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RESEARCH ARTICLE

Improved Sufficient Condition for $\ell_1 - \ell_2$-Minimization on Cumulative Coherence

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ABSTRACT The cumulative coherence condition serves as a valuable tool within the realm of Compressed Sensing. However, its application to the $\ell_1 - \ell_2$-minimization model lacks thorough discussion. This paper aims to address this gap by introducing a sufficient condition for stable recovery of sparse vectors under the $\ell_1 - \ell_2$-minimization model, building upon the cumulative coherence condition. Moreover, employing graphical analysis, this study contrasts our proposed sufficient condition with existing criteria for stable recovery using both $\ell_1 - \ell_2$-minimization and ℓ_1-minimization models. Experimental data illustrate that our proposed sufficiency conditions exhibit less stringent requirements compared to established conclusions.

INDEX TERMS $\ell_1 - \ell_2$-minimization, cumulative coherence, sparse signal, compressed sensing.

I. INTRODUCTION

In the domain of signal processing and information recovery, compressed sensing has emerged as a groundbreaking paradigm challenging traditional sampling methods. This technique leverages the inherent sparsity or compressibility of signals to accurately reconstruct them from a significantly reduced number of measurements. Its potential spans diverse disciplines, with applications ranging from imaging to wireless communications, including sampling theory [1], [2], model recognition [3], [4], and sensor networks [5], [6]. For further insights into compressive sensing, please consult [7], [8].

Compressed sensing primarily aims to reconstruct an unknown high-dimensional sparse signal $x \in \mathbb{R}^n$ from lower-dimensional $y = Ax$ measurements, where $A \in \mathbb{R}^{m\times n}$ with $m \ll n$. The most intuitive approach to reconstruct x is by finding the sparsest signal within the feasible set of solutions, leading to an ℓ_0-minimization model:

$$\min_{x\in R^n} \|x\|_0 \qquad subject \quad to \qquad \|y - Ax\|_2 \leq \epsilon, \tag{1}$$

where $\epsilon = 0$ indicates a noiseless case, and $\epsilon \neq 0$ indicates a noisy case.

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The ℓ_0-minimization model is NP-hard, and thus computationally it is not feasible in high-dimensional sets [9]. To solve this problem, various methods have been proposed such as ℓ_1-minimization model [9], [10], [11], [12], [13], [14], ℓ_p-minimization model [15], [16], ℓ_{1-2}-minimization model [17], [18], [19], [20], [21], [22], weighted ℓ_1-minimization model [23] and other methods [24], [25], [26].

There are numerous results on the ℓ_1-minimization model in the literature. These results are mainly based on the null space property [27], coherence [21], cumulative coherence [14], Restricted Isometry Property(RIP) [9], [11], [12], [13], [28] and restricted orthogonality constants [10].

Although the ℓ_1-minimization model yields considerable results, it is not exactly equivalent to the ℓ_0-minimization model [29], [30]. Hence, the $\ell_1-\ell_2$-minimization model [17], [19], [31] and ℓ_p-minimization model [15], [16] have been proposed to replace the ℓ_1-minimization model under circumstances where it underperforms. The $\ell_1 - \ell_2$-minimization model is as follows:

$$\min_{x\in R^n} \|x\|_1 - \|x\|_2 \qquad subject\ to \qquad \|y - Ax\|_2 \leq \epsilon, \tag{2}$$

where $\|x\|_1 = \sum_{i=1}^n |x_i|$, $\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$.

The ℓ_1 norm, tends to produce sparse solutions during the optimization process. This is because the ℓ_1 norm is not differentiable at zero, making it easier for optimization algorithms to shrink some coefficients precisely to zero during iterations, resulting in sparse solutions. On the other hand, the ℓ_2 norm, also known as the Euclidean norm, plays a smoothing role in the optimization process, helping to stabilize solutions and reduce overfitting. However, the ℓ_2 norm itself does not tend to produce sparse solutions. When combining the ℓ_1 norm and the ℓ_2 norm, i.e., the $\ell_1 - \ell_2$ minimization model, it is possible to balance the sparsity and stability of the solution to some extent. Specifically, the ℓ_1 norm encourages sparsity in the solution, while the ℓ_2 norm prevents the solution from becoming too complex or overfitting the data.

In the literature [15], [16], like the ℓ_1-minimization model, the $\ell_1 - \ell_2$-minimization model is also solved based on the null space property [17], coherence [21], restricted orthogonality constants [18], and restricted isometry property [18], [19].

The commonly used method to solve the $\ell_1 - \ell_2$-minimization model is to use the difference of convex function algorithm(DCA), which currently only ensures that the algorithm's sequence converges to the stable point of the objective function. Reference [19] also indicated that when appropriate parameter λ is selected, the clustering points generated by the algorithm are sparse vectors.

Our study's main contribution lies in employing the cumulative coherence to solve the $\ell_1 - \ell_2$-minimization model, a method scarcely explored by previous studies. We propose a sufficient condition enabling stable recovery of vectors under this model. Through graphical analysis, we demonstrate that our condition is weaker than that presented in [32, Theorem 1], representing the most extensive upper bound known to us.

We introduce related concepts in Section II, present our main results in Section III and its proofs are present in appendix, compare our findings with existing conclusions in Section IV, and conclude in Section V.

Notations: For $x \in R^n$, $supp(x) = \{i : x_i \neq 0\}$ and $\|x\|_0$ indicates the number of non-zero elements in x. $\|x\|_\infty = \max_{i\in[n]} |x_i|$ where $[n] = \{1, 2, 3, \cdots, n\}$. $s \in N^+$ and $x_{max(s)}$ is defined as the vector x with all but the largest s entries in absolute value set to zero, and $x_{-max(s)} = x - x_{max(s)}$. For $y \in R^n$, $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$. $T \subset [n]$, x_T is defined as the vector $(x_T)_i = x_i$, if $i \in T$ and otherwise $(x_T)_i = 0$.

II. PRELIMINARY

In this section, we will make some necessary preparations. Firstly we introduce the more general concept of ℓ_1-coherence function, which incorporates the usual coherence as the particular value $s = 1$ of its argument.

Definition 1 ([9]): Let $A \in R^{m\times n}$ be a matrix with ℓ_2-normalized columns $A_1, \ldots, A_n$ (that is, $\|A_i\|_2 = 1$ for all $i = 1, \ldots, n$). The cumulative coherence function $\mu_1(s) = \mu_1(A, s)$ of matrix A is defined for $s \in [n-1]$ by

$$\mu_1(s)=\max_{i\in[n]} \max\{\sum_{j\in S} |\langle A_i, A_j\rangle|, S \subset [n], card(S) = s, i \notin S\} \tag{3}$$

When the cumulative coherence of a matrix grows slowly, we can informally say that the dictionary is quasi-incoherent. The following lemmas are needed in the proof of our main results and we list them below. Lemma 1 provides the main properties of the cumulative coherence function, which has a form similar to the RIP property.

Lemma 1 ([9]): Let $A \in R^{m\times n}$ be a matrix with ℓ_2-normalized columns and $s \in [n]$. For all s-sparse vectors $x \in R^n$,

$$(1 - \mu_1(s-1))\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \mu_1(s-1))\|x\|_2^2. \tag{4}$$

Lemma 2 provides another property of the cumulative coherence function, which applies to two vectors where their support sets do not intersect.

Lemma 2 ([14]): Suppose that x is s-sparse and y is t-sparse; then,

$$|\langle Ax, Ay\rangle - \langle x, y\rangle| \leq \mu_1(s+t-1)\|x\|_2\|y\|_2. \tag{5}$$

Moreover, if $supp(x) \cap supp(y) = \emptyset$, then

$$|\langle Ax, Ay\rangle| \leq \mu_1(s+t-1)\|x\|_2\|y\|_2. \tag{6}$$

Lemma 3 provides the relationship between the ℓ_1-norm and the ℓ_2-norm, as well as the non zero maximum and minimum values of vector.

Lemma 3 ([11]): For any $x \in R^n$

$$\|x\|_2 - \frac{\|x\|_1}{\sqrt{n}} \leq \frac{\sqrt{n}}{4}(\max_{1\leq i\leq n} |x_i| - \min_{1\leq i\leq n} |x_i|). \tag{7}$$

III. MAIN RESULT

In this section, we propose a cumulative coherence condition, which can ensure that the $\ell_1 - \ell_2$-minimization model can stably restore vectors.

Theorem 1: Let s, s_1, s_2 be positive integers and let $s_1 \geq s$. if

$$\mu_1(s_1-1) + \frac{\eta}{8\sqrt{2}s_1s_2 - 4s_2\sqrt{s_1}}\mu_1(s_1+s_2-1) < 1, \tag{8}$$

where $\eta = 2\sqrt{2s_1s_2}(8s + s_2 - 4s_1) - \sqrt{s_2}(s_2 + 8s - 20s_1)$, then the solution $\overline{x}$ of minimization (2) and the original signal x satisfy

$$\|\overline{x} - x\|_2 \leq \frac{8s_1s_2\sqrt{1+\mu_1(s_1-1)}\epsilon}{\alpha(1-\mu_1(s_1-1)) - \beta\mu_1(s_1+s_2-1)} + \frac{2s_2\sqrt{s_1}(1-\mu_1(s_1-1))+\gamma\mu_1(s_1+s_2-1)}{\alpha(1-\mu_1(s_1-1))-\beta\mu_1(s_1+s_2-1)}\|x_{-max(s_1)}\|_1, \tag{9}$$

where $\alpha = 2\sqrt{2}s_1s_2 - s_2\sqrt{s_1}$, $\beta = 2\sqrt{2s_1s_2}(\frac{1}{4}s_2 - s_1 + 2s) - \sqrt{s_2}(\frac{1}{4}s_2 - 5s_1 + 2s)$, $\gamma = 8s_1\sqrt{s_2} + s_1 - \frac{1}{4}s_2 - 2s$.

According to Theorem 1, we can easily obtain the following theorem.

Theorem 2: Let s, s_1, s_2 be positive integers and let $s_1 \geq s$. if

$$\mu_1(s_1 - 1) + \frac{\eta}{8\sqrt{2}s_1 s_2 - 4s_2\sqrt{s_1}}\mu_1(s_1 + s_2 - 1) < 1, \tag{10}$$

where $\eta = 2\sqrt{2s_1 s_2}(8s + s_2 - 4s_1) - \sqrt{s_2}(s_2 + 8s - 20s_1)$, then (2) with $\epsilon = 0$ can accurately recover the s_1-sparse vector.

IV. COMPARISON OF BOUNDARY

In this section, we will indirectly compare condition (8) with condition (18) in [32, Theorem 1] which is the weakest condition we can find at present. First, as $\mu_1(s) \leq s\mu$, we transform cumulative coherence scaling into coherence. We know that

$$\mu < \frac{1}{s_1 - 1 + c(s_1 + s_2 - 1)}, \tag{11}$$

where

$$c = \frac{(2\sqrt{2s_1 s_2}(8s+s_2-4s_1)-\sqrt{s_2}(s_2+8s-20s_1))(s_1+s_2-1)}{8\sqrt{2}s_1 s_2 - 4s_2\sqrt{s_1}}$$

guarantees the condition (8) in Theorem 1. We should note that (11) is only a sufficient condition for (8), not a necessary condition. Similarly,

$$\mu < \frac{\sqrt{s} - 1}{(2\sqrt{s} + \sqrt{2} - 2)s - \sqrt{s} + 1} \tag{12}$$

guarantees condition (6) in [18, Theorem 1]. In addition,

$$\mu < \frac{1 - t}{3s - 1 + (2s - 1)t} \tag{13}$$

and

$$\mu < \frac{1}{s + a - 1 + \frac{(\sqrt{s}+1)(s+a+b-1)}{\sqrt{b}-1}}, \tag{14}$$

guarantee the conditions in Theorem 1 and Theorem 2 respectively in [31], where $t = (\frac{\sqrt{s}+1}{\sqrt{2s}-1})^2$.

$$\mu < \frac{4s - 1 - \sqrt{8s + 1}}{8s^2 - 8s}, \tag{15}$$

guarantees condition (18) in [32, Theorem 1]. Besides

$$\mu < \frac{1}{2s - 1}, \tag{16}$$

ensures that the ℓ_1-minimization model can recover sparse vectors. Next, we draw the relationship between the right side of inequalities (11) to (16) according to sparsity. We use $T1$ to $T6$ to represent the right-hand-side values of inequalities (11) to (16), respectively. In the data experiment, the parameters are set to $s_1 = 3s$, $s_2 = s$, $a = 2$, $b = 6$. We draw a data graph when sparsity s ranged from 100 to 2000. From Figures 1 to 4, it can be seen that when s increases, $T1$, to $T6$ gradually decrease; but $T1$ is always larger than the others, include $T5$ which is the largest upper bound that we can find at present.

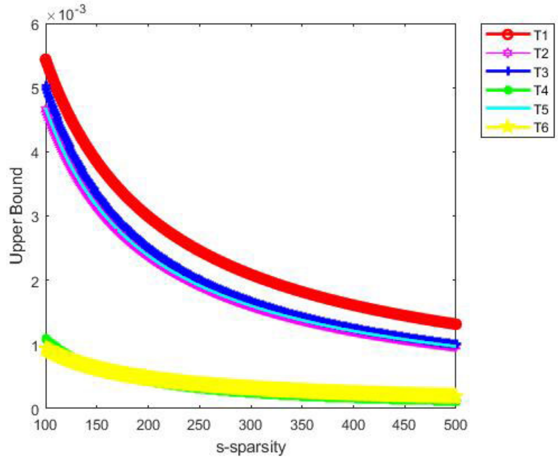

FIGURE 1. s from 100 to 500.

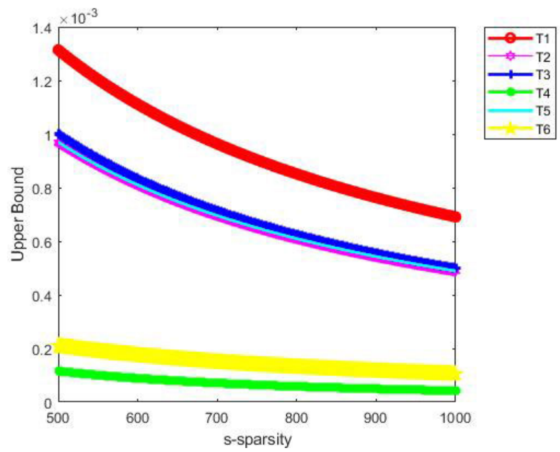

FIGURE 2. s from 500 to 1000.

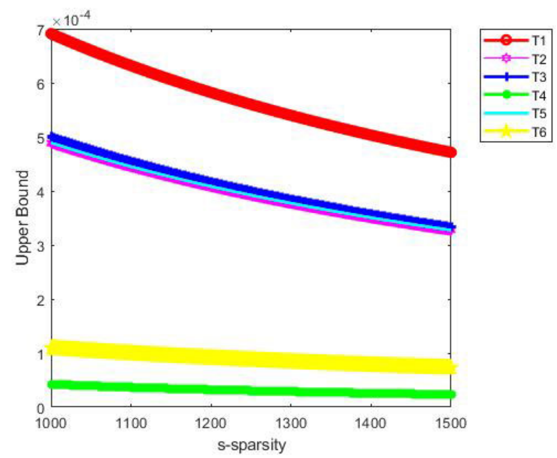

FIGURE 3. s from 1000 to 1500.

V. NUMERICAL EXPERIMENT

In this section, we conduct data experiments to verify that model (2) can stably recover sparse vectors. The algorithm we use can be found in reference [19].

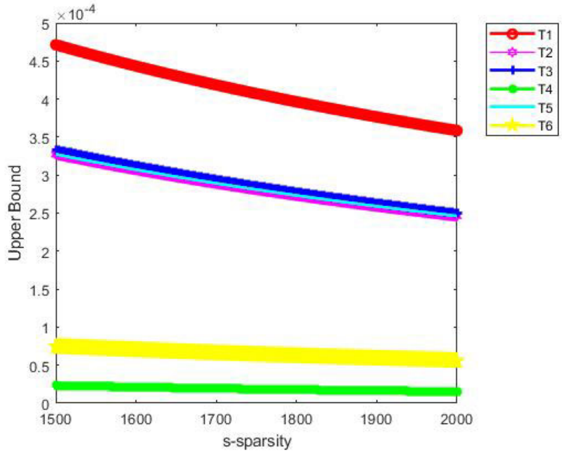

FIGURE 4. s from 1500 to 2000.

We conducted two sets of experiments on two different sizes of measurement matrices A. Each group of experiments conducted 100 experiments on different sparsity levels. Among these 100 experiments, the ratio of the number of successful restoration of sparse data to 100 was used as the success rate of experimental restoration.

TABLE 1. In 100 experiments, the frequency of model (2) stably recovered sparse data with measurement matrix $A \in R^{80\times 100}$ and different sparse s.

$s = 25$	$s = 30$	$s = 35$	$s = 40$	$s = 45$	$s = 50$
100%	100%	99%	95%	80%	42%

In the first set of experiments, we used a Gaussian matrix $A \in R^{80\times 100}$ and sparsity values as $s = 25, 30, 35, 40, 45$ and 50. Its experimental results are shown in Table 1. From Table 1, it can be seen that when the sparsity is 25, 30, model (2) can fully recover sparse data. When the sparsity is 35 and 40, the accuracy of model (2) in recovering sparse data is also very high. However, when the sparsity is 50, the ability of model (2) to recover sparse data is not ideal.

TABLE 2. In 100 experiments, the frequency of model (2) stably recovered sparse data with measurement matrix $A \in R^{60\times 100}$ and different sparse s.

$s = 10$	$s = 15$	$s = 20$	$s = 25$	$s = 30$	$s = 35$
100%	100%	100%	89%	47%	6%

In the second set of experiments, we used a Gaussian matrix $A \in R^{60\times 100}$ and sparsity values as $s = 10, 15, 20, 25, 30$ and 35. Its experimental results are shown in Table 2. From Table 2, it can be seen that when the sparsity is 10, 15, 20, model (2) can fully recover sparse data. When the sparsity is 25, the accuracy of model (2) in recovering sparse data is also very high. However, when the sparsity is 30 and 35, the ability of model (2) to recover sparse data is not ideal.

From these two sets of experiments, it can be inferred that model (2) can stably recover sparse data. The ability of model (2) to recover sparse data is related to the size of the measurement matrix and the sparsity of the data. The more measurement value there are, the smaller the sparsity, and the stronger the ability of model (2) to recover sparse data.

VI. CONCLUSION

Building upon the concept of cumulative coherence, this paper proposes a sufficient condition for the $\ell_1 - \ell_2$-minimization model to stably recover vectors. We illustrate through graphical representation that the upper bound of this sufficient condition is weaker than the condition presented in [32, Theorem 1], which currently stands as the weakest condition available.

APPENDIX

Here we provide a detailed proof of Theorem 1.

Proof: Set $h = \overline{x} - x = (h_1, h_2, \ldots .h_n)$. Let S_0 be the set of indices of the s_1 largest absolute value components of h, S_1 be the set of indices of the s_2 largest absolute value components of $h_{S_0^C}$, S_2 be the set of indices of the next s_2 largest absolute-value components of $h_{S_0^C}$, and so on. We assume that $[n]$ is divided into $S_0, S_1, \cdots, S_l$. From [18], we know

$$\|h_{-max(s_1)}\|_1 \leq \|h_{max(s_1)}\|_1 + 2\|x_{-max(s_1)}\|_1 + \|h\|_2. \quad (17)$$

Hence,

$$\begin{aligned}
&\|h_{-max(s_1)}\|_2^2 \\
&\leq \|h_{-max(s_1)}\|_1 \frac{\|h_{max(s_1)}\|_1}{s_1} \\
&\leq \frac{\|h_{max(s_1)}\|_1}{s_1}(\|h_{max(s_1)}\|_1 + 2\|x_{-max(s_1)}\|_1 + \|h\|_2) \\
&\leq \|h_{max(s_1)}\|_2^2 + \frac{\|h_{max(s_1)}\|_2}{\sqrt{s_1}}(2\|x_{-max(s_1)}\|_1 + \|h\|_2) \quad (18)
\end{aligned}$$

For $i \geq 1$, according to (3),

$$\|h_{S_i}\|_2 \leq \frac{\|h_{S_i}\|_1}{\sqrt{s_2}} + \frac{\sqrt{s_2}}{4}(\max_{j\in S_i}|h_j| - \min_{j\in S_i}|h_j|)$$

Therefor, we have

$$\begin{aligned}
&\sum_{i\geq 1}\|h_{S_i}\|_2 \\
&\leq \frac{1}{\sqrt{s_2}}\sum_{i\geq 1}\|h_{S_i}\|_1 + \sum_{i\geq 1}\frac{\sqrt{s_2}}{4}(\max_{j\in S_i}|h_j| - \min_{j\in S_i}|h_j|) \\
&= \frac{1}{\sqrt{s_2}}\sum_{i\geq 1}\|h_{S_i}\|_1 + \frac{\sqrt{s_2}}{4}\max_{j\in S_1}|h_j| \\
&\quad - \frac{\sqrt{s_2}}{4}\min_{j\in S_l}|h_j| + \sum_{i\geq 1}^{l-1}\frac{\sqrt{s_2}}{4}(\max_{j\in S_{i+1}}|h_j| - \min_{j\in S_i}|h_j|) \\
&\leq \frac{1}{\sqrt{s_2}}(\|h_{-max(s)}\|_1 - (s_1 - s)\max_{j\in S_1}|h_j|) + \frac{\sqrt{s_2}}{4}\max_{j\in S_1}|h_j|
\end{aligned} \quad (19)$$

Replacing s_1 with s in formula (17), we can get

$$\|h_{-max(s)}\|_1 \leq \|h_{max(s)}\|_1 + 2\|x_{-max(s)}\|_1 + \|h\|_2.$$

Substituting above formula into (19), we obtain

$$\begin{aligned}
&\sum_{i\geq 1}\|h_{S_i}\|_2 \\
&\leq \frac{1}{\sqrt{s_2}}(\|h_{max(s)}\|_1 + 2\|x_{-max(s)}\|_1 \\
&\quad + \|h\|_2 - (s_1 - s)\max_{j\in S_1}|h_j|) + \frac{\sqrt{s_2}}{4}\max_{j\in S_1}|h_j| \\
&\leq \frac{1}{\sqrt{s_2}}(\|h_{max(s)}\|_1 + 2\|x_{-max(s_1)}\|_1 + \|h\|_2) \\
&\quad + (\frac{\sqrt{s_2}}{4} - \frac{s_1 - s}{\sqrt{s_2}})\max_{j\in S_1}|h_j| \\
&\leq \frac{1}{\sqrt{s_2}}(\|h_{max(s_1)}\|_1 + 2\|x_{-max(s_1)}\|_1 + \|h\|_2) \\
&\quad + (\frac{\sqrt{s_2}}{4} - \frac{2(s_1 - s)}{\sqrt{s_2}})\max_{j\in S_1}|h_j| \\
&\leq \frac{1}{\sqrt{s_2}}(\sqrt{s_1}\|h_{max(s_1)}\|_2 + 2\|x_{-max(s_1)}\|_1 + \|h\|_2) \\
&\quad + (\frac{\sqrt{s_2}}{4} - \frac{2(s_1 - s)}{\sqrt{s_2}})\frac{\|h_{max(s_1)}\|_2}{\sqrt{s_1}} \\
&= (\sqrt{\frac{s_1}{s_2}} + \frac{\sqrt{s_2}}{4\sqrt{s_1}} - \frac{2(s_1 - s)}{\sqrt{s_1 s_2}})\|h_{max(s_1)}\|_2 \\
&\quad + \frac{1}{\sqrt{s_2}}(2\|x_{-max(s_1)}\|_1 + \|h\|_2) \\
&= t\|h_{max(s_1)}\|_2 + \frac{1}{\sqrt{s_2}}(2\|x_{-max(s_1)}\|_1 + \|h\|_2). \qquad (20)
\end{aligned}$$

It follows from Lemma 1, Lemma 2 and above inequality that

$$\begin{aligned}
&\langle Ah, Ah_{max(s_1)}\rangle \\
&= \langle Ah_{max(s_1)}, Ah_{max(s_1)}\rangle + \langle A\sum_{i\geq 1}h_{S_i}, Ah_{max(s_1)}\rangle \\
&\geq (1 - \mu_1(s_1 - 1))\|h_{max(s_1)}\|_2^2 \\
&\quad + \sum_{i\geq 1}(\langle h_{S_i}, h_{max(s_1)}\rangle - \mu_1(s_1+s_2-1)\|h_{max(s_1)}\|_2\|h_{S_i}\|_2) \\
&= (1 - \mu_1(s_1 - 1))\|h_{max(s_1)}\|_2^2 \\
&\quad - \mu_1(s_1 + s_2 - 1)\|h_{max(s_1)}\|_2\sum_{i\geq 1}\|h_{S_i}\|_2 \\
&\geq (1-\mu_1(s_1-1))\|h_{max(s_1)}\|_2^2 - \mu_1(s_1+s_2-1)\|h_{max(s_1)}\|_2 \\
&\quad \cdot (t\|h_{max(s_1)}\|_2 + \frac{1}{\sqrt{s_2}}(2\|x_{-max(s_1)}\|_1 + \|h\|_2)) \\
&= (1 - \mu_1(s_1 - 1) - t\mu_1(s_1 + s_2 - 1))\|h_{max(s_1)}\|_2^2 \\
&\quad - \frac{1}{\sqrt{s_2}}\mu_1(s_1+s_2-1)\|h_{max(s_1)}\|_2(2\|x_{-max(s_1)}\|_1 + \|h\|_2). \qquad (21)
\end{aligned}$$

From Cauchy-Schwarz inequality and Lemma 1 again, we get

$$\langle Ah, Ah_{max(s_1)}\rangle \leq 2\epsilon\sqrt{1 + \mu_1(s_1 - 1)}\|h_{max(s_1)}\|_2.$$

Combining the above two inequalities and applying condition (8), it holds that

$$\|h_{max(s_1)}\|_2 \leq \frac{2\sqrt{1 + \mu_1(s_1 - 1)}\epsilon}{1 - \mu_1(s_1 - 1) - t\mu_1(s_1 + s_2 - 1)} + \frac{\mu_1(s_1 + s_2 - 1)(2\|x_{-max(s_1)}\|_1 + \|h\|_2)}{\sqrt{s_2}(1 - \mu_1(s_1 - 1) - t\mu_1(s_1 + s_2 - 1))}.$$

Inequality (18) and the above inequality give

$$\begin{aligned}
\|h\|_2^2 &= \|h_{max(s_1)}\|_2^2 + \|h_{-max(s_1)}\|_2^2 \\
&\leq 2\|h_{max(s_1)}\|_2^2 + \frac{\|h_{max(s_1)}\|_2}{\sqrt{s_1}}(2\|x_{-max(s_1)}\|_1 + \|h\|_2) \\
&\leq (\sqrt{2}\|h_{max(s_1)}\|_2 + \frac{1}{2\sqrt{2s_1}}(2\|x_{-max(s_1)}\|_1 + \|h\|_2))^2 \\
&\leq (\frac{2\sqrt{2(1 + \mu_1(s_1 - 1))}\epsilon}{1 - \mu_1(s_1 - 1) - t\mu_1(s_1 + s_2 - 1)} \\
&\quad + (\frac{\sqrt{2}\mu_1(s_1 + s_2 - 1)}{\sqrt{s_2}(1-\mu_1(s_1-1)-t\mu_1(s_1+s_2-1))} + \frac{1}{2\sqrt{2s_1}}) \\
&\quad \cdot (2\|x_{-max(s_1)}\|_1 + \|h\|_2))^2. \qquad (22)
\end{aligned}$$

Hence,

$$\begin{aligned}
&(1 - \frac{\sqrt{2}\mu_1(s_1+s_2 - 1)}{\sqrt{s_2}(1-\mu_1(s_1-1) - t\mu_1(s_1+s_2 - 1))} - \frac{1}{2\sqrt{2s_1}})\|h\|_2 \\
&\leq \frac{2\sqrt{2(1 + \mu_1(s_1 - 1))}\epsilon}{1 - \mu_1(s_1 - 1) - t\mu_1(s_1 + s_2 - 1)} \\
&\quad + (\frac{\sqrt{2}\mu_1(s_1 + s_2 - 1)}{\sqrt{s_2}(1 - \mu_1(s_1 - 1) - t\mu_1(s_1 + s_2 - 1))} + \frac{1}{2\sqrt{2s_1}}) \\
&\quad \cdot (2\|x_{-max(s_1)}\|_1). \qquad (23)
\end{aligned}$$

From condition (8), we have

$$\begin{aligned}
\|h\|_2 &\leq \frac{8\sqrt{s_1 s_2(1 + \mu_1(s_1 - 1))}\epsilon}{a - b} \\
&\quad \frac{2\sqrt{s_2}(1 - \mu_1(s_1 - 1)) + e}{c - d}\|x_{-max(s_1)}\|_1 \\
&= \frac{8s_1 s_2\sqrt{1 + \mu_1(s_1 - 1)}\epsilon}{\alpha(1 - \mu_1(s_1 - 1)) - \beta\mu_1(s_1 + s_2 - 1)} \\
&\quad + \frac{2s_2\sqrt{s_1}(1-\mu_1(s_1-1))+\gamma\mu_1(s_1+s_2-1)}{\alpha(1-\mu_1(s_1-1)) - \beta\mu_1(s_1 + s_2 - 1)}\|x_{-max(s_1)}\|_1, \qquad (24)
\end{aligned}$$

where $a = (2\sqrt{2s_1 s_2} - \sqrt{s_2})(1-\mu_1(s_1-1))$, $b = ((2\sqrt{2s_1 s_2} - \sqrt{s_2})t + 4\sqrt{s_1})\mu_1(s_1 + s_2 - 1)$, $c = (2\sqrt{2s_1 s_2} - \sqrt{s_2})(1 - \mu_1(s_1 - 1))$, $d = ((2\sqrt{2s_1 s_2} - \sqrt{s_2})t + 4\sqrt{s_1})\mu_1(s_1 + s_2 - 1)$, $e = (8\sqrt{s_1} - 2\sqrt{s_2}t)\mu_1(s_1 + s_2 - 1)$ □

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