^{\overline{\overline{\mathcal{E}_1}}}}{(\pi-1)^{\overline{\overline{\mathcal{E}_1}}-1}}\right)\right)}, \\ 1-\log_\pi\left(1+\frac{\left(\pi^{1-\overline{\overline{\mathfrak{M}}}_{\overline{\overline{\mathbb{N}_1}}}}-1\right)^{\overline{\overline{\mathcal{E}_1}}}}{(\pi-1)^{\overline{\overline{\mathcal{E}_1}}-1}}\right), \\ \log_\pi\left(1+\frac{\left(\pi^{\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}_1}}}}-1\right)^{\overline{\overline{\mathcal{E}_1}}}}{(\pi-1)^{\overline{\overline{\mathcal{E}_1}}-1}}\right) \end{pmatrix}$$

$$\oplus \begin{pmatrix} \mathbb{I}_{\mathfrak{P}\left(1-\log_\pi\left(1+\frac{\left(\pi^{1-\frac{\overline{\overline{\mathcal{J}_1}}}{\mathfrak{P}}}-1\right)^{\overline{\overline{\mathcal{E}_2}}}}{(\pi-1)^{\overline{\overline{\mathcal{E}_2}}-1}}\right)\right)}, \\ 1-\log_\pi\left(1+\frac{\left(\pi^{1-\overline{\overline{\mathfrak{M}}}_{\overline{\overline{\mathbb{N}_1}}}}-1\right)^{\overline{\overline{\mathcal{E}_2}}}}{(\pi-1)^{\overline{\overline{\mathcal{E}_2}}-1}}\right), \\ \log_\pi\left(1+\frac{\left(\pi^{\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}_1}}}}-1\right)^{\overline{\overline{\mathcal{E}_2}}}}{(\pi-1)^{\overline{\overline{\mathcal{E}_2}}-1}}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \mathbb{I}_{\mathfrak{P}\left(1-\log_\pi\left(1+\frac{\left(\pi^{1-\frac{\overline{\overline{\mathcal{J}_1}}}{\mathfrak{P}}}-1\right)^{\left(\overline{\overline{\mathcal{E}_1}}+\overline{\overline{\mathcal{E}_2}}\right)}}{(\pi-1)^{\left(\overline{\overline{\mathcal{E}_1}}+\overline{\overline{\mathcal{E}_2}}\right)-1}}\right)\right)}, \\ 1-\log_\pi\left(1+\frac{\left(\pi^{1-\overline{\overline{\mathfrak{M}}}_{\overline{\overline{\mathbb{N}_1}}}}-1\right)^{\left(\overline{\overline{\mathcal{E}_1}}+\overline{\overline{\mathcal{E}_2}}\right)}}{(\pi-1)^{\left(\overline{\overline{\mathcal{E}_1}}+\overline{\overline{\mathcal{E}_2}}\right)-1}}\right), \\ \log_\pi\left(1+\frac{\left(\pi^{\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}_1}}}}-1\right)^{\left(\overline{\overline{\mathcal{E}_1}}+\overline{\overline{\mathcal{E}_2}}\right)}}{(\pi-1)^{\left(\overline{\overline{\mathcal{E}_1}}+\overline{\overline{\mathcal{E}_2}}\right)-1}}\right) \end{pmatrix}$$

$$= \left(\overline{\overline{\mathcal{E}_1}}+\overline{\overline{\mathcal{E}_2}}\right)\overline{\overline{\mathbb{N}_1}}$$

6. Omitted.

IV. FRANK AGGREGATION OPERATORS FOR ILNS

This section proposes the Frank aggregation operators based on ILNs, called ILFWA operator, ILFOWA operator, ILFWG operator, and ILFOWG operator. Further, we evaluate some basic properties for the above-evaluated operators. The advantages of these operators are listed below:

1) The averaging/geometric operators based on fuzzy sets (and their extensions) are the special cases of the proposed operators.
2) The Frank averaging/geometric operators based on fuzzy sets (and their extensions) are the special cases of the proposed operators.
3) The averaging/geometric operators based on linguistic sets (and their extensions) are the special cases of the proposed operators.
4) The Frank averaging/geometric operators based on linguistic sets (and their extensions) are the special cases of the proposed operators.

Similarly, we have a lot of advantages of the proposed operators, they can easily aggregate the collection of information into a singleton set.

Definition 7: Let $\overline{\overline{\mathbb{N}_z}} = \left(\mathbb{I}_{\mathcal{J}_z}, \overline{\overline{\mathfrak{M}}}_{\overline{\overline{\mathbb{N}_z}}}, \overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}_z}}}\right), z = 1, 2, \ldots, m$, be any family of ILNs. Then

$$\begin{aligned} ILFWA\left(\overline{\overline{\mathbb{N}_1}}, \overline{\overline{\mathbb{N}_2}}, \ldots, \overline{\overline{\mathbb{N}_m}}\right) &= \overline{\overline{\mathcal{E}_1\mathbb{N}_1}} \oplus \overline{\overline{\mathcal{E}_2\mathbb{N}_2}} \oplus \ldots \oplus \overline{\overline{\mathcal{E}_m\mathbb{N}_m}} \\ &= \sum_{z=1}^{m} \overline{\overline{\mathcal{E}_z\mathbb{N}_z}} \end{aligned} \tag{14}$$

Called the ILFWA operator, where the mathematical form of the weight vector is derived by: $\overline{\overline{\mathcal{E}_z}} \in [0, 1]$, $\sum_{z=1}^{m} \overline{\overline{\mathcal{E}_z}} = 1$.

Theorem 2: Let $\overline{\overline{\mathbb{N}_z}} = \left(\mathbb{I}_{\mathcal{J}_z}, \overline{\overline{\mathfrak{M}}}_{\overline{\overline{\mathbb{N}_z}}}, \overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}_z}}}\right), z = 1, 2, \ldots, m$, be any family of ILNs. Then, we prove that the aggregated shape of the ILFWA operator is again an ILN, such as

$$\begin{aligned} &ILFWA\left(\overline{\overline{\mathbb{N}_1}}, \overline{\overline{\mathbb{N}_2}}, \ldots, \overline{\overline{\mathbb{N}_m}}\right) \\ &= \begin{pmatrix} \mathbb{I}_{\mathfrak{P}\left(1-\log_\pi\left(1+\frac{\prod_{z=1}^{m}\left(\pi^{1-\frac{\overline{\overline{\mathcal{J}_z}}}{\mathfrak{P}}}-1\right)^{\overline{\overline{\mathcal{E}_z}}}}{\prod_{z=1}^{m}(\pi-1)^{\overline{\overline{\mathcal{E}_z}}-1}}\right)\right)}, \\ 1-\log_\pi\left(1+\frac{\prod_{z=1}^{m}\left(\pi^{1-\overline{\overline{\mathfrak{M}}}_{\overline{\overline{\mathbb{N}_z}}}}-1\right)^{\overline{\overline{\mathcal{E}_z}}}}{\prod_{z=1}^{m}(\pi-1)^{\overline{\overline{\mathcal{E}_z}}-1}}\right), \\ \log_\pi\left(1+\frac{\prod_{z=1}^{m}\left(\pi^{\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}_z}}}}-1\right)^{\overline{\overline{\mathcal{E}_z}}}}{\prod_{z=1}^{m}(\pi-1)^{\overline{\overline{\mathcal{E}_z}}-1}}\right) \end{pmatrix} \end{aligned} \tag{15}$$

Proof: Based on mathematical induction, we derive the ILFWA operator. For this, we consider $z=2$, then

$$\overline{\overline{\mathcal{E}_1\mathbb{N}_1}}=\begin{pmatrix}\mathbb{I}\;\mathfrak{P}\left(1-\log_{\pi}\left(1+\frac{\left(\pi^{1-\frac{\overline{\overline{\mathcal{J}_1}}}{\mathfrak{P}}}-1\right)^{\overline{\overline{\mathcal{E}_1}}}}{(\pi-1)^{\overline{\overline{\mathcal{E}_1}}-1}}\right)\right), \\ 1-\log_{\pi}\left(1+\frac{\left(\pi^{1-\overline{\overline{\mathfrak{nm}}}_{\overline{\overline{\mathbb{N}_1}}}}-1\right)^{\overline{\overline{\mathcal{E}_1}}}}{(\pi-1)^{\overline{\overline{\mathcal{E}_1}}-1}}\right), \\ \log_{\pi}\left(1+\frac{\left(\pi^{\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}_1}}}}-1\right)^{\overline{\overline{\mathcal{E}_1}}}}{(\pi-1)^{\overline{\overline{\mathcal{E}_1}}-1}}\right)\end{pmatrix}$$

$$\overline{\overline{\mathcal{E}_2\mathbb{N}_2}}=\begin{pmatrix}\mathbb{I}\;\mathfrak{P}\left(1-\log_{\pi}\left(1+\frac{\left(\pi^{1-\frac{\overline{\overline{\mathcal{J}_2}}}{\mathfrak{P}}}-1\right)^{\overline{\overline{\mathcal{E}_2}}}}{(\pi-1)^{\overline{\overline{\mathcal{E}_2}}-1}}\right)\right), \\ 1-\log_{\pi}\left(1+\frac{\left(\pi^{1-\overline{\overline{\mathfrak{nm}}}_{\overline{\overline{\mathbb{N}_2}}}}-1\right)^{\overline{\overline{\mathcal{E}_2}}}}{(\pi-1)^{\overline{\overline{\mathcal{E}_2}}-1}}\right), \\ \log_{\pi}\left(1+\frac{\left(\pi^{\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}_2}}}}-1\right)^{\overline{\overline{\mathcal{E}_2}}}}{(\pi-1)^{\overline{\overline{\mathcal{E}_2}}-1}}\right)\end{pmatrix}$$

$$ILFWA\left(\overline{\overline{\mathbb{N}_1}},\overline{\overline{\mathbb{N}_2}}\right)=\overline{\overline{\mathcal{E}_1\mathbb{N}_1}}\oplus\overline{\overline{\mathcal{E}_2\mathbb{N}_2}}$$

$$=\begin{pmatrix}\mathbb{I}\;\mathfrak{P}\left(1-\log_{\pi}\left(1+\frac{\left(\pi^{1-\frac{\overline{\overline{\mathcal{J}_1}}}{\mathfrak{P}}}-1\right)^{\overline{\overline{\mathcal{E}_1}}}}{(\pi-1)^{\overline{\overline{\mathcal{E}_1}}-1}}\right)\right), \\ 1-\log_{\pi}\left(1+\frac{\left(\pi^{1-\overline{\overline{\mathfrak{nm}}}_{\overline{\overline{\mathbb{N}_1}}}}-1\right)^{\overline{\overline{\mathcal{E}_1}}}}{(\pi-1)^{\overline{\overline{\mathcal{E}_1}}-1}}\right), \\ \log_{\pi}\left(1+\frac{\left(\pi^{\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}_1}}}}-1\right)^{\overline{\overline{\mathcal{E}_1}}}}{(\pi-1)^{\overline{\overline{\mathcal{E}_1}}-1}}\right)\end{pmatrix}$$

$$\oplus\begin{pmatrix}\mathbb{I}\;\mathfrak{P}\left(1-\log_{\pi}\left(1+\frac{\left(\pi^{1-\frac{\overline{\overline{\mathcal{J}_2}}}{\mathfrak{P}}}-1\right)^{\overline{\overline{\mathcal{E}_2}}}}{(\pi-1)^{\overline{\overline{\mathcal{E}_2}}-1}}\right)\right), \\ 1-\log_{\pi}\left(1+\frac{\left(\pi^{1-\overline{\overline{\mathfrak{nm}}}_{\overline{\overline{\mathbb{N}_2}}}}-1\right)^{\overline{\overline{\mathcal{E}_2}}}}{(\pi-1)^{\overline{\overline{\mathcal{E}_2}}-1}}\right), \\ \log_{\pi}\left(1+\frac{\left(\pi^{\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}_2}}}}-1\right)^{\overline{\overline{\mathcal{E}_2}}}}{(\pi-1)^{\overline{\overline{\mathcal{E}_2}}-1}}\right)\end{pmatrix}$$

$$=\begin{pmatrix}\mathbb{I}\;\mathfrak{P}\left(1-\log_{\pi}\left(1+\frac{\prod_{z=1}^{2}\left(\pi^{1-\frac{\overline{\overline{\mathcal{J}_z}}}{\mathfrak{P}}}-1\right)^{\overline{\overline{\mathcal{E}_z}}}}{\prod_{z=1}^{2}(\pi-1)^{\overline{\overline{\mathcal{E}_z}}-1}}\right)\right), \\ 1-\log_{\pi}\left(1+\frac{\prod_{z=1}^{2}\left(\pi^{1-\overline{\overline{\mathfrak{nm}}}_{\overline{\overline{\mathbb{N}_z}}}}-1\right)^{\overline{\overline{\mathcal{E}_z}}}}{\prod_{z=1}^{2}(\pi-1)^{\overline{\overline{\mathcal{E}_z}}-1}}\right), \\ \log_{\pi}\left(1+\frac{\prod_{z=1}^{2}\left(\pi^{\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}_z}}}}-1\right)^{\overline{\overline{\mathcal{E}_z}}}}{\prod_{z=1}^{2}(\pi-1)^{\overline{\overline{\mathcal{E}_z}}-1}}\right)\end{pmatrix}$$

The above result is hold for $z=2$. Further, if $z=q$, then

$$ILFWA\left(\overline{\overline{\mathbb{N}_1}},\overline{\overline{\mathbb{N}_2}},\ldots,\overline{\overline{\mathbb{N}_q}}\right)$$

$$=\begin{pmatrix}\mathbb{I}\;\mathfrak{P}\left(1-\log_{\pi}\left(1+\frac{\prod_{z=1}^{q}\left(\pi^{1-\frac{\overline{\overline{\mathcal{J}_z}}}{\mathfrak{P}}}-1\right)^{\overline{\overline{\mathcal{E}_z}}}}{\prod_{z=1}^{q}(\pi-1)^{\overline{\overline{\mathcal{E}_z}}-1}}\right)\right), \\ 1-\log_{\pi}\left(1+\frac{\prod_{z=1}^{q}\left(\pi^{1-\overline{\overline{\mathfrak{nm}}}_{\overline{\overline{\mathbb{N}_z}}}}-1\right)^{\overline{\overline{\mathcal{E}_z}}}}{\prod_{z=1}^{q}(\pi-1)^{\overline{\overline{\mathcal{E}_z}}-1}}\right), \\ \log_{\pi}\left(1+\frac{\prod_{z=1}^{q}\left(\pi^{\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}_z}}}}-1\right)^{\overline{\overline{\mathcal{E}_z}}}}{\prod_{z=1}^{q}(\pi-1)^{\overline{\overline{\mathcal{E}_z}}-1}}\right)\end{pmatrix}$$

Then, we exposed our result for $z=q+1$, such as

$$ILFWA\left(\overline{\overline{\mathbb{N}_1}},\overline{\overline{\mathbb{N}_2}},\ldots,\overline{\overline{\mathbb{N}_{q+1}}}\right)$$

$$= \overline{\overline{\mathcal{E}_1\mathbb{N}_1}} \oplus \overline{\overline{\mathcal{E}_2\mathbb{N}_2}} \oplus \ldots \oplus \overline{\overline{\mathcal{E}_q\mathbb{N}_q}} \oplus \overline{\overline{\mathcal{E}_{q+1}\mathbb{N}_{q+1}}}$$

$$= \sum_{z=1}^{q} \overline{\overline{\mathcal{E}_z\mathbb{N}_z}} \oplus \overline{\overline{\mathcal{E}_{q+1}\mathbb{N}_{q+1}}}$$

$$= \begin{pmatrix} \mathbb{I}_{\mathfrak{P}\left(1-\log_\pi\left(1+\frac{\prod_{z=1}^{q}\left(\pi^{1-\frac{\overline{\overline{\mathcal{J}}}_z}{\mathfrak{P}}}-1\right)^{\overline{\overline{\mathcal{E}}}_z}}{\prod_{z=1}^{q}(\pi-1)^{\overline{\overline{\mathcal{E}}}_z-1}}\right)\right)}, \\ 1-\log_\pi\left(1+\frac{\prod_{z=1}^{q}\left(\pi^{1-\overline{\overline{\mathbb{m}}}_{\overline{\overline{\mathbb{N}}}_z}}-1\right)^{\overline{\overline{\mathcal{E}}}_z}}{\prod_{z=1}^{q}(\pi-1)^{\overline{\overline{\mathcal{E}}}_z-1}}\right), \\ \log_\pi\left(1+\frac{\prod_{z=1}^{q}\left(\pi^{\overline{\overline{\mathbb{n}}}_{\overline{\overline{\mathbb{N}}}_z}}-1\right)^{\overline{\overline{\mathcal{E}}}_z}}{\prod_{z=1}^{q}(\pi-1)^{\overline{\overline{\mathcal{E}}}_z-1}}\right) \end{pmatrix} \oplus \overline{\overline{\mathcal{E}_{q+1}\mathbb{N}_{q+1}}}$$

$$= \begin{pmatrix} \mathbb{I}_{\mathfrak{P}\left(1-\log_\pi\left(1+\frac{\prod_{z=1}^{q}\left(\pi^{1-\frac{\overline{\overline{\mathcal{J}}}_z}{\mathfrak{P}}}-1\right)^{\overline{\overline{\mathcal{E}}}_z}}{\prod_{z=1}^{q}(\pi-1)^{\overline{\overline{\mathcal{E}}}_z-1}}\right)\right)}, \\ 1-\log_\pi\left(1+\frac{\prod_{z=1}^{q}\left(\pi^{1-\overline{\overline{\mathbb{m}}}_{\overline{\overline{\mathbb{N}}}_z}}-1\right)^{\overline{\overline{\mathcal{E}}}_z}}{\prod_{z=1}^{q}(\pi-1)^{\overline{\overline{\mathcal{E}}}_z-1}}\right), \\ \log_\pi\left(1+\frac{\prod_{z=1}^{q}\left(\pi^{\overline{\overline{\mathbb{n}}}_{\overline{\overline{\mathbb{N}}}_z}}-1\right)^{\overline{\overline{\mathcal{E}}}_z}}{\prod_{z=1}^{q}(\pi-1)^{\overline{\overline{\mathcal{E}}}_z-1}}\right) \end{pmatrix}$$

$$\oplus \begin{pmatrix} \mathbb{I}_{\mathfrak{P}\left(1-\log_\pi\left(1+\frac{\left(\pi^{1-\frac{\overline{\overline{\mathcal{J}}}_{q+1}}{\mathfrak{P}}}-1\right)^{\overline{\overline{\mathcal{E}}}_{q+1}}}{(\pi-1)^{\overline{\overline{\mathcal{E}}}_{q+1}-1}}\right)\right)}, \\ 1-\log_\pi\left(1+\frac{\left(\pi^{1-\overline{\overline{\mathbb{m}}}_{\overline{\overline{\mathbb{N}}}_{q+1}}}-1\right)^{\overline{\overline{\mathcal{E}}}_{q+1}}}{(\pi-1)^{\overline{\overline{\mathcal{E}}}_{q+1}-1}}\right), \\ \log_\pi\left(1+\frac{\left(\pi^{\overline{\overline{\mathbb{n}}}_{\overline{\overline{\mathbb{N}}}_{q+1}}}-1\right)^{\overline{\overline{\mathcal{E}}}_{q+1}}}{(\pi-1)^{\overline{\overline{\mathcal{E}}}_{q+1}-1}}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \mathbb{I}_{\mathfrak{P}\left(1-\log_\pi\left(1+\frac{\prod_{z=1}^{q+1}\left(\pi^{1-\frac{\overline{\overline{\mathcal{J}}}_z}{\mathfrak{P}}}-1\right)^{\overline{\overline{\mathcal{E}}}_z}}{\prod_{z=1}^{q+1}(\pi-1)^{\overline{\overline{\mathcal{E}}}_z-1}}\right)\right)}, \\ 1-\log_\pi\left(1+\frac{\prod_{z=1}^{q+1}\left(\pi^{1-\overline{\overline{\mathbb{m}}}_{\overline{\overline{\mathbb{N}}}_z}}-1\right)^{\overline{\overline{\mathcal{E}}}_z}}{\prod_{z=1}^{q+1}(\pi-1)^{\overline{\overline{\mathcal{E}}}_z-1}}\right), \\ \log_\pi\left(1+\frac{\prod_{z=1}^{q+1}\left(\pi^{\overline{\overline{\mathbb{n}}}_{\overline{\overline{\mathbb{N}}}_z}}-1\right)^{\overline{\overline{\mathcal{E}}}_z}}{\prod_{z=1}^{q+1}(\pi-1)^{\overline{\overline{\mathcal{E}}}_z-1}}\right) \end{pmatrix}$$

The initiated technique is held for all non-negative information.

Property 1: Let $\overline{\overline{\mathbb{N}}}_z = \left(\mathbb{I}_{\mathcal{J}_z}, \overline{\overline{\mathbb{m}}}_{\overline{\overline{\mathbb{N}}}_z}, \overline{\overline{\mathbb{n}}}_{\overline{\overline{\mathbb{N}}}_z}\right), z = 1, 2, \ldots, m$, be any family of ILNs. Then,

Idempotency: If $\overline{\overline{\mathbb{N}}}_z = \overline{\overline{\mathbb{N}}}, z = 1, 2, \ldots, m$, then

$$ILFWA\left(\overline{\overline{\mathbb{N}}}_1, \overline{\overline{\mathbb{N}}}_2, \ldots, \overline{\overline{\mathbb{N}}}_m\right) = \overline{\overline{\mathbb{N}}} \tag{16}$$

Monotonicity: If $\overline{\overline{\mathbb{N}}}_z \leq \overline{\overline{\mathbb{N}}}_z^{@}$, then

$$ILFWA\left(\overline{\overline{\mathbb{N}}}_1, \overline{\overline{\mathbb{N}}}_2, \ldots, \overline{\overline{\mathbb{N}}}_m\right) \leq ILFWA\left(\overline{\overline{\mathbb{N}}}_1^{@}, \overline{\overline{\mathbb{N}}}_2^{@}, \ldots, \overline{\overline{\mathbb{N}}}_m^{@}\right) \tag{17}$$

Boundedness: If $\overline{\overline{\mathbb{N}}}_z^{-} = \left(\mathbb{I}_{\min(\mathcal{J}_z)}, \min\left(\overline{\overline{\mathbb{m}}}_{\overline{\overline{\mathbb{N}}}_z}\right), \max\left(\overline{\overline{\mathbb{n}}}_{\overline{\overline{\mathbb{N}}}_z}\right)\right)$ and $\overline{\overline{\mathbb{N}}}_z^{+} = \left(\mathbb{I}_{\max(\mathcal{J}_z)}, \max\left(\overline{\overline{\mathbb{m}}}_{\overline{\overline{\mathbb{N}}}_z}\right), \min\left(\overline{\overline{\mathbb{n}}}_{\overline{\overline{\mathbb{N}}}_z}\right)\right), z = 1, 2, \ldots, m$, thus

$$\overline{\overline{\mathbb{N}}}_z^{-} \leq ILFWA\left(\overline{\overline{\mathbb{N}}}_1, \overline{\overline{\mathbb{N}}}_2, \ldots, \overline{\overline{\mathbb{N}}}_m\right) \leq \overline{\overline{\mathbb{N}}}_z^{+} \tag{18}$$

Definition 8: Let $\overline{\overline{\mathbb{N}}}_z = \left(\mathbb{I}_{\mathcal{J}_z}, \overline{\overline{\mathbb{m}}}_{\overline{\overline{\mathbb{N}}}_z}, \overline{\overline{\mathbb{n}}}_{\overline{\overline{\mathbb{N}}}_z}\right), z = 1, 2, \ldots, m$, be any family of ILNs. Then

$$\begin{aligned} &ILFOWA\left(\overline{\overline{\mathbb{N}}}_1, \overline{\overline{\mathbb{N}}}_2, \ldots, \overline{\overline{\mathbb{N}}}_m\right) \\ &= \overline{\overline{\mathcal{E}_1\mathbb{N}_{o(1)}}} \oplus \overline{\overline{\mathcal{E}_2\mathbb{N}_{o(2)}}} \oplus \ldots \oplus \overline{\overline{\mathcal{E}_m\mathbb{N}_{o(m)}}} \\ &= \sum_{z=1}^{m} \overline{\overline{\mathcal{E}_z\mathbb{N}_{o(z)}}} \end{aligned} \tag{19}$$

Called the ILFOWA operator, where the mathematical form of the weight vector is derived by: $\overline{\overline{\mathcal{E}}}_z \in [0, 1], \sum_{z=1}^{m} \overline{\overline{\mathcal{E}}}_z = 1$ with $o(z) \leq o(z-1)$.

Theorem 3: Let $\overline{\overline{\mathbb{N}}}_z = \left(\mathbb{I}_{\mathcal{J}_z}, \overline{\overline{\mathbb{m}}}_{\overline{\overline{\mathbb{N}}}_z}, \overline{\overline{\mathbb{n}}}_{\overline{\overline{\mathbb{N}}}_z}\right), z = 1, 2, \ldots, m$, be any family of ILNs. Then, we prove that the aggregated shape of the ILFOWA operator is again an ILN, such as

$$ILFOWA\left(\overline{\overline{\mathbb{N}}}_1, \overline{\overline{\mathbb{N}}}_2, \ldots, \overline{\overline{\mathbb{N}}}_m\right)$$

$$= \begin{pmatrix} \mathbb{I}_{\mathfrak{P}\left(1-\log_{\pi}\left(1+\frac{\prod_{z=1}^{m}\left(\pi^{1-\frac{\overline{\overline{\mathcal{J}_{o(z)}}}}{\mathfrak{P}}}-1\right)^{\overline{\overline{\mathcal{E}_z}}}}{\prod_{z=1}^{m}(\pi-1)^{\overline{\overline{\mathcal{E}_z}}-1}}\right)\right)}, \\ 1-\log_{\pi}\left(1+\frac{\prod_{z=1}^{m}\left(\pi^{1-\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_{o(z)}}}-1\right)^{\overline{\overline{\mathcal{E}_z}}}}{\prod_{z=1}^{m}(\pi-1)^{\overline{\overline{\mathcal{E}_z}}-1}}\right), \\ \log_{\pi}\left(1+\frac{\prod_{z=1}^{m}\left(\pi^{\overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}}}_{o(z)}}}-1\right)^{\overline{\overline{\mathcal{E}_z}}}}{\prod_{z=1}^{m}(\pi-1)^{\overline{\overline{\mathcal{E}_z}}-1}}\right) \end{pmatrix} \tag{20}$$

The proof of Theorem 3 is similar to the proof of Theorem 2.

Property 2: Let $\overline{\overline{\mathbb{N}}}_z = \left(\mathbb{I}_{\mathcal{J}_z}, \overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_z}, \overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}}}_z}\right), z = 1, 2, \ldots, m$, be any family of ILNs. Then,

Idempotency: If $\overline{\overline{\mathbb{N}}}_z = \overline{\overline{\mathbb{N}}}$, $z = 1, 2, \ldots, m$, then

$$ILFOWA\left(\overline{\overline{\mathbb{N}}}_1, \overline{\overline{\mathbb{N}}}_2, \ldots, \overline{\overline{\mathbb{N}}}_m\right) = \overline{\overline{\mathbb{N}}}. \tag{21}$$

Monotonicity: If $\overline{\overline{\mathbb{N}}}_z \leq \overline{\overline{\mathbb{N}}}_z^{@}$, then

$$ILFOWA\left(\overline{\overline{\mathbb{N}}}_1, \overline{\overline{\mathbb{N}}}_2, \ldots, \overline{\overline{\mathbb{N}}}_m\right) \leq ILFOWA\left(\overline{\overline{\mathbb{N}}}_1^{@}, \overline{\overline{\mathbb{N}}}_2^{@}, \ldots, \overline{\overline{\mathbb{N}}}_m^{@}\right) \tag{22}$$

Boundedness: If $\overline{\overline{\mathbb{N}}}_z^{-} = \left(\mathbb{I}_{\min(\mathcal{J}_z)}, \min\left(\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_z}\right), \max\left(\overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}}}_z}\right)\right)$ and $\overline{\overline{\mathbb{N}}}_z^{+} = \left(\mathbb{I}_{\max(\mathcal{J}_z)}, \max\left(\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_z}\right), \min\left(\overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}}}_z}\right)\right), z = 1, 2, \ldots, m$, thus

$$\overline{\overline{\mathbb{N}}}_z^{-} \leq ILFOWA\left(\overline{\overline{\mathbb{N}}}_1, \overline{\overline{\mathbb{N}}}_2, \ldots, \overline{\overline{\mathbb{N}}}_m\right) \leq \overline{\overline{\mathbb{N}}}_z^{+} \tag{23}$$

Definition 9: Let $\overline{\overline{\mathbb{N}}}_z = \left(\mathbb{I}_{\mathcal{J}_z}, \overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_z}, \overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}}}_z}\right), z = 1, 2, \ldots, m$, be any family of ILNs. Then

$$ILFWG\left(\overline{\overline{\mathbb{N}}}_1, \overline{\overline{\mathbb{N}}}_2, \ldots, \overline{\overline{\mathbb{N}}}_m\right) = \left(\overline{\overline{\mathbb{N}}}_1\right)^{\overline{\overline{\mathcal{E}_1}}} \otimes \left(\overline{\overline{\mathbb{N}}}_2\right)^{\overline{\overline{\mathcal{E}_2}}} \otimes \ldots \otimes \left(\overline{\overline{\mathbb{N}}}_m\right)^{\overline{\overline{\mathcal{E}_m}}} = \prod_{z=1}^{m}\left(\overline{\overline{\mathbb{N}}}_z\right)^{\overline{\overline{\mathcal{E}_z}}} \tag{24}$$

Called the ILFWG operator, where the mathematical form of the weight vector is derived by: $\overline{\overline{\mathcal{E}_z}} \in [0, 1], \sum_{z=1}^{m} \overline{\overline{\mathcal{E}_z}} = 1$.

Theorem 4: Let $\overline{\overline{\mathbb{N}}}_z = \left(\mathbb{I}_{\mathcal{J}_z}, \overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_z}, \overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}}}_z}\right), z = 1, 2, \ldots, m$, be any family of ILNs. Then, we prove that the aggregated shape of the ILFWG operator is again an ILN, such as

$$ILFWG\left(\overline{\overline{\mathbb{N}}}_1, \overline{\overline{\mathbb{N}}}_2, \ldots, \overline{\overline{\mathbb{N}}}_m\right)$$

$$= \begin{pmatrix} \mathbb{I}_{\mathfrak{P}\left(\log_{\pi}\left(1+\frac{\prod_{z=1}^{m}\left(\pi^{\frac{\overline{\overline{\mathcal{J}_z}}}{\mathfrak{P}}}-1\right)^{\overline{\overline{\mathcal{E}_z}}}}{\prod_{z=1}^{m}(\pi-1)^{\overline{\overline{\mathcal{E}_z}}-1}}\right)\right)}, \\ \log_{\pi}\left(1+\frac{\prod_{z=1}^{m}\left(\pi^{\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_z}}-1\right)^{\overline{\overline{\mathcal{E}_z}}}}{\prod_{z=1}^{m}(\pi-1)^{\overline{\overline{\mathcal{E}_z}}-1}}\right), \\ 1-\log_{\pi}\left(1+\frac{\prod_{z=1}^{m}\left(\pi^{1-\overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}}}_z}}-1\right)^{\overline{\overline{\mathcal{E}_z}}}}{\prod_{z=1}^{m}(\pi-1)^{\overline{\overline{\mathcal{E}_z}}-1}}\right) \end{pmatrix} \tag{25}$$

The proof of Theorem 4 is similar to the proof of Theorem 2.

Property 3: Let $\overline{\overline{\mathbb{N}}}_z = \left(\mathbb{I}_{\mathcal{J}_z}, \overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_z}, \overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}}}_z}\right), z = 1, 2, \ldots, m$, be any family of ILNs. Then,

Idempotency: If $\overline{\overline{\mathbb{N}}}_z = \overline{\overline{\mathbb{N}}}$, $z = 1, 2, \ldots, m$, then

$$ILFWG\left(\overline{\overline{\mathbb{N}}}_1, \overline{\overline{\mathbb{N}}}_2, \ldots, \overline{\overline{\mathbb{N}}}_m\right) = \overline{\overline{\mathbb{N}}}. \tag{26}$$

Monotonicity: If $\overline{\overline{\mathbb{N}}}_z \leq \overline{\overline{\mathbb{N}}}_z^{@}$, then

$$ILFWG\left(\overline{\overline{\mathbb{N}}}_1, \overline{\overline{\mathbb{N}}}_2, \ldots, \overline{\overline{\mathbb{N}}}_m\right) \leq ILFWG\left(\overline{\overline{\mathbb{N}}}_1^{@}, \overline{\overline{\mathbb{N}}}_2^{@}, \ldots, \overline{\overline{\mathbb{N}}}_m^{@}\right) \tag{27}$$

Boundedness: If $\overline{\overline{\mathbb{N}}}_z^{-} = \left(\mathbb{I}_{\min(\mathcal{J}_z)}, \min\left(\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_z}\right), \max\left(\overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}}}_z}\right)\right)$ and $\overline{\overline{\mathbb{N}}}_z^{+} = \left(\mathbb{I}_{\max(\mathcal{J}_z)}, \max\left(\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_z}\right), \min\left(\overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}}}_z}\right)\right), z = 1, 2, \ldots, m$, thus

$$\overline{\overline{\mathbb{N}}}_z^{-} \leq ILFWG\left(\overline{\overline{\mathbb{N}}}_1, \overline{\overline{\mathbb{N}}}_2, \ldots, \overline{\overline{\mathbb{N}}}_m\right) \leq \overline{\overline{\mathbb{N}}}_z^{+} \tag{28}$$

Definition 10: Let $\overline{\overline{\mathbb{N}}}_z = \left(\mathbb{I}_{\mathcal{J}_z}, \overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_z}, \overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}}}_z}\right), z = 1, 2, \ldots, m$, be any family of ILNs. Then

$$ILFOWG\left(\overline{\overline{\mathbb{N}}}_1, \overline{\overline{\mathbb{N}}}_2, \ldots, \overline{\overline{\mathbb{N}}}_m\right) = \left(\overline{\overline{\mathbb{N}}}_{o(1)}\right)^{\overline{\overline{\mathcal{E}_1}}} \otimes \left(\overline{\overline{\mathbb{N}}}_{o(2)}\right)^{\overline{\overline{\mathcal{E}_2}}} \otimes \ldots \otimes \left(\overline{\overline{\mathbb{N}}}_{o(m)}\right)^{\overline{\overline{\mathcal{E}_m}}} = \prod_{z=1}^{m}\left(\overline{\overline{\mathbb{N}}}_{o(z)}\right)^{\overline{\overline{\mathcal{E}_z}}} \tag{29}$$

Called the ILFOWG operator, where the mathematical form of the weight vector is derived by: $\overline{\overline{\mathcal{E}_z}} \in [0, 1], \sum_{z=1}^{m} \overline{\overline{\mathcal{E}_z}} = 1$ with order $o(z) \leq o(z-1)$.

Theorem 5: Let $\overline{\overline{\mathbb{N}}}_z = \left(\mathbb{I}_{\mathcal{J}_z}, \overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_z}, \overline{\overline{\mathfrak{n}}}_{\overline{\overline{\mathbb{N}}}_z}\right), z = 1, 2, \ldots, m$, be any family of ILNs. Then, we prove that the aggregated shape of the ILFOWG operator is again an ILN, such as

$$ILFOWG\left(\overline{\overline{\mathbb{N}}}_1, \overline{\overline{\mathbb{N}}}_2, \ldots, \overline{\overline{\mathbb{N}}}_m\right)$$

$$= \begin{pmatrix} \left(\mathbb{I}, \mathfrak{P}\left(\log_\pi\left(1+\frac{\prod_{z=1}^{m}\left(\pi^{\frac{\overline{\overline{\mathcal{J}}}_{o(z)}}{\mathfrak{P}}}-1\right)^{\overline{\overline{\mathcal{E}}}_z}}{\prod_{z=1}^{m}(\pi-1)^{\overline{\overline{\mathcal{E}}}_z-1}}\right)\right)\right), \\ \log_\pi\left(1+\frac{\prod_{z=1}^{m}\left(\pi^{\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_{o(z)}}}-1\right)^{\overline{\overline{\mathcal{E}}}_z}}{\prod_{z=1}^{m}(\pi-1)^{\overline{\overline{\mathcal{E}}}_z-1}}\right), \\ 1-\log_\pi\left(1+\frac{\prod_{z=1}^{m}\left(\pi^{1-\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_{o(z)}}}-1\right)^{\overline{\overline{\mathcal{E}}}_z}}{\prod_{z=1}^{m}(\pi-1)^{\overline{\overline{\mathcal{E}}}_z-1}}\right) \end{pmatrix} \tag{30}$$

The proof of Theorem 5 is similar to the proof of Theorem 2.

Property 4: Let $\overline{\overline{\mathbb{N}}}_z = \left(\mathbb{I}_{\mathcal{J}_z}, \overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_z}, \overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_z}\right), z = 1, 2, \ldots, m$, be any family of ILNs. Then,

Idempotency: If $\overline{\overline{\mathbb{N}}}_z = \overline{\overline{\mathbb{N}}}$, $z = 1, 2, \ldots, m$, then

$$ILFOWG\left(\overline{\overline{\mathbb{N}}}_1, \overline{\overline{\mathbb{N}}}_2, \ldots, \overline{\overline{\mathbb{N}}}_m\right) = \overline{\overline{\mathbb{N}}}. \tag{31}$$

Monotonicity: If $\overline{\overline{\mathbb{N}}}_z \leq \overline{\overline{\mathbb{N}}}_z^{@}$, then

$$ILFOWG\left(\overline{\overline{\mathbb{N}}}_1, \overline{\overline{\mathbb{N}}}_2, \ldots, \overline{\overline{\mathbb{N}}}_m\right) \leq ILFOWG\left(\overline{\overline{\mathbb{N}}}_1^{@}, \overline{\overline{\mathbb{N}}}_2^{@}, \ldots, \overline{\overline{\mathbb{N}}}_m^{@}\right) \tag{32}$$

Boundedness: If $\overline{\overline{\mathbb{N}}}_z^{-} = \left(\mathbb{I}_{\min(\mathcal{J}_z)}, \min\left(\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_z}\right), \max\left(\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_z}\right)\right)$ and $\overline{\overline{\mathbb{N}}}_z^{+} = \left(\mathbb{I}_{\max(\mathcal{J}_z)}, \max\left(\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_z}\right), \min\left(\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_z}\right)\right), z = 1, 2, \ldots, m$, thus

$$\overline{\overline{\mathbb{N}}}_z^{-} \leq ILFOWG\left(\overline{\overline{\mathbb{N}}}_1, \overline{\overline{\mathbb{N}}}_2, \ldots, \overline{\overline{\mathbb{N}}}_m\right) \leq \overline{\overline{\mathbb{N}}}_z^{+} \tag{33}$$

The initiated ILFWA operator, ILFOWA operator, ILFWG operator, and ILFOWG operator are superior to the averaging and geometric operators based on FSs, IFSs, LSs, and their combinations.

V. FRANK-BASED TOPSIS METHOD

In this section, we derive the TOPSIS technique based on Frank aggregation operators, called ILFWA operator and ILFWG operator to enhance the worth of the derived theory. The major steps of the TOPSIS technique are listed below:

Step 1: Calculate some known and unknown IL information in the shape of a matrix.

Step 2: Evaluate the positive ideal solution (PIS) and negative ideal solution (NIS) under the presence of the IL information, such as

$$\overline{\overline{\mathbb{N}}}_z^{+} = \begin{Bmatrix} \left(\mathbb{I}_{\max(\mathcal{J}_{i1})}, \max\left(\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_{i1}}\right), \min\left(\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_{i1}}\right)\right), \\ \left(\mathbb{I}_{\max(\mathcal{J}_{i2})}, \max\left(\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_{i2}}\right), \min\left(\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_{i2}}\right)\right) \\ , \ldots, \left(\mathbb{I}_{\max(\mathcal{J}_{im})}, \max\left(\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_{im}}\right), \min\left(\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_{im}}\right)\right) \end{Bmatrix},$$

$$z = 1, 2, \ldots, m \tag{34}$$

$$\overline{\overline{\mathbb{N}}}_z^{-} = \begin{Bmatrix} \left(\mathbb{I}_{\min(\mathcal{J}_{i1})}, \min\left(\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_{i1}}\right), \max\left(\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_{i1}}\right)\right), \\ \left(\mathbb{I}_{\min(\mathcal{J}_{i2})}, \min\left(\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_{i2}}\right), \max\left(\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_{i2}}\right)\right) \\ , \ldots, \left(\mathbb{I}_{\min(\mathcal{J}_{im})}, \min\left(\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_{im}}\right), \max\left(\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_{im}}\right)\right) \end{Bmatrix},$$

$$z = 1, 2, \ldots, m \tag{35}$$

Step 3: Evaluate the aggregated value by using our considered unknown values and PIS and NIS, such as

$$ILFWA^{+}\left(\overline{\overline{\mathbb{N}}}_1, \overline{\overline{\mathbb{N}}}_2, \ldots, \overline{\overline{\mathbb{N}}}_m\right), ILFWA^{-}\left(\overline{\overline{\mathbb{N}}}_1, \overline{\overline{\mathbb{N}}}_2, \ldots, \overline{\overline{\mathbb{N}}}_m\right)$$

$$= \begin{pmatrix} \left(\mathbb{I}, \mathfrak{P}\left(1-\log_\pi\left(1+\frac{\prod_{z=1}^{m}\left(\pi^{1-\frac{\overline{\overline{\mathcal{J}}}_z}{\mathfrak{P}}}-1\right)^{\overline{\overline{\mathcal{E}}}_z}}{\prod_{z=1}^{m}(\pi-1)^{\overline{\overline{\mathcal{E}}}_z-1}}\right)\right)\right), \\ 1-\log_\pi\left(1+\frac{\prod_{z=1}^{m}\left(\pi^{1-\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_z}}-1\right)^{\overline{\overline{\mathcal{E}}}_z}}{\prod_{z=1}^{m}(\pi-1)^{\overline{\overline{\mathcal{E}}}_z-1}}\right), \\ \log_\pi\left(1+\frac{\prod_{z=1}^{m}\left(\pi^{\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_z}}-1\right)^{\overline{\overline{\mathcal{E}}}_z}}{\prod_{z=1}^{m}(\pi-1)^{\overline{\overline{\mathcal{E}}}_z-1}}\right) \end{pmatrix} \tag{36}$$

$$ILFWG^{+}\left(\overline{\overline{\mathbb{N}}}_1, \overline{\overline{\mathbb{N}}}_2, \ldots, \overline{\overline{\mathbb{N}}}_m\right), ILFWG^{-}\left(\overline{\overline{\mathbb{N}}}_1, \overline{\overline{\mathbb{N}}}_2, \ldots, \overline{\overline{\mathbb{N}}}_m\right)$$

$$= \begin{pmatrix} \left(\mathbb{I}, \mathfrak{P}\left(\log_\pi\left(1+\frac{\prod_{z=1}^{m}\left(\pi^{\frac{\overline{\overline{\mathcal{J}}}_z}{\mathfrak{P}}}-1\right)^{\overline{\overline{\mathcal{E}}}_z}}{\prod_{z=1}^{m}(\pi-1)^{\overline{\overline{\mathcal{E}}}_z-1}}\right)\right)\right), \\ \log_\pi\left(1+\frac{\prod_{z=1}^{m}\left(\pi^{\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_z}}-1\right)^{\overline{\overline{\mathcal{E}}}_z}}{\prod_{z=1}^{m}(\pi-1)^{\overline{\overline{\mathcal{E}}}_z-1}}\right), \\ 1-\log_\pi\left(1+\frac{\prod_{z=1}^{m}\left(\pi^{1-\overline{\overline{\mathfrak{m}}}_{\overline{\overline{\mathbb{N}}}_z}}-1\right)^{\overline{\overline{\mathcal{E}}}_z}}{\prod_{z=1}^{m}(\pi-1)^{\overline{\overline{\mathcal{E}}}_z-1}}\right) \end{pmatrix} \tag{37}$$

Step 4: Evaluate the closeness measures by using the aggregated values, such as

$$G_i^{CM} = \frac{ILFWA^{-}\left(\overline{\overline{\mathbb{N}}}_1, \overline{\overline{\mathbb{N}}}_2, \ldots, \overline{\overline{\mathbb{N}}}_m\right)}{ILFWA^{+}\left(\overline{\overline{\mathbb{N}}}_1, \overline{\overline{\mathbb{N}}}_2, \ldots, \overline{\overline{\mathbb{N}}}_m\right)+ILFWA^{-}\left(\overline{\overline{\mathbb{N}}}_1, \overline{\overline{\mathbb{N}}}_2, \ldots, \overline{\overline{\mathbb{N}}}_m\right)} \tag{38}$$

G_i^{CM}

$$=\frac{ILFWG^{-}\left(\overline{\overline{\mathbb{N}_1}},\overline{\overline{\mathbb{N}_2}},\ldots,\overline{\overline{\mathbb{N}_m}}\right)}{ILFWG^{+}\left(\overline{\overline{\mathbb{N}_1}},\overline{\overline{\mathbb{N}_2}},\ldots,\overline{\overline{\mathbb{N}_m}}\right)+ILFWG^{-}\left(\overline{\overline{\mathbb{N}_1}},\overline{\overline{\mathbb{N}_2}},\ldots,\overline{\overline{\mathbb{N}_m}}\right)} \tag{39}$$

Step 5: Derive the ranking measures according to their closeness measure and try to evaluate the best one.

VI. MADM: DECISION-MAKING METHODS

In this section, we are doing to compute the technique of MADM method under the initiated techniques such as the ILFWA operator and ILFWG operator to evaluate the supremacy and validity of the derived theory.

Consider $\overline{\overline{\mathbb{N}_1}},\overline{\overline{\mathbb{N}_2}},\ldots,\overline{\overline{\mathbb{N}_m}}$ be any family of alternatives, where for each alternative we have the collection of attributes such as $\overline{\overline{\mathbb{N}_1}}^{AT},\overline{\overline{\mathbb{N}_2}}^{AT},\ldots,\overline{\overline{\mathbb{N}_n}}^{AT}$ with weight vectors $\sum_{z=1}^{m}\overline{\overline{\mathcal{E}_z}}=1,\overline{\overline{\mathcal{E}_z}}\in[0,1]$. Further, we compute the matrix by using the values of ILNs, the linguistic information is derived in the shape: $\mathbb{I}_{\mathcal{J}\left(\overline{\overline{\mathbb{x}}}\right)}\in S=\{\mathbb{I}_{\mathcal{J}_z\left(\overline{\overline{\mathbb{x}}}\right)}:z=1,2,\ldots,2\mathfrak{P}\}$, where $\overline{\overline{\mathbb{m}_{\overline{\overline{\mathbb{N}_{IL}}}}}}\left(\overline{\overline{\mathbb{x}}}\right)$ and $\overline{\overline{\mathbb{n}_{\overline{\overline{\mathbb{N}_{IL}}}}}}\left(\overline{\overline{\mathbb{x}}}\right)$ describes the supporting and supporting against grades with a strong condition: $0\le\overline{\overline{\mathbb{m}_{\overline{\overline{\mathbb{N}_{IL}}}}}}\left(\overline{\overline{\mathbb{x}}}\right)+\overline{\overline{\mathbb{n}_{\overline{\overline{\mathbb{N}_{IL}}}}}}\left(\overline{\overline{\mathbb{x}}}\right)\le 1$. Furthermore, the information $\overline{\overline{R_{\overline{\overline{\mathbb{N}_{IL}}}}}}\left(\overline{\overline{\mathbb{x}}}\right)=1-\left(\overline{\overline{\mathbb{m}_{\overline{\overline{\mathbb{N}_{IL}}}}}}\left(\overline{\overline{\mathbb{x}}}\right)+\overline{\overline{\mathbb{n}_{\overline{\overline{\mathbb{N}_{IL}}}}}}\left(\overline{\overline{\mathbb{x}}}\right)\right)$ uses as neutral information and the simple shape of ILNs is derived from the shape: $\overline{\overline{\mathbb{N}_z}}=\left(\mathbb{I}_{\mathcal{J}_z},\overline{\overline{\mathbb{m}_{\overline{\overline{\mathbb{N}_z}}}}},\overline{\overline{\mathbb{n}_{\overline{\overline{\mathbb{N}_z}}}}}\right),z=1,2,\ldots,m$. Finally, we derive the procedure of the decision-making technique, whose major cases are listed below:

Case 1: Compute the matrix by putting the ILNs. Further, we normalize the matrix, if we have cost-type information, such as

$$N=\begin{cases}\left(\mathbb{I}_{\mathcal{J}_z},\overline{\overline{\mathbb{m}_{\overline{\overline{\mathbb{N}_z}}}}},\overline{\overline{\mathbb{n}_{\overline{\overline{\mathbb{N}_z}}}}}\right) & benefit\\ \left(\mathbb{I}_{\mathcal{J}_z},\overline{\overline{\mathbb{n}_{\overline{\overline{\mathbb{N}_z}}}}},\overline{\overline{\mathbb{m}_{\overline{\overline{\mathbb{N}_z}}}}}\right) & cost\end{cases} \tag{40}$$

We are not required to normalize the matrix if we have benefit-type data.

Case 2: Compute the aggregated values by using the data in a matrix based on ILFWA operator and ILFWG operator, such as

$$ILFWA\left(\overline{\overline{\mathbb{N}_1}},\overline{\overline{\mathbb{N}_2}},\ldots,\overline{\overline{\mathbb{N}_m}}\right)=\begin{pmatrix}\mathbb{I}_{\mathfrak{P}\left(1-\log_\pi\left(1+\frac{\prod_{z=1}^{m}\left(\pi^{1-\frac{\overline{\overline{\mathcal{J}_z}}}{\mathfrak{P}}}-1\right)^{\overline{\overline{\mathcal{E}_z}}}}{\prod_{z=1}^{m}(\pi-1)^{\overline{\overline{\mathcal{E}_z}}-1}}\right)\right)},\\ 1-\log_\pi\left(1+\frac{\prod_{z=1}^{m}\left(\pi^{1-\overline{\overline{\mathbb{m}_{\overline{\overline{\mathbb{N}_z}}}}}}-1\right)^{\overline{\overline{\mathcal{E}_z}}}}{\prod_{z=1}^{m}(\pi-1)^{\overline{\overline{\mathcal{E}_z}}-1}}\right),\\ \log_\pi\left(1+\frac{\prod_{z=1}^{m}\left(\pi^{\overline{\overline{\mathbb{n}_{\overline{\overline{\mathbb{N}_z}}}}}}-1\right)^{\overline{\overline{\mathcal{E}_z}}}}{\prod_{z=1}^{m}(\pi-1)^{\overline{\overline{\mathcal{E}_z}}-1}}\right)\end{pmatrix} \tag{41}$$

$$ILFWG\left(\overline{\overline{\mathbb{N}_1}},\overline{\overline{\mathbb{N}_2}},\ldots,\overline{\overline{\mathbb{N}_m}}\right)=\begin{pmatrix}\mathbb{I}_{\mathfrak{P}\left(\log_\pi\left(1+\frac{\prod_{z=1}^{m}\left(\pi^{\frac{\overline{\overline{\mathcal{J}_z}}}{\mathfrak{P}}}-1\right)^{\overline{\overline{\mathcal{E}_z}}}}{\prod_{z=1}^{m}(\pi-1)^{\overline{\overline{\mathcal{E}_z}}-1}}\right)\right)},\\ \log_\pi\left(1+\frac{\prod_{z=1}^{m}\left(\pi^{\overline{\overline{\mathbb{m}_{\overline{\overline{\mathbb{N}_z}}}}}}-1\right)^{\overline{\overline{\mathcal{E}_z}}}}{\prod_{z=1}^{m}(\pi-1)^{\overline{\overline{\mathcal{E}_z}}-1}}\right),\\ 1-\log_\pi\left(1+\frac{\prod_{z=1}^{m}\left(\pi^{1-\overline{\overline{\mathbb{n}_{\overline{\overline{\mathbb{N}_z}}}}}}-1\right)^{\overline{\overline{\mathcal{E}_z}}}}{\prod_{z=1}^{m}(\pi-1)^{\overline{\overline{\mathcal{E}_z}}-1}}\right)\end{pmatrix} \tag{42}$$

Case 3: Compute the score values of the aggregated information, such as

$$\mathbb{S}\left(\overline{\overline{\mathbb{N}_z}}\right)=\frac{\mathcal{J}_z}{\mathfrak{P}}*\left(\overline{\overline{\mathbb{m}_{\overline{\overline{\mathbb{N}_z}}}}}-\overline{\overline{\mathbb{n}_{\overline{\overline{\mathbb{N}_z}}}}}\right)\in[-1,1] \tag{43}$$

Case 4: Compute the ranking values based on score values for addressing the best one.

A. DevOps CHALLENGES BASED ON PROPOSED OPERATORS

In this sub-section, we discuss the DevOps challenges under the consideration of the initiated operators, called ILFWA operators and ILFWG operators. DevOps system aims to enhance collaboration and communication among software development and information technology operations teams. The major theme of this DevOps is to rationalize and computerize the software delivery and organization management technique to attain quicker and more massive dominant releases. For evaluating the major key components or categories within the DevOps ecosystems, such as

1. Version Control Systems "$\overline{\overline{\mathbb{N}_1}}$".
2. Configuration Management "$\overline{\overline{\mathbb{N}_2}}$".
3. Continuous Integration Tools "$\overline{\overline{\mathbb{N}_3}}$".
4. Continuous Deployment "$\overline{\overline{\mathbb{N}_4}}$".
5. Collaboration and Communications "$\overline{\overline{\mathbb{N}_5}}$".
6. Infrastructure as Code "$\overline{\overline{\mathbb{N}_6}}$".

To select the best one, we have the following weight vectors, such as $(0.2, 0.1, 0.3, 0.1, 0.3)^T$ with attributes such as growth analysis, social impact, political impact, environmental impact, and internet version. Finally, we derive the procedure of the decision-making technique, whose major cases are listed below:

Case 1: Compute the matrix by putting the ILNs, see Table 2. Further, we normalize the matrix, if we have cost-type information, such as

$$N=\begin{cases}\left(\mathbb{I}_{\mathcal{J}_z},\overline{\overline{\mathbb{m}_{\overline{\overline{\mathbb{N}_z}}}}},\overline{\overline{\mathbb{n}_{\overline{\overline{\mathbb{N}_z}}}}}\right) & benefit\\ \left(\mathbb{I}_{\mathcal{J}_z},\overline{\overline{\mathbb{n}_{\overline{\overline{\mathbb{N}_z}}}}},\overline{\overline{\mathbb{m}_{\overline{\overline{\mathbb{N}_z}}}}}\right) & cost\end{cases}$$

TABLE 2. IL decision matrix.

$\overline{\overline{\mathbb{N}}}_1$	$(\mathbb{I}_3, 0.8, 0.1)$	$(\mathbb{I}_{3.1}, 0.81, 0.11)$	$(\mathbb{I}_{3.2}, 0.82, 0.12)$	$(\mathbb{I}_{3.3}, 0.83, 0.13)$	$(\mathbb{I}_{34}, 0.84, 0.14)$
$\overline{\overline{\mathbb{N}}}_2$	$(\mathbb{I}_1, 0.5, 0.2)$	$(\mathbb{I}_{1.1}, 0.51, 0.21)$	$(\mathbb{I}_{1.2}, 0.52, 0.22)$	$(\mathbb{I}_{1.3}, 0.53, 0.23)$	$(\mathbb{I}_{1.4}, 0.54, 0.24)$
$\overline{\overline{\mathbb{N}}}_3$	$(\mathbb{I}_4, 0.3, 0.1)$	$(\mathbb{I}_{4.1}, 0.31, 0.11)$	$(\mathbb{I}_{4.2}, 0.32, 0.12)$	$(\mathbb{I}_{4.3}, 0.33, 0.13)$	$(\mathbb{I}_{4.4}, 0.34, 0.14)$
$\overline{\overline{\mathbb{N}}}_4$	$(\mathbb{I}_4, 0.7, 0.2)$	$(\mathbb{I}_{4.1}, 0.71, 0.21)$	$(\mathbb{I}_{4.2}, 0.72, 0.22)$	$(\mathbb{I}_{4.3}, 0.73, 0.23)$	$(\mathbb{I}_{4.4}, 0.74, 0.24)$
$\overline{\overline{\mathbb{N}}}_5$	$(\mathbb{I}_2, 0.6, 0.2)$	$(\mathbb{I}_{2.1}, 0.61, 0.21)$	$(\mathbb{I}_{2.2}, 0.62, 0.22)$	$(\mathbb{I}_{2.3}, 0.63, 0.23)$	$(\mathbb{I}_{2.4}, 0.64, 0.24)$
$\overline{\overline{\mathbb{N}}}_6$	$(\mathbb{I}_3, 0.2, 0.1)$	$(\mathbb{I}_{3.1}, 0.21, 0.11)$	$(\mathbb{I}_{3.2}, 0.22, 0.12)$	$(\mathbb{I}_{3.3}, 0.23, 0.13)$	$(\mathbb{I}_{3.4}, 0.24, 0.14)$

TABLE 3. IL aggregated values.

	ILFWA Operator	ILFWG Operator
$\overline{\overline{\mathbb{N}}}_1$	$(\mathbb{I}_{3.2253}, 0.8225, 0.1211)$	$(\mathbb{I}_{3.2173}, 0.8219, 0.1220)$
$\overline{\overline{\mathbb{N}}}_2$	$(\mathbb{I}_{1.2221}, 0.5221, 0.2215)$	$(\mathbb{I}_{1.2116}, 0.5218, 0.2221)$
$\overline{\overline{\mathbb{N}}}_3$	$(\mathbb{I}_{4.2331}, 0.3221, 0.1211)$	$(\mathbb{I}_{4.2181}, 0.3217, 0.1220)$
$\overline{\overline{\mathbb{N}}}_4$	$(\mathbb{I}_{4.2331}, 0.7223, 0.2215)$	$(\mathbb{I}_{4.2181}, 0.7218, 0.2221)$
$\overline{\overline{\mathbb{N}}}_5$	$(\mathbb{I}_{2.2231}, 0.6222, 0.2215)$	$(\mathbb{I}_{2.2158}, 0.6218, 0.2221)$
$\overline{\overline{\mathbb{N}}}_6$	$(\mathbb{I}_{3.2252}, 0.2221, 0.1211)$	$(\mathbb{I}_{3.2173}, 0.2215, 0.1220)$

TABLE 4. IL score information.

	ILFWA Operator	ILFWG Operator
$\overline{\overline{\mathbb{N}}}_1$	0.45247	0.4503
$\overline{\overline{\mathbb{N}}}_2$	0.07349	0.07263
$\overline{\overline{\mathbb{N}}}_3$	0.17017	0.16839
$\overline{\overline{\mathbb{N}}}_4$	0.424	0.42163
$\overline{\overline{\mathbb{N}}}_5$	0.17817	0.17716
$\overline{\overline{\mathbb{N}}}_6$	0.06513	0.064

We are not required to normalize the matrix if we have benefit-type data. Table 2 does not aim to normalize.

Case 2: Compute the aggregated values by using the data in a matrix based on ILFWA operator and ILFWG operator, see Table 3.

Case 3: Compute the score values of the aggregated information, see Table 4.

Case 4: Compute the ranking values based on score values for addressing the best one, see Table 5.

Finally, after a long assessment, we have the best decision is $\overline{\overline{\mathbb{N}}}_1$ based on the ILFWA operator and ILFWG operator. Further, we simplify the supremacy and flexibility of the derived operators with the help of comparative analysis by using some prevailing techniques.

TABLE 5. Representation of ranking values.

Methods	Ranking values	Best optimal
ILFWA Operator	$\overline{\overline{\mathbb{N}}}_1 > \overline{\overline{\mathbb{N}}}_4 > \overline{\overline{\mathbb{N}}}_5 > \overline{\overline{\mathbb{N}}}_3 > \overline{\overline{\mathbb{N}}}_2 > \overline{\overline{\mathbb{N}}}_6$	$\overline{\overline{\mathbb{N}}}_1$
ILFWG Operator	$\overline{\overline{\mathbb{N}}}_1 > \overline{\overline{\mathbb{N}}}_4 > \overline{\overline{\mathbb{N}}}_5 > \overline{\overline{\mathbb{N}}}_3 > \overline{\overline{\mathbb{N}}}_2 > \overline{\overline{\mathbb{N}}}_6$	$\overline{\overline{\mathbb{N}}}_1$

TABLE 6. Representation of the comparative analysis.

Methods	Ranking values	Best optimal
Frank [35]	Bounded Features	Bounded Features
Iancu [36]	Bounded Features	Bounded Features
Zhang et al. [37]	Bounded Features	Bounded Features
Xia et al. [38]	Bounded Features	Bounded Features
ILFWA Operator	$\overline{\overline{\mathbb{N}}}_1 > \overline{\overline{\mathbb{N}}}_4 > \overline{\overline{\mathbb{N}}}_5 > \overline{\overline{\mathbb{N}}}_3 > \overline{\overline{\mathbb{N}}}_2 > \overline{\overline{\mathbb{N}}}_6$	$\overline{\overline{\mathbb{N}}}_1$
ILFWG Operator	$\overline{\overline{\mathbb{N}}}_1 > \overline{\overline{\mathbb{N}}}_4 > \overline{\overline{\mathbb{N}}}_5 > \overline{\overline{\mathbb{N}}}_3 > \overline{\overline{\mathbb{N}}}_2 > \overline{\overline{\mathbb{N}}}_6$	$\overline{\overline{\mathbb{N}}}_1$

VII. COMPARATIVE ANALYSIS

In this section, we compare the initiated ranking techniques with some of the ranking techniques of existing methods by using the information in Table 2. Further, the comparison between proposed and existing operators plays an important role in the existence of the supremacy and validity of the evaluated operators. For this, we have the following prevailing technique, for instance, the modified version of the triangular norms was initiated by Frank [35], called Frank norms in 1979. Additionally, the Frank aggregation operators for intuitionistic fuzzy measures were derived by Iancu [36]. Zhang et al. [37] derived the Frank power operators for IFSs. In 2012, Xia et al. [38] evaluated the aggregation operators based on Archimedean norms for IFSs. Therefore, using the data in Table 2, the comparative analysis is listed in Table 6.

Finally, after a long assessment, we have the best decision is $\overline{\overline{\mathbb{N}}}_1$ based on the ILFWA operator and ILFWG operator. Further, the existing techniques are not able to resolve the data in Table 2, because the existing techniques computed based on FSs, IFSs, and LSs, but the data in Table 2 is given in the shape of ILSs, therefore, for these existing techniques, it is not possible to cope with it, because up to date no one can derive any kind of operators, methods, and measures based on intuitionistic linguistic sets. Hence, the initiated techniques are superior then existing information.

VIII. CONCLUSION

The technique of intuitionistic linguistic set is a very meaningful and reliable technique to cope with uncertain vague information in genuine life problems. After all, we conclude the following remarks about the proposed theory, such as: for the construction of the frank aggregation operators, we computed the Frank operational laws based on IL variables and also derived their fundamental properties. Further, we analyzed the ILFWA operator, ILFOWA operator, ILFWG operator, and ILFOWG operator, and discussed their basic properties. Additionally, we evaluated the TOPSIS technique based on the initiated operators to enhance the worth of the explored information. Moreover, we introduced the MADM methods based on initiated operators for evaluating the major tools and technology that are commonly utilized in DevOps workflows. Finally, we selected some existing techniques and tried to compare their ranking results with our obtained ranking results to show the supremacy and validity of the presented techniques.

No doubt, the technique of intuitionistic linguistic set is very reliable, but to consider the problem of election they are not working feasibly, because during the election we faced the following information, for instance, if someone cast his vote in the fever, against, abstinence and refusal of candidates, which are four possibilities, for managing such kind of problems, the technique of IL set has been failed due to vagueness and uncertainty. For this, we aim to propose the technique of picture fuzzy linguistic sets.

In the future, we will propose the technique of Pythagorean linguistic sets, q-rung orthopair fuzzy sets, and many others and we try to discuss their application in the field of artificial intelligence, machine learning, neural networks, shortest path problems, inference systems, and decision-making techniques based on proposed operators for Pythagorean linguistic sets and their extensions to enhance the flexibility and validity of the initiated operators.

ABBREVIATIONS

FS: Fuzzy Sets, IFS: Intuitionistic Fuzzy Sets, DevOps: Development and Operations, IL: Intuitionistic Linguistic, FTN: Frank T-norm, FTCN: Frank T-conorm, ILFWA: Intuitionistic Linguistic Frank Weighted Averaging, ILFOWA: Intuitionistic Linguistic Frank Weighted Ordered Averaging, ILFWG: Intuitionistic Linguistic Frank Weighted Geometric, ILFOWG: Intuitionistic Linguistic Frank Weighted Ordered Geometric, TOPSIS: Technique for Order of Preference by Similarity to the ideal solution, MADM: multi-attribute decision-making.

ACKNOWLEDGMENT

The authors would like to thank the contributions of the different team members of the VOTE-TRA project [https://www.sfi.ie/challenges/digital-for-resilience/VOTE_TRA/ (Accessed on 27 April 2024)].

REFERENCES

[1] A. Qumer Gill, A. Loumish, I. Riyat, and S. Han, "DevOps for information management systems," *VINE J. Inf. Knowl. Manage. Syst.*, vol. 48, no. 1, pp. 122–139, Feb. 2018.

[2] L. E. Lwakatare, T. Kilamo, T. Karvonen, T. Sauvola, V. Heikkilä, J. Itkonen, P. Kuvaja, T. Mikkonen, M. Oivo, and C. Lassenius, "DevOps in practice: A multiple case study of five companies," *Inf. Softw. Technol.*, vol. 114, pp. 217–230, Oct. 2019.

[3] L. A. Zadeh, "Fuzzy sets," *Inf. Control*, vol. 8, no. 3, pp. 338–353, 1965.

[4] K. Atanassov, "Intuitionistic fuzzy sets," (In Bulgarian), in *Proc. VII ITKR's Session*, Sofia, Bulgaria, Jun. 1983, pp. 1–9.

[5] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets Syst.*, vol. 20, no. 1, pp. 87–96, Aug. 1986.

[6] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning-I," *Inf. Sci.*, vol. 8, no. 3, pp. 199–249, 1975.

[7] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning-III," *Inf. Sci.*, vol. 9, no. 1, pp. 43–80, Jan. 1975.

[8] A. Mardani, M. Nilashi, E. K. Zavadskas, S. R. Awang, H. Zare, and N. M. Jamal, "Decision making methods based on fuzzy aggregation operators: Three decades review from 1986 to 2017," *Int. J. Inf. Technol. Decis. Making*, vol. 17, no. 2, pp. 391–466, Mar. 2018.

[9] M. Akram, A. Adeel, and J. C. R. Alcantud, "Fuzzy N-soft sets: A novel model with applications," *J. Intell. Fuzzy Syst.*, vol. 35, no. 4, pp. 4757–4771, Oct. 2018.

[10] J. M. Merigó and M. Casanovas, "Fuzzy generalized hybrid aggregation operators and its application in fuzzy decision making," *Int. J. Fuzzy Syst.*, vol. 12, no. 1, pp. 23–45, 2010.

[11] M. Akram, A. Adeel, and J. C. R. Alcantud, "Hesitant fuzzy N-soft sets: A new model with applications in decision-making," *J. Intell. Fuzzy Syst.*, vol. 36, no. 6, pp. 6113–6127, Jun. 2019.

[12] M. Xia, Z. Xu, and N. Chen, "Some hesitant fuzzy aggregation operators with their application in group decision making," *Group Decis. Negotiation*, vol. 22, no. 2, pp. 259–279, Mar. 2013.

[13] F. Fatimah and J. C. R. Alcantud, "The multi-fuzzy N-soft set and its applications to decision-making," *Neural Comput. Appl.*, vol. 33, no. 17, pp. 11437–11446, Sep. 2021.

[14] J. M. Merigó and A. M. Gil-Lafuente, "Fuzzy induced generalized aggregation operators and its application in multi-person decision making," *Expert Syst. Appl.*, vol. 38, no. 8, pp. 9761–9772, Aug. 2011.

[15] T. Mahmood and Z. Ali, "Fuzzy superior Mandelbrot sets," *Soft Comput.*, vol. 26, no. 18, pp. 9011–9020, Sep. 2022.

[16] T. M. Al-shami, J. C. R. Alcantud, and A. Mhemdi, "New generalization of fuzzy soft sets: (a, b)-Fuzzy soft sets," *AIMS Math.*, vol. 8, no. 2, pp. 2995–3025, 2023.

[17] X. Shi, Z. Ali, T. Mahmood, and P. Liu, "Power aggregation operators of interval-valued Atanassov-intuitionistic fuzzy sets based on Aczel–Alsina t-norm and t-conorm and their applications in decision making," *Int. J. Comput. Intell. Syst.*, vol. 16, no. 1, p. 43, Mar. 2023.

[18] A. Patel, S. Jana, and J. Mahanta, "Construction of similarity measure for intuitionistic fuzzy sets and its application in face recognition and software quality evaluation," *Expert Syst. Appl.*, vol. 237, Mar. 2024, Art. no. 121491.

[19] K. Deveci and O. Güler, "Ranking intuitionistic fuzzy sets with hypervolume-based approach: An application for multi-criteria assessment of energy alternatives," *Appl. Soft Comput.*, vol. 150, Jan. 2024, Art. no. 111038.

[20] X. Wu, Z. Zhu, G. Chen, W. Pedrycz, L. Liu, and M. Aggarwal, "Generalized TODIM method based on symmetric intuitionistic fuzzy Jensen–Shannon divergence," *Expert Syst. Appl.*, vol. 237, Mar. 2024, Art. no. 121554.

[21] C. Kahraman, S. Cebi, B. Oztaysi, and S. Cevik Onar, "Intuitionistic fuzzy sets with ordered pairs and their usage in multi-attribute decision making: A novel intuitionistic fuzzy TOPSIS method with ordered pairs," *Mathematics*, vol. 11, no. 18, p. 3867, Sep. 2023.

[22] C. Xu and Y. Wen, "New measure of circular intuitionistic fuzzy sets and its application in decision making," *AIMS Math.*, vol. 8, no. 10, pp. 24053–24074, 2023.

[23] T. Mahmood, Z. Ali, S. Baupradist, and R. Chinram, "TOPSIS method based on Hamacher choquet-integral aggregation operators for atanassov-intuitionistic fuzzy sets and their applications in decision-making," *Axioms*, vol. 11, no. 12, p. 715, Dec. 2022.

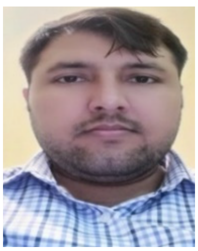
[24] A. K. Das and C. Granados, "IFP-intuitionistic multi fuzzy N-soft set and its induced IFP-hesitant N-soft set in decision-making," *J. Ambient Intell. Humanized Comput.*, vol. 14, no. 8, pp. 10143–10152, Aug. 2023.

[25] F. Ecer, "An extended MAIRCA method using intuitionistic fuzzy sets for coronavirus vaccine selection in the age of COVID-19," *Neural Comput. Appl.*, vol. 34, no. 7, pp. 5603–5623, Apr. 2022.

[26] F. Herrera, E. Herrera-Viedma, and L. Martinez, "A fuzzy linguistic methodology to deal with unbalanced linguistic term sets," *IEEE Trans. Fuzzy Syst.*, vol. 16, no. 2, pp. 354–370, Apr. 2008.

[27] H. Zhang, "Linguistic intuitionistic fuzzy sets and application in MAGDM," *J. Appl. Math.*, vol. 2014, pp. 1–11, Apr. 2014.

[28] S. Yu, J. Wang, and J. Wang, "An extended TODIM approach with intuitionistic linguistic numbers," *Int. Trans. Oper. Res.*, vol. 25, no. 3, pp. 781–805, May 2018.
[29] K. P. Yoon and W. K. Kim, "The behavioral TOPSIS," *Expert Syst. Appl.*, vol. 89, pp. 266–272, Dec. 2017.
[30] Y. Kim, E.-S. Chung, S.-M. Jun, and S. U. Kim, "Prioritizing the best sites for treated wastewater instream use in an urban watershed using fuzzy TOPSIS," *Resour., Conservation Recycling*, vol. 73, pp. 23–32, Apr. 2013.
[31] B. Yatsalo, A. Korobov, B. Öztayşi, C. Kahraman, and L. Martínez, "A general approach to fuzzy TOPSIS based on the concept of fuzzy multicriteria acceptability analysis," *J. Intell. Fuzzy Syst.*, vol. 38, no. 1, pp. 979–995, Jan. 2020.
[32] S. S. Gautam, S. R. Singh, and S. R. Singh, "TOPSIS for multi criteria decision making in intuitionistic fuzzy environment," *Int. J. Comput. Appl.*, vol. 156, no. 8, pp. 42–49, Dec. 2016.
[33] B. D. Rouyendegh, A. Yildizbasi, and P. Üstünyer, "Intuitionistic fuzzy TOPSIS method for green supplier selection problem," *Soft Comput.*, vol. 24, no. 3, pp. 2215–2228, Feb. 2020.
[34] E. P. Klement, R. Mesiar, and E. Pap, *Triangular Norms*, vol. 13. Slovakia: Tatra Mountains Mathematical Publications, 1997, pp. 169–193.
[35] M. J. Frank, "On the simultaneous associativity of $F(x, y)$ and $x + y - F(x, y)$," *Aequationes Mathematicae*, vol. 19, no. 1, pp. 194–226, Dec. 1979.
[36] I. Iancu, "Intuitionistic fuzzy similarity measures based on Frank t-norms family," *Pattern Recognit. Lett.*, vol. 42, pp. 128–136, Jun. 2014.
[37] X. Zhang, P. Liu, and Y. Wang, "Multiple attribute group decision making methods based on intuitionistic fuzzy Frank power aggregation operators," *J. Intell. Fuzzy Syst.*, vol. 29, no. 5, pp. 2235–2246, Sep. 2015.
[38] M. Xia, Z. Xu, and B. Zhu, "Some issues on intuitionistic fuzzy aggregation operators based on Archimedean t-conorm and t-norm," *Knowl.-Based Syst.*, vol. 31, pp. 78–88, Jul. 2012.
[39] F. Ecer, G. Haseli, R. Krishankumar, and M. Hajiaghaei-Keshteli, "Evaluation of sustainable cold chain suppliers using a combined multi-criteria group decision-making framework under fuzzy ZE-numbers," *Expert Syst. Appl.*, vol. 245, Jul. 2024, Art. no. 123063.
[40] M. M. Salih, B. B. Zaidan, A. A. Zaidan, and M. A. Ahmed, "Survey on fuzzy TOPSIS state-of-the-art between 2007 and 2017," *Comput. Oper. Res.*, vol. 104, pp. 207–227, Apr. 2019.



ZEESHAN ALI received the B.S. degree in mathematics from Abdul Wali Khan University, Mardan, Pakistan, in 2016, the M.S. degree in pure mathematics from International Islamic University Islamabad, Pakistan, in 2018, and the Ph.D. degree in mathematics from International Islamic University Islamabad, in 2023, under the supervision of Dr. Tahir Mahmood. He is an Assistant Professor of mathematics with the Department of Information Management, National Yunlin University of Science and Technology, Douliou, Yunlin, Taiwan. He was a Researcher (IRSIP) with KERMIT, Department of Mathematical Modeling, Statistics and Bioinformatics, Ghent University, Ghent, Belgium. He was also with International Islamic University Islamabad, as a Visiting Lecturer in mathematics from Fall 2019 to Spring 2022. From 2023 to 2024, he was an Assistant Professor with the Department of Mathematics and Statistics, Riphah International University, Islamabad. He has published more than 185 articles in national and international journals. More than 180 research publications on his credit with more than 3400 citations, more than 500 impact factor, an H-index of 30, and an i-index of 80. According to Stanford University and Scopus, he is among the World's top 2% scientists as career-long impact and also single year impact of 2021, 2022, 2023, and 2024. His research interests include the applications of statistics, fuzzy clustering, soft computing, pattern recognition, machine learning, aggregation operators, fuzzy logic, fuzzy decision making, fuzzy superior Mandelbrot sets, type-2 fuzzy sets, fuzzy groups, fuzzy rings, fuzzy modules, research optimization, fuzzy fixed-point theory, and fuzzy differential equations and their applications.

ZOHAIB AHMAD KHAN received the M.S. degree in electrical engineering from the University of Engineering and Technology, Taxila, Pakistan, and the Ph.D. degree from Riphah International University, Islamabad, Pakistan. Currently, he is an Assistant Professor with Riphah International University. His research interests include adaptive signal processing, biomedical signal processing, stochastic processes, neural networks, and self-mixing laser interferometry.

TAPAN SENAPATI received the B.Sc., M.Sc., and Ph.D. degrees in mathematics from Vidyasagar University, India, in 2006, 2008, and 2013, respectively. He is currently a Postdoctoral Fellow with the School of Mathematics and Statistics, Southwest University, Chongqing, China. He is also an Assistant Teacher of mathematics with the Government of West Bengal, India. He has published three books and more than 80 articles in reputed international journals. He is also an editor of the book *Real Life Applications of Multiple Criteria Decision-Making Techniques in Fuzzy Domain* (Springer). His research results have been published in *Fuzzy Sets and Systems*, IEEE Transactions on Fuzzy Systems, *Expert Systems with Applications*, *Applied Soft Computing*, *Engineering Applications of Artificial Intelligence*, *International Journal of Intelligent Systems*, and *International Journal of General Systems*. His main research interests include fuzzy sets, fuzzy optimization, soft computing, multi attribute decision making, and aggregation operators. He is a reviewer of several international journals and an Academic Editor of *Discrete Dynamics in Nature and Society* (SCIE), *Computational Intelligence and Neuroscience*, and *Mathematical Problems in Engineering*. His name has been listed in the "World's Top 2 %Scientists' list" (Stanford University), from 2020 to 2021.

SARBAST MOSLEM received the Ph.D. degree in transport and vehicle engineering from Budapest University of Technology and Economics, in 2020. He is currently a Postdoctoral Research Fellow with the School of Architecture Planning and Environmental Policy, University College Dublin (UCD), and a PI of VOTE-TRA Project, Ireland. He has been involved in several research projects on national and international level and the author or co-author of several conference papers and articles. He has published more than 50 papers from the area of the research interest. His research interests include transport planning, traffic engineering, logistics, supply chain management, soft computing, decision policy, fuzzy set theory, sustainability, and citizen's science. He is a gusting editor in some international journals in addition to a number of journals and conferences.