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RESEARCH ARTICLE

Robust Set Stability for Switched Boolean Networks Under One-Bit Function Perturbation

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ABSTRACT This paper investigates robust set stability for the switched Boolean networks (SBNs) with arbitrary switching signal affected by one-bit function perturbation (OBFP). Firstly, the dynamics of these networks are converted into algebraic forms utilizing semi-tensor product (STP) method. Secondly, OBFP impact on the set stability of SBNs is divided into four cases. Then, by constructing a state set and defining an index vector, several necessary and sufficient conditions are provided to detect whether an SBN with arbitrary switching signal is still stable to the given set after OBFP. Finally, a biological example is proposed to demonstrate the effectiveness of the obtained theoretical results.

INDEX TERMS Switched Boolean networks, semi-tensor product of matrices, set stability, one-bit function perturbation, arbitrary switching signal.

I. INTRODUCTION

A Boolean network (BN) is a typical binary-valued discrete-time system, which was initially proposed by Kauffman to predict and approximate gene regulatory networks [1]. In a BN, each node has a value of 1 or 0, representing active or inactive, respectively. The evolution of each node is related to a Boolean function which is assigned by its neighboring nodes, itself and some basic logical operators. Now BNs have attracted extensive attention and have become an effective model of many other complex networks, including wireless sensor networks, neural networks and networked evolutionary games, ect [2], [3], [4].

With the development of practical research problems, BNs have been generalized in many aspects [5], [6]. In order to describe the main features of switching phenomena in biological networks, BNs whose dynamics are governed by distinct switching models are named switched Boolean networks (SBNs). In terms of theoretical development viewpoints, SBNs are the natural extension of BNs, that is, BNs can be regarded as the special form of SBNs. From the perspective of practical applications, SBNs can better model and analyse the interaction and evolution of genes. For instance, the genes of bacteriophage λ contain two different models: lysis and lysogeny [7]. Therefore, when modeling this network as a BN, the dynamics becomes an SBN. In the last several decades, thanks to the appearance of semi-tensor product (STP) of matrices introduced by Cheng et al. [8], the logical form of an SBN can be converted to an equivalent algebraic representation. Up to now, a multitude of fundamental and important problems on SBNs have been explored by STP approach, such as controllability and observability [9], [10], stability and stabilization [11], [12], optimal control [13], etc..

The set stability issue of system is a basic and meaningful issue of classical control theory [14]. Set stability of SBNs means that all the initial states can eventually converge to an attractor subset of given set under arbitrary switching signal. In recent years, numerous landmark results about set stability of SBNs have been derived [15], [16]. Using the STP method, Guo et al. first discussed the set stability of an SBN with arbitrary switching signal, and gave an efficient criterion for

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set stability based on invariant subsets [17]. Then, Li and Tang generalized the results to switched Boolean control networks, and developed an algorithm to design state-feedback controls which make system set stabilizable [18]. Actually, many analysis problems are highly relevant to the set stability of SBNs such as synchronization [19], partial stability [20] and output tracking [21] for SBNs.

It is noted that the logical values of BNs may be changed by reason of gene mutation, which can be viewed as function perturbation [22]. Recently, the effect of function perturbation on the dynamics of logical networks has become a hot research topic. Li et al. studied function perturbations impact on stability of BNs, and some criteria which can keep BNs stability were obtained [23]. Reference [5] discussed the influence of OBFP on the finite-time stability of probabilistic BNs. The other fundamental problems of logical networks, such as optimal control [24], observability [25], detectability [26], have been investigated. It should be pointed out that function perturbation may change the attractors of the original SBNs. In fact, the set stability problem of SBNs is closely related to the attractors of SBNs, which may also be influenced due to function perturbation. A question that comes up naturally, how will the function perturbation affect the set stability of SBNs? Note that Wu et al. considered stability of SBNs with function perturbation, and a criterion under which the global stability of SBNs maintained unchanged was established [27]. Since the given set is a single point, we do not need to consider whether the affected state is an equilibrium point. This also forms the main difficulty in studying the function perturbation impact on set stability of SBNs. To our best knowledge, there exist few results about the set stability for SBNs subject to function perturbation at present.

In this paper, we intend to give some methods to verify the robust set stability for SBNs under OBFP. The main contributions are as follows. (i) OBFP impact on the set stability of SBNs with arbitrary switching signal is studied for the first time. Based on the possible relationship among the affected state, perturbed state, and the largest invariant set, the robustness analysis of set stability is divided into four cases. On the basis of these four cases, several necessary and sufficient conditions are derived for the robust set stability of SBNs under OBFP. Compared with [28] which assumed that the perturbed state was not included in the invariant set, the above classification is more reasonable since the location of genetic mutation is arbitrary in practical biological networks. (ii) Our results can be seen as an extension of [27]. When the given set becomes a single point, our results will degenerate into the results of robust stability of SBNs. However, the methods proposed in this article are easier to understand and detect than previous techniques.

The rest of this paper is arranged as follows. Section II shows some notations and definitions about STP. Section III gives the SBN model and problem formulation. In Section IV, the main results are presented. Section V gives a biological example to describe the validity of the proposed method, and Section VI is a brief conclusion.

II. PRELIMINARIES

This section gives some necessary symbols and definitions, which will be used throughout the article.

- $\mathbb{R}$ and $\mathbb{Z}_+$ denote the set of real numbers and positive integers, respectively.
- $\mathbb{R}_{n\times s}$ is the set of $n \times s$ real matrices.
- $[a, b] := \{a, a+1, \cdots, b\}$.
- Set $\mathcal{D} = \{0, 1\}$ and $\mathcal{D}^n = \underbrace{\mathcal{D}\times\mathcal{D}\times\ldots\times\mathcal{D}}_{n}$.
- $\mathrm{Col}_i(A)$ is the ith column of matrix A. The set of columns of A is denoted by $\mathrm{Col}(A)$.
- $\Delta_n := \{\delta_n^i \mid i \in [1, n]\}$, where $\delta_n^i = \mathrm{Col}_i(I_n)$. For compactness, $\Delta := \Delta_2$.
- If $\mathrm{Col}(A) \subseteq \Delta_n$, then $A \in \mathbb{R}_{n\times s}$ is called a logical matrix. Denote the set of all $n \times s$ logical matrices by $\mathcal{L}_{n\times s}$.
- If $A \in \mathcal{L}_{n\times s}$, denote A briefly by $A = \delta_n[i_1\ i_2\ \cdots\ i_s]$.
- $[A]_{i,j}$ is the element on the (i, j) entry of matrix A.
- $\mathbf{0}_n := [\underbrace{0, 0, \ldots, 0}_{n}]^T$, $\mathbf{1}_n := [\underbrace{1, 1, \ldots, 1}_{n}]^T$.
- A matrix $A \in \mathbb{R}_{n\times s}$ is called a Boolean matrix, if all its entries are either 0 or 1. Denote the set of $n \times s$ Boolean matrices by $\mathcal{B}_{m\times n}$.
- Assume $A = (a_{ij}), B = (b_{ij}) \in \mathcal{B}_{m\times n}$, then $A +_{\mathcal{B}} B = (a_{ij} \vee b_{ij})$, where "$\vee$" represents the logical operator "or". $(\mathcal{B}) \sum_{i=1}^{n} L_i := L_1 +_{\mathcal{B}} L_2 +_{\mathcal{B}} \cdots +_{\mathcal{B}} L_n$.
- Assume $A \in \mathcal{B}_{m\times n}$ and $B \in \mathcal{B}_{n\times p}$, then $A \times_{\mathcal{B}} B := C = (c_{ij})_{m\times p}$, where $c_{ij} = (\mathcal{B}) \sum_{k=1}^{n} a_{ik} \wedge b_{kj}$, and "$\wedge$" represents the logical operator "and". $A^{(k)} = A^{(k-1)} \times_{\mathcal{B}} A$, where k is a positive integer.

Definition 1 [8]: The STP of two matrices $A \in \mathbb{R}_{m\times n}$ and $B \in \mathbb{R}_{s\times t}$ is defined as

$$A \ltimes B = (A \otimes I_{\frac{\lambda}{n}})(B \otimes I_{\frac{\lambda}{s}}),$$

where λ denotes the least common multiple of n and s, and $\otimes$ denotes the Kronecker product.

STP is a generalization of ordinary matrix product, and it retains almost all the basic properties of ordinary matrix product. Throughout this article, we omit the symbol "$\ltimes$".

Definition 2 [8]: let $M \in \mathbb{R}_{m\times s}$ and $N \in \mathbb{R}_{n\times s}$. Define the Khatri-Rao product of M and N, denoted by $M * N$, as $M * N = [Col_1(M) \ltimes Col_1(N)\quad Col_2(M) \ltimes Col_2(N)\quad \cdots\quad Col_s(M) \ltimes Col_s(N)] \in \mathbb{R}_{mn\times s}$.

For a logical variable $x \in \mathcal{D}$, we define its vector form as $(x, 1-x)^T$, then there is an equivalence relationship between $\mathcal{D}$ and Δ. It is easy to see that if x_i is the vector form of logical variable X_i, then there is a one-to-one correspondence between $X = (X_1, X_2, \cdots, X_n)^T \in \mathcal{D}^n$ and $x = \ltimes_{i=1}^{n} x_i \in \Delta_{2^n}$. We call x the vector form of X.

Lemma 1 [8]: Consider a logical mapping $f : \mathcal{D}^n \to \mathcal{D}$. There exists a unique matrix $M_f \in \mathcal{L}_{2\times 2^n}$, called the structure matrix of f such that

$$f(X_1, X_2, \cdots, X_n) \sim M_f \ltimes_{i=1}^{n} x_i,$$

where $x_i \in \Delta$ is the vector form of $X_i \in \mathcal{D}$, $i = 1, 2, \cdots, n$, and " $\sim$" stands for the equivalence relation.

III. PROBLEM FORMULATION

An SBN with n nodes and ω subnetworks can be described as

$$\begin{cases} X_1(t+1) = f_1^{\sigma(t)}(X_1(t), \ldots, X_n(t)), \\ X_2(t+1) = f_2^{\sigma(t)}(X_1(t), \ldots, X_n(t)), \\ \quad \vdots \\ X_n(t+1) = f_n^{\sigma(t)}(X_1(t), \ldots, X_n(t)). \end{cases} \tag{1}$$

where $X_i \in \mathcal{D}$, $i \in [1,\ n]$ are Boolean variables, $\sigma(t)$ denotes the switching signal taking values from a finite set $[1, \omega]$. $f_i^{\sigma}(t) : \mathcal{D}^n \to \mathcal{D}$, $i \in [1,\ n]$ are Boolean functions.

Denote the vector form of X_i by x_i, that is, $x_i = (X_i, 1 - X_i)^T$. In the light of Lemma 1, SBN (1) can be expressed as

$$\begin{cases} x_1(t+1) = M_1^{\sigma(t)} x(t), \\ x_2(t+1) = M_2^{\sigma(t)} x(t), \\ \quad \vdots \\ x_n(t+1) = M_n^{\sigma(t)} x(t), \end{cases} \tag{2}$$

where $x(t) = \ltimes_{i=1}^n x_i(t)$, and $M_i^{\sigma(t)}$ represents the structure matrix of $f_i^{\sigma(t)}$, $i \in [1,\ n]$.

Multiplying the n equations in (2) together yields

$$x(t+1) = L_{\sigma(t)} x(t), \tag{3}$$

where $L_{\sigma(t)} = M_1^{\sigma(t)} * M_2^{\sigma(t)} * \cdots * M_n^{\sigma(t)} \in \mathcal{L}_{2^n \times 2^n}$, and $*$ is the Khatri-Rao product of matrices. System (3) is called the algebraic form of SBN (1).

Identify the k-th switching signal by δ_ω^k, $k \in [1, \omega]$, the algebraic representation of SBN (3) can be equivalently rewritten as

$$x(t+1) = L\sigma(t)x(t), \tag{4}$$

where $\sigma(t) \in \Delta_\omega$. The matrix $L := [L_1\ L_2\ \ldots\ L_\omega] \in \mathcal{L}_{2^n \times \omega 2^n}$ is called the state transition matrix of SBN (4), where $L_k := \delta_{2^n}[\alpha_1^k\ \alpha_2^k\ \cdots\ \alpha_{2^n}^k] \in \mathcal{L}_{2^n \times 2^n}$, $k \in [1, \omega]$.

For initial state $x(0) \in \Delta_{2^n}$, under switching signal sequence $\sigma := \{\sigma(t), t \in [0, \tau]\}$, the state of SBN (4) at time $\tau + 1$ is indicated as $x(\tau + 1; x(0), \sigma)$. According to the algebraic representation (4), the concept of $\mathcal{M}$-stable for SBNs is reviewed below [17].

Definition 3 [17]: Given a subset $\mathcal{M} \subseteq \Delta_{2^n}$. SBN (4) is said to be $\mathcal{M}$-stable, if for any initial state $x(0) \in \Delta_{2^n}$, there exists $\tau \in \mathbb{Z}_+$ such that $x(t; x(0), \sigma) \in \mathcal{M}$ for any $t \geqslant \tau$ and arbitrary switching signal sequence σ.

As is well known, the largest invariant subset of a given set plays an important role in set stability. The following criterion is reviewed for the set stability analysis of SBNs.

Lemma 2 [17]: SBN (4) is $\mathcal{M}$-stable if and only if SBN (4) can be stable to its largest invariant subset.

There are many methods to calculate the largest invariant subset, thus we will not elaborate on them further. For detailed details, please refer to [14] and [17]. This paper assumes that the largest invariant subset of $\mathcal{M}$ is $I_s(\mathcal{M})$.

When using SBNs to model gene regulatory networks, gene mutation is considered as function perturbation. Specifically, some truth value in the logical function f of SBN (1) is flipped. Correspondingly, some column in the state transition matrix L of (4) is changed.

In order to study the $\mathcal{M}$-stable problem of SBNs subject to OBFP, we give a natural assumption and definition as below.

Assumption 1: Before OBFP occurs, SBN (4) is $\mathcal{M}$-stable under arbitrary switching signal.

Definition 4: SBN (4) is robustly stable to set $\mathcal{M}$ under arbitrary switching signal, if SBN (4) is still $\mathcal{M}$-stable after OBFP.

IV. MAIN RESULTS

In this section, we will give some necessary and sufficient conditions to detect robust set stability of SBNs after OBFP.

We first propose the following assumption.

Assumption 2: Given $l \in \{1, \cdots, \omega 2^n\}$. After OBFP occurs, $\mathrm{Col}_l(L)$ is altered from $\delta_{2^n}^{\gamma}$ to $\delta_{2^n}^{\gamma^*}$, where $\gamma \neq \gamma^*$.

In fact, OBFP only changes the l-th column of L, and there are no changes in the other columns of L. Obviously, the state transition matrix L of SBN (4) is changed into a new matrix $\hat{L} \in \mathcal{L}_{2^n \times \omega 2^n}$, where

$$\mathrm{Col}_i(\hat{L}) = \begin{cases} \delta_{2^n}^{\gamma^*}, & \text{if } i = l; \\ \mathrm{Col}_i(L), & \text{otherwise}. \end{cases} \tag{5}$$

Hence, SBN (4) under OBFP becomes the following form:

$$x(t+1) = \hat{L}\sigma(t)x(t). \tag{6}$$

Lemma 3: ([8]) For any integer $1 \leq i \leq \omega 2^n$, there exist unique positive integers $i_1 \in [1, \omega]$ and $i_2 \in [1, 2^n]$ such that

$$\delta_{\omega 2^n}^i = \delta_\omega^{i_1} \ltimes \delta_{2^n}^{i_2}, \tag{7}$$

where $i = (i_1 - 1)2^n + i_2$.

Set $l = (k^* - 1)2^n + \varphi^*$. By Assumption 2, we know that OBFP only affects state $\delta_{2^n}^{\varphi^*}$ in the k^*-th subnetwork, named the affected state. Therefore, for any $x = \delta_{2^n}^{\varphi}$ and any $\sigma = \delta_\omega^k$, if $\varphi \neq \varphi^*$, it follows that

$$L\sigma x = \delta_{2^n}^{\alpha_\varphi^k} = \hat{L}\sigma x. \tag{8}$$

If $\varphi = \varphi^*$ and $k = k^*$, one has

$$L\sigma x = \delta_{2^n}^{\gamma} \neq \delta_{2^n}^{\gamma^*} = \hat{L}\sigma x. \tag{9}$$

According to the possible relationship among the affected state $\delta_{2^n}^{\varphi *}$, perturbed state $\delta_{2^n}^{\gamma^*}$, and the largest invariant set $I_s(\mathcal{M})$, the robustness analysis of set stability is divided into four cases as blew.

Case 1: $\delta_{2^n}^{\varphi *} \notin I_s(\mathcal{M})$ and $\delta_{2^n}^{\gamma^*} \notin I_s(\mathcal{M})$;
Case 2: $\delta_{2^n}^{\varphi *} \in I_s(\mathcal{M})$ and $\delta_{2^n}^{\gamma^*} \notin I_s(\mathcal{M})$;
Case 3: $\delta_{2^n}^{\varphi *} \notin I_s(\mathcal{M})$ and $\delta_{2^n}^{\gamma^*} \in I_s(\mathcal{M})$;
Case 4: $\delta_{2^n}^{\varphi *} \in I_s(\mathcal{M})$ and $\delta_{2^n}^{\gamma^*} \in I_s(\mathcal{M})$.

Next, we analyse how OBFP affects the set stability of SBN (4). We first construct $A = L \ltimes_{\mathcal{B}} \mathbf{1}_\omega$ and $\hat{A} = \hat{L} \ltimes_{\mathcal{B}} \mathbf{1}_\omega$. Then, set $\Gamma := (\mathcal{B})\sum_{i=1}^{2^n} A^i$. It follows from

Proposition 3.1 in [29] that $[\Gamma]_{\varphi,\theta} > 0$ means that there must exist one path from $\delta_{2^n}^{\theta}$ to $\delta_{2^n}^{\varphi}$ for SBN (4) before OBFP.

Denote the index vector of a given set $\mathcal{M}$ as $J_{\mathcal{M}}$, where

$$(J_{\mathcal{M}})_i = \begin{cases} 1, & \text{if } \delta_{2^n}^{i} \notin \mathcal{M}, \\ 0, & \text{if } \delta_{2^n}^{i} \in \mathcal{M}, \end{cases}$$

and $(J_{\mathcal{M}})_i$ is the i-th element of column vector $J_{\mathcal{M}}$.

In the light of the set Γ and index vector $J_{\mathcal{M}}$, we provide several criteria to detect whether an SBN with arbitrary switching signal is still stable to the set $\mathcal{M}$ after OBFP.

Theorem 1: Under Assumption 1, when OBFP in Assumption 2 is Case 1, SBN (4) is robustly stable to the set $\mathcal{M}$ under arbitrary switching signal, if and only if one of the following two conditions holds

(i) $[\Gamma]_{\varphi*,\gamma^*} = 0$,

(ii) $[\Gamma]_{\varphi*,\gamma^*} > 0$, $J_{\mathcal{M}}^T \mathrm{Col}_{\varphi^*}(\hat{A}^{2^n}) = 0$.

Proof: (Necessary) We prove the necessary by contradiction. Suppose that $[\Gamma]_{\varphi*,\gamma^*} > 0$ and $J_{\mathcal{M}}^T \mathrm{Col}_{\varphi^*}(\hat{A}^{2^n}) \neq 0$. By Assumption 1, it derives that $\delta_{2^n}^{\gamma^*}$ can reach set $I_s(\mathcal{M})$ under arbitrary switching signal sequence before OBFP, which together with $[\Gamma]_{\varphi*,\gamma^*} > 0$ shows that there must exist at least one path from $\delta_{2^n}^{\gamma^*}$ to $I_s(\mathcal{M})$ including $\delta_{2^n}^{\varphi^*}$ before OBFP. Without loss of generality, we suppose that $\delta_{2^n}^{\gamma^*}$ can be steered to $\delta_{2^n}^{\varphi^*}$ at the s_1th step and $\delta_{2^n}^{\gamma^*}$ can be steered to $I_s(\mathcal{M})$ at the sth step, where $s_1 < s$. Then, the path from $\delta_{2^n}^{\gamma^*}$ to $I_s(\mathcal{M})$ can be described as

$$\delta_{2^n}^{\gamma^*} \xrightarrow{\sigma(0)} \cdots \xrightarrow{\sigma(s_1-1)} \delta_{2^n}^{\varphi^*} \rightarrow \cdots \xrightarrow{\sigma(s-1)} I_s(\mathcal{M}), \tag{10}$$

where the corresponding switching signal sequence is $\{\delta_{\omega}^{k_0}, \cdots, \delta_{\omega}^{k_{s_1-1}}, \cdots, \delta_{\omega}^{k_{s-1}}\}$.

According to Eq. (9), one has $\delta_{2^n}^{\gamma^*} = \hat{L}\delta_{\omega}^{k^*}\delta_{2^n}^{\varphi^*}$, that is, $\delta_{2^n}^{\varphi^*}$ can reach $\delta_{2^n}^{\gamma^*}$ in one step under switching signal $\sigma = \delta_{\omega}^{k^*}$ after OBFP. By selecting switching signal sequence $\sigma_1 =: \{\delta_{\omega}^{k^*}, \delta_{\omega}^{k_0}, \cdots, \delta_{\omega}^{k_{s_1-1}}\}$, we can obtain the following path

$$\delta_{2^n}^{\varphi^*} \rightarrow \delta_{2^n}^{\gamma^*} \rightarrow \cdots \rightarrow \delta_{2^n}^{\varphi^*}. \tag{11}$$

There forms a new cycle (11) for SBN (4) after OBFP.

Since $J_{\mathcal{M}}^T \mathrm{Col}_{\varphi^*}(\hat{A}^{2^n}) \neq 0$, then there exists at least one state $x(2^n; \delta_{2^n}^{\varphi^*}, \sigma) = \delta_{2^n}^{\nu} \notin \mathcal{M}$ in the above cycle (11). Thereby state $\delta_{2^n}^{\varphi^*}$ cannot stay in set $\mathcal{M}$ forever under arbitrary switching signal sequence after OBFP, which is a contradiction to the fact that SBN (4) is robustly stable to set $\mathcal{M}$.

(Sufficiency) First, we suppose condition (i) holds. From Assumption 1, SBN (4) is $\mathcal{M}$-stable under arbitrary switching signal before OBFP. Hence, for any state $\delta_{2^n}^{\theta} \in \Delta_{2^n}$, the paths from $\delta_{2^n}^{\theta}$ to set $I_s(\mathcal{M})$ have the following two situations.

- Case 1: $[\Gamma]_{\varphi^*,\theta} = 0$, which implies that $\delta_{2^n}^{\theta}$ can reach set $I_s(\mathcal{M})$, and there exists no path from $\delta_{2^n}^{\theta}$ to set $I_s(\mathcal{M})$ containing $\delta_{2^n}^{\varphi^*}$, simultaneously.
- Case 2: $[\Gamma]_{\varphi^*,\theta} > 0$, which implies that $\delta_{2^n}^{\theta}$ can reach set $I_s(\mathcal{M})$, and there exists at least one path from $\delta_{2^n}^{\theta}$ to set $I_s(\mathcal{M})$ containing $\delta_{2^n}^{\varphi^*}$, simultaneously.

For Case 1, one path from $\delta_{2^n}^{\theta}$ to set $I_s(\mathcal{M})$ is arbitrarily selected and supposed as

$$\delta_{2^n}^{\theta} \rightarrow \cdots \rightarrow x(t) \rightarrow \cdots \rightarrow I_s(\mathcal{M}), \tag{12}$$

where $\sigma := \{\sigma(t) = \delta_{\omega}^{k_t}, t \in [0, \tau-1]\} \subseteq \Delta_{\omega}$, τ denotes the number of steps from $\delta_{2^n}^{\theta}$ to $I_s(\mathcal{M})$, and $\{x(1), \cdots, x(\tau-1)\}$ is a sequence of states in the path from $\delta_{2^n}^{\theta}$ to $I_s(\mathcal{M})$. Clearly, $x(t) \neq \delta_{2^n}^{\varphi^*}$, $t \in [1, \tau-1]$.

After OBFP, it follows from (8) that

$$\begin{aligned} x(\tau; \delta_{2^n}^{\theta}, \sigma) &= \hat{L}\sigma(\tau-1)x(\tau-1) \\ &= \hat{L}\sigma(\tau-1)\hat{L}\sigma(\tau-2)x(\tau-2) \\ &= \cdots \\ &= \ltimes_{t=\tau-1}^{0}(\hat{L}\sigma(t))\delta_{2^n}^{\theta} \\ &= L\sigma(\tau-1)x(\tau-1) \\ &= L\sigma(\tau-1)L\sigma(\tau-2)x(\tau-2) \\ &= \cdots \\ &= \ltimes_{t=\tau-1}^{0}(L\sigma(t))\delta_{2^n}^{\theta} \\ &\in I_s(\mathcal{M}). \end{aligned}$$

Thus, OBFP has no effect on the path (12), which together with Assumption 1 means that $I_s(\mathcal{M})$ is reachable from every state $\delta_{2^n}^{\theta} \in \Delta_{2^n}$ under arbitrary switching signal.

For Case 2, we select an arbitrary path from $\delta_{2^n}^{\theta}$ to $I_s(\mathcal{M})$ as

$$\begin{aligned} \delta_{2^n}^{\theta} \rightarrow \cdots \rightarrow x(t_1) \rightarrow \cdots \rightarrow \delta_{2^n}^{\varphi^*}\delta_{2^n}^{\eta} \rightarrow \\ \rightarrow \cdots \rightarrow x(t_2) \rightarrow \cdots \rightarrow I_s(\mathcal{M}), \end{aligned} \tag{13}$$

where the corresponding switching signal sequence is $\sigma := \{\sigma(t) = \delta_{\omega}^{k_t} : t \in [0, \tau_1+\tau_2-1]\} \subseteq \Delta_{\omega}$, $\tau_1+\tau_2$ denotes the number of time steps from $\delta_{2^n}^{\theta}$ to $I_s(\mathcal{M})$, and $\{x(t_1) : t_1 \in [1, \tau_1-1]\}$ is the states in the path from $\delta_{2^n}^{\theta}$ to $\delta_{2^n}^{\eta}$, $\{x(t_2) : t_2 \in [\tau_1+1, \tau_1+\tau_2-1]\}$ represents the states in the path from $\delta_{2^n}^{\eta}$ to $I_s(\mathcal{M})$. We can easily obtain that $x(\tau_1-1) = \delta_{2^n}^{\varphi^*}$, $x(\tau_1) = \delta_{2^n}^{\eta}$ and $x(t) \neq \delta_{2^n}^{\varphi^*}$, $t \in \{1, \ldots, \tau_1+\tau_2-1\} \backslash \{\tau_1-1\}$.

Two situations of the relationship between the state $\delta_{2^n}^{\eta}$ and perturbed state $\delta_{2^n}^{\gamma^*}$ may appear in path (13): (i) $\delta_{2^n}^{\eta} \neq \delta_{2^n}^{\gamma^*}$; (ii) $\delta_{2^n}^{\eta} = \delta_{2^n}^{\gamma^*}$. For $\delta_{2^n}^{\eta} \neq \delta_{2^n}^{\gamma^*}$, similar to the analysis of path (12), OBFP has no effect on the path (13), and every state $\delta_{2^n}^{\theta} \in \Delta_{2^n}$ can still reach to $I_s(\mathcal{M})$ under arbitrary switching signal sequence after OBFP. For $\delta_{2^n}^{\eta} = \delta_{2^n}^{\gamma^*}$, after OBFP occurs, it follows that

$$\begin{cases} x(\tau_1-1; \delta_{2^n}^{\theta}, \sigma) = \ltimes_{t=\tau_1-2}^{0}(\hat{L}\delta_{\omega}^{k_t})\delta_{2^n}^{\theta} \\ = \ltimes_{t=\tau_1-2}^{0}(L\delta_{\omega}^{k_t})\delta_{2^n}^{\theta} = \delta_{2^n}^{\varphi^*}, \\ x(1; \delta_{2^n}^{\varphi^*}, \sigma) = \hat{L}\delta_{\omega}^{k^*}\delta_{2^n}^{\varphi^*} = \delta_{2^n}^{\gamma^*}. \end{cases} \tag{14}$$

Considering $[\Gamma]_{\varphi*,\gamma^*} = 0$, one has

$$\begin{aligned} x(\tau_3-1; \delta_{2^n}^{\gamma^*}, \sigma) &= \ltimes_{t=\tau_3-1}^{0}(\hat{L}\delta_{\omega}^{j_t})\delta_{2^n}^{\gamma^*} \\ &= \ltimes_{t=\tau_3-1}^{0}(L\delta_{\omega}^{j_t})\delta_{2^n}^{\gamma^*} \\ &\in I_s(\mathcal{M}), \end{aligned} \tag{15}$$

where $\sigma := \{\sigma(t) = \delta_{\omega}^{j_t} : t \in [0, \tau_3-1]\} \subseteq \Delta_{\omega}$ and τ_3 denotes the number of time steps from $\delta_{2^n}^{\gamma^*}$ to $I_s(\mathcal{M})$.

Combining (14) with (15), one has

$$\begin{aligned} & x(\tau_1+\tau_3;\delta_{2^n}^{\theta},\sigma) \\ & = \ltimes_{t=\tau_3-1}^{0}(\hat{L}\delta_{\omega}^{j_t})\hat{L}\delta_{\omega}^{k^*}\ltimes_{t=\tau_1-2}^{0}(\hat{L}\delta_{\omega}^{i_t})\delta_{2^n}^{\theta} \\ & = \ltimes_{t=\tau_1+\tau_3-1}^{0}(\hat{L}\sigma(t))\delta_{2^n}^{\theta} \\ & \in I_s(\mathcal{M}), \end{aligned}$$

which shows that path (13) changes to

$$\begin{aligned} & \delta_{2^n}^{\theta} \to \cdots \to x(t_1) \to \cdots \\ & \quad \to \delta_{2^n}^{\varphi^*} \not\to \delta_{2^n}^{\gamma} \to \cdots \to x(t_2) \to \cdots \to I_s(\mathcal{M}) \\ & \quad \downarrow \\ & \quad \delta_{2^n}^{\gamma^*} \to \cdots \to x(t_3) \to \cdots \to I_s(\mathcal{M}), \end{aligned} \tag{16}$$

where the corresponding switching signal sequence is $\sigma := \{\sigma(t)=\delta_{\omega}^{i_t}: t\in[0,\tau_1-2]\}\cup\{\sigma(t)=\delta_{\omega}^{k^*}: t=\tau_1-1\}\cup\{\sigma(t)=\delta_{\omega}^{j_{t-\tau_1}}: t\in[\tau_1,\tau_1+\tau_3-1]\}$, $\tau_1+\tau_3$ denotes the number of time steps from $\delta_{2^n}^{\theta}$ to set $I_s(\mathcal{M})$, and $\{x(t_3): t_3\in[\tau_1+1,\tau_1+\tau_3-1]\}$ is a sequence of states in the path from $\delta_{2^n}^{\gamma^*}$ to set $I_s(\mathcal{M})$. We obtain that $\delta_{2^n}^{\theta}$ can reach to $I_s(\mathcal{M})$ if $\delta_{2^n}^{\eta}=\delta_{2^n}^{\gamma^*}$. On the basis of above analysis, for case 2, it holds that set $I_s(\mathcal{M})$ is still reachable from every state $\delta_{2^n}^{\theta}\in\Delta_{2^n}$ under arbitrary switching signal sequence after OBFP occurs.

To sum up, we have proved that SBN (4) is robustly stable to the set $I_s(\mathcal{M})$ when condition (i) holds.

Next, we suppose that condition (ii) holds. For any state $\delta_{2^n}^{\theta}\in\Delta_{2^n}$, we just discuss the situation: $[\Gamma]_{\varphi^*,\theta}>0$ and $\delta_{2^n}^{\eta}=\delta_{2^n}^{\gamma^*}$. The analysis of other situations is similar to the proof in condition (i). If $[\Gamma]_{\varphi^*,\theta}>0$ and $\delta_{2^n}^{\eta}=\delta_{2^n}^{\gamma^*}$, without loss of generality, the path from $\delta_{2^n}^{\theta}$ to $\delta_{2^n}^{\gamma^*}$ can be described as

$$\delta_{2^n}^{\theta} \to \cdots \to x(t_1) \to \cdots \delta_{2^n}^{\varphi^*} \to \delta_{2^n}^{\gamma^*}, \tag{17}$$

where the corresponding switching signal sequence is $\sigma := \{\sigma(t)=\delta_{\omega}^{i_t}: t\in[0,\tau_1-2]\}\cup\{\sigma(t)=\delta_{\omega}^{k^*}: t=\tau_1-1\}$, τ_1 denotes the number of time steps from $\delta_{2^n}^{\theta}$ to $\delta_{2^n}^{\gamma^*}$. This together with $[\Gamma]_{\varphi*,\gamma^*}>0$ shows that a new cycle as (11) is formed for SBN (4) after OBFP. Denote the cycle (11) by $\Omega=\{\delta_{2^n}^{\varphi^*},\delta_{2^n}^{\gamma^*},\delta_{2^n}^{\gamma_1^*}\cdots\delta_{2^n}^{\gamma_l^*}\}$.

Since $J_{\mathcal{M}}^T\mathrm{Col}_{\varphi^*}(\hat{A}^{2^n})=0$, one has $\Omega\subseteq\mathcal{M}$, which means every state $\delta_{2^n}^{\theta}\in\Delta_{2^n}$ can reach to set $\mathcal{M}$ under arbitrary switching signal after OBFP occurs. By Definition 4, SBN (4) is robustly stable to set $\mathcal{M}$ under arbitrary switching signal. ■

On the basis of Theorem 1, the following corollary can be derived immediately, so the proof is omitted.

Corollary 1: Under Assumption 1, when OBFP in Assumption 2 is Case 2, SBN (4) is robustly stable to the set $\mathcal{M}$ under arbitrary switching signal, if and only if one of the following two conditions holds

(i) $[\Gamma]_{\varphi*,\gamma^*}=0$,

(ii) $[\Gamma]_{\varphi*,\gamma^*}>0$, $J_{\mathcal{M}}^T\mathrm{Col}_{\varphi^*}(\hat{A}^{2^n})=0$.

Below we discuss the case 3 of Assumption 2. The following theorem can be drawn.

Theorem 2: Under Assumption 1, when OBFP in Assumption 2 is Case 3, SBN (4) is robustly stable to the set $\mathcal{M}$ under arbitrary switching signal.

Proof: There are two cases between the affected state $\delta_{2^n}^{\varphi*}$ and perturbed state $\delta_{2^n}^{\gamma^*}$, that is $[\Gamma]_{\varphi*,\gamma^*}=0$ and $[\Gamma]_{\varphi*,\gamma^*}>0$. Next, we prove that the above two situations have no impact on the set stability of SBN (4) after OBFP occurs.

From the proof of the sufficiency of Theorem 1, we can obtain that if $[\Gamma]_{\varphi*,\gamma^*}=0$, the path from $\delta_{2^n}^{\theta}$ to set $I_s(\mathcal{M})$ can be described as (12). Moreover, OBFP as Assumption 2 does not affect the path (12). By Assumption 1, we know that every $\delta_{2^n}^{\theta}\in\Delta_{2^n}$ can still evolve into set $\mathcal{M}$ under arbitrary switching signal sequence.

If $[\Gamma]_{\varphi*,\gamma^*}>0$, the path from $\delta_{2^n}^{\theta}$ to set $I_s(\mathcal{M})$ can be described as (13). When $\sigma(\tau_1-1)\neq\delta_{\omega}^{k^*}$, the path (13) is not affected by OBFP. When $\sigma(\tau_1-1)=\delta_{\omega}^{k^*}$, after OBFP occurs, the path (13) changes to be

$$\delta_{2^n}^{\theta} \to \cdots \to x(t_1) \to \cdots \delta_{2^n}^{\varphi^*} \to \delta_{2^n}^{\gamma^*}. \tag{18}$$

It follows from Case 3 that $\delta_{2^n}^{\gamma^*}\in I_s(\mathcal{M})$. Based on the property of the largest invariant set, we know that $\delta_{2^n}^{\theta}$ can evolve into set $I_s(\mathcal{M})$ and stay in set $I_s(\mathcal{M})$ forever under arbitrary switching signal after OBFP occurs.

According to the above discussion, one has that SBN (4) with arbitrary switching signal is robustly stable to the set $\mathcal{M}$ under Assumption 1 and Case 3 of Assumption 2. ■

Similarly, from Theorem 2, we can derive the following corollary.

Corollary 2: Under Assumption 1 and Case 4 of Assumption 2, SBN (4) is robustly stable to the set $\mathcal{M}$ under arbitrary switching signal.

V. ILLUSTRATIVE EXAMPLE

We use the following example to verify the effectiveness of the results obtained.

Example 1: The following SBN originates from a biological example: a reduced *E. coli* lactose operon network [18]. The five genes, termed *lac* mRNA, the high-concentration lactose, medium-concentration lactose, the high exolactose, and the medium exolactose are denoted by state X_1, state X_2, state X_3, input variable U_1 and input variable U_2, respectively. If we suppose that the values of input variables U_1 and U_2 are consistent with the values of state X_3, then this network model can be described as follows.

$$\begin{cases} X_1(t+1)=f_1^{\sigma(t)}(X_1(t),X_2(t),X_3(t)), \\ X_2(t+1)=f_2^{\sigma(t)}(X_1(t),X_2(t),X_3(t)), \\ X_3(t+1)=f_3^{\sigma(t)}(X_1(t),X_2(t),X_3(t)), \end{cases} \tag{19}$$

where $f_1^1=f_1^2=X_2(t)\vee X_3(t)$, $f_2^1=f_2^2=X_3(t)\vee X_1(t)$, $f_3^1=X_3(t)\vee(X_3(t)\wedge X_1(t))$, $f_3^2=X_3(t)$.

Identify $\sigma(t)=k\sim\delta_2^k$, $k\in\{1,2\}$, we can convert (19) into the algebraic form as

$$x(t+1)=L\sigma(t)x(t), \tag{20}$$

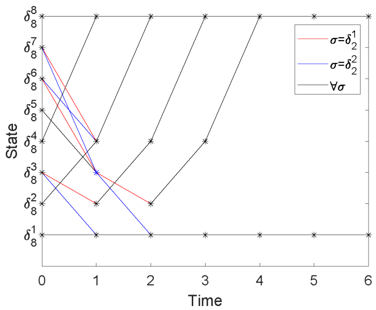

FIGURE 1. The state trajectory of SBN (20) before function perturbation .

where $x(t)$, $\sigma(t)$ are the vector forms of $(X_1(t), X_2(t), X_3(t))$, $\sigma(t)$ respectively. The state transition matrix of SBN (20) is $L = \delta_8[1, 4, 1, 8, 3, 4, 3, 8, 1, 4, 2, 8, 3, 3, 4, 8]$.

Given a set $\mathcal{M} = \{\delta_8^1, \delta_8^8\}$. A direct calculation shows that the largest invariant subset of set $\mathcal{M}$ for SBN (20) is $I_s(\mathcal{M}) = \{\delta_8^1, \delta_8^8\}$. The state trajectory graph of SBN (20) before OBFP is shown in Fig. 1.

Then we calculate

$$A = L \ltimes_{\mathcal{B}} \mathbf{1}_2 = \begin{bmatrix} 1&0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&1&1&0 \\ 0&1&0&0&0&1&1&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&1 \end{bmatrix},$$

$$\Gamma = (\mathcal{B})\sum_{i=1}^{8} A^i = \begin{bmatrix} 1&0&1&0&1&1&1&0 \\ 0&0&1&0&1&1&1&0 \\ 0&0&0&0&1&1&1&0 \\ 0&1&1&0&1&1&1&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&1&1&1&1&1&1&1 \end{bmatrix}.$$

(1) After OBFP, $\mathrm{Col}_{11}(L)$ undergoes perturbation, which is changed from δ_8^2 to δ_8^4. By Lemma 3, we obtain $k^* = 2$ and $\varphi^* = 3$, that is, the affected state and perturbed state are δ_8^3 and δ_8^4, respectively. Hence, SBN (20) becomes

$$x(t+1) = \hat{L}\sigma(t)x(t). \tag{21}$$

Here $\hat{L} = \delta_8[1, 4, 1, 8, 3, 4, 3, 8, 1, 4, 4, 8, 3, 3, 4, 8]$.

Since $\delta_8^3 \notin I_s(\mathcal{M})$, $\delta_8^4 \notin I_s(\mathcal{M})$ and $[\Gamma]_{3,4} = 0$, it follows from Theorem 1 that SBN (20) is robustly stable to set $\mathcal{M}$ after OBFP. The corresponding state trajectory graph of dynamics can be described by Fig. 2.

(2) After OBFP, $\mathrm{Col}_1(L)$ is changed from δ_8^1 to δ_8^5. By Lemma 3, we obtain $k^* = 1$ and $\varphi^* = 1$, that is, the affected state and perturbed state are δ_8^1 and δ_8^5, respectively. The state transition matrix of SBN (20) is changed to be

$$\hat{L} = \delta_8[5, 4, 1, 8, 3, 4, 3, 8, 1, 4, 2, 8, 3, 3, 4, 8].$$

Since $\delta_8^1 \in I_s(\mathcal{M})$, $\delta_8^5 \notin I_s(\mathcal{M})$, we know that OBFP is Case 2. A sequence of calculations yield that $[\Gamma]_{1,5} > 0$,

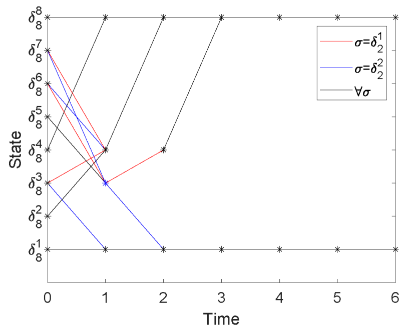

FIGURE 2. State trajectory graph of dynamics after function perturbation in Case 1 .

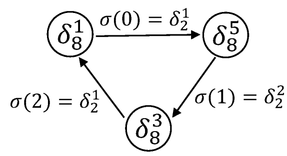

FIGURE 3. Dynamics of initial state $x(0) = \delta_8^1$ under switching sequence $\sigma := \{\sigma(0) = \delta_2^1, \sigma(1) = \delta_2^2, \sigma(2) = \delta_2^1\}$.

$J_{\mathcal{M}}^T \mathrm{Col}_1(\hat{A}^{2^n}) \neq 0$. Therefore, we can know from Corollary 1 that SBN (20) is not robustly stable to set $\mathcal{M}$ under arbitrary switching sequence after OBFP. For example, if we select switching sequence $\sigma = \{\sigma(0) = \delta_2^1, \sigma(1) = \delta_2^2, \sigma(2) = \delta_2^1\}$, the trajectory of SBN (20) with initial state $x(0) = \delta_8^1$ can be expressed as $x(1) = \delta_8^5$, $x(2) = \delta_8^3$, and $x(3) = \delta_8^1$. There forms a new cycle (see Fig. 3), which is not contained in set $\mathcal{M}$.

VI. CONCLUSION

We have investigated robust set stability about SBNs affected by OBFP. Based on the algebraic representation of an SBN, we have provided several necessary and sufficient conditions to detect whether an SBN with arbitrary switching signal is still stable to the given set after OBFP. Robust set stability can be applied to many other problems of logical networks, such as robust synchronization of SBNs and robust optimization of games etc., which remain for further study. Furthermore, gene mutations often occur in a stochastic manner in practical GRNs. Hence, future work can study stochastic function perturbations impact on the behavior of SBNs.

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