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RESEARCH ARTICLE

Picture Fuzzy Directed Hypergraphs With Applications Toward Decision-Making and Managing Hazardous Chemicals

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ABSTRACT Many real-life problems present multiple options necessitating modern researchers to identify the most suitable choice or decision. In this context, fuzzy sets have proven to be valuable tools for addressing such problems. Picture fuzzy sets (PFSs), due to their increased flexibility and broader domain as compared to other extensions of fuzzy sets (FSs) have been extensively used in solving various decision-making problems. To achieve optimal decisions, various fuzzy graphs and fuzzy hypergraph structures are employed. Picture fuzzy directed hypergraphs (PFDHGs) offer greater utility in addressing problems with multiple options or uncertainties. In this study, we introduce novel concepts of PFDHGs, along with their applications in decision-making theory and hazardous chemicals management. Initially, we present the foundational concepts of PFDHGs, including essential terminologies such as hyper-edge and height. Subsequently, we explore various types of PFDHGs, such as simple PFDHGs and support simple PFDHGs. Additionally, we introduce the concept of $(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma})$-level hyperarcs and elucidate the fundamental sequence in PFDHGs based on these arcs. The study further examines elementary PFDHGs, partial PFDHGs, and ordered PFDHGs, investigating the relationships among them. Structural properties of PFDHGs are analyzed through different types of homomorphisms of PFDHGs. We also define the order and size of PFDHGs. Finally, we present the applications of PFDHGs in decision-making and managing hazardous chemicals, emphasizing the innovative contributions and practical implications of our work.

INDEX TERMS Decision making, hazardous chemicals, management, partial PFDHGs, PFDHGs.

I. INTRODUCTION

The concepts of FSs was first introduced by Zadeh [1] in 1965. FSs has been proven to be a useful tool in solving various daily life problems with uncertainties. Following this, numerous generalized forms of FSs have been introduced in the literature. Firstly, interval-valued fuzzy sets (IVFSs) as a generalization of FSs was initiated by Zadeh [2]. Many other direct extensions of FSs and their applications have been explored from time to time. Some of the well known extensions of FSs include intuitionistic fuzzy sets (IFSs), bipolar fuzzy sets (BPFSs), picture fuzzy sets (PFSs) etc. IFSs was introduced by Atanassov [3]. IFSs allocates the membership and non-membership degrees to each of its entity. The notion of BPFSs was added to the literature by Zhang [4]. BPFSs express each of its entity using positive membership and negative membership degrees. Recently, Arif et al. [5] investigated the application of BPFSs towards decision-making. Cuong and Kreinovich [6] proposed the concepts of PFSs, where each member was assigned three values i.e., membership, neutral and non membership values. PFSs was more generalized than both FSs and IFSs.

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TABLE 1. Abbreviations used in this manuscript.

Terms	Notations
Decision making	DM
Membership degree	MD
Fuzzy sets	FSs
Intuitionistic fuzzy sets	IFSs
Picture fuzzy	PF
Picture fuzzy sets	PFSs
Fuzzy graphs	FGs
Intuitionistic fuzzy graphs	IFGs
Picture fuzzy graphs	PFGs
Hypergraphs	HGs
Fuzzy hypergraphs	FHGs
Fuzzy directed hypergraphs	FDHGs
Intuitionistic fuzzy directed hypergraphs	IFDHGs
Picture fuzzy hypergraphs	PFHGs
Picture fuzzy directed hypergraphs	PFDHGs

We refer [7] for basic definitions and applications of PFSs in DM. Afterwards, Phong et al. [8] studied many operations and relations on PFSs. Different types of PFSs such as interval-valued picture fuzzy sets (IVPFSs) [9], bipolar picture fuzzy sets [10] etc have been introduced in the literature. Overall, FSs and its extensions have played a key role in solving many real-world problems related to network analysis, medical sciences, decision-making theory etc. FSs and its generalizations has played a crucial role towards decision making specially multi criteria decision analysis (MCDA). One can consult [11] for applications and trends of MCDA.

The idea of fuzzy graphs (FGs) was firstly given by Rosenfeld [12]. Through FGs, more effective and efficient models have been explored for explaining real world problems as compared to those of crisp graphs. Like FSs, many generalizations of FGs have been introduced. An extension of FGs termed Interval-valued fuzzy graphs (IVFGs) was initiated in [13]. Shannon and Atanassov [14] proposed the idea of intuitionistic fuzzy graphs (IFGs) which was further elaborated in [15]. Interval-valued intuitionistic fuzzy graphs (IVIFGs) was proposed in [16]. Zuo et al. [17] added the term picture fuzzy graphs (PFGs) as an extension FGs and IFGs. Recently, many types of PFGs with applications in various fields have been explored. For instance, Khan et al. introduced bipolar PFGs with application towards social networks [18], Cayley PFGs with application in networking [19], Arif et al. introduced some indices for PFGS with applications in site selection and networking [20], and interval-valued picture (S, T)-fuzzy graphs with application in MADM [21]. Similarly, Shi et al. introduced the concepts of interval-valued picture fuzzy graphs and provided its applications in social networks and transmission control protocol [22]. For more work on PFGs, one may consult [23], [24], [25]

A hypergraph is also a mathematical structure consisting of vertices and edges. Different types of hypergraphs and their applications in numerous areas of sciences have been investigated. Basically, an hypergraph structure is the pair

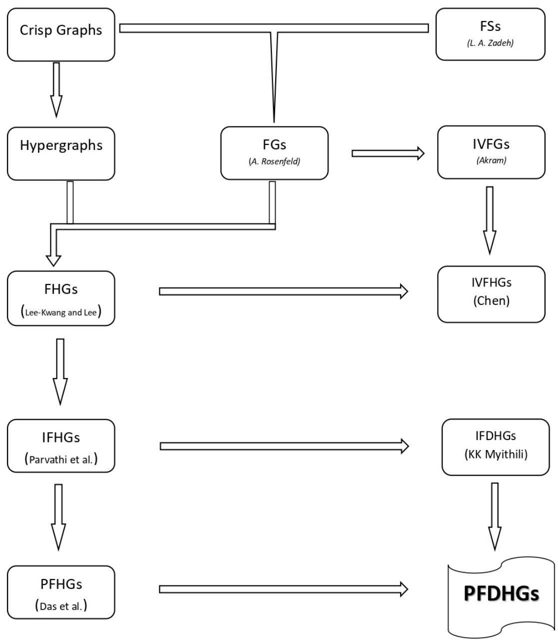

FIGURE 1. Generalizations of Fuzzy hypergraphs.

$\bar{H} = (\bar{V}, \bar{E})$, where $\bar{V} = \{\bar{v}_1, \bar{v}_2, \bar{v}_3, \dots \bar{v}_n\}$ and $\bar{E} = \{\bar{E}_1, \bar{E}_2, \bar{E}_3, \dots \bar{E}_r\}$ are the sets of non-empty set of vertices and edges, respectively with $\bigcup_k \bar{E}_k = \bar{V}_k$, for $k = 1, 2, 3, \dots r$. A hypergraph is a simple, if $\bar{E}_i \subseteq \bar{E}_j$ implies $i = j$. A linear hypergraph is a simple hypergraph with $| \bar{E}_i \bigcap \bar{E}_j | \leq 1$, for each $\bar{E}_i, \bar{E}_j \in \bar{E}$. Alternatively, Lee-Kwang and Lee [26] introduced the notion of fuzzy hypergraphs (FHGs). After this, various generalizations of FHGs were introduced like interval-valued fuzzy hypergraphs (IVFHGs) [27], intuitionistic fuzzy hypergraphs (IFHGs) [28], [29], interval-valued intuitionistic fuzzy hypergraphs (IVIFHGs) [30] etc. The term Intutitionistic fuzzy directed hypergraphs (IFDHGs) was initiated in [31] and [32]. Recently, the notion of interval-valued picture fuzzy hypergraph was explored in [33]. Sequentially, by incorporating the term PFDHGs, we bridge the gap in the literature as highlighted in Fig. 1. Some abbreviations used in this manuscript are enlisted in Table 1.

PFSs is the extension of both the FSs and IFSs that's why it is more capable to deal uncertainties properly as compared to the other generalizations of FSs. Likewise, PFG is the generalization of both FGs and IFGs. On the same pattern, FHGs, FDHGS, IFHGS, IFDHGS and PFHGS have been introduced in the literature. However, the notion PFDHGs is missing in the existing literature. To fill this gap, we initiate the idea of PFDHGs along with its applications towards decision making and chemical graph theory. The capability of PFDHGs to deal uncertain information using its vast unique structure makes it one of the best tool to deal many problems occur in daily life. PFDHGs can deal uncertain information and help in achieving the perfect option or solution for the problem.

Novelty: The novelty of our work can be described in the following steps.

1) Initially, we shift many important characterizations of FDHGs and IFDHGs towards PFDHGs.
2) We present useful terminologies like hyper-edge, height and $(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma})$-level hyperarcs of PFDHGs.
3) Various types of PFDHGs like simple PFDHGs, support simple PFDHGs, elementary PFDHGs, partial PFDHGs, ordered PFDHGs are also explored.
4) We also define different types of structure preserving maps regarding PFDHGs and apply them collectively in theorems 1 and 2.
5) Finally, applications of PFDHGs towards decision making and most proficient arrangement for hazardous wastes (chemicals) are presented.

Motivations: The motivations of this study are as follows.

1) Mainly, the existing notions like FDHGs and IFDHGs motivated us to initiate the concept of PFDHGs.
2) PFDHGs being advanced form of FDHGs increases the options of expressing the uncertainties.
3) The PFDHGs are more reliable and flexible than the crisp DHGs and FDHGs and are easy to apply to any system. Thus PFDHGs provide more accuracy to the system.
4) PFDHGs can be twisted towards FDHGs and IFDHGs by assigning different membership values. Hence, PFDHGs combines the qualitative characteristics of both the FDHGs and IFDHGs.

Remaining manuscript is organized as: In section II, some useful terminologies are provided. In section III, we start our discussion by defining PFDHGs by using PF-relations. We present various types of IVPFHGs and discuss different terms like height of PFDHGs, support and support simple, partial PFDHGs and $(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma})$-level cuts of PFDHGs. Homomorphism, weak and co homomorphism and isomorphism of PFDHGs are also discussed. Order and size of PFDHGs are also defined. In section IV, we provide two applications of IVPFHGs. Section IV-A contains an application of PFDHGs towards decision making theory along with numerical computations, and in section IV-B, we provide an application for the most proficient arrangement of hazardous wastes (chemicals). We provide superiority and comparative analysis of our study with the existing ones in section V. Finally, we provide the conclusion and future prospects of our study.

II. PRELIMINARIES

Definition 1 [1] A pair $(\bar{\xi}, V)$, where V is any nonempty set and $\bar{\xi} : V \longrightarrow [0, 1]$ denotes the MD, is a FS.

Definition 2 [3]: An IFS $\mathcal{I}$ is the collection which can be expressed as $\mathcal{I} = \{(v, \vartheta_{\mathcal{I}}(v), \upsilon_{\mathcal{I}}(v)) : v \in V\}$ where $\vartheta_{\mathcal{I}}(v) \in [0, 1]$ is the MD of v in $\mathcal{I}$, $\upsilon_{\mathcal{I}}(v) \in [0, 1]$ is the non-MD of v in I such that $\vartheta_{\mathcal{I}}(v) + \upsilon_{\mathcal{I}}(v) \leq 1$, for all $v \in V$.

Definition 3 [6]: A PFS $\mathcal{P}$ is a collection written as $\mathcal{P} = \{(s, \phi_{\mathcal{P}}(s), \chi_{\mathcal{P}}(s), \psi_{\mathcal{P}}(s)) : s \in V\}$, where $\phi_{\mathcal{P}}(s) \in [0, 1]$ is the MD, $\chi_{\mathcal{P}}(s) \in [0, 1]$ is the neutral-MD and $\psi_{\mathcal{P}}(s) \in [0, 1]$ denotes the non-MD of s in $\mathcal{P}$ such that $(\phi_{\mathcal{P}}(s) + \chi_{\mathcal{P}}(s) + \psi_{\mathcal{P}}(s) \leq 1)$.

Here, $(1-(\phi_{\mathcal{P}}(s)+\chi_{\mathcal{P}}(s)+\psi_{\mathcal{P}}(s))$ is the degree of refusal MD of s in $\mathcal{P}$.

Definition 4 [12]: Let V be a nonempty set. Then the FG $\mathcal{G}$ on V is presented as a pair of functions $\omega_{\bar{p}}$ and $\omega_{\bar{q}}$ i.e., $G = (\omega_{\bar{p}}, \omega_{\bar{p}})$. Where $\omega_{\bar{p}}$ is the fuzzy subset of V and $\omega_{\bar{q}}$ is a symmetric fuzzy relation on $V \times V$ i.e., $\omega_{\bar{p}}$: $V \rightarrow [0, 1]$ and $\omega_{\bar{q}}$: $V \times V \rightarrow [0, 1]$ satisfying $\omega_{\bar{q}}(x, y) \leq \omega_{\bar{p}}(x) \wedge \omega_{\bar{p}}(y)$, $\forall x, y \in V$.

Definition 5 [14]: Let $G^* = (Q^*, R^*)$ be a graph. Then, $G = (Q, R)$ is an IFG over a graph G^*, where $Q = (\phi_Q, \psi_Q)$ is an IFS on $(Q^*$ and $R = (\phi_R, \psi_R)$ is an IFS over $R^* \subseteq Q^* \times Q^*$. such that for each edge $e_1e_2 \in R^*$,

$$\phi_R(e_1e_2) \leq min\phi_Q(e_1), \phi_Q(e_2)$$
$$\psi_R(e_1e_2) \geq max\psi_Q(e_1), \psi_Q(e_2).$$

Definition 6 [17]: Let $G^* = (Q^*, R^*)$ be a graph. Then, $G = (Q, R)$ is a PFG on $G^* = (Q^*, R^*)$, where $Q = (\phi_Q, \chi_Q, \psi_Q)$ is a PFS on Q^* and $R = (\phi_R, \chi_R, \psi_R)$ is a PFS over $R^* \subseteq Q^* \times Q^*$ such that for each edge $e_1e_2 \in R^*$,

$$\phi_R(e_1e_2) \leq min\phi_Q(e_1), \phi_Q(e_2)$$
$$\chi_R(e_1e_2) \leq min\chi_Q(e_1), \chi_Q(e_2)$$
$$\psi_R(e_1e_2) \geq max\psi_Q(e_1), \psi_Q(e_2).$$

We refer [17] for further discussions on PFGs.

III. PICTURE FUZZY DIRECTED HYPERGRAPHS (PFDHGs)

In this section, firstly we introduce the notions PF directed hyperarc and PFDHGs. In the context of PFDHGs, we define height of PF directed hyperarc. Moreover, we describe various types of PFDHGs such as simple, support simple and elementary PFDHGs. Further to this, we also introduce the concept of $(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma})$-level hyperarcs, partial PFDHG, transversal of PFDHG and fundamental sequence of PFDHGs. Finally, we discuss the concept of homomorphism and isomorphism of PFDHGs.

Definition 7: On a non-empty set of vertices V, a directed hyperarc can be described as a couple of $\overrightarrow{F} = (t(\overrightarrow{F}), h(\overrightarrow{F}),$

where $t(\overrightarrow{F})$ and $h(\overrightarrow{F})$ are the disjoint subsets of V. A vertex x is called a source vertex in V, if $x \notin h(\overrightarrow{F})$. A vertex d is called destination vertex in V, if $d \notin t(\overrightarrow{F})$. A directed hyperarc is called a picture fuzzy directed hyperarc, if it contains a pair $\overrightarrow{\zeta} = (t(\overrightarrow{\zeta}), h(\overrightarrow{\zeta}))$ of disjoint PF subsets of vertices such that $t(\overrightarrow{\zeta})$ is the tail of ζ and $h(\overrightarrow{\zeta})$ is head of $\overrightarrow{\zeta}$, respectively.

Definition 8: A PFDHG on a non-empty set V is $\overrightarrow{\mathcal{H}} = (P, R)$, where $P = \{\overrightarrow{\underline{\eta}}_1, \overrightarrow{\underline{\eta}}_2, \dots \overrightarrow{\underline{\eta}}_r\}$ is collection of ordered pairs of $\overrightarrow{\underline{\eta}}_k = (t(\overrightarrow{\underline{\eta}}_k), h(\overrightarrow{\underline{\eta}}_k))$, where $t(\overrightarrow{\underline{\eta}}_k)$ and $h(\overrightarrow{\underline{\eta}}_k)$ are disjoint PF subsets on V and R is defined as PF relation on $\overrightarrow{\underline{\eta}}_k$ satisfying

1) $\phi_R(\overrightarrow{F}) = \phi_R(t(\overrightarrow{F}), h(\overrightarrow{F})) \le min\{\wedge_{i=1}^{m} \phi_{t(\overrightarrow{\underline{\eta}}_k)}(u_i), \wedge_{i=1}^{n} \phi_{h(\overrightarrow{\underline{\eta}}_k)}(v_i)\}$
2) $\chi_R(\overrightarrow{F}) = \chi_R(t(\overrightarrow{F}), h(\overrightarrow{F})) \le min\{\wedge_{i=1}^{m} \chi_{t(\overrightarrow{\underline{\eta}}_k)}(u_i), \wedge_{i=1}^{n} \chi_{h(\overrightarrow{\underline{\eta}}_k)}(v_i)\}$
3) $\psi_R(\overrightarrow{F}) = \psi_R(t(\overrightarrow{F}), h(\overrightarrow{F})) \ge max\{\wedge_{i=1}^{m} \psi_{t(\overrightarrow{\underline{\eta}}_k)}(u_i), \wedge_{i=1}^{n} \psi_{h(\overrightarrow{\underline{\eta}}_k)}(v_i)\}$
4) $\phi_R(\overrightarrow{F}) + \chi_R(\overrightarrow{F}) + \psi_R(\overrightarrow{F}) \le 1$, for each $\overrightarrow{F}_k$, $1 \le k \le r$, where $t(\overrightarrow{F}) = \{u_1, u_2, \dots u_m\} \subset V$ and $h(\overrightarrow{F}) = \{v_1, v_2, \dots v_m\} \subset V$
5) $\bigcup_k \text{supp}(t(\overrightarrow{\underline{\eta}}_k) \cup \bigcup_k \text{supp}(h(\overrightarrow{\underline{\eta}}_k)) = V, k = 1, 2, 3, \dots r$

Definition 9: Let $\overrightarrow{\mathcal{H}} = (P, R)$ is a PFDHG then height $(h(\overrightarrow{\underline{\eta}}))$ of the PF directed hyperarc $\overrightarrow{\underline{\eta}}$ is defined as

$$h(\overrightarrow{\underline{\eta}}) = (\phi_{h(\overrightarrow{\underline{\eta}}_k)}, \chi_{h(\overrightarrow{\underline{\eta}}_k)}, \psi_{h(\overrightarrow{\underline{\eta}}_k)}) = (\max\{\wedge_{x\in V} \phi_{t(\overrightarrow{\underline{\eta}})}(u), \wedge_{x\in V} \phi_{h(\overrightarrow{\underline{\eta}})}(u)\}, \max\{\wedge_{x\in V} \chi_{t(\overrightarrow{\underline{\eta}})}(u), \wedge_{x\in V} \chi_{h(\overrightarrow{\underline{\eta}})}(u)\}, \min\{\wedge_{x\in V} \psi_{t(\overrightarrow{\underline{\eta}})}(u), \wedge_{x\in V} \psi_{h(\overrightarrow{\underline{\eta}})}(u)\})$$

Definition 10: A PFDHG will be simple if for every $\overrightarrow{\underline{\eta}}_i$, $\overrightarrow{\underline{\eta}}_j \in P$, $supp(t(\overrightarrow{\underline{\eta}}_i)) \subseteq supp(t(\overrightarrow{\underline{\eta}}_i))$ and $supp(h(\overrightarrow{\underline{\eta}}_i)) \subseteq supp(h(\overrightarrow{\underline{\eta}}_i))$ then $i = j$.

Definition 11: A PFDHG $\overrightarrow{\mathcal{H}} = (P, R)$ will be support simple if whenever $\overrightarrow{\underline{\eta}}_i$, $\overrightarrow{\underline{\eta}}_j \in P$, $t(\overrightarrow{\underline{\eta}}_i) \subseteq t(\overrightarrow{\underline{\eta}}_i)$, $(h\overrightarrow{\underline{\eta}}_i) \subseteq h(\overrightarrow{\underline{\eta}}_i)$ also $supp(t(\overrightarrow{\underline{\eta}}_i)) = supp(t(\overrightarrow{\underline{\eta}}_i))$ and $supp(h(\overrightarrow{\underline{\eta}}_i)) = supp(h(\overrightarrow{\underline{\eta}}_i))$ then $\overrightarrow{\underline{\eta}}_i = \overrightarrow{\underline{\eta}}_j$ i and j.

Definition 12: Let us consider $\overrightarrow{\mathcal{H}} = (P, R)$ is a PFDHG on V. For any $\widetilde{\alpha}$, $\widetilde{\beta}$ and $\widetilde{\gamma} \in [0, 1]$, the $(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma})$- level hyperarc of a PF directed hyperarc $\overrightarrow{\underline{\eta}}$ is defined as

$$\begin{aligned}\overrightarrow{\underline{\eta}}_{(\widetilde{\alpha},\widetilde{\beta},\widetilde{\gamma})} &= (t(\overrightarrow{\underline{\eta}}_{(\widetilde{\alpha},\widetilde{\beta},\widetilde{\gamma})}), (h(\overrightarrow{\underline{\eta}}_{(\widetilde{\alpha},\widetilde{\beta},\widetilde{\gamma})}))) \\ &= ((p \in V \mid \phi_t(\overrightarrow{\underline{\eta}})(p) \ge \widetilde{\alpha}, \chi_t(\overrightarrow{\underline{\eta}}) \ge \widetilde{\beta}, \\ &\quad \psi_t(\overrightarrow{\underline{\eta}})(p) \le \widetilde{\gamma}), (q \in V \mid \phi_h(\overrightarrow{\underline{\eta}})(q) \\ &\quad \ge \widetilde{\alpha}, \chi_h(\overrightarrow{\underline{\eta}}) \ge \widetilde{\beta}, \psi_h(\overrightarrow{\underline{\eta}})(q) \le \widetilde{\gamma})).\end{aligned}$$

$\overrightarrow{\mathcal{H}}(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}) = (P(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}), R(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}))$ is called a $(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma})$-level directed hypergraph of $\overrightarrow{\mathcal{H}}$ where, $P(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma})$ is defined as

$$P(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}) = \{(\bigcup_{k=1}^{r}(h(\overrightarrow{\underline{\eta}}_{k(\widetilde{\alpha},\widetilde{\beta},\widetilde{\gamma})}))) \bigcup (\bigcup_{k=1}^{r}(t(\overrightarrow{\underline{\eta}}_{k(\widetilde{\alpha},\widetilde{\beta},\widetilde{\gamma})})) \quad 1 \le k \le r)\},$$

Definition 13: Let us consider $\overrightarrow{\mathcal{H}} = (P, R)$ is a PFDHG on V. The fundamental sequence$(f_s(\overrightarrow{\mathcal{H}}))$ of $\overrightarrow{\mathcal{H}}$ is a triplet $(\widetilde{\alpha}$, $\widetilde{\beta}$ and $\widetilde{\gamma}) \in [0, 1] \times [0, 1] \times [0, 1]$, $0 \le \widetilde{\alpha}_i + \widetilde{\beta}_i + \widetilde{\gamma}_i \le 1$, $1 \le i \le n$, such that $\widetilde{\alpha}_1 > \widetilde{\alpha}_2 > \widetilde{\alpha}_3 > \dots \widetilde{\alpha}_n$, $\widetilde{\beta}_1 > \widetilde{\beta}_2 > \widetilde{\beta}_3 > \dots \widetilde{\beta}_n$ and $\widetilde{\gamma}_1 < \widetilde{\gamma}_2 < \widetilde{\gamma}_3 < \dots \widetilde{\gamma}_n$ such that

1) If $1 \ge \widetilde{\alpha} > \widetilde{\alpha}_1$, $1 \ge \widetilde{\beta} > \widetilde{\beta}_1$, and $0 \le \widetilde{\gamma} < \widetilde{\gamma}_1$ then $R_{(\widetilde{\alpha},\widetilde{\beta},\widetilde{\gamma})} = \phi$
2) If $\widetilde{\alpha}_{i+1} < \widetilde{\alpha} \le \widetilde{\alpha}_i$, $\widetilde{\beta}_{i+1} < \widetilde{\beta} \le \widetilde{\beta}_i$ and $\widetilde{\gamma}_1 \le \widetilde{\gamma} < \widetilde{\gamma}_{i+1}$ then $R_{(\widetilde{\alpha},\widetilde{\beta},\widetilde{\gamma})} = R_{(\widetilde{\alpha}_i,\widetilde{\beta}_i,\widetilde{\gamma}_i)}$
3) $R_{(\widetilde{\alpha}_i,\widetilde{\beta}_i,\widetilde{\gamma}_i)} \sqsubset R_{(\widetilde{\alpha}_{i+1},\widetilde{\beta}_{i+1},\widetilde{\gamma}_{i+1})}$

The core set$(\$(\overrightarrow{\mathcal{H}}))$ of $\overrightarrow{\mathcal{H}}$ is the corresponding sequence of $(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma})$-level directed hypergraphs $\overrightarrow{\mathcal{H}}_{(\widetilde{\alpha}_1,\widetilde{\beta}_1,\widetilde{\gamma}_1)}$, $\overrightarrow{\mathcal{H}}_{(\widetilde{\alpha}_2,\widetilde{\beta}_2,\widetilde{\gamma}_2)}$, . . ., $\overrightarrow{\mathcal{H}}_{(\widetilde{\alpha}_n,\widetilde{\beta}_n,\widetilde{\gamma}_n)}$. The $(\widetilde{\alpha}_n, \widetilde{\beta}_n, \widetilde{\gamma}_n)$-level directed hypergraphs $\overrightarrow{\mathcal{H}}_{(\widetilde{\alpha}_n,\widetilde{\beta}_n,\widetilde{\gamma}_n)}$ is called support level of $\overrightarrow{\mathcal{H}}$.

Definition 14: Let us consider $\overrightarrow{\mathcal{H}} = (P, R)$ is a PFDHG on V then $supp(P) = \{(supp(t(\overrightarrow{\underline{\eta}}), supp(h(\overrightarrow{\underline{\eta}}))) \mid \overrightarrow{\underline{\eta}} \in P\}$. The collection of PF directed hyperarcs P is called elementary if P is single-valued on $supp(P)$. A PFDHG $\overrightarrow{\mathcal{H}}$ is elementary if P and R are elementary otherwise it is non-elementary.

Definition 15: Let $\overrightarrow{\acute{\mathcal{H}}} = (\acute{P}, \acute{R})$ be a PFDHG. Then, a subgraph $\overrightarrow{\mathcal{H}} = (P, R)$ of $\overrightarrow{\acute{\mathcal{H}}}$ is called partial PFDHG if

1) $supp(P) \subseteq supp(\acute{P})$ and $supp(R) \subseteq supp(\acute{R})$,
2) if $supp(\overrightarrow{\underline{\eta}}_i) \in supp(P)$ and $supp(\overrightarrow{\underline{\acute{\eta}}}_i) \in supp(\acute{P})$ such that

$$supp(\overrightarrow{\underline{\eta}}_i) = supp(\overrightarrow{\underline{\acute{\eta}}}_i)$$

then $\overrightarrow{\underline{\eta}}_i = \overrightarrow{\underline{\acute{\eta}}}_i$.

It is represented as $\overrightarrow{\mathcal{H}} \subseteq \overrightarrow{\acute{\mathcal{H}}}$.

Definition 16: A PFDHG $\overrightarrow{\mathcal{H}} = (P, R)$ is called ordered, if the core set $\$(\overrightarrow{\mathcal{H}}) = \{\overrightarrow{\mathcal{H}}_{(\widetilde{\alpha}_1,\widetilde{\beta}_1,\widetilde{\gamma}_1)}, \overrightarrow{\mathcal{H}}_{(\widetilde{\alpha}_2,\widetilde{\beta}_2,\widetilde{\gamma}_2)}, \dots \overrightarrow{\mathcal{H}}_{(\widetilde{\alpha}_n,\widetilde{\beta}_n,\widetilde{\gamma}_n)}\}$ is ordered that is $\overrightarrow{\mathcal{H}}_{(\widetilde{\alpha}_1,\widetilde{\beta}_1,\widetilde{\gamma}_1)} \subseteq \overrightarrow{\mathcal{H}}_{(\widetilde{\alpha}_2,\widetilde{\beta}_2,\widetilde{\gamma}_2)} \subseteq \dots \subseteq \overrightarrow{\mathcal{H}}_{(\widetilde{\alpha}_n,\widetilde{\beta}_n,\widetilde{\gamma}_n)}$. $\overrightarrow{\mathcal{H}}$ is simply ordered if $\overrightarrow{\mathcal{H}}$ is ordered and whenever $\acute{R} \subset R_{(\widetilde{\alpha}_{i+1},\widetilde{\beta}_{i+1},\widetilde{\gamma}_{i+1})} \setminus R_{(\widetilde{\alpha}_i,\widetilde{\beta}_i,\widetilde{\gamma}_i)}$ then $\acute{R} \subseteq R_{(\widetilde{\alpha}_i,\widetilde{\beta}_i,\widetilde{\gamma}_i)}$

Proposition 1: Let $\overrightarrow{\mathcal{H}}$ be an elementary PFDHG then $\overrightarrow{\mathcal{H}}$ is ordered. If $\overrightarrow{\mathcal{H}}$ is ordered PFDHG and support level $\overrightarrow{\mathcal{H}}_{(\widetilde{\alpha}_n,\widetilde{\beta}_n,\widetilde{\gamma}_n)}$ is simple then $\overrightarrow{\mathcal{H}}$ is an elementary PFDHG.

Remark 1:
1) If $\overrightarrow{\mathcal{H}} = (P, R)$ is a PFDHG with $P = \{\overrightarrow{\underline{\eta}}_1, \overrightarrow{\underline{\eta}}_2, \dots \overrightarrow{\underline{\eta}}_r\}$ then $P^* = \{\overrightarrow{\underline{\eta}}_1^*, \overrightarrow{\underline{\eta}}_2^*, \dots \overrightarrow{\underline{\eta}}_r^*\}$ is collection of crisp directed hyperarcs corresponding to P.
2) In PFDHG $\overrightarrow{\mathcal{H}} = (P, R)$, if x is a vertex of tail of any PF directed hyperarc $\overrightarrow{\underline{\eta}}$ then $\overrightarrow{\underline{\eta}}_1(x) = (\psi_{t(\overrightarrow{\underline{\eta}})}, \chi_{t(\overrightarrow{\underline{\eta}})}, \phi_{t(\overrightarrow{\underline{\eta}})})$. If $x \in h(\overrightarrow{\underline{\eta}})^*$ then $\overrightarrow{\underline{\eta}}(x) = (\psi_{h(\overrightarrow{\underline{\eta}})}, \chi_{h(\overrightarrow{\underline{\eta}})}, \phi_{h(\overrightarrow{\underline{\eta}})})$

Definition 17: Let $\overrightarrow{\mathcal{H}} = (P, R)$ be a PFDHG. The PF transversal of $\overrightarrow{\mathcal{H}}$ is a set of PF directed hyperarcs $\widetilde{\mathcal{T}}$ such that $\widetilde{\mathcal{T}}_h(\overrightarrow{\underline{\eta}}_i) \cap \overrightarrow{\underline{\eta}}_{ih}(\overrightarrow{\underline{\eta}}_i) \neq \phi$, for each $\overrightarrow{\underline{\eta}}_i \in P$. $\widetilde{\mathcal{T}}$ is said to be minimal PF transversal if whenever $\epsilon \subset \widetilde{\mathcal{T}}$, ϵ is not a PF transversal of $\overrightarrow{\mathcal{H}}$. The collection of all minimal PF transversals of $\overrightarrow{\mathcal{H}}$ is expressed as $\widetilde{\mathcal{T}}_{\tilde{r}}(\overrightarrow{\mathcal{H}})$.

Lemma 1: $\overrightarrow{\mathcal{H}} = (P, R)$ be a PFDHG with fundamental sequence $f_s(\overrightarrow{\mathcal{H}}) = \{(\widetilde{\alpha}_1, \widetilde{\beta}_1, \widetilde{\gamma}_1), (\widetilde{\alpha}_2, \widetilde{\beta}_2, \widetilde{\gamma}_2)..., (\widetilde{\alpha}_n, \widetilde{\beta}_n, \widetilde{\gamma}_n)\}$. If $\widetilde{\mathcal{T}}$ is a PF transversal of $\overrightarrow{\mathcal{H}}$ then $\phi_{h(\widetilde{\mathcal{T}})} \geq \phi_{h(\overrightarrow{\underline{\eta}_i})}$, $\chi_{h(\widetilde{\mathcal{T}})} \geq \chi_{h(\overrightarrow{\underline{\eta}_i})}$ and $\psi_{h(\widetilde{\mathcal{T}})} \leq \psi_{h(\overrightarrow{\underline{\eta}_i})}$. If $\widetilde{\mathcal{T}}$ is a minimal PF transversal of $\overrightarrow{\mathcal{H}}$ then $h(\widetilde{\mathcal{T}}) = (max\{\phi_{h(\overrightarrow{\underline{\eta}_i})} \mid \overrightarrow{\underline{\eta}_i} \in P\}, max\{\chi_{h(\overrightarrow{\underline{\eta}_i})} \mid \overrightarrow{\underline{\eta}_i} \in P\}, min\{\psi_{h(\overrightarrow{\underline{\eta}_i})} \mid \overrightarrow{\underline{\eta}_i} \in P\}) = (\widetilde{\alpha}_1, \widetilde{\beta}_1, \widetilde{\gamma}_1)$

Proposition 2: Let $\overrightarrow{\mathcal{H}}$ be a PFDHG, then following are equivalent.

1) $\widetilde{\mathcal{T}}$ is a PF transversal of $\overrightarrow{\mathcal{H}}$
2) For every $\overrightarrow{\underline{\eta}_i} \in P$, $(\widetilde{\alpha}_1, \widetilde{\beta}_1, \widetilde{\gamma}_1) \in [0, 1] \times [0, 1] \times [0, 1]$, $0 \leq \widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma} \leq 1$ with $\widetilde{\alpha} < \phi_{h(\overrightarrow{\underline{\eta}_i})}$, $\widetilde{\beta} < \chi_{h(\overrightarrow{\underline{\eta}_i})}$ and $\widetilde{\gamma} > \phi_{h(\overrightarrow{\underline{\eta}_i})}$ then $\widetilde{\mathcal{T}}_{(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma})} \bigcap \underline{\eta}_{i(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma})} \neq \phi$
3) $\widetilde{\mathcal{T}}_{(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma})}$ is a transversal of $\overrightarrow{\mathcal{H}}_{(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma})}$

Graph (Hypergraph) homomorphism play an important role in understanding the relationships and structural characteristics among two different graphs. By preserving the connections between vertices, graph homomorphism analyze how one graph can be expressed within another graph. It play an important role in various fields like networking, optimization, computer science etc. Here, we will discuss about the homomorphism and isomorphism of PFDHGs.

Definition 18: Let $\overrightarrow{\mathcal{H}} = (P, R)$ and $\overrightarrow{\acute{\mathcal{H}}} = (\acute{P}, \acute{R})$ be any two PFDHGs on X and $\acute{X}$, respectively, where $P = \{\overrightarrow{\underline{\eta}_1}, \overrightarrow{\underline{\eta}_2}, \ldots \overrightarrow{\underline{\eta}_r}\}$ and $P^* = \{\overrightarrow{\underline{\eta}_1}^*, \overrightarrow{\underline{\eta}_2}^*, \ldots \overrightarrow{\underline{\eta}_r}^*\}$. A homomorphism of PFDHGs $\overrightarrow{\mathcal{H}}$ and $\overrightarrow{\acute{\mathcal{H}}}$ is a mapping $\varphi : X \longrightarrow \acute{X}$ that satisfies

1) $\wedge_{j=1}^r \psi_{\overrightarrow{\underline{\eta}_j}} \leq \wedge_{j=1}^r \phi_{\overrightarrow{\underline{\eta}_j}^*}(\varphi(x))$, $\wedge_{j=1}^r \chi_{\overrightarrow{\underline{\eta}_j}} \leq \wedge_{j=1}^r \chi_{\overrightarrow{\underline{\eta}_j}^*}(\varphi(x))$and $\vee_{j=1}^r \phi_{\overrightarrow{\underline{\eta}_j}} \geq \vee_{j=1}^r \phi_{\overrightarrow{\underline{\eta}_j}^*}(\varphi(x))$, $\forall x \in X$
2) $\psi_R(\{t_1, \ldots t_s\}, \{h_1, \ldots h_m\}) \leq \psi_{\acute{R}}(\{\varphi(t_1), \ldots, \varphi(t_s)\}, \{\varphi(h_1), \ldots, \varphi(h_m)\})\chi_R(\{t_1, \ldots t_s\}, \{h_1, \ldots h_m\}) \leq \chi_{\acute{R}}(\{\varphi(t_1), \ldots, \varphi(t_s)\}, \{\varphi(h_1), \ldots, \varphi(h_m)\})$ $\phi_R(\{t_1, \ldots t_s\}, \{h_1, \ldots h_m\}) \geq \phi_{\acute{R}}(\{\varphi(t_1), \ldots, \varphi(t_s)\}, \{\varphi(h_1), \ldots, \varphi(h_m)\})$, $\forall t_1, t_2, \ldots, t_s, h_1, h_2, \ldots, h_m \in X$

Definition 19: Let $\overrightarrow{\mathcal{H}}$ and $\overrightarrow{\acute{\mathcal{H}}}$ be PFDHGs. A weak Ismorphism is a bijective homomorphism $\varphi : X \longrightarrow \acute{X}$ satisfying

$$\wedge_{j=1}^r \psi_{\overrightarrow{\underline{\eta}_j}} = \wedge_{j=1}^r \phi_{\overrightarrow{\underline{\eta}_j}^*}(\varphi(x)),$$
$$\wedge_{j=1}^r \chi_{\overrightarrow{\underline{\eta}_j}} = \wedge_{j=1}^r \chi_{\overrightarrow{\underline{\eta}_j}^*}(\varphi(x)) \quad \text{and} \quad \vee_{j=1}^r \phi_{\overrightarrow{\underline{\eta}_j}} = \vee_{j=1}^r \phi_{\overrightarrow{\underline{\eta}_j}^*}(\varphi(x))$$

for all $x \in X$.

Definition 20: Let $\overrightarrow{\mathcal{H}}$ and $\overrightarrow{\acute{\mathcal{H}}}$ be PFDHGs. A co-weak Ismorphism is a bijective homomorphism $\varphi : X \longrightarrow \acute{X}$ satisfying

$$\psi_R(\{t_1, \ldots t_s\}, \{h_1, \ldots h_m\}) = \psi_{\acute{R}}(\{\varphi(t_1), ..., \varphi(t_s)\}, \quad \{\varphi(h_1), ..., \varphi(h_m)\})$$
$$\chi_R(\{t_1, \ldots t_s\}, \{h_1, \ldots h_m\}) = \chi_{\acute{R}}(\{\varphi(t_1), ..., \varphi(t_s)\}, \quad \{\varphi(h_1), ..., \varphi(h_m)\})$$
$$\phi_R(\{t_1, \ldots t_s\}, \{h_1, \ldots h_m\}) = \phi_{\acute{R}}(\{\varphi(t_1), ..., \varphi(t_s)\}, \quad \{\varphi(h_1), ..., \varphi(h_m)\})$$
$$\forall t_1, , ..., t_s, h_1, .., h_m \in X.$$

Definition 21: An isomorphism of PFDHGs $\overrightarrow{\mathcal{H}}$ and $\overrightarrow{\acute{\mathcal{H}}}$ is a bijective mapping $\varphi : \overrightarrow{\mathcal{H}} \longrightarrow \overrightarrow{\acute{\mathcal{H}}}$ that satisfies

1) $\wedge_{j=1}^r \psi_{\overrightarrow{\underline{\eta}_j}} = \wedge_{j=1}^r \phi_{\overrightarrow{\underline{\eta}_j}^*}(\varphi(x))$, $\wedge_{j=1}^r \chi_{\overrightarrow{\underline{\eta}_j}} = \wedge_{j=1}^r \chi_{\overrightarrow{\underline{\eta}_j}^*}(\varphi(x))$ and $\vee_{j=1}^r \phi_{\overrightarrow{\underline{\eta}_j}} = \vee_{j=1}^r \phi_{\overrightarrow{\underline{\eta}_j}^*}(\varphi(x))$, $\forall x \in X$
2) $\psi_R(\{t_1, \ldots t_s\}, \{h_1, \ldots h_m\}) = \psi_{\acute{R}}(\{\varphi(t_1), \ldots, \varphi(t_s)\}, \{\varphi(h_1), \ldots, \varphi(h_m)\})\chi_R(\{t_1, \ldots t_s\}, \{h_1, \ldots h_m\})$ $\chi_{\acute{R}}(\{\varphi(t_1), \ldots, \varphi(t_s)\}, \{\varphi(h_1), \ldots, \varphi(h_m)\})$ $\phi_R(\{t_1, \ldots t_s\}, \{h_1, \ldots h_m\}) = \phi_{\acute{R}}(\{\varphi(t_1), \ldots, \varphi(t_s)\}, \{\varphi(h_1), \ldots, \varphi(h_m)\})\forall t_1, t_2, \ldots, t_s, h_1, h_2, \ldots, h_m \in X$

Definition 22: Let $\overrightarrow{\mathcal{H}} = (P, R)$ be a PFDHG then order $O(\overrightarrow{\mathcal{H}})$ of $\mathcal{H}$ is

$$O(\overrightarrow{\mathcal{H}}) = \Big(\sum_{x \in X} \wedge_{j=1} \psi_{\overrightarrow{\underline{\eta}_j}}(x), \sum_{x \in X} \vee_{j=1} \chi_{\overrightarrow{\underline{\eta}_j}}(x), \sum_{x \in X} \vee_{j=1} \phi_{\overrightarrow{\underline{\eta}_j}}(x)\Big)$$

Definition 23: Let $\overrightarrow{\mathcal{H}} = (P, R)$ be a PFDHG then size $S(\overrightarrow{\mathcal{H}})$ of $\mathcal{H}$ is

$$S(\overrightarrow{\mathcal{H}}) = \Big(\sum\nolimits_{\overrightarrow{F_i} \in P^*} \psi_R(\overrightarrow{F_i}), \sum\nolimits_{\overrightarrow{F_i} \in P^*} \chi_R \overrightarrow{F_i}, \sum\nolimits_{\overrightarrow{F_i} \in P^*} \phi_R(\overrightarrow{F_i})\Big)$$

Theorem 1: $O(\overrightarrow{\mathcal{H}})$ and $S(\overrightarrow{\mathcal{H}})$ of isomorphic PFDHGs are same.

Proof: Let $\overrightarrow{\mathcal{H}_1} = (P_1, R_1)$ and $\overrightarrow{\mathcal{H}_2} = (P_2, R_2)$ be two PFDHGs on X_1 and X_2, respectively, where $P_1 = \{\underline{\eta}_{11}, \underline{\eta}_{12}, \cdots \underline{\eta}_{1r}\}$ and $P_2 = \{\underline{\eta}_{21}, \underline{\eta}_{22}, \cdots \underline{\eta}_{2r}\}$ be the classes of PF directed hyperarcs. Let $\varphi : X_1 \longrightarrow X_2$ be an isomorphism from $\overrightarrow{\mathcal{H}_1}$ to $\overrightarrow{\mathcal{H}_2}$ then using

$$O(\overrightarrow{\mathcal{H}_1}) = \Big(\sum_{x_1 \in X_1} \wedge_{j=1} \psi_{\overrightarrow{\underline{\eta}_{1j}}}(x_1), \sum_{x_1 \in X_1} \vee_{j=1} \chi_{\overrightarrow{\underline{\eta}_{1j}}}(x_1), \sum_{x_1 \in X_1} \vee_{j=1} \phi_{\overrightarrow{\underline{\eta}_{1j}}}(x_1)\Big)$$
$$= \Big(\sum_{x_1 \in X_1} \wedge_{j=1} \psi_{\overrightarrow{\underline{\eta}_{1j}}}(\varphi(x_1)), \sum_{x_1 \in X_1} \vee_{j=1} \chi_{\overrightarrow{\underline{\eta}_{1j}}}(\varphi(x_1)), \sum_{x_1 \in X_1} \vee_{j=1} \phi_{\overrightarrow{\underline{\eta}_{1j}}}(\varphi(x_1))\Big)$$
$$= \Big(\sum_{x_2 \in X_2} \wedge_{j=1} \psi_{\overrightarrow{\underline{\eta}_{1j}}}(\varphi(x_2)), \sum_{x_2 \in X_2} \vee_{j=1} \chi_{\overrightarrow{\underline{\eta}_{1j}}}(\varphi(x_2)), \sum_{x_2 \in X_2} \vee_{j=1} \phi_{\overrightarrow{\underline{\eta}_{1j}}}(\varphi(x_2))\Big) = O(\overrightarrow{\mathcal{H}_2})$$

$$S(\overrightarrow{\mathcal{H}_1}) = \Big(\sum_{\overrightarrow{F_{1i}} \in P_1^*} \psi_{R_1}(\overrightarrow{F_{1i}}), \sum_{\overrightarrow{F_{1i}} \in P_1^*} \chi_{R_1} \overrightarrow{F_{1i}}, \sum_{\overrightarrow{F_{1i}} \in P_1^*} \phi_{R_1}(\overrightarrow{F_{1i}})\Big)$$

$$= \Big(\sum_{\overrightarrow{F_{1i}} \in P_1^*} \psi_{R_2}(\varphi(\overrightarrow{F_{1i}})), \sum_{\overrightarrow{F_{1i}} \in P_1^*} \chi_{R_2}(\varphi(\overrightarrow{F_{1i}})), \sum_{\overrightarrow{F_{1i}} \in P_1^*} \phi_{R_2}(\varphi(\overrightarrow{F_{1i}})) \Big)$$
$$= \Big(\sum_{\overrightarrow{F_{2i}} \in P_2^*} \psi_{R_2}(\overrightarrow{F_{2i}}), \sum_{\overrightarrow{F_{2i}} \in P_2^*} \chi_{R_2} \overrightarrow{F_{2i}}, \sum_{\overrightarrow{F_{2i}} \in P_2^*} \phi_{R_2}(\overrightarrow{F_{2i}}) \Big)$$
$$= S(\overrightarrow{\mathcal{H}_2})$$

■

Remark 2: The $O(\overrightarrow{\mathcal{H}})$ and $S(\overrightarrow{\mathcal{H}})$ of weak and co-weak isomorphic PFDHGs, respectively are same.

Theorem 2: The isomorphism relation between PFDHGs is an equivalence relation.

Proof: Let $\overrightarrow{\mathcal{H}_1} = (P_1, R_1)$, $\overrightarrow{\mathcal{H}_2} = (P_2, R_2)$ and $\overrightarrow{\mathcal{H}_3} = (P_3, R_3)$ be three PFDHGs on X_1, X_2 and X_3, respectively, where $P_1 = \{\underline{\eta}_{11}, \underline{\eta}_{12}, \cdots \underline{\eta}_{1r}\}$, $P_2 = \{\underline{\eta}_{21}, \underline{\eta}_{22}, \cdots \underline{\eta}_{2r}\}$ and $P_3 = \{\underline{\eta}_{31}, \underline{\eta}_{32}, \cdots \underline{\eta}_{3r}\}$.

1) **Reflexive**
Let $P : X_1 \longrightarrow X_1$ is an isomorphism such that by $P(x_1) = x_1$, for all $x_1 \in X_1$. Then,
   a) $(\wedge_j \psi_{\overrightarrow{\underline{\eta}_{1j}}}(x_1), \vee_j \chi_{\overrightarrow{\underline{\eta}_{1j}}}(x_1), \vee_j \phi_{\overrightarrow{\underline{\eta}_{1j}}}(x_1))$ $= (\wedge_j \psi_{\overrightarrow{\underline{\eta}_{1j}}}(P(x_1)), \vee_j \chi_{\overrightarrow{\underline{\eta}_{1j}}}(P(x_1)), \vee_j \phi_{\overrightarrow{\underline{\eta}_{1j}}}(P(x_1)))$
   b) $(\psi_{R_1}(\overrightarrow{F_{1i}}), \chi_{R_1}\overrightarrow{F_{1i}}, \phi_{R_1}(\overrightarrow{F_{1i}})) = (\psi_{R_1}(P(\overrightarrow{F_{1i}})), \chi_{R_1} (P(\overrightarrow{F_{1i}})), \phi_{R_1}(P(\overrightarrow{F_{1i}}))), \forall x_1 \in X_1, t(\overrightarrow{F_{1i}}) \subset X_1, h(\overrightarrow{F_{1i}}) \subset X_1$.
   P is an isomorphism of a PFDHG to itself.
2) **Symmetric**
Let $\varphi : X_1 \longrightarrow X_2$ is an isomorphism such that $\varphi(x_1) = x_2$. Since, φ is a bijective therefore, $\varphi^{-1} : X_2 \longrightarrow X_1$ exists and $\varphi^{-}1(x_2) = x_1$, for all $x_2 \in X_2$. Then
$$(\wedge_{j=1} \psi_{\overrightarrow{\underline{\eta}_{2j}}}(x_2), \vee_j \chi_{\overrightarrow{\underline{\eta}_{2j}}}(x_2), \vee_j \phi_{\overrightarrow{\underline{\eta}_{2j}}}(x_2))$$
$$= (\wedge_j \psi_{\overrightarrow{\underline{\eta}_{2j}}}(\varphi(x_1)), \vee_j \chi_{\overrightarrow{\underline{\eta}_{2j}}}(\varphi(x_1)), \vee_j \phi_{\overrightarrow{\underline{\eta}_{2j}}}(\varphi(x_1)))$$
$$(\wedge_{j=1} \psi_{\overrightarrow{\underline{\eta}_{1j}}}(x_1), \vee_j \chi_{\overrightarrow{\underline{\eta}_{1j}}}(x_1), \vee_j \phi_{\overrightarrow{\underline{\eta}_{1j}}}(x_1))$$
$$= (\wedge_j \psi_{\overrightarrow{\underline{\eta}_{1j}}}(\varphi^{-}1(x_2)), \vee_j \chi_{\overrightarrow{\underline{\eta}_{1j}}}(\varphi^{-}1(x_2)), \vee_j \phi_{\overrightarrow{\underline{\eta}_{1j}}}(\varphi^{-}1(x_2)))$$
$$R_2(\overrightarrow{F_{2j}}) = R_2(\varphi(\overrightarrow{F_{1j}})) = R_1(\overrightarrow{F_{1j}})$$
$$= R_1(\varphi^{-}1(\overrightarrow{F_{2j}})), t(\overrightarrow{F_{2j}}) \subseteq X_2, h(\overrightarrow{F_{2j}}) \subseteq X_2.$$
Hence, $\varphi^{-}1$ is an isomorphism.
3) **Transitive**
Let $\varphi_1 : X_1 \longrightarrow X_2$ and $\varphi_2 : X_2 \longrightarrow X_3$ are two isomorphisms i.e. $\overrightarrow{\mathcal{H}_1}$ onto $\overrightarrow{\mathcal{H}_2}$ and $\overrightarrow{\mathcal{H}_2}$ onto $\overrightarrow{\mathcal{H}_3}$, which are defined as $\varphi_1(x_1) = x_2$ and $\varphi_2(x_2) = x_3$ respectively. By definition
$$(\wedge_j \psi_{\overrightarrow{\underline{\eta}_{1j}}}(x_1), \vee_j \chi_{\overrightarrow{\underline{\eta}_{1j}}}(x_1), \vee_j \phi_{\overrightarrow{\underline{\eta}_{1j}}}(x_1))$$
$$= (\wedge_j \psi_{\overrightarrow{\underline{\eta}_{2j}}}(x_2), \vee_j \chi_{\overrightarrow{\underline{\eta}_{2j}}}(x_2), \vee_j \phi_{\overrightarrow{\underline{\eta}_{2j}}}(x_2))$$
$$= (\wedge_j \psi_{\overrightarrow{\underline{\eta}_{3j}}}\varphi_2((x_2)), \vee_j \chi_{\overrightarrow{\underline{\eta}_{3j}}}\varphi_2((x_2)), \vee_j \phi_{\overrightarrow{\underline{\eta}_{3j}}}\varphi_2((x_2)))$$
$$= (\wedge_j \psi_{\overrightarrow{\underline{\eta}_{3j}}}\varphi_2(\varphi_1((x_1))), \vee_j \chi_{\overrightarrow{\underline{\eta}_{3j}}}\varphi_2(\varphi_1((x_1))), \vee_j \phi_{\overrightarrow{\underline{\eta}_{3j}}}\varphi_2(\varphi_1((x_1))))$$
$$= (\wedge_j \psi_{\overrightarrow{\underline{\eta}_{3j}}}\varphi_2 \circ \varphi_1((x_1)), \vee_j \chi_{\overrightarrow{\underline{\eta}_{3j}}}\varphi_2 \circ \varphi_1((x_1)), \vee_j \phi_{\overrightarrow{\underline{\eta}_{3j}}}\varphi_2 \circ \varphi_1((x_1)))$$
$$R_1(\overrightarrow{F_{1j}}) = R_2(\varphi(\overrightarrow{F_{2j}})) = R_3(\varphi_2(\overrightarrow{F_{1j}}))$$
$$= R_3(\varphi_2 \circ \varphi_1(\overrightarrow{F_{1j}})),$$
where $\overrightarrow{F_{ij}} = (t(\overrightarrow{F_{ij}}), h(\overrightarrow{F_{ij}})), t(\overrightarrow{F_{ij}}) \subset X_1, h(\overrightarrow{F_{ij}}) \subset X_1$. Clearly, $\varphi_2 \circ \varphi_1$ is isomorphism from $\overrightarrow{\mathcal{H}_1}$ to $\overrightarrow{\mathcal{H}_3}$. Hence, isomorphism of PFDHGs is an equivalence relation.

■

Remark 3: The weak isomorphism relation between PFDHGs is a partial order relation.

IV. APPLICATIONS

A. APPLICATION OF PFDHGS IN DECISION MAKING

Now a days, many daily life problems have been solved by using various approaches such as FSs theory, FGs theory, probability etc. FGs and FHGs have also been utilized for modeling complex daily life problems with uncertainties. Here, we propose an algorithm based on PFHGs for solving the problem occurring in DM. A flowchart is also provided to the support of proposed algorithm.

ALGORITHM

1) Input the picture fuzzy values of all PFDHG edges.
2) Find the MD, neutral-MD and non-MD values of PFDH edges such that
$$\chi_R\{m, n\} \leq \bigwedge \{\chi_S(m), \chi_S(n)\}$$
$$\psi_R\{m, n\} \leq \bigwedge \{\psi_S(m), \psi_S(n)\}$$
$$\phi_R\{m, n\} \geq \bigvee \{\phi_S(m), \phi_S(n)\}.$$
3) Calculate the $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$-cuts $\tau_j^{((\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}))}$ of PFDH edges such that
$$\chi_t(p) \geq \tilde{\alpha}$$
$$\psi_t(p) \geq \tilde{\beta}$$
$$\phi_t(p) \leq \tilde{\gamma}$$
for all $t = 1, 2, \ldots, k$.
4) Determine the PFSs describing the parameters according to the decision maker satisfaction levels.

A flow chart given in Fig.2 further elaborates our proposed algorithm.

Let us consider a daily life problem in which a university student Mr. Z wants to buy a laptop manufactured by different companies such as Dell, HP, Apple etc. He needs to purchase a perfect laptop for his study according to his requirements. In order to purchase a perfect laptop, Mr. Z considers four companies namely $O = \{o_1, o_2, o_3, o_4\}$. By utilizing

TABLE 2. Incidence matrix.

$\mathcal{I}$	τ_1	τ_2	τ_3	τ_4
o_1	$\langle 0.6, 0.3, 0.1\rangle$	$\langle 0.3, 0.4, 0.2\rangle$	$\langle 0.9, 0.05, 0.03\rangle$	$\langle 0.7, 0.1, 0.1\rangle$
o_2	$\langle 0.7, 0.1, 0.1\rangle$	$\langle 0, 0, 0\rangle$	$\langle 0, 0, 0\rangle$	$\langle 0.5, 0.1, 0.1\rangle$
o_3	$\langle 0, 0, 0\rangle$	$\langle 0.5, 0.1, 0.3\rangle$	$\langle 0.5, 0.4, 0.1\rangle$	$\langle 0.1, 0.8, 0.1\rangle$
o_4	$\langle 0, 0, 0\rangle$	$\langle 0, 0, 0\rangle$	$\langle 0.7, 0.1, 0.2\rangle$	$\langle 0.4, 0.3, 0.3\rangle$

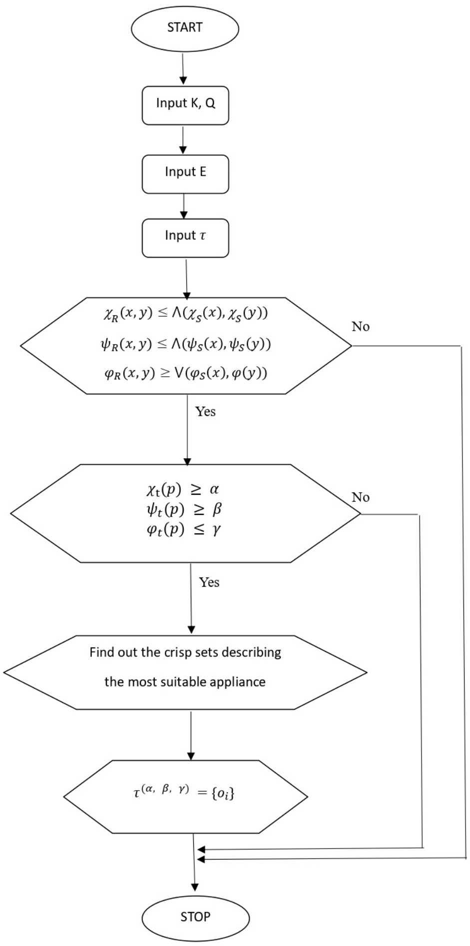

FIGURE 2. Flowchart.

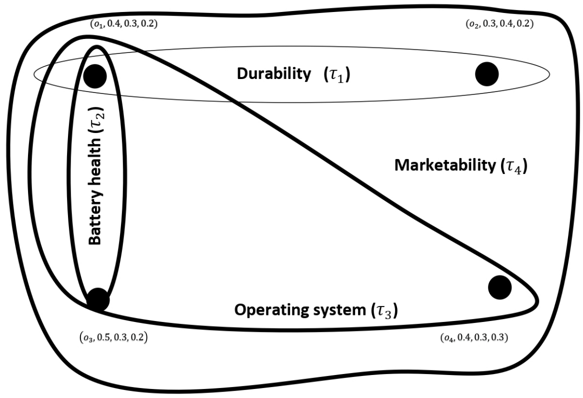

FIGURE 3. Picture fuzzy hypergraph.

$\langle\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}\rangle$-level hyper cuts in PFDHG, we can determine the best option to purchase.

Consider the vertices of a PFDHG as companies, with hyper-edges representing the common features of those laptops belonging to the specific hyperedge as in Fig. 3.

For vertices, the MD represents the laptop companies and depicts the extent to which the laptop of that company fulfills the features required by a costumer, neutral-MD indicates how much the laptop features do not affect the customer's need, and non-MD reflects much the laptop lacks the desired features. These are also depicted in Table. 2.

The attributes are hyperedges $Y = \{\tau_1, \tau_2, \tau_3, \tau_4\}$ of PFDHGs, representing the features of different companies like durability τ_1, battery health τ_2, operating system τ_3 and marketability τ_4. Since τ_2 is considered as battery health, the MD values $(0.6, 0.3, 0.1)$ of τ_1 denote that 60% of the laptops from some company o_1 has a good battery health as per customer's demand, 30% of laptops have normal battery health which doesn't effect the customer's desire, and 10% of the company's laptops lack the battery health as per customers wish. Similarly, the values of all vertices describe the features of all laptops manufactured by various companies.

Further, in choosing a perfect laptop, we determine the $\langle\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}\rangle$-level hypercuts of all hyperedges. Then by selecting the specific values $\tilde{\alpha}, \tilde{\beta}$ and $\tilde{\gamma}$ by the customer, the laptop of his desired features will be considered. Assume that the costumers desired figures are given by $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}) = (0.5, 0.1, 0.1)$. It means the customer will be satisfied if the laptop has 50% or more of the mentioned features, 10% of the laptop features have no effect on customer's demand, and 10% of the features are deficient in the laptop. The $\langle\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}\rangle$-level sets of all hyperedges are in Table. 3.

Here, $\tau_1^{(0.5,0.1,0.1)}$-level set represents that laptops of company o_1 and o_2 are most durable compared to the laptops of other companies, $\tau_2^{(0.5,0.1,0.1)}$-level set denotes that laptop of company o_3 has the best battery health, $\tau_3^{(0.5,0.1,0.1)}$-level set denotes that the company's laptop o_3 has best operating system and $\tau_4^{(0.5,0.1,0.1)}$-level set denotes that laptops of companies o_1 and o_2 have the best marketability.

TABLE 3. $\langle \widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma} \rangle$-level sets.

$\tau^{(0.5,0.1,0.1)}$-level sets	Appliances
$\tau_1^{(0.5,0.1,0.1)}$	o_1, o_2
$\tau_2^{(0.5,0.1,0.1)}$	o_3
$\tau_3^{(0.5,0.1,0.1)}$	o_3
$\tau_4^{(0.5,0.1,0.1)}$	o_1, o_2

Thus, by considering different $\langle \widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma} \rangle$-level hypercuts corresponding to the needs of Mr. Z, the most suitable laptop fulfilling his demands can be selected.

B. MANAGEMENT OF HAZARDOUS CHEMICALS THROUGH PFDHGS

Medical field is one of important field of science, with its main aim being the study of living beings. In this context, the first step to diagnose the disease and then search for its treatment. No doubt, chemicals play a crucial role in all areas of medical science. The primary concern of the scientists is how to preserve various chemicals. For this purpose, scientists make efforts to store hazardous chemicals ($\mathcal{HC}s$) in a safe and suitable environment. Many studies have been conducted on this matter, and many effective ways for handling $\mathcal{HC}s$ have been suggested overtime. Under certain complex circumstances, the traditional methods of storing the chemicals in a safe environment are no longer effective. So, there is a need for new methods to avoid the adverse effects of certain chemicals on living beings. As $\mathcal{HC}s$ pose a threat to the environment and living beings. So, some $\mathcal{HC}s$ like sludges, solids, liquids, gases etc can be produced during different activities/experiments. Thus, if such chemicals are not properly handled, then they can cause serious health issues like cancer, nerve damage, birth defects etc. Hence to ensure well being of living beings, protection of environment, the proper management for dumping $\mathcal{HC}s$ is extremely important. We also encounter uncertainties in almost every procedure for preserving $\mathcal{HC}s$. Since, FSs has been proven the best tool for dealing with vague and uncertain situations. Evidently, PFSs being the most generalized form of FSs with MD, neutral-MD and non-MD values of any entity, can play a crucial role in properly preserving $\mathcal{HC}s$. Consequentially, PFDHGs would be an efficient tool for managing the system of storing $\mathcal{HC}s$. In this context, we propose a PFDHG model for some compatible and incompatible elements, as shown in Fig. 4. The $\mathcal{HC}s$ discussed in our study are elaborated in Table. 4.

The oval vertices $G = \{G_1, G_2, G_3, G_4, G_5\}$ of a PFDHG represent the elements adjacent to them. Details of these vertices are provided in Table. 5. Fill containers upto 75% capacity for cost efficient and safe management of wastes.

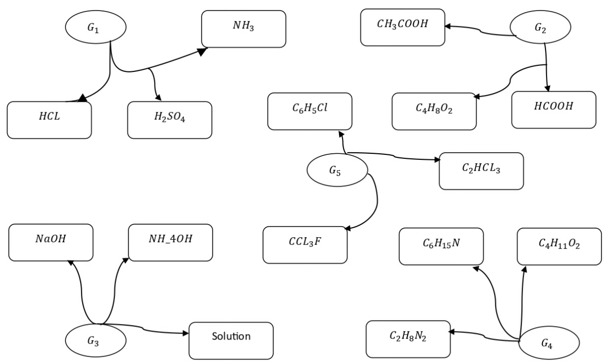

FIGURE 4. Directed picture fuzzy hypergraph model.

Additionally, the quality of containers material should be such that it is compatible with the $\mathcal{HC}s$ stored in it. As we know that when two or more different elements are combined sometimes it may result in severe reaction if these elements are not compatible to each other. No reaction occur if they are compatible to each other. So only those elements that are compatible and are safe to store together are joined using PF directed hyperedges. For the perfect and safe management of waste one must be aware of the properties of these $\mathcal{HC}s$ like toxicity, corrosivity, reactivity etc. A PFS can be used to express the characteristic of any element in a best possible way in terms of MD, neutral-MD and non-MD. So, here we consider corrosivity of elements and express them using PFSs as

$$D = \{(HNO_3, 65, 10, 25), (H_2SO_4, 65, 10, 25), (HCl, 65, 10, 25), (CH_3COOH, 85, 05, 10), (C_4H_8O_2, 85, 05, 10), (HCOOH, 85, 05, 10), (NaOH, 80, 08, 12), (NH_4OH, 80, 08, 12), (Solution, 80, 08, 12), (C_6H_{15}N, 75, 10, 15), (C_4H_{11}O_2, 70, 10, 20), (C_2H_8N_2, 80, 15, 05), (C_6H_5Cl, 65, 15, 20), (C_2HCl_3, 65, 15, 20), (CCl_3F, 75, 15, 10)\}$$

Table. 6 describes the importance of defining this picture fuzzy set.

The material used for making containers must also be compatible with the waste stored in them, and they must be non-leaking. Avoid using containers made of

TABLE 4. Chemicals.

Compounds	Formula
Nitric Acid	HNO_3
Sulphuric Acid	H_2SO_4
Hydrochloric Acid	HCL
Acitic Acid	CH_3COOH
Butyric Acid	$C_4H_8O_2$
Formic Acid	$HCOOH$
Sodium Hydroxide	$NaOH$
Ammonium Hydroxide	NH_4OH
Solution	
Triethylamine	$C_6H_{15}N$
Diethanolamine	$C_4H_{11}O_2$
Ethylenediamine	$C_2H_8N_2$
Chlorobenzene	C_6H_5Cl
Trichloroethylene	C_2HCL_3
Trichlorofluoromethane	CCl_3F

TABLE 5. Oval Vertices Discription.

	Category	Membership values	Proficiency (%)	Impartialness (%)	Ineptness (%)
G_1	Inorganic Acids	(0.65, 0.20, 0.15)	65%	20%	15%
G_2	Organic Acids	(0.75, 0.15, 0.1)	75%	15%	1%
G_3	Caustics	(0.50, 0.25, 0.25)	50%	25%	25%
G_4	Amines and Alkanolamines	(0.70, 0.25, 0.15)	70%	25%	15%
G_5	Halogenated Compounds	(0.65, 0.10, 0.25)	65%	10%	25%

TABLE 6. Square vertices discription.

Square Vertices	Highly Corrosive (%)	Fortifying (%)	Slightly Corrosive (%)
HNO_3	65	10	25
H_2SO_4	65	10	25
HCL	65	10	25
CH_3COOH	85	05	10
$C_4H_8O_2$	85	05	10
$HCOOH$	85	05	10
$NaOH$	80	08	12
NH_4OH	80	08	12
Solution	80	08	12
$C_6H_{15}N$	75	10	15
$C_4H_{11}O_2$	70	10	20
$C_2H_8N_2$	80	15	05
C_6H_5Cl	65	15	20
C_2HCL_3	65	15	20
CCl_3F	75	15	10

incompatible material for storing such $\mathcal{HC}s$, For instance, avoid using metal containers for keeping acids, polyethylene containers(light weighted) for storing solvents, and glass containers for storing HCl. Thus, one should ensure that containers must be highly compatible with chemicals stored in them. Now, we consider a containers/cabinets set $C = \{C_1, C_2, C_3, C_4, C_5\}$, and define five PFSs on it, keeping in mind their compatibility with these elements. The MD, neutral-MD and non-MD of these PFSs refer to highly compatible, slightly compatible and incompatible, respectively. For example, the MDs $C_1(G_2) = (0.310, 0.220, 0.455)$ imply that the C_1 container is made up of such material that is incompatible for storing inorganic acids and suitable for storing organic acids, as $C_1(G_1) = (0.650, 0.230, 0.100)$. Similarly, using the same assumptions, we define other PFSs as given in Table. 7.

It can be noted from Table. 7 that container C_1 is THE best choice to store inorganic acids or halogenated compounds, but it will be used to store inorganic acids because it is highly compatible to them i.e., 65%. Likewise, the quality of material of C_2 is compatible with organic acids, caustics, and halogenated compounds but it will be used to store organic acids because the MD is greatest in this case. Similarly, we find that C_3, C_4 and C_5 are ideal containers for storing amines and alkanolamines, halogenated compounds and caustics, respectively. The results are demonstrated in Table. 8. The graphical representations of these storages are shown in Fig.5. Thus, by utilizing the proposed model, $\mathcal{HC}s$ can be

TABLE 7. Compatibility and incompatibility levels of containers to chemical substances.

C	G_1	G_2	G_3	G_4	G_5
C_1	(0.650,0.230, 0.100)	(0.310,0.220, 0.455)	(0.220,0.110, 0.650)	(0.010,0.330, 0.590)	(0.400,0.100, 0.500)
C_2	(0.100,0.530, 0.320)	(0.650,0.230, 0.100)	(0.230,0.120, 0.560)	(0.150,0.310, 0.550)	(0.220,0.650, 0.110)
C_3	(0.220,0.110, 0.650)	(0.220,0.110, 0.650)	(0.130,0.420, 0.455)	(0.620,0.110, 0.250)	(0.550,0.055, 0.420)
C_4	(0.310,0.220, 0.455)	(0.310,0.220, 0.455)	(0.920,0.010, 0.050)	(0.320,0.410, 0.450)	(0.890,0.099, 0.150)
C_5	(0.001,0.250, 0.690)	(0.220,0.110, 0.650)	(0.650,0.230, 0.200)	(0.120,0.210, 0.650)	(0.900,0.010, 0.050)

TABLE 8. Results.

Containers	Chemicals
C_1	Inorganic acids
C_2	Organic acids
C_3	Amines & akanolamines
C_4	Halogenated compounds
C_5	Caustics

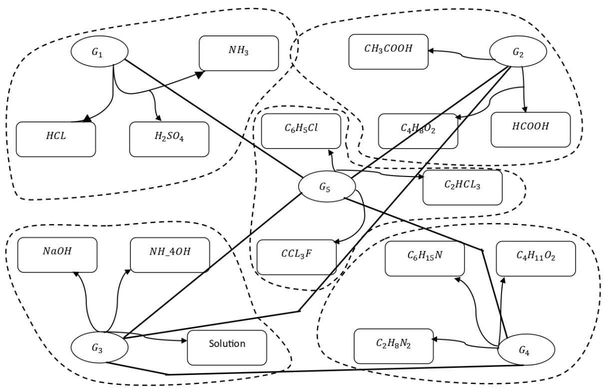

FIGURE 5. Graphical expression for the storages of chemicals.

systemized in a more appropriate way to reduce their precarious risks to the health of living beings and the environment.

V. COMPARATIVE ANALYSIS AND SUPERIORITY

Zadeh initiated the idea of FSs in 1965. Since then FSs has been generalized in many different ways. Some famous extensions of FSs are IFSs, PFSs, BPFSs etc. These generalizations of FSs have their own importance under specific environments. IFSs and PFSs are the main direct extensions of FSs and among the other extensions. Basically, FSs use only membership degree to express each of its entity, IFSs has a MD and non-MD for each of its element, but PFSs allocate MD, neutral-MD and non-MD to each of its entity. Consequently, PFSs become more generalized form of FSs. However, PFSs can be reduced to IFSs and FSs by choosing different MDs. Thus the PFSs is the best tool to deal the problems with uncertainties. Based on these generalizations, the concepts of FGs, IFGs an PFGs have been added in the literature. It has also proven that PFGs are more useful as compared to IFGs and FGs. PFGs can model the complex problems which cannot be modeled through FGs and IFGs. Likewise, the notion of PFHGs has its own importance and appear more useful in modeling the real world problems with uncertainties. The notions of FDHGs and IFDHGs have been described in the literature and also applied to different circumstances. However, sometimes to handle complex problems through these are impossible. Hence, we need a vast structure like PFDHGs with more options to address such situations. Our proposed structure i.e., PFDHGs is more flexible and easy to apply as compared to the others found in [33], [34], [35], [36], [37], and [38]. In Table.9, we provide the details about some existing FHGs and the notions associated with them. Overall, we have observed that our newly established structure appears more useful than the existing ones, which proves its superiority.

TABLE 9. Generalizations of fuzzy graphs.

Authors	Reference	Notions introduced
Akram and Luqman	[36]	Hypergraphs in intuitionistic fuzzy environment
Luqman et al.	[34]	q-Rung orthopair fuzzy directed hypergraphs
Luqman et al.	[38]	q-Rung orthopair fuzzy hypergraphs
Samantha and Pal	[37]	Bipolar fuzzy hypergraphs
Akram	[35]	m-Polar Fuzzy Hypergraphs
Akram and Luqman	[39]	Bipolar neutrosophic directed hypergraphs
Khan et al.	[33]	Interval-valued picture fuzzy hypergraphs

VI. CONCLUSION

FSs and its generalizations were used widely to solve different types of daily life problems. Likewise, FGs and its generalizations were used to model real world problems. PFSs is an extension of FSs, three values MD, neutral-MD and non-MD were allocated to each of its entity. FHGs and its extensions were used to study more complex problems. In this manuscript, we have introduced a direct extension of FDHGs named PFDHGs. We have also provided its applications towards decision making theory and in managing the hazardous chemicals. In the start, we have introduced the concept of PFDHGs along with different useful terminologies like hyper-edge, height etc related to it. Afterwards, we have initiated various types of PFDHGs such as simple PFDHGs, support simple PFDHGs etc. We introduce the notion of $(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma})$-level hyperarcs and describe the term fundamental sequence in PFDHGs based on these arcs. We have also initiated the ideas of elementary PFDHGs, partial PFDHGs, ordered PFDHGs etc and explored few relations among them. Some structural properties of PFDHGS have also been explored by introducing different kinds of homomorphisms on PFDHGs. The terms order and size of PFDHGs have been examined through homomorphism of PFDHGs. To ensure the impact of this study, we have provided two different applications of PFDHGs. The first application of PFDHGs is related to decision making, in which we have proposed a case study for buying the best laptop. Secondly, we proposed an affective and appropriate method for storing the $\mathcal{HC}s$ in a most affective and appropriate way. Furthermore, one could extend the ideas presented in this study towards other extensions of FHGs like T-spherical fuzzy hypergraphs, T-spherical directed fuzzy hypergraphs, bipolar picture fuzzy hypergraphs etc.

CONFLICT OF INTEREST

The authors declare that they have no conflict of interest regarding the publication of this article.

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