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RESEARCH ARTICLE

Modeling and Sliding Mode Control of Supply Chains With Limited Product Shelf Life

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ABSTRACT Recently, much work has been devoted to the management of perishable inventories. However, the available results are obtained under various simplifying assumptions. It is most often assumed that the amount of deteriorating goods is proportional to the total amount of goods stored in the warehouse. Since this assumption is quite unrealistic, in this article we explicitly take into account the fact that every product has a limited shelf life (represented by an expiration date). Therefore we are introducing a non-linear, state space model of supply chains with a finite shelf life. In this model, the remaining shelf lives are represented by individual state variables. In the first place, stored products with the shortest shelf life are sold, and in the event of insufficient demand, the products are disposed of. We then apply a sliding mode control (SMC) strategy to control the flow of goods in the system under consideration. It has been shown that the proposed SM control strategy prevents the loss of products due to the end of their shelf life, and ensures full satisfaction of unpredictable consumer demand. Moreover, the strategy requires limited storage capacity and respects the capabilities of suppliers. These properties are then verified in a simulation example.

INDEX TERMS Inventory management, perishable inventory, sliding mode control.

I. INTRODUCTION

Effective management of logistic chains is a crucial concern of many businesses in current times. With the growth of population, the consumption of consumer goods increases as well. This creates the need to devise more efficient strategies for managing limited storage space. As such, supply chain management is an important and relevant topic in the control engineering community.

Many works on modeling of supply chains have been published, with different assumptions about the underlying system. Most common models consider products that either do not lose quality over time, or such ones where that loss is negligible and can be ignored. Those models seek efficient control of the plant, where the available storage space is often constrained or expensive [1], [2]. The product delivered to the warehouse might become delayed [3], defective or lost in shipping [4]. In multi-echelon logistic chains, undesirable phenomena such as the bullwhip effect might occur [5]. If not taken into account, this effect can potentially lead to unfulfilled customers' demand in multi-echelon systems. Some research explores cases where this demand can be backlogged [1], [3], [6], and others consider complex cases where it is represented as a random variable [2], [7].

While discussing inventory management strategies, one must often consider the problem of wares with limited shelf life. Many of the goods kept in warehouses cannot be stored indefinitely. There are several works that consider how degradation affects the problem of inventory management. Early studies, such as [3], focused on streamlined cases where the product has to be sold immediately after being delivered, with no ability to remain in the warehouse. Works such as [6] expand that idea for perishable inventories. Majority of such papers define a fraction that specifies how much of the product stored in the warehouse degrades in each discrete period [8], [9], [10] or the rate at which the product degrades in continuous time [11], [12], [13]. Some works consider a situation where this fraction specifies the rate of quality loss (which lowers the price of the product [14], [15] or the demand for it [16], [17]) and attempt to slow down the decay [14], [18].

Established literature on inventory management rarely takes the amount of time the product has already been stored in the warehouse into account. Authors of [14] and [16]

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touch upon this idea, but they focus on devising policies that help minimize the operating costs of the warehouse. Authors of [19] are also somewhat close to this approach, as they consider a shop with two shelves, where ordering fresh product triggers an event where product gets moved between full-price and discount shelves. Similarly, the Author of [20] considers that the age of the product lowers its price. Both of these approaches focus on a demand model with the sales dependent on the freshness/age of the product. The Authors of [21] consider a warehouse with a single, perishable product with a known lifetime. All the wares that exceed it are removed from the warehouse and can no longer be sold. The paper also considers that some product expires earlier, according to the Weibull distribution. These ideas are also considered in [22], where the same model is used with a neural network controller. This allows one to improve the robustness of the system and improve the associated costs. In another work [23] Authors consider a network of machines that manufacture a product that completely perishes after a defined time, but they focus on maximizing profits with machines that might break down during production. In conclusion, although various approaches to products with limited shelf life have been considered, the problem of modeling this phenomenon in a practical manner, where wares have a finite shelf life and it is in warehouse manager's best interest to sell the oldest wares first, has not been explored fully.

A crucial aspect of effective inventory management is the control algorithm, which determines the size of orders based on the available data. Indeed, many different authors have successfully applied control theory methods in inventory management problems in the past [1], [5], [7], [24]. In [1] Authors create an optimal control policy for a production-inventory system with the demand and production time modeled as a Markovian arrival processes. The goal is to minimize the costs of holding the inventory and backlogging the demand. Another approach can be seen in [7], where the Authors consider a model predictive control strategy based on a Laguerre function. They propose a warehouse model with multiple products, each with a lead time required before delivery. On the other hand, Authors of [5] propose a control strategy for a continuous, nonlinear three-stage production-distribution model. In particular, an adaptive SMC strategy is used to counter numerical chattering. Authors further consider adaptive fractional-order SMC, which carries even greater benefits. In article [24], a non-switching type reaching law based SMC strategy is proposed for supply chains with non-negligible lead time and imperfect supply lines with a certain loss factor. Sliding mode control strategies [25] are particularly significant in the context of this paper. Compared to alternatives such as LQ-optimal control or model predictive control, they are easy to design and tune for any dynamical system with full state information. Their most discerning feature is the ability to reject the effect of disturbance on system dynamics by driving its state onto a specific hyperplane [26] or to its immediate vicinity [27]. Since the considered inventory management problem involves uncertainties with potentially large magnitude and rate of change, it is natural to consider SMC strategies, which are well suited to counteract such disturbances. Such strategies are not devoid of shortcomings, but their most prominent disadvantages, such as undesirable chattering or the need for full state information, do not become a concern in the considered class of inventory management systems. A robust way of designing SMC strategies for discrete-time systems involves the use of the so-called reaching law [28], [29]. This approach allows one to define a function which specifies the desired evolution of the system representative point and then apply it to design the desired control signal.

In this work we introduce a new model of logistic chains in which limited shelf life of stored wares is *explicitly* taken into account. This makes an essential difference when compared to previous results based on simplified or indirect modeling of commodity deterioration. In particular, in this model the full amount of goods stored in the warehouse is split between multiple state variables. Each variable defines the amount of products with a particular remaining shelf life. This allows one to accurately represent products with a finite lifespan and to satisfy the demand in a way that aims to always make use of wares before they expire. Indeed, the goal of the control process discussed in our work is to satisfy the unpredictable (but bounded) consumers' demand while completely preventing the loss of wares. This objective needs to be achieved while taking limited warehouse capacity into account. To that end, the novel non-linear model of a supply chain is controlled with a reaching law based sliding mode control strategy, which is shown to achieve all aforementioned goals.

The remainder of the paper is organized as follows. In Section II we present a new approach to modeling of inventory management systems in the state space. To the best of our knowledge, the proposed approach is the first one which explicitly takes into account finite shelf life of the stored goods. Then, in Section III we describe the proposed sliding mode control algorithm. Section IV contains formal proofs of several important properties of the system provided by the proposed control scheme. This includes ensuring that all wares are sold before the end of their shelf life and making sure that the warehouse is capable of fully satisfying the unpredictable demand. Finally, in Section V we showcase these properties via simulation and in Section VI we give concluding remarks.

II. INVENTORY MANAGEMENT SYSTEM

In this paper we consider logistic chains in which multiple suppliers deliver goods to a common warehouse. As opposed to many previous research works, in this paper limited shelf life of these goods is explicitly taken into account. Stored wares are then sold according to a largely unpredictable consumers' demand. Ideally, the objective of the control process for such delivery chains is to ensure full satisfaction of the demand while preventing the stored goods from becoming too old and unsuitable for use or consumption. This objective needs to be achieved taking the limited warehouse capacity into consideration. In this section, notation related to the considered class of logistic systems will be introduced, and dynamics of such systems will be modeled in the state space.

We introduce the following notation regarding the supply chains considered in this paper. Let r denote the number of suppliers who deliver goods to the common warehouse with limited capacity y_d. For each $i = 1, 2, \ldots, r$ the i–th supplier:

- Fulfills the part $s_i \in (0, 1)$ of the order generated by the controller.
- Has an individual delivery time $m_i \in \mathbb{N}$.
- Sends products with a set lifespan $\lambda_i > m_i$ after which these products can no longer be sold (i.e. λ_i is the length of time from the moment of manufacturing till the moment the products get unusable or unfit for consumption). Thus, the supplier delivers wares that can be stored at the warehouse for $n_i = \lambda_i - m_i$ time instants.
- Has certain maximum transport capabilities equal to constant $u_{i\max}$.

In order to streamline future analysis, we further define combined transport capabilities of all suppliers as

$$u_{\max} = \sum\nolimits_{i=1}^{r} u_{i\max}. \tag{1}$$

One easily concludes that the maximum order size $u_{\max}$ exceeds capabilities of each individual supplier, which is why the order is distributed between them according to constants s_i. These constants are typically defined so that each supplier receives a part of the order proportional to its capabilities, which implies $s_i = u_{i\max}/u_{\max}$ for all $i = 1, 2, \ldots, r$. Then, one easily notices that these constants satisfy

$$\sum\nolimits_{i=1}^{r} s_i = \frac{1}{u_{\max}} \sum\nolimits_{i=1}^{r} u_{i\max} = 1. \tag{2}$$

In the considered system, we denote the longest delivery time and the longest time wares can spend at the warehouse as

$$m = \max_{i=1,\ldots,r} (m_i), \quad n = \max_{i=1,\ldots,r} (n_i). \tag{3}$$

The considered system is reviewed periodically, with constant T being the review period. The replenishment orders for the warehouse are generated at regular time instants kT with $k = 0, 1, 2, \ldots$. In order to streamline notation, in the remainder of the paper we will enumerate instants kT simply with k. The objective of the control process is to ensure that the unpredictable consumers' demand $\tilde{d}(k)$ is always fully satisfied. Although this demand is not explicitly known, it is assumed to be bounded in the following way for all time instants

$$0 < d_{\min} \leq \tilde{d}(k) \leq d_{\max} \leq u_{\max}. \tag{4}$$

In particular, it is assumed that maximum capabilities of the suppliers $u_{\max} \geq d_{\max}$, since otherwise it would be impossible to satisfy the consumers' demand regardless of the applied controller. In order to design a successful control strategy for such supply chains, their dynamics will first be modeled in the $n + m$ dimensional state space, where n states of the system will represent stored goods, while m remaining ones will model deliveries in progress.

A. STATE SPACE DYNAMICS REPRESENTATION OF THE CONSIDERED PLANT

In order to accurately control the distribution of wares in the considered supply chain, it is desirable to obtain its delay-free representation in the extended state space. In particular, dynamics of this inventory management system will be expressed as

$$\begin{aligned} \boldsymbol{x}(k+1) &= \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{b}u(k) - \boldsymbol{d}(k) \\ y(k) &= \boldsymbol{q}^{\mathrm{T}}\boldsymbol{x}(k), \end{aligned} \tag{5}$$

where state vector $\boldsymbol{x}$ contains information about the amount of stored goods with different remaining shelf life as well as deliveries that are already underway, output y denotes the total amount of goods in the warehouse, which is naturally limited by the warehouse capacity y_d. Control signal u represents the amount of ordered wares at a given time, $\boldsymbol{d}$ denotes the amount of sold goods and $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{q}$ are of appropriate dimensions. All of these elements will now be properly defined.

1) VECTOR x

State vector in the considered model will be divided into two distinct parts $\boldsymbol{x}_\alpha(k)$ and $\boldsymbol{x}_\beta(k)$ so that

$$\boldsymbol{x}(k) = \begin{bmatrix} \boldsymbol{x}_\alpha(k) \\ \boldsymbol{x}_\beta(k) \end{bmatrix} \in \mathbb{R}^{n+m}. \tag{6}$$

Elements of sub-vector

$$\boldsymbol{x}_\alpha(k) = \begin{bmatrix} x_1(k) \\ \vdots \\ x_n(k) \end{bmatrix} \in \mathbb{R}_+^n \tag{7}$$

contain information about the amount of goods stored at the warehouse, and having different remaining shelf life. In this way, state variable $x_1(k)$ denotes wares which are one time instant away from becoming unsuitable for use and, if not sold, will have to be disposed of at time $k + 1$. State variable x_n represents stored goods that can spend the longest possible time n at the warehouse before getting too old to be sold. Naturally, the amount of stored wares is always non-negative. The second sub-vector

$$\boldsymbol{x}_\beta(k) = \begin{bmatrix} x_{n+1}(k) \\ \vdots \\ x_{n+m}(k) \end{bmatrix} \in \mathbb{R}^m \tag{8}$$

contains information about the orders that have already been placed, but have not necessarily yet arrived at the warehouse. In this way, state variable $x_{n+1}(k)$ denotes wares which were ordered m time instants ago. Likewise, state variable $x_{n+m}(k)$

represents wares which were ordered in the previous time instant, i.e. $x_{n+m}(k) = u(k-1)$. More generally

$$x_{n+j}(k) = u(k-m+j-1) \tag{9}$$

for any $j = 1, \ldots, m$. Assuming that initially the warehouse is empty and no orders have been placed at any time $k < 0$, $\boldsymbol{x}(0) = \boldsymbol{0}_{n+m}$.

2) MATRIX A

The state matrix in model (5) must be defined to reflect the following characteristics of the supply chain:

- warehouse contains goods with different expiry dates, that gradually approach their end of shelf life in each time instant,
- it takes time to deliver ordered goods, and individual suppliers have different delivery times,
- deliveries from each supplier arrive with a particular time until expiration date.

With this in mind, matrix $\boldsymbol{A}$ is divided into the following four sub-matrices

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix}. \tag{10}$$

Sub-matrix $\boldsymbol{A}_{11}$ reflects the fact that stored wares gradually approach their expiry date. Since the maximum time goods can be stored at the warehouse is n, this matrix is defined as

$$\boldsymbol{A}_{11} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{n \times n}. \tag{11}$$

This implies that, in the absence of sales or new goods arrival, the amount of goods $x_j(k)$ with remaining shelf life j, after each discretization period becomes $x_{j-1}(k+1)$. Similarly, sub-matrix $\boldsymbol{A}_{22}$ representing delayed deliveries is expressed as

$$\boldsymbol{A}_{22} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{m \times m}. \tag{12}$$

This sub-matrix helps keep track of the past m orders that have not yet fully arrived at the warehouse. Matrix $\boldsymbol{A}_{12} \in \mathbb{R}^{n \times m}$ is responsible for ensuring that the correct part of past orders arrives at the warehouse from each supplier. Furthermore, it guarantees that the received goods are added to determine the appropriate state variable of vector $\boldsymbol{x}_\alpha$, according to the remaining time until their expiry date. In order to ensure these properties, for all $i = 1, \ldots, r$ constants s_i are added to the element at row n_i and column $m - m_i + 1$ of matrix $\boldsymbol{A}_{12}$, while the remaining elements of this matrix are equal to zero. Finally, sub-matrix $\boldsymbol{A}_{21} = \boldsymbol{0}_{m \times n}$.

It is worth noting that in this paper a very general case of logistic chains is considered, where goods with different remaining shelf life can arrive at the warehouse. However, in many practical cases it is reasonable to assume that all wares have an identical shelf life upon arriving, in which case matrix $\boldsymbol{A}_{12}$ can be simplified. Indeed, when shelf life of arriving goods $n_i = n$ for all $i = 1, \ldots, r$, then all rows of $\boldsymbol{A}_{12}$ other than the n-th one will consist of zeros.

3) VECTOR d

Before this vector is properly defined, we introduce the following scalar variable

$$d_0(k) = \min\left(\tilde{d}(k), x_1(k)\right), \tag{13}$$

where $\tilde{d}(k)$ is the current demand. This variable represents the amount of wares sold at time k that would otherwise be lost due to their end of shelf life in the next time instant (i.e. they would leave the warehouse regardless of demand). On the other hand, vector $\boldsymbol{d}(k)$ denotes the amount of wares sold at time k that would have otherwise remained in the warehouse until $k+1$. This $n+m$ dimensional vector is expressed as

$$\boldsymbol{d}(k) = [d_1(k) \quad \ldots \quad d_{n-1}(k) \quad 0 \quad \ldots \quad 0]^{\mathrm{T}}, \tag{14}$$

where $d_1, \ldots, d_{n-1}$ denote sales of goods with different remaining shelf life. With the inclusion of $d_0(k)$ defined by (13), for any $j = 0, \ldots n-1$ variable $d_j(k)$, representing the amount of goods with the remaining shelf life equal to j units and sold at time k, can be expressed as

$$d_j(k) = \min\left(p_j(k), x_{j+1}(k)\right), \tag{15}$$

where $p_0(k) = \tilde{d}(k)$ and

$$p_j(k) = p_{j-1}(k) - d_{j-1}(k) \tag{16}$$

for $j = 1, \ldots, n-1$. In this equation $p_j(k)$ represents the remaining part of the demand which cannot be satisfied by these goods whose shelf life is smaller than or equal to $j-1$. Variables $d_j(k)$ defined in this way reflect the fact that wares closest to their expiry date are always sold first. At the same time, total sales will never exceed the demand $\tilde{d}(k)$ at a given time k. However, sales can actually be lower than the demand if there is not enough products to sell, which implies that

$$\sum_{j=0}^{n-1} d_j(k) \leq \tilde{d}(k). \tag{17}$$

Finally, since state variables $x_{n+1}(k), \ldots, x_{n+m}(k)$ represent ordered goods that have not yet arrived at the warehouse and cannot be sold, elements of vector (14) corresponding to these variables are zeros.

4) VECTORS b AND q

Input distribution vector $\boldsymbol{b}$ in the considered logistic system is defined as

$$\boldsymbol{b} = [0 \quad 0 \quad \ldots \quad 0 \quad 1]^{\mathrm{T}}, \tag{18}$$

which implies that the control signal $u(k)$ denoting the amount of ordered goods becomes the final state variable $x_{n+m}(k+1)$.

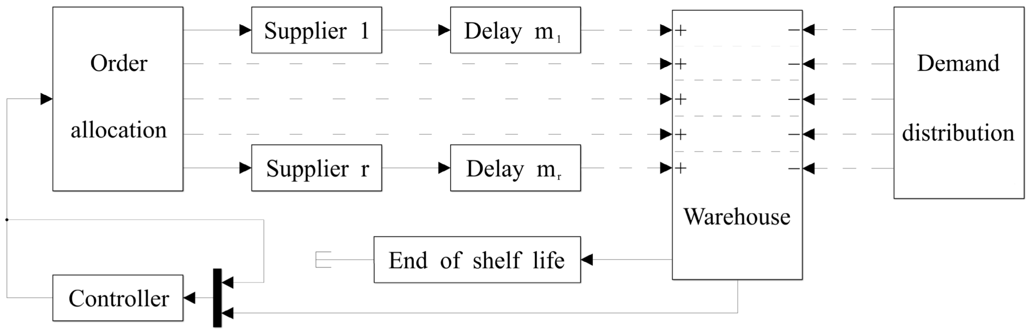

FIGURE 1. Flow of wares in the considered logistic chain.

Vector $\boldsymbol{q}$ must be selected to ensure that output $y(k)$ in relation (5) represents all stored wares, but not ones that are still in transit. Consequently, this vector becomes

$$\boldsymbol{q} = [\underbrace{1 \quad \dots \quad 1}_{n} \quad \underbrace{0 \quad \dots \quad 0}_{m}]^{\mathrm{T}}. \tag{19}$$

The model introduced in this section has never been discussed before and presents a new approach to the supply chain management problem. In order to better illustrate the flow of wares in this model, a generalized block diagram is given in Figure 1. Distinct "levels" of the warehouse in that diagram represent wares with different times until their end of shelf life. The model will now be illustrated in an example system with three suppliers.

B. EXAMPLE SYSTEM WITH THREE SUPPLIERS

In order to make the structure of the considered plants more clear, an example inventory management system will now be presented. This particular system will also be used in a simulation example later in this paper. This system involves three suppliers that:

- fulfill parts of the order $s_1 = 0.5$, $s_2 = 0.3$ and $s_3 = 0.2$,
- have delivery times $m_1 = 5$, $m_2 = 2$ and $m_3 = 2$,
- send products with remaining lifespan $\lambda_1 = 13$, $\lambda_2 = 10$ and $\lambda_3 = 7$,
- ultimately deliver wares that can be stored for $n_1 = 8$, $n_2 = 8$ and $n_3 = 5$.

This yields $n = 8$ and $m = 5$, which means that state-space representation (5) of this supply chain will be a 13-th order system. Consequently, state vector (6) for this system becomes $\boldsymbol{x}(k) = [\boldsymbol{x}_\alpha^{\mathrm{T}}(k) \;\; \boldsymbol{x}_\beta^{\mathrm{T}}(k)]^{\mathrm{T}}$, where

$$\boldsymbol{x}_\alpha(k) = \begin{bmatrix} x_1(k) \\ \vdots \\ x_8(k) \end{bmatrix}, \quad \boldsymbol{x}_\beta(k) = \begin{bmatrix} x_9(k) \\ \vdots \\ x_{13}(k) \end{bmatrix}. \tag{20}$$

Furthermore, elements of state matrix (10) are expressed as

$$\boldsymbol{A}_{11} = \begin{bmatrix} 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0 \end{bmatrix}, \tag{21}$$

$$\boldsymbol{A}_{12} = \begin{bmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0.2&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0.5&0&0&0.3&0 \end{bmatrix}, \tag{22}$$

$$\boldsymbol{A}_{22} = \begin{bmatrix} 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \end{bmatrix} \tag{23}$$

and $\boldsymbol{A}_{21} = \boldsymbol{0}_{5\times 8}$. Finally, $\boldsymbol{b}$ and $\boldsymbol{q}$ have the following form

$$\begin{aligned} \boldsymbol{b} &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]^{\mathrm{T}}, \\ \boldsymbol{q} &= [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^{\mathrm{T}}. \end{aligned} \tag{24}$$

A visual diagram describing the flow of wares in the considered system can be seen from Figure 2.

III. SLIDING MODES IN LOGISTIC CHAINS

The design process of a sliding mode control strategy for the considered inventory management system will now be described. This process begins with the selection of the so-called sliding hyperplane, which is typically described as

$$\sigma(k) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}_d - \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(k) = \sigma_d - \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(k) = 0, \tag{25}$$

where $\boldsymbol{x}_d$ is the target state and vector $\boldsymbol{c}$ consists of constants selected to ensure a stable response of the system. Selection

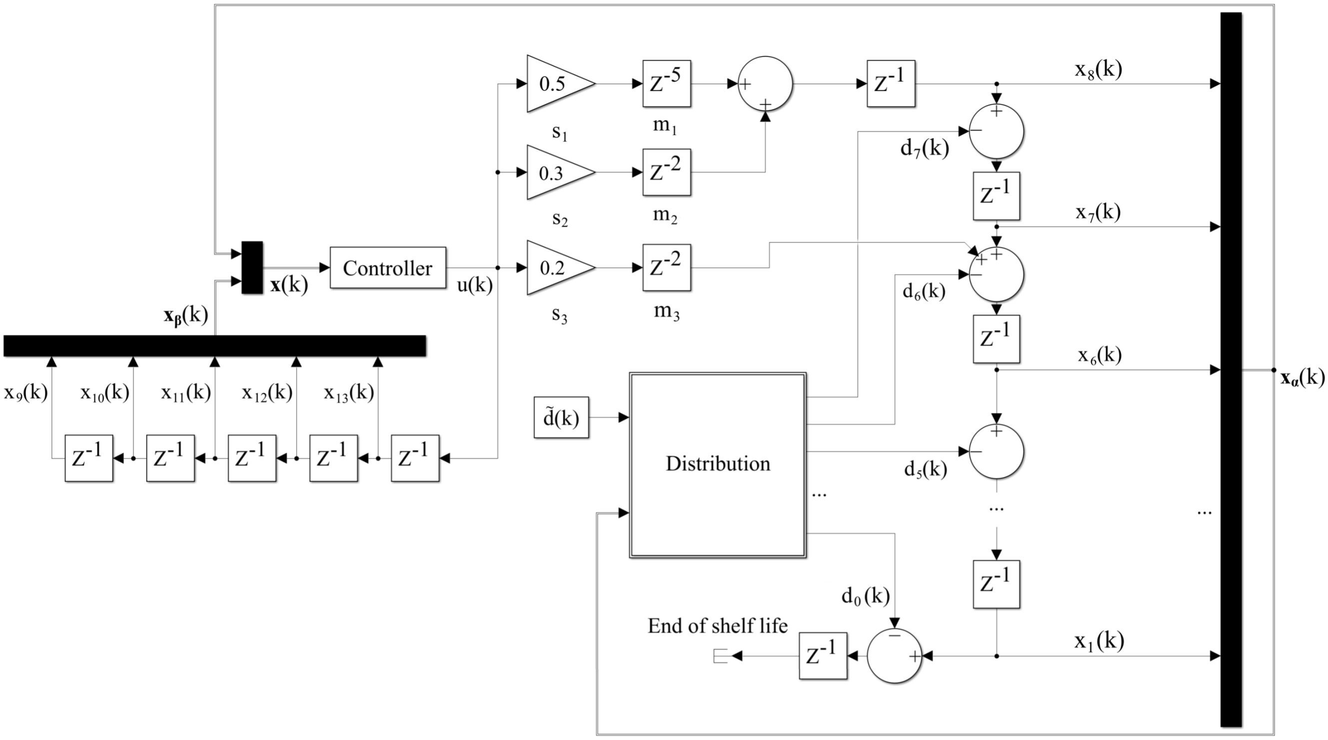

FIGURE 2. Example logistic chain. The left side represents delayed orders, i.e. the x_β part of the state vector. The right side depicts the amount of goods in the warehouse, denoted by x_α. Order allocation between the suppliers and demand distribution can be seen in the middle.

of these two elements will now be described in greater detail. Constant σ_d in relation (25) is defined in order to streamline the analysis presented in the later part of the paper.

1) VECTOR x_d

Target state for the controlled plant must be selected in a way that puts constraints on the amount of goods stored in the warehouse. At the same time, the vector must take into account the layout of matrix (10), which specifies when and how the wares are actually delivered. In order to formulate this vector properly, let us first define R_i as the sum of all elements of the i-th row of matrix $\boldsymbol{A}_{12}$. Then, $\boldsymbol{x}_d$ can be expressed as

$$\boldsymbol{x}_d = \Big[\underbrace{\frac{\gamma_1 y_d}{\gamma^*} \quad \dots \quad \frac{\gamma_n y_d}{\gamma^*}}_{n} \quad \underbrace{\frac{y_d}{\gamma^*} \quad \dots \quad \frac{y_d}{\gamma^*}}_{m} \Big]^{\mathrm{T}}, \tag{26}$$

where y_d is the constant warehouse capacity and

$$\gamma_j = \sum\nolimits_{i=j}^{n} R_i, \quad \gamma^* = \sum\nolimits_{j=1}^{n} \gamma_j. \tag{27}$$

In other words, target vector $\boldsymbol{x}_d$ is designed to reflect a steady state of the system taking into account the form of matrix $\boldsymbol{A}$. It is easy to notice that γ_1 is always equal to 1 and that the sum of first n elements of this vector yields exactly y_d. Particular choice of warehouse capacity y_d will be discussed later in this paper.

2) VECTOR $\boldsymbol{c} \in \mathbb{R}^{n+m}$

In discrete-time sliding mode control, this vector is selected so that all closed-loop poles of the controlled plant are placed inside the unit circle. This involves analyzing the eigenvalues of the closed-loop system state matrix

$$\boldsymbol{A}_{cl} = \boldsymbol{A} - \boldsymbol{b}(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b})^{-1}\boldsymbol{c}^{\mathrm{T}}\boldsymbol{A}, \tag{28}$$

which immediately implies that $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b} \neq 0$. Considering state matrix (10) and input distribution vector (18), this vector can be divided into two parts just like the state vector itself. In particular

$$\boldsymbol{c} = \begin{bmatrix} \boldsymbol{c}_\alpha \\ \boldsymbol{c}_\beta \end{bmatrix} \in \mathbb{R}^{n+m}. \tag{29}$$

In this work, elements of sub-vectors $\boldsymbol{c}_\alpha$ and $\boldsymbol{c}_\beta$ are chosen to place all eigenvalues of matrix (28) at zero. In order to achieve this, one selects

$$\boldsymbol{c}_\alpha = [\underbrace{1 \quad \dots \quad 1}_{n}]^{\mathrm{T}}, \quad \boldsymbol{c}_\beta = [\underbrace{c_{\beta 1} \quad \dots \quad c_{\beta m}}_{m}]^{\mathrm{T}}. \tag{30}$$

Constants $c_{\beta i}$ for $i = 1, \dots, m$ in this relation are expressed as

$$c_{\beta i} = \sum\nolimits_{j=1}^{i} S_j \tag{31}$$

where S_j is the sum of all elements of the j-th column of matrix $\boldsymbol{A}_{12}$. With this in mind, vector $\boldsymbol{c}_\beta$ becomes

$$\boldsymbol{c}_\beta = \Big[S_1 \quad S_1 + S_2 \quad \dots \quad \sum\nolimits_{j=1}^{m} S_j \Big]^{\mathrm{T}}. \tag{32}$$

and it is easy to verify that this vector places all eigenvalues of closed-loop system state matrix (28) at zero. In particular, for the example system described in Section II-B, the entire vector $\boldsymbol{c}$ has the following form

$$\boldsymbol{c} = \begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0.5 & 0.5 & 0.5 & 1 & 1\end{bmatrix}^{\mathrm{T}} \tag{33}$$

In the next section, the selected sliding hyperplane will be used to design an effective control strategy for the considered logistic chains.

IV. PROPOSED SLIDING MODE CONTROL STRATEGY

The control strategy proposed in this work is based on the reaching law approach, in which the desired evolution of sliding variable $\sigma(k)$ is first stated in the form of a function. Considering the unpredictable effect of disturbance on this variable, the proposed reaching law [29] is expressed as

$$\sigma(k+1) = \sigma(0) \max\left\{1 - \frac{k}{k^*}, 0\right\} + \boldsymbol{c}^{\mathrm{T}}\boldsymbol{d}(k), \tag{34}$$

where k^* is a positive integer selected by the designer. The objective of this reaching law is to drive $\sigma(k)$ towards zero in exactly k^* steps of equal magnitude and confine it to the vicinity of zero in each step afterwards. This property will now be stated and proven in the following lemma.

Lemma 1: If the reaching law is designed according to (34) and the disturbance affecting the plant is bounded according to (4), then for all $k \geq k^*$ sliding variable $\sigma(k)$ satisfies

$$0 \leq \sigma(k) \leq d_{\max} - d_0(k-1), \tag{35}$$

where $d_0(k-1)$ is defined according to (13).

Proof: For any $k \geq k^*$ reaching law (34) is effectively reduced to $\sigma(k+1) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{d}(k)$. Considering (14) and (30), one gets

$$\sigma(k+1) = d_1(k) + d_2(k) + \ldots + d_{n-1}(k). \tag{36}$$

Relations (15) and (16) directly imply that $d_i(k) \geq 0$ for all $i = 1 \ldots n-1$, which gives

$$\sigma(k+1) \geq 0 \tag{37}$$

for all k. Furthermore, from (36) using inequalities (4) and (17) one can obtain

$$\begin{aligned}\sigma(k+1) &= d_0(k) - d_0(k) + d_1(k) + \ldots + d_{n-1}(k)\\ &\leq \tilde{d}(k) - d_0(k) \leq d_{\max} - d_0(k),\end{aligned} \tag{38}$$

which concludes the proof. □

It is further worth noticing that, since reaching law (34) has no switching term depending on $\operatorname{sgn}[\sigma(k)]$, it will not produce undesirable chattering in the sliding phase. In the next step of sliding mode controller design, formula (34) denoting the desired values of the sliding variable is applied to design an appropriate control signal. First, substitution of (5) into the right hand side of (25) yields

$$\sigma(k+1) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}_d - \boldsymbol{c}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}(k) - \boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}u(k) + \boldsymbol{c}^{\mathrm{T}}\boldsymbol{d}(k). \tag{39}$$

Then, considering reaching law (34), one obtains

$$\begin{aligned}&\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}_d - \boldsymbol{c}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}(k) - \boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}u(k) + \boldsymbol{c}^{\mathrm{T}}\boldsymbol{d}(k) =\\ &= \sigma(0)\max\left\{1 - \frac{k}{k^*}, 0\right\} + \boldsymbol{c}^{\mathrm{T}}\boldsymbol{d}(k).\end{aligned} \tag{40}$$

Then, since $\sigma(0) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}_d$, the reaching law based control signal can be obtained as

$$u(k) = \left(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}\right)^{-1}\left[\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}_d \min\left\{\frac{k}{k^*}, 1\right\} - \boldsymbol{c}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}(k)\right]. \tag{41}$$

Then, the obtained control strategy can be applied to generate delivery requests in the considered logistic chain. In order to ensure full efficiency of the supply chain, the control strategy should guarantee that:

- The stored goods are always sold before the end of their shelf life.
- The unpredictable consumers' demand is always satisfied despite limited warehouse capacity.
- The amount of goods ordered at any time is limited, and the control strategy does not generate returns.

These properties will now be formally demonstrated in the following three subsections. It is also important to mention that, since this approach allows one to stay inside certain bounds of the control signal, we do not use any conventional saturation in the system.

A. PREVENTING LOSS OF WARES DUE TO END OF SHELF LIFE

It will now be shown that, with the right choice of design parameters in the proposed control scheme, all goods can be sold before the end of their shelf life.

Theorem 1: If the control signal for system (5) is described by (41) and constant σ_d satisfies the following inequality

$$\sigma_d \leq \sum\nolimits_{l=1}^{n} lR_l d_{\min} + \sum\nolimits_{j=1}^{m} c_{\beta j} d_{\min}, \tag{42}$$

where for any $l = 1, \ldots, n$ constant R_l is the sum of all elements of l-th row of matrix $\boldsymbol{A}_{12}$, then for every $k \geq k^* + m$ loss of wares due to end of their shelf life will be avoided.

Proof: Let $k \geq k^*$. One can notice that reaching law (34) is reduced to $\sigma(k+1) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{d}(k)$ for such k. Two cases will be considered. First, suppose that $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{d}(k) < \tilde{d}(k)$. This implies that all goods available at the warehouse have been sold to consumers. Since empty warehouse poses no risk of losing wares, this case needs no further analysis.

Suppose now that $\sigma(k+1) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{d}(k) = \tilde{d}(k) \geq d_{\min}$. Then, since the amount of stored goods is reduced by at least $d_{\min}$ in each step due to demand, control signal (41) will have to compensate for that amount to keep $\sigma(k)$ close to zero. In other words, we have

$$u(k) \geq d_{\min} \tag{43}$$

for $k \geq k^*$. Then, matrix $\boldsymbol{A}_{22}$ in (23) implies that elements of vector $\boldsymbol{x}_\beta$ in (6) are delayed control signals. Thus, after m time instants, i.e. for $k \geq k^* + m$ one obtains

$$\boldsymbol{x}_{\beta i} \geq d_{\min} \quad \text{for} \quad i = 1, \ldots, m. \tag{44}$$

Considering (6), (25) and (29), one obtains

$$\sigma(k) = \sigma_d - \boldsymbol{c}_\alpha^{\mathrm{T}} \boldsymbol{x}_\alpha(k) - \boldsymbol{c}_\beta^{\mathrm{T}} \boldsymbol{x}_\beta(k). \tag{45}$$

Element $\boldsymbol{c}_\alpha^{\mathrm{T}} \boldsymbol{x}_\alpha(k)$ in this relation represents the amount of goods stored in the warehouse at a given time, as evident from (6), (29) and (30). Then, Lemma 1 implies that $\sigma(k) \geq 0$ for all k, which further gives

$$\boldsymbol{c}_\alpha^{\mathrm{T}} \boldsymbol{x}_\alpha(k) \leq \sigma_d - \boldsymbol{c}_\beta^{\mathrm{T}} \boldsymbol{x}_\beta(k). \tag{46}$$

Then, for any $k \geq k^* + m$, inequalities (44) lead to

$$\boldsymbol{c}_\alpha^{\mathrm{T}} \boldsymbol{x}_\alpha(k) \leq \sigma_d - \sum\nolimits_{j=1}^{m} c_{\beta j} d_{\min}. \tag{47}$$

If σ_d satisfies inequality (42), then the relation above leads to

$$\boldsymbol{c}_\alpha^{\mathrm{T}} \boldsymbol{x}_\alpha(k) \leq \sum\nolimits_{l=1}^{n} l R_l d_{\min} \tag{48}$$

This implies that the largest possible delivery at any given time $k \geq k^* + m$ equals $\sum_{l=1}^{n} l R_l d_{\min}$. In particular, this sum represents the maximum amount of product that can be delivered to the warehouse that will not cause degradation to occur, even in the worst case scenario of minimum demand. This delivery is then distributed among elements of vector $\boldsymbol{x}_\alpha$ according to coefficients in matrix $\boldsymbol{A}_{12}$. Since minimum demand at any given time is equal to $d_{\min}$, it is easy to notice that wares distributed in such a way will always be sold in time. One concludes that if constant σ_d is selected according to (42), then for all $k \geq k^* + m$ loss of wares is prevented. □

Remark 1: It should be reminded that constant $\sigma_d = \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x}_d$ in relation (42) is not equal to the required warehouse capacity, but represents a value connected with sliding motion of the system, as evident from (25). The actual warehouse capacity needed to achieve the property described in Theorem 1 is

$$y_d = \sigma_d - \sum\nolimits_{j=1}^{m} c_{\beta j} d_{\min}, \tag{49}$$

as it excludes variables of vector $\boldsymbol{x}_\beta(k)$ representing goods that are not yet delivered. As seen from (44), these variables are lower bounded by $d_{\min}$ for all $k \geq k^* + m$.

B. LIMITING THE MAGNITUDE OF ORDERS

In this section it will be demonstrated that the proposed control scheme does not generate negative control signals and never requires the suppliers to exceed their transport capabilities. These properties will be summarized in the following theorem.

Theorem 2: If the control signal for system (5) is described by (41) and constant σ_d conforms to (42), then for all $k \geq k^* + m$ the control signal satisfies

$$0 \leq u(k) \leq u_{\max}. \tag{50}$$

Proof: First, lower bounds of the control signal will be investigated. As evident from reaching law (34), the control signal is designed to always drive sliding variable $\sigma(k)$ towards zero. Since $\sigma(0) = \sigma_d > 0$, Lemma 1 implies that the sliding variable never changes its sign. Thus, control signal $u(k)$ will always maintain non-negative values.

Let us now analyze upper bounds of the control signal. If σ_d satisfies (42), then Theorem 1 implies that no wares are lost for any $k \geq k^* + m$. Thus, since control signal $u(k)$ is non-negative, value of sliding variable $\sigma(k)$ can be increased only by the sales $\boldsymbol{d}(k)$ specified in (14). After k^* initial time instants, control strategy (41) drives the sliding variable to zero (plus the effect of most recent disturbance) in each step. Therefore, controller $u(k)$ will strictly generate signals that compensate for sales $\boldsymbol{c}^{\mathrm{T}} \boldsymbol{d}(k-1)$ from the previous time instant. In other words

$$u(k) = \boldsymbol{c}^{\mathrm{T}} \boldsymbol{d}(k-1) \tag{51}$$

for $k \geq k^*$. Then, considering (4), (14) and (17) one gets

$$u(k) \leq d_{\max} \leq u_{\max}, \tag{52}$$

which concludes the proof. □

The theorem has demonstrated that the control signal is bounded for all k after $k^* + m$ initial time instants. In order to ensure that $u(k)$ is also limited at the beginning of the control process, one needs to choose a sufficiently large k^* with respect to $\sigma(0) = \sigma_d$. In particular, it is recommended to select

$$k^* \geq \left\lceil \frac{\sigma_d}{u_{\max}} \right\rceil, \tag{53}$$

where $\lceil \cdot \rceil$ is the ceiling function. Such a choice ensures that $u(k)$ will also be bounded by $u_{\max}$ in the initial stages of the control process.

C. SATISFYING THE UNPREDICTABLE DEMAND

In the final subsection of this chapter, it will be shown that when the warehouse capacity is sufficiently large, the consumers' demand is always fully satisfied. This will be achieved by demonstrating that, after a finite number of initial time instants, the warehouse is never completely emptied.

Theorem 3: If the control signal for system (5) is described by (41) and constant σ_d satisfies the following inequality

$$\sigma_d > \sum\nolimits_{j=1}^{m} c_{\beta j} d_{\max} + 2 d_{\max}, \tag{54}$$

then for all $k \geq k^* + 2m$ the amount of stored wares $y(k) > 0$.

Proof: Let us first use (25) and (30) to express sliding variable $\sigma(k)$ as

$$\begin{aligned} \sigma(k) = \quad & \sigma_d - \big[x_1(k) + \ldots + x_n(k)\big] - \\ & - \sum\nolimits_{j=1}^{m} c_{\beta j} x_{n+j}(k) \\ = \quad & \sigma_d - y(k) - \sum\nolimits_{j=1}^{m} c_{\beta j} x_{n+j}(k). \end{aligned} \tag{55}$$

Lemma 1 implies that for all $k \geq k^*$

$$d_{\max} - d_0(k-1) \geq \sigma_d - y(k) - \sum\nolimits_{j=1}^{m} c_{\beta j} x_{n+j}(k). \tag{56}$$

Since $d_0(k-1) \geq 0$, this further implies

$$y(k) \geq \sigma_d - d_{\max} - \sum\nolimits_{j=1}^{m} c_{\beta j} x_{n+j}(k). \tag{57}$$

Theorem 2 guarantees that for any time instant starting from k^*+m, control signal will be bounded by $u_{\max}$. However, one can further notice from relation (52) that the actual bound of the control signal for $k \geq k^*+m$ is equal to $d_{\max}$. Thus, since elements of vector $\boldsymbol{x}_\beta(k) \in \mathbb{R}^m$ are delayed control signals, one concludes that for all $k \geq k^* + 2m$

$$x_{\beta j}(k) \leq d_{\max} \quad \text{for} \quad j = 1, \ldots, m. \tag{58}$$

With this in mind, relation (57) becomes

$$y(k) \geq \sigma_d - d_{\max} - \sum_{j=1}^{m} c_{\beta j} d_{\max}. \tag{59}$$

Since $y(k)$ represents the amount of stored wares before sales at time k occur, we require it to be greater than $d_{\max}$ to always satisfy the unpredictable demand. In order to ensure this property, one must select

$$\sigma_d > \sum_{j=1}^{m} c_{\beta j} d_{\max} + d_{\max} + d_{\max}, \tag{60}$$

which is consistent with (54). □

Remark 2: It is worth pointing out that in some logistic chains it might become impossible to satisfy inequalities (42) and (54) at the same time. Thus, one would have to accept either the risk of losing wares or potentially unfulfilled demand. However, the gap between lower and upper bounds of y_d becomes wide when n is significantly larger than m. This is the case in most practical applications, since time spent in the warehouse is typically much longer than time spent in transit. Therefore, satisfying (42) and (54) simultaneously does not pose a challenge.

V. SIMULATION RESULTS

In this section properties demonstrated in Theorems 1-3 will be verified in a simulation. In particular, we consider a 13-th order system described in Section II-B of this paper. The objective is to ensure all aforementioned properties of the system while the amount of ordered goods never exceeds $u_{\max} = 12$. The consumers' demand is expressed as

$$\tilde{d}(k) = 7.5 + 2.5 * (-1)^{\lfloor k/50 \rfloor} \tag{61}$$

where only its minimum value $d_{\min} = 5$ and maximum value $d_{\max} = 10$ are available to the controller. An appropriate sliding hyperplane (25) for the considered control strategy is first defined. Particularly, vector $\boldsymbol{c}$ is chosen according to (29) and equals

$$\boldsymbol{c} = [1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 0.5 \;\; 0.5 \;\; 0.5 \;\; 1 \;\; 1]^{\mathrm{T}} \tag{62}$$

Then, warehouse capacity for the considered problem will be selected according to Theorems 1 and 3, taking Remark 1 into account. Inequality (42) implies that in order to sell all ordered goods before the end of their shelf life, σ_d must be upper bounded by 55.5. On the other hand, (54) suggests that in order to fully satisfy demand (61), σ_d must be strictly greater than 55. With this in mind, we define

$$\sigma_d = 55.5. \tag{63}$$

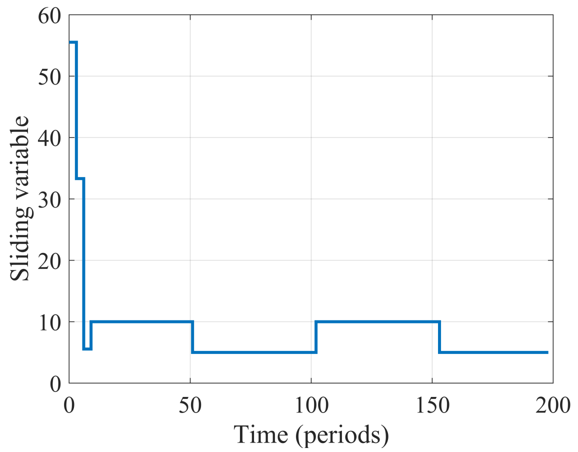

FIGURE 3. Sliding variable.

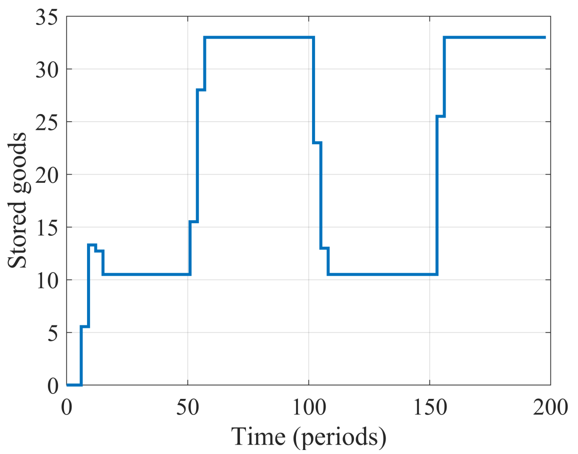

FIGURE 4. Amount of stored goods.

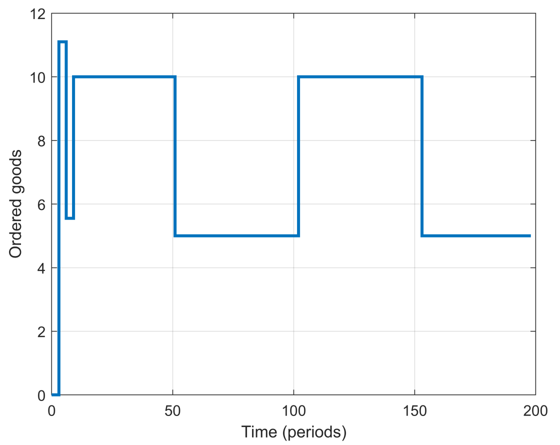

FIGURE 5. Amount of ordered goods.

Considering Remark 1, physical warehouse capacity needed to achieve the desired properties of the system is $y_d = 38$. Wares in the considered logistic system are ordered according to control strategy (41). Constant k^* in this strategy is selected according to (53) and equals 5, which ensures that capabilities of the suppliers ($u_{\max} = 12$) will never be exceeded.

Figures 3-6 illustrate the results of the simulation. Figure 3 demonstrates that sliding variable $\sigma(k)$ is driven to the

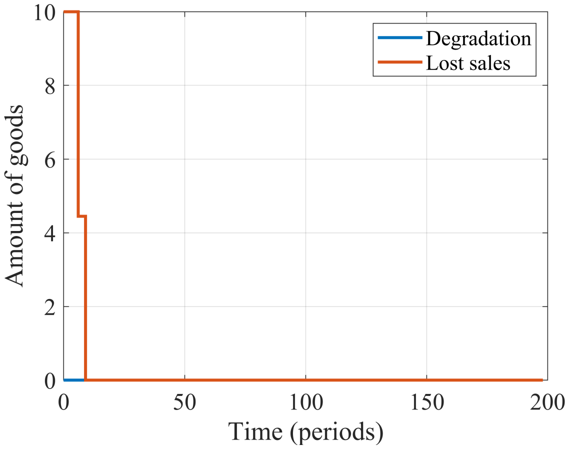

FIGURE 6. Lost goods and unfulfilled demand.

vicinity of zero in a small number of initial steps and remains in that vicinity for all future time instants. Furthermore, consistently with Lemma 1, this variable never assumes negative values. Figure 4 shows the amount of goods stored in the warehouse at a given time. It can be seen that, after a finite number of initial steps, this amount is strictly positive, which means the demand is always satisfied. Furthermore, the available warehouse capacity of 38 is never exceeded. Figure 5 illustrates the control signal. It can be clearly seen that the amount of ordered goods never exceeds the capabilities of the suppliers and never creates the need to return stored wares. Finally, Figure 6 shows the amount of goods that exceeded their shelf life as well as the amount of unfulfilled demand. It can be seen that no goods are ever lost due to their end of shelf life and that after a finite number of initial time instants, customers' demand is always fully satisfied. This is consistent with Theorems 1 and 3, respectively. It is worth noticing that, in the considered example, the admissible range of constant y_d was very narrow as we wish to satisfy (42) and (54), which makes it close to the fringe case described in Remark 2.

VI. CONCLUSION

In this paper a control algorithm for management of inventories with limited shelf life has been proposed. A logistic chain in which a single type of product is being transported from multiple suppliers to a common warehouse has first been described in detail. Each supplier has a non-negligible delivery time and the product can then be stored at the warehouse for a specific amount of review periods. When the product's shelf life ends, it needs to be disposed of. Goods stored at the warehouse are then sold according to unpredictable consumers' demand and the product closest to the end of its shelf life is sold first. The model of a supply chain introduced in this work reflects loss of wares in a realistic manner as opposed to former works on the subject, which have not strictly considered limited shelf life. Indeed, former works have typically considered degradation expressed as a fraction of stored wares, while in this paper loss of wares is appropriately expressed as a finite expiration date for each of the delivered products.

It has been demonstrated that, using the proposed sliding mode control strategy, one can completely prevent loss of wares after a finite number of initial review periods, while also fully satisfying the consumers' demand. As such, the proposed sliding mode controller imposes practical bounds on the considered plant, which can be seen in the theorems presented in the paper. These properties are achieved with the right selection of warehouse capacity. It has been further shown that the size of orders generated by the proposed control scheme is always lower and upper bounded, which means it can comply with capabilities of the suppliers. Increasing the order of the considered system for products with a longer shelf life is also easy to achieve, due to the simplicity of state matrix (10) and high computational efficiency of the proposed sliding mode controller, as well as the ease of its design.

Another interesting development one could consider is the option of lowering price of products closer to the end of their shelf life to incentivize their sales in a wider variety of situations. However, since the main objective of this particular paper is to present the novel state-space model of a logistic chain with perishable products, these extensions will be relegated to future works.

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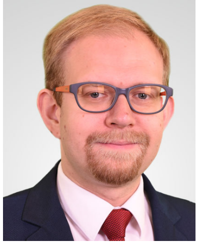

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