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RESEARCH ARTICLE

Tracking Differentiators for Both the Real Time Signals and the Time Delayed Signals

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ABSTRACT The differential tracking for a given signal is a well-known and challenging problem in control theory and practice. In this note, we introduce a novel method to design the differentiators. The proposed differentiator, in contrast to existing differentiators that take the form of a dynamical system, is represented through convolutions. This idea is primarily inspired by the mollifier technique, which is well-known in the theory of partial differential equations. Both the weighted moving average technique and the mollifier technique are used in the differentiator's design. By the proper choice of the kernel function, we can obtain the derivative of the given signal by integrating rather than differentiating the signal itself directly. As a result, the proposed tracking differentiators can be robust to the high-frequency signals. Although our approach is simple, it is very effective, both for the real time signals and the time delayed signals.

INDEX TERMS High-gain, mollifier function, tracking differentiator, weighted moving average.

I. INTRODUCTION

The differential tracking for a given signal is a well-known and challenging problem in control theory and practice. The differential tracking for a given signal is a well-known and challenging problem in control theory and practice. There are a lot of practical applications of the tracking differentiator in control problems. For instance, although Proportional Integral Derivative (PID) controller is the most popular approach in industrial applications, the derivative action "D" is seldom used in engineering applications due to its sensitivity to high frequency noise. One feasible method is to apply the action "D" by the virtue of the tracking differentiator [2].

There are various methods for differentiating a signal. For instance, in [3], a finite-time differentiator was proposed to estimate the inverse period and its derivative, along with the period and reactivity of the reactor. In [4], the problem of accurately differentiating a signal with a bounded second derivative was addressed. It introduced a class of fixed-time convergent differentiators capable of differentiating any signal with a Lipschitz continuous time derivative within a predefined finite time. To overcome the limitations of exact differentiators, [5] presented new implicit and semi-implicit discretization schemes aimed at mitigating digital chattering induced by incorrect time discretization of set-valued functions. Additionally, there are several other approaches to designing differentiators, such as the linear differentiator [6], the sliding-mode based differentiators [7] and the adaptive differentiators [8].

Actually, the numerous researches have been done on differentiation trackers like the super-twisting second-order sliding-mode algorithm [9], linear time-derivative tracker [10], robust exact differentiation [11], [12], nonlinear differentiator [13], [14], [15], the high-gain differentiator [16], [17], [18], and the recent linear differentiator based on the extended dynamics approach [19], to name just a few. However, all the tracking differentiators we cite above were designed for the given signals without time delay.

On the other hand, time delay is everywhere in real world systems. In [20], due to the inherent process structure and different positions of sampling instruments, time-delays

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commonly exist between process variables and quality variables, which may distort the original distribution and relationship in collected data. Therefore, the authors proposed a novel data-driven industrial quality predictor to overcome the problem caused by time-delay. And time-delay occurs in autonomous underwater vehicles [21], a six-phase induction motor [22], motion tracking of cell puncture mechanism [23] and so on. Therefore, it is very necessary to design a tracking differentiator for the signals with time delay.

In this technical note, we will propose the tracking differentiators for both the real time signals and the time delayed signals. Our idea, which is different from the tracking differentiator we cite above, is inspired by the weighted moving average technical and the mollifier technical. The weighted moving average has been extensively used by smoothing random fluctuations in statistics [24], and the mollifier technical has been extensively used by smooth approximation in PDE [25]. What we will do is bring the two technicals together to design the tracking differentiator. Although our approach is simple, it is very effective, both for the real time signals and for the time delayed signals.

Throughout this note, $v^{(i)}(t)$ represents the i-th order derivative of $v(t)$ at time t. Let $\mathbb{R}$ and $\mathbb{Z}$ be the set of real numbers and the set of integer numbers, respectively. We note

$$\mathbb{R}^+ := \{s \in \mathbb{R} \mid s \geq 0\}, \tag{1.1}$$

$$\mathbb{Z}_+ := \{s \in \mathbb{Z} \mid s > 0\} \tag{1.2}$$

and

$$\|f(t)\|_\infty := \sup_{t\in[0,\infty)} |f(t)|. \tag{1.3}$$

$\mathcal{S}_\lambda[f(t)]$ represents the *dilation scaling* of the function $f(t)$, which is defined by

$$\mathcal{S}_\lambda[f(t)] := \lambda f(\lambda t), \quad \forall \lambda > 0. \tag{1.4}$$

The convolution used in this paper is given by

$$f_1(t) * f_2(t) := \int_0^t f_1(s) f_2(t-s)ds. \tag{1.5}$$

The remainder of this article is structured as follows. In section II and section III, we will discuss the tracking differentiators for the real time signals and the time delayed signals, respectively. The proposed differentiator, in contrast to existing differentiators that take the form of a dynamical system, is represented through convolutions. Both the weighted moving average technique and the mollifier technique are used in the differentiator's design. In section IV, numerical simulations are presented to illustrate our theoretical results. Some concluding remarks are given in Section V.

II. TRACKING DIFFERENTIATOR FOR THE REAL TIME SIGNALS

Motivated by [25], our tracking differentiator considered in this note is based on a special weight function, which is given by

$$J(t) = \begin{cases} C_0 \exp\left[\dfrac{1}{t(t-1)}\right], & t \in (0,1), \\ 0, & \text{else}, \end{cases} \tag{2.1}$$

where C_0 is a positive constant such that

$$\int_0^\infty J(t)dt = 1. \tag{2.2}$$

It is easy to verify that $\text{supp} J(t) \subset (0,1)$ and

$$\frac{d^i}{dt^i} J(t)\Big|_{t=0} = \frac{d^i}{dt^i} J(t)\Big|_{t=1} = 0, \quad i = 1, 2, \cdots. \tag{2.3}$$

Before giving our main results, we first have a lemma, which is very important to the design of the tracking differentiator.

Lemma 2.1: For any $i \in \mathbb{Z}_+$, suppose that the input $v(t) \in C^i(\mathbb{R}^+)$ and $\|v^{(i)}(t)\|_\infty < \infty$. Then, for any $t \in [1/\lambda, \infty)$, it follows that

$$\left|\frac{d^{i-1}}{dt^{i-1}} \{\mathcal{S}_\lambda[J(t)]\} * v(t) - v^{(i-1)}(t)\right| \leq \frac{\|v^{(i)}(t)\|_\infty}{\lambda}. \tag{2.4}$$

Proof: See Appendix A.

A. THE DESIGN OF THE REAL TIME TRACKING DIFFERENTIATOR

Our tracking differentiator takes on the following form:

$$\begin{cases} z_{1\lambda}(t) = \mathcal{S}_\lambda[J(t)] * v(t), \\ z_{2\lambda}(t) = \dfrac{d}{dt} \{\mathcal{S}_\lambda[J(t)]\} * z_{1\lambda}(t), \\ z_{3\lambda}(t) = \dfrac{d}{dt} \{\mathcal{S}_\lambda[J(t)]\} * z_{2\lambda}(t), \\ \quad\vdots \\ z_{n\lambda}(t) = \dfrac{d}{dt} \{\mathcal{S}_\lambda[J(t)]\} * z_{(n-1)\lambda}(t), \end{cases} \tag{2.5}$$

where $v(t)$ is the input and $z_{i\lambda}(t)$ $(i = 1, 2, \cdots, n)$ is the output. The main idea of tracking differentiator (2.5) is that, the output $z_{i\lambda}(t)$ can be, through regulating λ, considered as the approximations of the corresponding $v^{(i-1)}(t)$.

Theorem 2.1: For any $i \in \mathbb{Z}_+$, suppose that $v(t) \in C^i(\mathbb{R}^+)$ and $\|v^{(i)}(t)\|_\infty < \infty$. Then, for any $t > i/\lambda$, we have

$$\left|z_{i\lambda}(t) - v^{(i-1)}(t)\right| \leq \frac{i\|v^{(i)}(t)\|_\infty}{\lambda}. \tag{2.6}$$

Hence, for any given $a > 0$, we have

$$z_{i\lambda}(t) \to v^{(i-1)}(t) \text{ as } \lambda \to +\infty \text{ uniformly in } [a, \infty). \tag{2.7}$$

Proof: Applying Lemma 2.1 we have

$$|z_{1\lambda}(t) - v(t)| \leq \frac{\|\dot{v}(t)\|_\infty}{\lambda}, \quad \forall t > 1/\lambda. \tag{2.8}$$

A straightforward computation shows that, for $t > i/\lambda$,

$$z_{i\lambda}(t) = \frac{d}{dt} \{\mathcal{S}_\lambda[J(t)]\} * z_{(i-1)\lambda}(t)$$

$$= \underbrace{\int_0^{1/\lambda}\cdots\int_0^{1/\lambda}}_{i}\prod_{j=1}^{i}\mathcal{S}_\lambda[J(s_j)]$$
$$v^{(i-1)}\left(t-\sum_{k=1}^{i}s_k\right)ds_1\cdots ds_i. \tag{2.9}$$

According to (2.2), we have

$$\underbrace{\int_0^{1/\lambda}\cdots\int_0^{1/\lambda}}_{i}\prod_{j=1}^{i}\mathcal{S}_\lambda[J(s_j)]ds_1\cdots ds_i = 1. \tag{2.10}$$

It then follows from Lagrange's mean value theorem that

$$\begin{aligned}
&\left|z_{i\lambda}(t)-v^{(i-1)}(t)\right| \\
&= \Bigg|\underbrace{\int_0^{1/\lambda}\cdots\int_0^{1/\lambda}}_{i}\prod_{j=1}^{i}\mathcal{S}_\lambda[J(s_j)] \\
&\quad \left[v^{(i-1)}\left(t-\sum_{k=1}^{i}s_k\right)-v^{(i-1)}(t)\right]ds_1\cdots ds_i\Bigg| \\
&\le \underbrace{\int_0^{1/\lambda}\cdots\int_0^{1/\lambda}}_{i}\prod_{j=1}^{i}\mathcal{S}_\lambda[J(s_j)] \\
&\quad \left|v^{(i-1)}\left(t-\sum_{k=1}^{i}s_k\right)-v^{(i-1)}(t)\right|ds_1\cdots ds_i \\
&\le \frac{i\|v^{(i)}(t)\|_\infty}{\lambda}\underbrace{\int_0^{1/\lambda}\cdots\int_0^{1/\lambda}}_{i}\prod_{j=1}^{i}\mathcal{S}_\lambda[J(s_j)]ds_1\cdots ds_i,
\end{aligned}$$

which, together with (2.10) and (2.8), leads to (2.6). So the proof is complete. □

Remark 2.1: Using (2.9), we have

$$z_{i\lambda}(t) = \underbrace{\frac{d}{dt}\mathcal{S}_\lambda[J(t)]*\cdots*\frac{d}{dt}\mathcal{S}_\lambda[J(t)]}_{i-1}*\mathcal{S}_\lambda[J(t)]*v(t).$$

So we can get the outputs $z_{i\lambda}(t)$ by i-times straightforward convolution. Hence, we can choose the starting time of the derivative tracking at i/λ instead of zero. Therefore, (2.6) means that the proposed tracking differentiator is a non-peaking tracking differentiator and the differentiation error only depends on one tuning parameter λ.

Remark 2.2: Since (2.2), it is easy to obtain that

$$\int_0^{1/\lambda}\mathcal{S}_\lambda[J(s)]ds = 1 \text{ for } \forall\ \lambda > 0. \tag{2.11}$$

Hence, $\mathcal{S}_\lambda[J(t)]$ can be regarded as a weight function. For any function $f(t)$, the convolution $\mathcal{S}_\lambda[J(t)]*f(t)$ is actually a weighted moving average of $f(t)$ over the interval $[t-1/\lambda, t]$. On the other hand, it follows from (2.9) that

$$z_{i\lambda}(t) = \underbrace{\mathcal{S}_\lambda[J(t)]*\cdots*\mathcal{S}_\lambda[J(t)]}_{i}*v^{(i-1)}(t). \tag{2.12}$$

Therefore, the tracking output $z_{i\lambda}(t)$ is the i-times weighted moving average of $v^{(i-1)}(t)$.

Remark 2.3: From the another point of view, $J(t)$ can be regarded as a mollifier function, which is used to approximate the functions in $L^p_{loc}(\mathbb{R}^+)$, $1 \le p < \infty$. In PDE, for each function $f \in L^p_{loc}(\mathbb{R}^+)$, $\mathcal{S}_\lambda[J(t)]*f(t)$ is usually used to approximate $f(t)$, provided λ is large enough (Theorem 6, [25], p. 630). Consequently, $z_{i\lambda}(t)$ also can be regarded as the i-times mollification of $v^{(i-1)}(t)$ by the mollifier function $J(t)$.

B. ROBUSTNESS OF THE REAL TIME TRACKING DIFFERENTIATOR

Just as the statement in Remark 2.2, the outputs $z_{i\lambda}(t)$ are actually several special exponentially weighted moving averages, which has been extensively used by smoothing random fluctuations. Hence, tracking differentiator (2.5) is still robust to the high-frequency noise because the high-frequency noise can be "average out" by several integral loops. In this subsection, we will study the robustness of the tracking differentiator (2.5) strictly. For this purpose, we choose sinusoid signal $v(t) = A\sin(\omega t+\phi)$. Define

$$\int_0^1 |J^{(i)}(t)|dt := \beta_i,\quad i = 1, 2, \cdots. \tag{2.13}$$

A straightforward computation shows that

$$\int_0^{1/\lambda}\left|\frac{d^i}{dt^i}\mathcal{S}_\lambda[J(t)]\right|dt = \lambda^i\beta_i. \tag{2.14}$$

Using the properties of convolution and (2.5), we deduce, for $t > i/\lambda$,

$$\begin{aligned}
&z_{i\lambda}(t) \\
&= \underbrace{\frac{d}{dt}\mathcal{S}_\lambda[J(t)]*\cdots*\frac{d}{dt}\mathcal{S}_\lambda[J(t)]}_{i-1}*\mathcal{S}_\lambda[J(t)]*v(t) \\
&= -\frac{A}{\omega}\underbrace{\int_0^{1/\lambda}\cdots\int_0^{1/\lambda}}_{i}\prod_{j=1}^{i}\frac{d}{dt}\mathcal{S}_\lambda[J(s_j)] \\
&\quad \cos\left[\omega\left(t-\sum_{k=1}^{i}s_k\right)+\phi\right]ds_1\cdots ds_i.
\end{aligned} \tag{2.15}$$

Consequently, from (2.14) we have

$$\begin{aligned}
&|z_{i\lambda}(t)| \\
&\le \frac{A}{\omega}\underbrace{\int_0^{1/\lambda}\cdots\int_0^{1/\lambda}}_{i}\prod_{j=1}^{i}\left|\frac{d}{dt}\mathcal{S}_\lambda[J(s_j)]\right|\cdot ds_1\cdots ds_i \\
&= \frac{A}{\omega}\prod_{j=1}^{i}\int_0^{1/\lambda}\left|\frac{d}{dt}\mathcal{S}_\lambda[J(s_j)]\right|ds_j \\
&= \frac{A}{\omega}\lambda^i\beta_1^i \to 0 \quad \text{as} \quad \omega\to\infty,
\end{aligned} \tag{2.16}$$

which means that the tracking differentiator (2.5) is robust to the high-frequency signals.

Remark 2.4: From (2.6) and (2.16) we see that the tuning parameter λ in differentiator (2.5) plays a significant role in convergence and noise tolerance: the larger the λ is, the more accurate the tracking effect would be, but the more sensitive the noise would be. This suggests that the choice of parameter λ in (2.5) is a tradeoff between tracking accuracy and noise tolerance in practice.

Example 2.1: We choose $v(t) = \cos t + \xi(t)$, where $\xi(t)$ is also the noise of the standard normal distribution with intension 0.05%. The tracking result by differentiator (2.5) is plotted in Figure 1, where the parameter is chosen by $\lambda = 25$. It is seen that the high-frequency noise is suppressed effectively.

III. TRACKING DIFFERENTIATOR FOR THE SIGNALS WITH TIME DELAY

In this section, we will introduce a tracking differentiator for the time delayed signals by a directly method. Lemma 2.1 and the Taylor expansion technical will be used in the proposed tracking differentiator.

A. DESIGN OF THE TIME DELAYED TRACKING DIFFERENTIATOR

For any $n \in \mathbb{Z}_+$ and $i = 1, 2, \cdots, n-1$, we define

$$\hat{z}_{in\alpha\tau}(t) := \sum_{k=1}^{n-i} \frac{1}{(k-1)!} \left\{ \frac{d^{k-1+i}}{dt^{k-1+i}} \mathcal{S}_{\alpha n}[J(t)] * v(t-\tau) \right\} \tau^{k-1}, \tag{3.1}$$

where $\tau > 0$ is the time delay, and $\alpha > 0$ is a gain constant. The main idea of our tracking differentiator is that, with the time delayed input $v(t-\tau)$, the output $\hat{z}_{in\alpha\tau}(t)$ can be, through regulating α and n, considered as the approximations of the corresponding $v^{(i)}(t)$.

Theorem 3.1: For any given $\tau > 0$. Assume that $v(t) \in C^n(\mathbb{R}^+)$ and

$$M_n := \sup \left\{ \|v^{(j)}(t)\|_\infty \; j = 0, 1, \cdots, n \right\} < +\infty. \tag{3.2}$$

Then, for any $t \in [1/\alpha n + \tau, +\infty)$, it follows that

$$\left| \hat{z}_{in\alpha\tau}(t) - v^{(i)}(t) \right| \leq \frac{M_n}{n\alpha} \cdot e^\tau + \frac{\|v^{(n)}(t)\|_\infty}{(n-i)!} \tau^{n-i}. \tag{3.3}$$

Proof: A straightforward computation shows that, for $t > 1/\alpha n + \tau$,

$$\begin{aligned} &\left| \hat{z}_{in\alpha\tau}(t) - v^{(i)}(t) \right| \\ &= \Big| \sum_{k=1}^{n-i} \frac{1}{(k-1)!} \left\{ \frac{d^{k-1}}{dt^{k-1}} \Big(\mathcal{S}_{\alpha n}[J(t)] \Big) * v^{(i)}(t-\tau) \right\} \tau^{k-1} \\ &\quad - v^{(i)}(t) \Big| \\ &\leq \Big| \sum_{k=1}^{n-i} \frac{1}{(k-1)!} \left\{ \frac{d^{k-1}}{dt^{k-1}} \Big(\mathcal{S}_{\alpha n}[J(t)] \Big) * v^{(i)}(t-\tau) \right\} \tau^{k-1} \\ &\quad - \left(\sum_{k=1}^{n-i} \frac{1}{(k-1)!} v^{(k-1+i)}(t-\tau)\tau^{k-1} \right) \Big| \\ &\quad + \left| \left(\sum_{k=1}^{n-i} \frac{1}{(k-1)!} v^{(k-1+i)}(t-\tau)\tau^{k-1} \right) - v^{(i)}(t) \right| \\ &\leq: I_1 + I_2. \end{aligned} \tag{3.4}$$

From Lemma 2.1 we obtain that, for $t > 1/\alpha n + \tau$,

$$\begin{aligned} &\left| \frac{d^{k-1}}{dt^{k-1}} \{\mathcal{S}_{\alpha n}[J(t)]\} * v^{(i)}(t-\tau) - v^{(k-1+i)}(t-\tau) \right| \\ &\quad < \frac{\|v^{(k+i)}(t)\|_\infty}{n\alpha}; \end{aligned} \tag{3.5}$$

Then,

$$\begin{aligned} I_1 &= \Big| \sum_{k=1}^{n-i} \frac{1}{(k-1)!} \Big\{ \frac{d^{k-1}}{dt^{k-1}} \Big(\mathcal{S}_{\alpha n}[J(t)] \Big) * v^{(i)}(t-\tau) \\ &\quad - v^{(k-1+i)}(t-\tau) \Big\} \tau^{k-1} \Big| \\ &< \left| \sum_{k=1}^{n-i} \frac{1}{(k-1)!} \frac{\|v^{(k+i)}(t)\|_\infty}{n\alpha} \tau^{k-1} \right| \\ &\leq \frac{M_n}{n\alpha} \sum_{k=1}^{n-i} \frac{1}{(k-1)!} \tau^{k-1} < \frac{M_n}{n\alpha} \cdot e^\tau. \end{aligned} \tag{3.6}$$

On the other hand, it follows from Taylor expansion that

$$\begin{aligned} I_2 &= \left| \left[\sum_{k=1}^{n-i} \frac{1}{(k-1)!} v^{(k-1+i)}(t-\tau)\tau^{k-1} \right] - v^{(i)}(t) \right| \\ &\leq \frac{\|v^{(n)}(t)\|_\infty}{(n-i)!} \tau^{n-i}, \end{aligned} \tag{3.7}$$

which, together with (3.4) and (3.6), leads easily to (3.3). So the proof is complete. □

From Theorem 3.1, we are able to obtain the following Corollary immediately.

Corollary 3.1: For any given $\tau > 0$ and $a > \tau$. Assume that $v(t) \in C^\infty(\mathbb{R}^+)$ and

$$M := \sup \left\{ \|v^{(j)}(t)\|_\infty \; j \in \mathbb{Z}_+ \right\} < +\infty. \tag{3.8}$$

Then, for each given positive constant α, it has

$$\lim_{n\to\infty} \left| \hat{z}_{in\alpha\tau}(t) - v^{(i)}(t) \right| = 0 \text{ uniformly in } [a, +\infty). \tag{3.9}$$

B. ROBUSTNESS FOR TIME DELAYED TRACKING DIFFERENTIATOR

In this subsection, we will study the robustness of the tracking differentiator (3.1). For this purpose, we choose sinusoid signal $v(t) = A\sin(\omega t + \phi)$ again. Using the properties of convolution, we deduce from (2.14) that

$$\frac{d^{i-1}}{dt^{i-1}} \{\mathcal{S}_{\alpha n}[J(t)]\} * v(t-\tau)$$

$$
\begin{aligned}
&= \frac{d^i}{dt^i}\{\mathcal{S}_{\alpha n}[J(t)]\} * \int_0^t v(a-\tau)da \\
&= \frac{d^i}{dt^i}\{\mathcal{S}_{\alpha n}[J(t)]\} * \frac{A}{\omega}[\cos(\phi-\omega\tau)-\cos(\omega t+\phi-\omega\tau)] \\
&= \frac{d^i}{dt^i}\{\mathcal{S}_{\alpha n}[J(t)]\} * \left[-\frac{A}{\omega}\cos(\omega t+\phi-\omega\tau)\right] \\
&\le \frac{A}{\omega}\int_0^t \left|\frac{d^i}{dt^i}\{\mathcal{S}_{\alpha n}[J(s)]\}\cos(\omega(t-s)+\phi-\omega\tau)\right| ds \\
&\le \frac{A}{\omega}\int_0^{1/\alpha n}\left|\frac{d^i}{dt^i}\{\mathcal{S}_{\alpha n}[J(s)]\}\right| ds \\
&= \frac{A}{\omega}\alpha^i n^i \beta_i.
\end{aligned}
$$

That is, for $t > 1/\alpha n + \tau$, we have

$$
\frac{d^{i-1}}{dt^{i-1}}\{\mathcal{S}_{\alpha n}[J(t)]\} * v(t-\tau) \le \frac{A}{\omega}\alpha^i n^i \beta_i. \tag{3.10}
$$

On the other hand, it follows from (3.1) and (3.10) that, for $t > \tau + 1/\alpha n$,

$$
\begin{aligned}
&\left|\hat{z}_{in\alpha\tau}(t)\right| \\
&\quad \le \frac{A}{\omega}\sum_{k=1}^{n-i}\frac{1}{(k-1)!}(n\alpha)^{k+i}\beta_{k+i}\tau^{k-1} \\
&\quad \le \frac{A}{\omega}(n\alpha)^{i+1}\beta_* \sum_{k=1}^{n-i}\frac{1}{(k-1)!}(n\alpha)^{k-1}\tau^{k-1} \\
&\quad \le \frac{A}{\omega}(n\alpha)^{i+1}\beta_* e^{n\alpha\tau} \to 0 \quad \text{as} \quad \omega\to\infty,
\end{aligned} \tag{3.11}
$$

where

$$
\beta_* := \max\{\beta_k \;\; k = 1, 2, \cdots, n\}.
$$

(3.11) means that the tracking differentiator (3.1) is robust to the small high-frequency signals.

Remark 3.1: Here we emphasize that the larger n is, the weaker the robustness of (3.1) wound be. In fact, with the increasing of n, β_* grows rapidly (For example, by a simple computation we get $\beta_0 = 1$, $\beta_1 \approx 5.2$, $\beta_2 \approx 44.1$, $\beta_3 \approx 502.9$, $\beta_4 \approx 8.4 \times 10^3$). Therefore, it follows from (3.11) that the robustness would become quite small provided n is large.

Remark 3.2: From (3.3) we see that the error of approximation depends only on $\frac{M_n}{n\alpha}\cdot e^{\tau}$ and $\frac{\|v^{(n)}(t)\|_\infty}{(n-i)!}\tau^{n-i}$. When the time delay τ is small, $\frac{\|v^{(n)}(t)\|_\infty}{(n-i)!}\tau^{n-i}$ may become much more smaller even if n is not very large. For example, if $\tau = 0.01$, $i = 0$ and $n = 3$, $\frac{\|v^{(n)}(t)\|_\infty}{(n-i)!}\tau^{n-i} = 10^{-6}\cdot\|v^{(3)}(t)\|_\infty\cdot\frac{1}{6}$. On the other hand, $\frac{M_n}{n\alpha}\cdot e^{\tau}$ can be absorbed by choosing α sufficiently large. Consequently, the tracking differentiator is, at least, effective to the signal with small time lag.

Example 3.1: We choose $v(t) = \cos t + \xi(t)$ again, where $\xi(t)$ is also the noise of the standard normal distribution with intension 0.05%. The tracking result by differentiator (3.1) is plotted in Figure 2, where the parameters are chosen by $\tau = 0.05$ and $\alpha = 10$. It is seen that the larger n is, the weaker the robustness is. Although the high-frequency noise is also suppressed effectively while $n = 1$, the robustness of tracking differentiator (3.1) is weaker than (2.5).

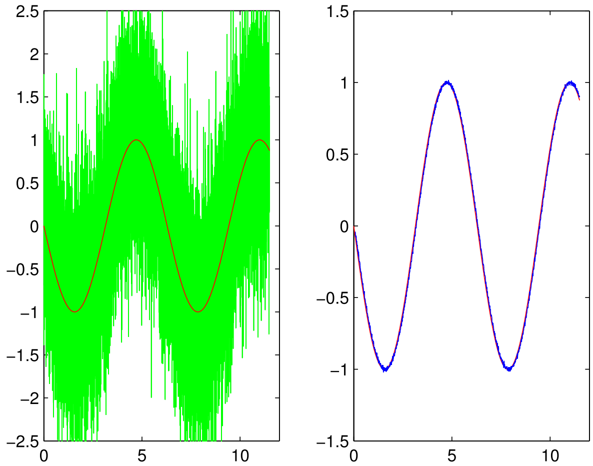

FIGURE 1. Red, Ideal derivative signal; Green, Directly differentiate by Matlab; Blue, Tracking by (2.5).

TABLE 1. Comparisons between Real-Time and High-gain.

	Real-Time	High-gain
Tuning parameters	$\lambda = 150$	$\varepsilon = 0.01$
Absolute error	0.0255	0.0666
Relative error	0.52%	1.39%

IV. NUMERICAL SIMULATION

In this section, we make some numerical simulations for the proposed differentiators to illustrate the theoretical results. The numerical code is programmed by Matlab. The step of time is chosen as $dt = 10^{-4}$. We compare the proposed differentiators (2.5) (denoted by **Real-Time**) and (3.1) (denoted by **Time-Delay** with a linear high-gain tracking differentiator (denoted by **High-gain**), which is given by

$$
\begin{cases}
\dot{z}_{1\varepsilon}(t) = z_{2\varepsilon}(t), \\
\dot{z}_{2\varepsilon}(t) = z_{3\varepsilon}(t), \\
\dot{z}_{3\varepsilon}(t) = \dfrac{6}{\varepsilon^3}[v(t) - z_{1\varepsilon}(t)] - \dfrac{6}{\varepsilon^2}z_{2\varepsilon}(t) - \dfrac{3}{\varepsilon}z_{1\varepsilon}(t), \\
z_{1\varepsilon}(0) = z_{2\varepsilon}(0) = z_{3\varepsilon}(0) = 0.
\end{cases} \tag{4.1}
$$

In the simulation, the input signal is chosen as $v(t) = \sin t + \cos 5t$.

The derivative tracking results by differentiator **Real-Time** and **High-gain** are plotted in Figure 3 and Figure 4, respectively. The performance comparisons between the real-time differentiator and the high-gain tracking differentiator are given in the following table.

From Figures 3, 4 and the table 1, we see that our tracking differentiator is much better than the high-gain differentiator. More concretely, there is no peaking phenomenon takes place in Figure 3, while, in Figure 4, a serious peaking phenomenon takes place. In fact, under the premise of ensuring accuracy, the peak phenomenon of **High-gain** is inevitable, no matter how we choose the parameter ε.

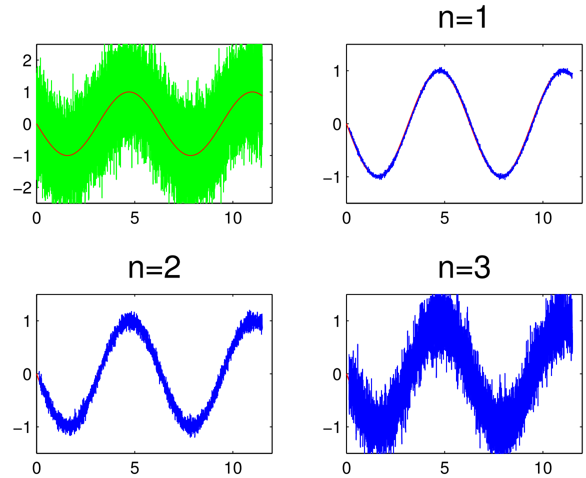

FIGURE 2. Red, Ideal derivative signal; Green, Directly differentiate by Matlab; Blue, Tracking by (3.1).

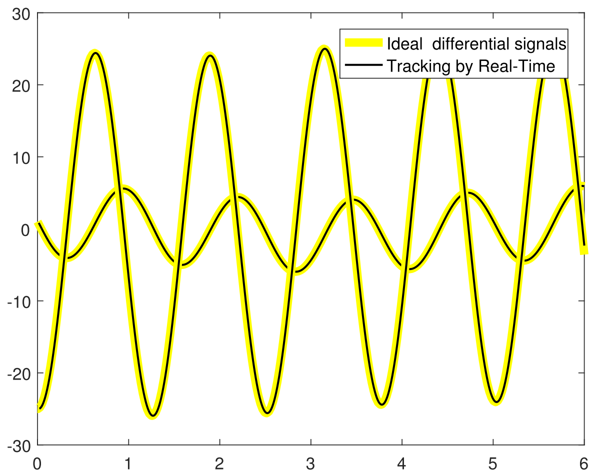

FIGURE 3. First and second derivatives tracking by Real-Time: Yellow, Ideal derivative signal; Black,Tracking by (2.5).

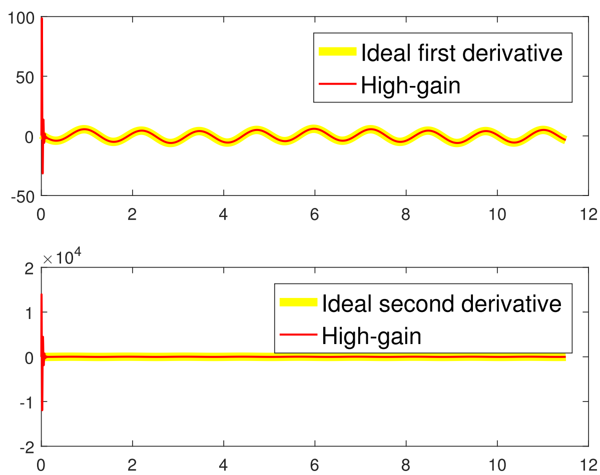

FIGURE 4. First and second derivatives tracking by High-gain: Yellow, Ideal derivative signal; Red,Tracking by (4.1).

In Figure 5 and Figure 6, $v(t)$ and $\dot{v}(t)$ is tracked by **Time-Delay** respectively. The parameters are taken by $\alpha = 125$, $\tau = 0.1$ and $n = 2$.

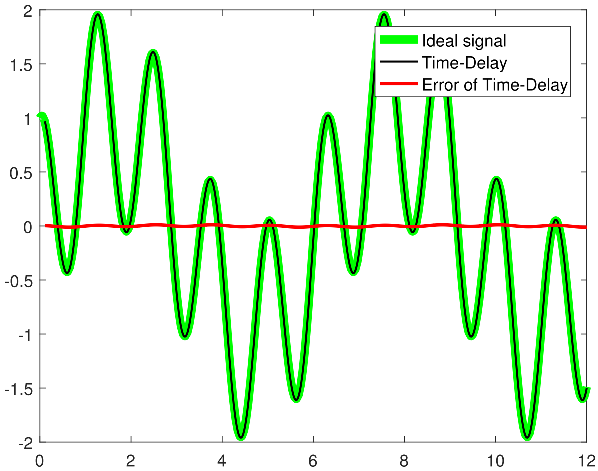

FIGURE 5. Signal tracking by Time-Delay: Green, Ideal signal; Black, Time-Delay; Red, error of Time-Delay.

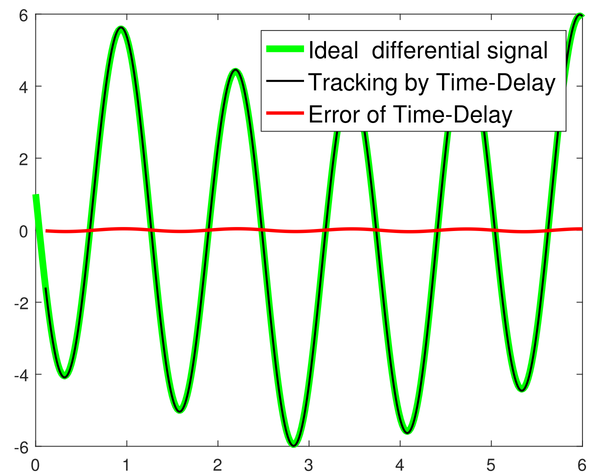

FIGURE 6. First derivative tracking by Time-Delay: Green, Ideal differential signal; Black, Tracking by Time-Delay; Red, error of Time-Delay.

It is seen that there is still no peaking phenomenon take place. Although there is a time delay $\tau = 0.1$, the tracking performance is satisfactory.

V. CONCLUDING REMARKS

In this paper, two tracking differentiators are proposed by combining the weighted moving average technical and the mollifier technical. The first one is a real time tracking differentiator, and the second one is designed for the signals with time delay. Both of them are non-peaking tracking differentiators, and they are robust to the high-frequency signals. Differentiators are primarily utilized for filtering and differentiating signals, with myriad applications in physics. They play pivotal roles in systems like electric motor control systems, as well as in control systems for automobiles, aircraft, radar, and GPS positioning systems. However, the robustness becomes quite small when it is used to track higher derivatives. What's more, the tracking accuracy can be improved by adjusting the parameters of the tracking differentiators. Although it is simple, the tracking performance is very effective for the tracking of first derivative.

Throughout the entire note, the weight function $J(t)$ plays a crucial role in all our findings. The effectiveness of the method depends on the proper selection of the kernel function, which may be a challenging task. We have deferred this task to our future work.

APPENDIX

Proof of Lemma 2.1: It follows from (1.4), (2.1) and (2.3) that $\mathrm{supp}\mathcal{S}_\lambda[J(t)] \subset (0, 1/\lambda)$ and

$$\frac{d^i}{dt^i}\mathcal{S}_\lambda[J(t)]\Big|_{t=0,1/\lambda} = 0,\ i = 1, 2, \cdots. \tag{5.1}$$

It then follows that

$$\begin{aligned}
&\frac{d^{i-1}}{dt^{i-1}}\{\mathcal{S}_\lambda[J(t)]\} * v(t)\\
&= \frac{d^{i-1}}{dt^{i-1}}\{\mathcal{S}_\lambda[J(t)] * v(t)\}\\
&= \frac{d^{i-2}}{dt^{i-2}}\left\{v(0)\mathcal{S}_\lambda[J(t)] + \mathcal{S}_\lambda[J(t)] * \dot{v}(t)\right\}\\
&= \sum_{k=1}^{i-1} v^{(k-1)}(0)\cdot\frac{d^{i-1-k}}{dt^{i-1-k}}\{\mathcal{S}_\lambda[J(t)]\}\\
&\quad + \mathcal{S}_\lambda[J(t)] * v^{(i-1)}(t).
\end{aligned} \tag{5.2}$$

Therefore, for $t > 1/\lambda$, (5.2) becomes

$$\frac{d^{i-1}}{dt^{i-1}}\{\mathcal{S}_\lambda[J(t)]\} * v(t) = \mathcal{S}_\lambda[J(t)] * v^{(i-1)}(t). \tag{5.3}$$

Taking (2.2) into account, we obtain that, for $t > 1/\lambda$,

$$\int_0^t \mathcal{S}_\lambda[J(s)]ds = \int_0^{1/\lambda} \mathcal{S}_\lambda[J(s)]ds = 1, \tag{5.4}$$

where the variable substitution $\lambda s = \alpha$ is used. Combining (5.3) and (5.4), we have

$$\begin{aligned}
&\left|\frac{d^{i-1}}{dt^{i-1}}\{\mathcal{S}_\lambda[J(t)]\} * v(t) - v^{(i-1)}(t)\right|\\
&= \left|\mathcal{S}_\lambda[J(s)] * v^{(i-1)}(t) - v^{(i-1)}(t)\right|\\
&= \left|\int_0^t \mathcal{S}_\lambda[J(s)]v^{(i-1)}(t-s)ds - \int_0^t \mathcal{S}_\lambda[J(s)]v^{(i-1)}(t)ds\right|\\
&\le \int_0^{1/\lambda} \mathcal{S}_\lambda[J(s)]\left|v^{(i-1)}(t-s) - v^{(i-1)}(t)\right| ds.
\end{aligned} \tag{5.5}$$

Let $\lambda s = \alpha$ again. From Lagrange's mean value theorem, we have

$$\begin{aligned}
&\int_0^{1/\lambda} \mathcal{S}_\lambda[J(s)]\left|v^{(i-1)}(t-s) - v^{(i-1)}(t)\right| ds\\
&= \int_0^1 J(\alpha)\left|v^{(i-1)}\left(t - \frac{\alpha}{\lambda}\right) - v^{(i-1)}(t)\right| d\alpha\\
&\le \frac{\|v^{(i)}(t)\|_\infty}{\lambda}\int_0^1 \alpha J(\alpha)\, d\alpha\\
&\le \frac{\|v^{(i)}(t)\|_\infty}{\lambda}.
\end{aligned} \tag{5.6}$$

Combining (5.5) and (5.6), we complete the proof. □

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